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Finite and Infinite Dimensional Complex Geometry and Representation Theory

THE GENERALIZED CAYLEY MAP FROM A LIE GROUP TO ITS LIE ALGEBRA

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This talk is mainly based on the paper [5].

Let $\pi : G \to \mathrm{End}(V)$ be an infinitesimally faithful complex representation of a connected Lie group G. Consider $(A, B) \mapsto \mathrm{tr}(AB)$ on $\mathrm{End}(V)$ and suppose that it is non-degenerate on the linear subspace $\pi'(\mathfrak{g}) \subseteq \mathrm{End}(V)$. Then the orthogonal projection $\mathrm{pr}_\pi : \mathrm{End}(V) \to \pi'(\mathfrak{g})$ is defined:

$$\begin{array}{ccc} G & \xrightarrow{\text{representation } \pi} & \mathrm{End}(V) \\ \Phi_\pi \Big\downarrow \text{ Cayley map} & & \mathrm{pr}_\pi \Big\downarrow \text{ orthoproj.} \\ \mathfrak{g} & \xrightarrow{\text{infinites. repr. } \pi'} & \pi'(\mathfrak{g}) \end{array} \qquad \begin{aligned} \Psi_\pi(g) &= \Psi(g) \\ &:= \det(d\Phi(g)) \end{aligned}$$

The Cayley mapping Φ has the following simple properties:

(1) $\Phi(bxb^{-1}) = \mathrm{Ad}_b(\Phi(x))$.
(2) We have $\Phi(g) \in \mathrm{Cent}(\mathfrak{g}^g) \subset Z_{\mathfrak{g}}(\mathfrak{g}^g)$.
(3) $d\Phi(e) : \mathfrak{g} \to \mathfrak{g}$ is the identity mapping.
(4) $H \subset G$ be a Cartan subgroup with Cartan algebra $\mathfrak{h} \subset \mathfrak{g}$. Then $\Phi(H) \subset \mathfrak{h}$.
(5) For the character $\chi_\pi(g) = \mathrm{tr}(\pi(g))$ of π we have
$d\chi_\pi(g)(T_e(\mu_g)X) = \mathrm{tr}(\pi'(\Phi_\pi(g))\pi'(X))$

Further results are:

• Let $\pi : G \to \mathrm{Aut}(V)$ be a representation admitting a Cayley mapping. Let $H = (\bigcap_{a\in A} G^a)_o = (G^A)_o \subseteq G$ be a subgroup which is the connected centralizer of a subset $A \subseteq G$ and suppose that H is itself reductive. Then $\pi|H : H \to \mathrm{End}(V)$ admits a Cayley mapping and $\Phi_\pi|H = \Phi_{\pi|H} : H \to \mathfrak{h}$.

• Let G be a semisimple real or complex Lie group, let $\pi : G \to \mathrm{Aut}(V)$ be an infinitesimally effective representation. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ be the decomposition into the simple ideals $\mathfrak{g}_i$. Let $G_1, \dots, G_k$ be the corresponding connected subgroups of G. Then $\Phi_\pi|G_i = \Phi_{\pi|G_i}$ for $i = 1, \dots, k$.

• G a simple Lie group, for direct sum and tensor product representations

$$\Phi_{\pi_1\oplus\pi_2}(g) = \frac{j_{\pi_1}}{j_{\pi_1\oplus\pi_2}}\,\Phi_{\pi_1}(g) + \frac{j_{\pi_2}}{j_{\pi_1\oplus\pi_2}}\,\Phi_{\pi_2}(g) \in \mathfrak{g}.$$

$$\Phi_{\pi_1\otimes\pi_2}(g) = \frac{j_{\pi_1}\chi_{\pi_2}(g)}{j_{\pi_1\otimes\pi_2}}\,\Phi_{\pi_1}(g) + \frac{\chi_{\pi_1}(g) j_{\pi_2}}{j_{\pi_1\otimes\pi_2}}\,\Phi_{\pi_2}(g) \in \mathfrak{g}.$$

Results for algebraic groups. Now let G be a reductive complex algebraic group and π a rational representation. We have $A(\mathfrak{g}) = A(\mathfrak{g})^G \otimes \operatorname{Harm}(\mathfrak{g})$ by [Kostant, 1963], where $\operatorname{Harm}(\mathfrak{g})$ is the space of all regular functions killed by all invariant differential operators with constant coefficients. We define $\operatorname{Harm}_\pi(G) := \Phi_\pi^*(\operatorname{Harm}(\mathfrak{g}))$. It is a G-module.
• For the localization at Ψ we have $A(G)_\Psi = A(G)_\Psi^G \otimes \operatorname{Harm}_\pi(G)$. Moreover, we have $A(G) = A(G)^G \otimes \operatorname{Harm}_\pi(G)$ if and only if $\Phi : G \to \mathfrak{g}$ maps regular orbits in G to regular orbits in $\mathfrak{g}$.
• If $\Phi(e) = 0 \in \mathfrak{g}$ then for the G-equivariant extension of the rational function fields $\Phi^* : Q(\mathfrak{g}) \to Q(G)$ the degrees satisfy $[Q(G) : Q(\mathfrak{g})] = [Q(G)^G : Q(\mathfrak{g})^G]$.
• Let $a \in G$ be regular. Assume that $d\Phi(a)$ is invertible. Then Φ restricts to an isomorphism $\Phi : \overline{\operatorname{Conj}_G(a)} \to \overline{\operatorname{Ad}_G(\Phi(a))}$ of affine varieties.
• Let $a \in G$. Then for the semisimple parts we have $\Phi(a_s) = \Phi(a)_s$ and $\Phi(a) = \Phi(a_s) + \Phi(a)_n \in \mathfrak{g}^a$ is the Jordan decomposition.
• Let G be a connected reductive complex algebraic group and let $\Phi : G \to \mathfrak{g}$ be the Cayley mapping of a rational representation with $\Phi(e) = 0$. Then $\Phi : G_{\text{pos}} \to \mathfrak{g}_{\text{real}}$ is bijective and a fiber respecting isomorphism of real algebraic varieties, where G_{pos} is the set of all $a \in G$ whose semisimple part has positive eigenvalues, and $\mathfrak{g}_{\text{real}}$ is the set of all $X \in \mathfrak{g}$ whose semisimple part has only real eigenvalues.

Relation to the classical Cayley mapping. Let $T : \operatorname{Spin}(n, \mathbb{C}) \to SO(n, \mathbb{C})$ be the double cover. We consider the spin representation $\operatorname{Spin} : \operatorname{Spin}(n, \mathbb{C}) \to \operatorname{Aut}(S_n)$.
• There is a choice of the sign of the square root so that $\chi(g) := \sqrt{\det(1 + T(g))}$ satisfies

$$\Phi_{\text{Spin}}(g) = -\frac{2}{2^{n/2}}\,\chi(g)\,\Gamma(T(g)) \in \mathfrak{so}(n, \mathbb{C}).$$

for all $g \in \operatorname{Spin}(n, \mathbb{C})$. Moreover, $\chi \in A(\operatorname{Spin}(n, \mathbb{C}))$ and we have for the rational function fields

$$Q(\operatorname{Spin}(n))^{\operatorname{Spin}(n)} = Q(\mathfrak{so}(n, \mathbb{C}))^{\operatorname{Spin}(n)}[\chi],$$
$$Q(\operatorname{Spin}(n)) = Q(\mathfrak{so}(n, \mathbb{C}))[\chi].$$

Thus the generalized Cayley mapping $\Phi_{\text{Spin}} : \operatorname{Spin}(n, \mathbb{C}) \to \mathfrak{so}(n, \mathbb{C})$ factors to the classical Cayley transform $\Gamma : SO(n, \mathbb{C})^* \to \operatorname{Lie}\operatorname{Spin}(n, \mathbb{C})^{(*)}$, up to multiplication by a function, via the natural identifications.

Relation to Poisson structures. For a representation π of a Lie group G we can try to pull back the Poisson structure on $\mathfrak{g}^*$ via the derivative of the character $d\chi_\pi : G \to \mathfrak{g}^*$. This pullback is a rational Poisson structure on G which in fact is an integrable Dirac structure in the sense of [1], [2], [3]. Let us explain this a little:

Let M be a smooth manifold of dimension m. A *Dirac structure* on M is a vector subbundle $D \subset TM \times_M T^*M$ with the following two properties:

(1) Each fiber D_x is maximally isotropic with respect to the metric of signature (m, m) on $TM \times_M T^*M$ given by $\langle (X, \alpha), (X', \alpha') \rangle_+ = \alpha(X') + \alpha'(X)$. So D is of fiber dimension m.
(2) The space of sections of D is closed under the non-skew-symmetric version of the Courant-bracket $[(X, \alpha), (X', \alpha')] = ([X, X'], \mathcal{L}_X\alpha' - i_{X'}d\alpha)$.

Natural examples of Dirac structures are the following: Symplectic structures ω on M, where $D = D^\omega = \{(X, \omega(X)) : X \in TM\}$ is just the graph of $\omega : TM \to T^*M$; these are precisely the Dirac structures D with $TM \cap D = \{0\}$. Poisson structures P on M where $D = D^P = \{(P(\alpha), \alpha) : \alpha \in T^*M\}$ is the graph of $P : T^*M \to TM$; these are precisely the Dirac structures D which are transversal to T^*M.

Given a Dirac structure D on M we consider its range $R(D) = \mathrm{pr}_{TM}(D) = \{X \in TM : (X, \alpha) \in D \text{ for some } \alpha \in T^*M\}$. There is a skew symmetric 2-form Θ_D on $R(D)$ which is given by $\Theta_D(X, X') = \alpha(X')$ where $\alpha \in T^*M$ is such that $(X, \alpha) \in D$. The range $R(D)$ is an integrable distribution of non-constant rank in the sense of Stefan and Sussmann, see [4], so M is foliated into maximal integral submanifolds L of $R(D)$ of varying dimension, which are all initial submanifolds. The form Θ_D induces a closed 2-form on each leaf L and (L, Θ_D) is thus a presymplectic manifold (Θ_D might be degenerate on L). If the Dirac structure corresponds to a Poisson structure then the (L, Θ_D) are exactly the symplectic leaves of the Poisson structure.

The main advantage of Dirac structures is that one can apply arbitrary push forwards and pull backs to them. So if $f : N \to M$ is a smooth mapping and D_M is a Dirac structure on M then the pull back is defined by $f^*D_M = \{(X, f^*\alpha) \in TN \times_N T^*N : (Tf.X, \alpha) \in D_M\}$. Likewise the push forward of a Dirac structure D_N on N is given by $f_*D_N = \{(Tf.X, \alpha) \in TM \times_M T^*M : (X, f^*\alpha) \in D_N\}$.

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