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A COMMON GENERALIZATION OF THE FRÖLICHER-NIJENHUIS BRACKET AND THE SCHOUTEN BRACKET FOR SYMMETRIC MULTIVECTOR FIELDS

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Abstract. There is a canonical mapping from the space of sections of the bundle $\Lambda T^*M \otimes STM$ to $\Omega(T^*M; T(T^*M))$. It is shown that this is a homomorphism on $\Omega(M; TM)$ for the Frölicher-Nijenhuis brackets, and also on $\Gamma(STM)$ for the Schouten bracket of symmetric multivector fields. But the whole image is not a subalgebra for the Frölicher-Nijenhuis bracket on $\Omega(T^*M; T(T^*M))$.

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1. Introduction

It is well known that there are several extensions of the bracket of vector fields on a smooth manifold M. In particular, the Frölicher-Nijenhuis bracket extends the bracket of vector fields to all vector valued differential forms on M, i.e. to $\Omega(M; TM)$. Another classical extension is the Schouten bracket, this is an extension of the bracket of vector fields to all symmetric multivector fields, i.e. to $\Gamma(STM)$. The Schouten bracket has a natural interpretation in terms of Poisson bracket. Indeed, there is an obvious isomorphism π^* of the algebra $\Gamma(STM)$ on the algebra of smooth functions on T^*M which are polynomial on the fiber. On the other hand there is a natural symplectic structure on T^*M and the Schouten bracket corresponds just to the Poisson bracket under the above isomorphism.

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It is very natural, and it is the aim of this paper, to try to find a common generalization of the two above brackets.

Let us give an example of problem where such an extension could be welcome. Suppose that M is equipped with a Riemannian metric g and let $\underline{g}$ denote the corresponding contravariant symmetric two-tensor field. Then, $\pi^*(\underline{g})$ is the Hamiltonian of the geodesic flow on T^*M and the symmetric tensor fields S satisfying $[\underline{g}, S] = 0$ correspond to functions on T^*M which are invariant by the geodesic flow; such symmetric tensor fields are called Killing tensors. These Killing tensors form a Poisson subalgebra of $\Gamma(STM)$. Now if S is a Killing tensor of order k; then it is not hard to show that only its covariant derivatives of order not greater than k are independent, i.e. its covariant derivatives of order greater than k are linear combination of those of order not greater than k with coefficients which are covariant expressions in the curvature tensor. This implies in particular that the equations $[\underline{g}, S] = 0$ have a lot of integrability conditions and, since these integrability conditions are always consequence of $d^2 = 0$, it is natural to introduce the algebra $\Omega(M; STM)$ of symmetric multivector valued forms to analyse them. This algebra is a graded-commutative algebra for the graduation given by the form-degree and on this algebra there is a nice algebra of graded derivations associated with the metric. It is generated by three antiderivations, $\nabla, \delta_g, \delta'_g$, where ∇ is the exterior covariant differential corresponding to the Levi-Civita connection, δ_g is the unique $C^\infty(M)$-linear antiderivation such that $\delta_g X \in \Omega^1(M)$ for $X \in \Gamma(TM)$ is the one-form $Y \mapsto \delta_g X(Y) = g(X, Y)$ and $\delta_g \Omega^1(M) = 0$, δ'_g is the unique $C^\infty(M)$-linear antiderivation such that $\delta'_g \Gamma(TM) = 0$ and $\delta'_g \omega \in \Gamma(TM)$ for $\omega \in \Omega^1(M)$ is the vector field obtained by the contraction of $\underline{g}$ with ω. One has: $\delta_g^2 = 0$, ${\delta'}_g^2 = 0$, $\delta_g \delta'_g + \delta'_g \delta_g$ equals the total degree in form and tensor, $\nabla \delta_g + \delta_g \nabla = 0$ (because ∇ is torsion free) and the derivation $D = \nabla \delta'_g + \delta'_g \nabla$ is an extension to $\Omega(M; STM)$ of the Schouten bracket with $\underline{g}$. So it is natural to try to construct a bracket on $\Omega(M; STM)$ extending the Schouten bracket for which D is the bracket with $\underline{g}$. It is not difficult to construct such a bracket namely

$$[\alpha \otimes F, \beta \otimes G]_\nabla = L^\nabla_{\alpha \otimes F}(\beta) G - (-1)^{ab} L^\nabla_{\beta \otimes G}(\alpha) F + \alpha \wedge \beta \otimes [F, G]$$

for $\alpha \in \Omega^a(M)$, $\beta \in \Omega^b(M)$, F, $G \in \Gamma(STM)$ with

$$L^\nabla_{\alpha \otimes F}(\omega) = i_{\alpha \otimes F} \nabla \omega + (-1)^a \nabla i_{\alpha \otimes F} \omega$$

for $\omega \in \Omega(M)$ and where the generalised insertion i is defined by

$$i_{\alpha \otimes X_1 \vee \dots \vee X_k}(\omega) = \sum_{r=1}^{k} \alpha \wedge i_{X_r}(\omega) \otimes X_1 \vee \dots \widehat{X_r} \dots \vee X_r$$

(the hat meaning omission of this element).

More generally if ∇ is any torsion free linear connection on M, the above formula defines a bracket $[\quad,\quad]_\nabla$ which is an extension to $\Omega(M; STM)$ of both the Schouten bracket and the Frölicher-Nijenhuis bracket. Furthermore, this bracket is a graded derivation in each variable, it is also graded antisymmetric but unfortunately it does not satisfy the graded Jacobi identity.

In this paper we shall follow another way: we first send $\Omega(M;STM)$ in $\Omega(T^*M)$ by using the isomorphism π^*, then we use a construction introduced by one of us [5] to send it in $\Omega(T^*M;T(T^*M))$ in which there is the Frölicher-Nijenhuis bracket and we show that this gives injective homomorphisms of graded Lie algebras for the Frölicher-Nijenhuis bracket on $\Omega(M;TM)$ and the Schouten bracket on $\Gamma(STM)$. But the common generalization of these two brackets does not exist on the space $\Omega(M;STM)$, only on $\Omega(T^*M;T(T^*M))$. This is similar to the common generalization found by Vinogradov [14,1] of the Frölicher-Nijenhuis bracket and the skew symmetric Schouten bracket on $\Gamma(\Lambda TM)$, which exist only on a quotient of a certain space of 'superdifferential operators' on $\Omega(M)$.

2. The Poisson bracket for differential forms

2.1. Frölicher-Nijenhuis bracket. For the convenience of the reader we review here the theory of graded derivations of the graded commutative algebra of differential form on a smooth manifold M. See [2] and [3] for the original source, and [6] or [4], sections 8–11, as a convenient reference, whose notation we follow here.

The space $\operatorname{Der}(\Omega(M))$ of all graded derivations of the graded commutative algebra of differential forms on M is a graded Lie algebra with the graded commutator as bracket. In the following formulas we will always assume that $K \in \Omega^k(M;TM) = \Gamma(\Lambda^k T^*M \otimes TM)$, $L \in \Omega^l(M;TM)$, $\omega \in \Omega^q(M)$. The formula

$$(i_K\omega)(X_1,\dots,X_{k+q-1}) = \tfrac{1}{k!(q-1)!}\sum_\sigma \operatorname{sign}(\sigma)\omega(K(X_{\sigma1},\dots,X_{\sigma k}),X_{\sigma(k+1)},\dots)$$

for $X_i \in \mathfrak{X}(M)$ (or T_xM) defines an graded derivation $i_K \in \operatorname{Der}_{k-1}\Omega(M)$ and any derivation D with $D|\Omega^0(M)=0$ is of this form. On $\Omega^{*+1}(M,TM)$ (with the grading $*$) a graded Lie bracket is given by $[K,L]^\wedge = i_K L - (-1)^{(k-1)(l-1)} i_L K$ where $i_K(\psi\otimes X) := i_K(\psi)\otimes X$, which satifies $i([K,L]^\wedge) := [i_K,i_L]$. It is called the *Nijenhuis-Richardson bracket*, see [11].

The exterior derivative d is an element of $\operatorname{Der}_1\Omega(M)$. We define the *Lie derivation* $\mathcal{L}_K = \mathcal{L}(K) \in \operatorname{Der}_k\Omega(M)$ by $\mathcal{L}_K := [i_K,d]$. For any graded derivation $D \in \operatorname{Der}_k\Omega(M)$ there are unique $K \in \Omega^k(M;TM)$ and $L \in \Omega^{k+1}(M;TM)$ such that $D = \mathcal{L}_K + i_L$. We have $L=0$ if and only if $[D,d]=0$, and $D|\Omega^0(M)=0$ if and only if $K=0$. Clearly $[[\mathcal{L}_K,\mathcal{L}_L],d]=0$, so we have $[\mathcal{L}(K),\mathcal{L}(L)] = \mathcal{L}([K,L])$ for a uniquely defined $[K,L]\in\Omega^{k+l}(M;TM)$. This vector valued form $[K,L]$ is called the *Frölicher-Nijenhuis bracket* of K and L. It is well behaved with respect to the obvious relation of f-relatedness of tangent bundle valued differential forms, where $f: M\to N$. For $k=l=0$ it coincides with the Lie bracket of vector fields. Let the degree of ω be q, of φ be k, and of ψ be ℓ. Let the other degrees be as indicated. Then the following formulas hold

$$[\mathcal{L}_K, i_L] = i([K,L]) - (-1)^{k\ell}\mathcal{L}(i_L K) \tag{1}$$

$$i(\omega\wedge L) = \omega\wedge i(L). \tag{2}$$

$$\mathcal{L}(\omega\wedge K) = \omega\wedge\mathcal{L}_K - (-1)^{q+k-1} i(d\omega\wedge K). \tag{3}$$

$$\begin{aligned}[\omega\wedge K_1, K_2] = \omega\wedge[K_1,K_2] &- (-1)^{(q+k_1)k_2}\mathcal{L}(K_2)\omega\wedge K_1\\ &+ (-1)^{q+k_1} d\omega\wedge i(K_1)K_2.\end{aligned} \tag{4}$$

(5) $$[\varphi\otimes X,\psi\otimes Y]=\varphi\wedge\psi\otimes[X,Y]+\varphi\wedge\mathcal{L}_X\psi\otimes Y-\mathcal{L}_Y\varphi\wedge\psi\otimes X + (-1)^k\left(d\varphi\wedge i_X\psi\otimes Y+i_Y\varphi\wedge d\psi\otimes X\right).$$

2.2. Poisson manifolds. Let (M,ρ) be a Poisson manifold, that is a smooth manifold M together with a 2-field $\rho\in\Gamma(\Lambda^2TM)$ satisfying $[\rho,\rho]=0$, where $[\quad,\quad]$ is the Schouten-Nijenhuis bracket on $\Gamma(\Lambda^{*-1}TM)$, see [7] and [12]. Then ρ induces a skew symmetric differential concomitant on $C^\infty(M,\mathbb{R})$ given by $\{f,g\}_\rho=\rho(df,dg)$. The Jacobi identity for this bracket is equivalent to $[\rho,\rho]=0$, see [7], 1.4 for a nice proof. Here we view ρ as a skew symmetric bilinear form on T^*M, but also as a vector bundle homomorphism $\rho:T^*M\to TM$.

It is well known that for a symplectic manifold (M,ω) with associated Poisson structure $\rho=\omega^{-1}:T^*M\to TM$ we have the following exact sequence of Lie algebras:

(1) $$0\to H^0(M)\to C^\infty(M,\mathbb{R})\xrightarrow{H}\mathfrak{X}_\omega(M)\xrightarrow{\gamma}H^1(M)\to 0$$

Here $H^*(M)$ is the real De Rham cohomology of M, the space $C^\infty(M,\mathbb{R})$ is equipped with the Poisson bracket $\{\quad,\quad\}_\rho$, $\mathfrak{X}_\omega(M)$ consists of all vector fields ξ with $\mathcal{L}_\xi\omega=0$ (the locally Hamiltonian vector fields), which is a Lie algebra for the Lie bracket. Also H_f is the Hamiltonian vector field for $f\in C^\infty(M,\mathbb{R})$ given by $H_f=\rho(df)$, and $\gamma(\xi)$ is the cohomology class of $i_\xi\omega$. The spaces $H^0(M)$ and $H^1(M)$ are equipped with the zero bracket.

2.3. The graded Poisson bracket for differential forms. In [5] the exact sequence 2.2.(1) has been generalized in the following way. It was stated there for symplectic manifolds, but the proofs there work without any change also for Poisson manifolds.

We consider first the space $\Omega(M;TM)=\Gamma(\Lambda^*T^*M\otimes TM)$ of tangent bundle valued differential forms on M, equipped with the Frölicher-Nijenhuis bracket $[\quad,\quad]$. We first extend $\rho:T^*M\to TM$ to a module valued graded derivation of degree -1 by

(1) $$\begin{aligned}&\rho:\Omega(M)\to\Omega(M;TM),\\ &\rho|\Omega^0(M)=0,\qquad\text{and for }\varphi_i\in\Omega^1(M)\text{ by}\\ &\rho(\varphi_1\wedge\dots\wedge\varphi_k)=\sum_{i=1}^k(-1)^{i-1}\varphi_1\wedge\dots\widehat{\varphi_i}\dots\wedge\varphi_k\otimes\rho(\varphi_i).\end{aligned}$$

Then we have the *Hamiltonian mapping*

(2) $$\begin{aligned}&H:\Omega(M)\to\Omega(M;TM),\\ &H(\psi):=\rho(d\psi),\\ &H(f_0\,df_1\wedge\dots\wedge df_k)=\sum_{i=0}^k(-1)^i df_0\wedge\dots\widehat{df_i}\dots\wedge df_k\otimes H_{f_i}.\end{aligned}$$

Theorem. [5]. *Let (M,ρ) be a Poisson manifold. Then on the space $\Omega(M)/B(M)$ of differential forms modulo exact forms there exists a unique graded Lie bracket $\{\ ,\ \}_\rho$, which is given by the quotient modulo $B(M)$ of*

$$\{\varphi,\psi\}^1_\rho = i(H_\varphi)d\psi, \qquad \text{or}$$

$$\{f_0\,df_1\wedge\dots\wedge df_k, g_0\,dg_1\wedge\dots\wedge dg_l\}^1_\rho = \tag{3}$$
$$= \sum_{i,j}(-1)^{i+j}\{f_i,g_j\}_\rho\,df_0\wedge\dots\widehat{df_i}\dots\wedge df_k\wedge dg_0\wedge\dots\widehat{dg_j}\dots\wedge dg_k,$$

such that $H:\Omega(M)/B(M)\to\Omega(M;TM)$ is a homomorphism of graded Lie algebras.

If $\rho=\omega^{-1}$ for a symplectic structure ω on M then we have a short exact sequence of vector spaces

$$0\to H^*(M)\to\Omega(M)/B(M)\xrightarrow{H}\Omega_{\mathcal{L}\omega=0}(M;TM)\to \tag{4}$$
$$\to H^{*+1}(M)\oplus\Gamma(E_\omega)\to 0$$

where $\Gamma(E_\omega)$ is a space of sections of a certain vector bundle and where the space $\Omega_{\mathcal{L}\omega=0}(M;TM)$ is the graded Lie subalgebra of all $K\in\Omega(M;TM)$ such that for the Lie derivative along K we have $\mathcal{L}_K\omega=0$. We also have the exact sequence of graded Lie algebras

$$0\to H^*(M)\to\Omega(M)/B(M)\xrightarrow{H}\Omega_\omega(M;TM)\to H^{*+1}(M)\to 0 \tag{5}$$

where now $\Omega_\omega(M;TM)$ is the graded Lie subalgebra of all $K\in\Omega^k(M;TM)$ such that for the Lie derivative along K we have $\mathcal{L}_K\omega=0$ and $K+\frac{(-1)^{k+1}}{k+1}\rho(i_K\omega)=0$, and where on the De Rham cohomology spaces we put the brackets 0.

See [5] for the proof of this theorem and for more information. The step from the sequence (4) to (5) was noticed in [8]. Parts of this theorem were reproved by a different method in [1]. We just note here that on $\Omega(M)$ itself the bracket $\{\ ,\ \}^1_\rho$ is graded anticommutative, but does not satisfy the graded Jacobi identity, whereas a second form, $\{\varphi,\psi\}^2_\rho=\mathcal{L}_{H(\varphi)}\psi$, satisfies the graded Jacobi identity but is not graded anticommutative, and they differ by something exact.

3. The Frölicher-Nijenhuis bracket on $\Omega(T^*M;T(T^*M))$

3.1. Let M be a smooth manifold. We consider the cotangent bundle $\pi:T^*M\to M$, the Liouville form $\Theta_M\in\Omega^1(T^*M)$, given by $\Theta_M(\xi)=\langle\pi_{T^*M}\xi,T(\pi_M).\xi\rangle_{TM}$, and the canonical symplectic form $\omega_M=-d\Theta_M$.

The space $\Gamma(STM)$ of symmetric contravariant tensor fields carries a natural differential concomitant which was found by Schouten [13] and which for $X_i,Y_j\in\mathfrak{X}(M)$ and for $f,g\in C^\infty(M,\mathbb{R})$ is given by (see [7])

$$[f,g]=0 \tag{1}$$
$$[X_1\vee\dots\vee X_k,Y_1\vee\dots\vee Y_l]=$$
$$=\sum_{i,j}[X_i,Y_j]\vee X_1\vee\dots\widehat{X_i}\dots\vee X_k\vee Y_1\vee\dots\widehat{Y_j}\dots\vee Y_l,$$
$$[f,Y_1\vee\dots\vee Y_l]=\sum_j df(Y_j).Y_1\vee\dots\widehat{Y_j}\dots\vee Y_l.$$

Obviously $\Gamma(S^{*+1}TM)$ is a Lie algebra (with grading $*$, but not a graded Lie algebra). Any symmetric multivector field $U \in \Gamma(S^kTM)$ may be viewed as a function on T^*M which is homogeneous of degree k on each fiber. So we have a linear injective mapping

$$\pi^* : \Gamma(S^kTM) \to C^\infty(T^*M, \mathbb{R})$$
$$(\pi^*U)(\varphi) = \langle \varphi^k, U\rangle_{TM}.$$

It is well known that π^* is a homomorphism of Lie algebras, where on $C^\infty(T^*M, \mathbb{R})$ we consider the canonical Poisson bracket $\{\ \ ,\ \ \}$ induced by $\rho = \omega_M^{-1}$. See also 3.5.(2).

3.2. We consider the pullback $\pi^* : \Omega(M) \to \Omega(T^*M)$, and we extend it to the linear mapping

$$\pi^* : \Gamma(\Lambda^kT^*M \otimes S^lTM) \to \Omega^k(T^*M),$$
$$(\pi^*A)_\varphi(\xi_1, \dots, \xi_k) = \langle \varphi \vee \dots \vee \varphi, A(T\pi.\xi_1, \dots, T\pi.\xi_k)\rangle_{TM}. \tag{1}$$

The space $\Gamma(\Lambda T^*M \otimes STM) = \bigoplus_{k,l} \Gamma(\Lambda^kT^*M \otimes S^lTM)$ is a graded commutative algebra with respect to the degree k, and $\pi^* : \Gamma(\Lambda T^*M \otimes STM) \to \Omega(T^*M)$ is obviously a homomorphism with respect to the 'wedge' products. In the following we will always write π^* in front of any tensor field on M which contains vector field components, but we will suppress it if we consider pullbacks of functions or differential forms to T^*M.

Lemma.

(2) *For each $k \geq 0$ and for $l > 0$ the mapping*

$$h : \Gamma(\Lambda^kT^*M \otimes S^lTM) \xrightarrow{\pi^*} \Omega^k(T^*M) \to \frac{\Omega^k(T^*M)}{B^k(T^*M)} \xrightarrow{H} \Omega^k(T^*M; T(T^*M))$$

is injective.

(3) *For $l = 0$ the mapping*

$$\Omega(M) \xrightarrow{\pi^*} \Omega(T^*M) \to \frac{\Omega(T^*M)}{B(T^*M)}$$

induces an injective linear mapping

$$\frac{\Omega(M)}{B(M)} \to \frac{\Omega(T^*M)}{B(T^*M)}.$$

(4) *Let $I \in \mathfrak{X}(T^*M)$ be the vertical homothetic vector field on T^*M, given by $I(\varphi) = \frac{\partial}{\partial t}|_1 t\varphi$. Then for each $l \geq 0$ the image of the linear mapping*

$$\pi^* : \Gamma(\Lambda^kT^*M \otimes S^lTM) \to \Omega^k(T^*M)$$

is the subspace consisting of all horizontal differential forms $\Phi \in \Omega(T^*M)$ which satisfy $\mathcal{L}_I\Phi = l.\Phi$

Proof. Since $\pi : T^*M \to M$ is a homotopy equivalence with homotopy inverse the zero section, the pullback operator induces an injective linear mapping $\pi^* : \Omega(M)/B(M) \to \Omega(T^*M)/B(T^*M)$. This proves (3).

Now let $0 \neq A \in \Gamma(\Lambda^k T^*M \otimes S^l TM)$. We consider the vertical vector field $I \in \mathfrak{X}(T^*M)$, $I(\varphi) = vl(\varphi, \varphi) = \frac{\partial}{\partial t}|_1 t\varphi$. The flow of I is given by the vertical homotheties $\mathrm{Fl}^I_t(\varphi) = e^t\varphi$, we have $(\mathrm{Fl}^I_t)^*\pi^*A = e^{lt}\pi^*A$, and thus

$$i_I d\pi^*A + 0 = \mathcal{L}_I \pi^*A = \frac{d}{dt}|_0 (\mathrm{Fl}^I_t)^*\pi^*A = \frac{d}{dt}|_0 e^{lt}\pi^*A = l\pi^*A$$

which is not 0 for $l > 0$. Since $\rho : \Omega^{>0}(T^*M) \to \Omega(T^*M; T(T^*M))$ is injective, (2) follows.

We also conclude the inclusion $\subseteq$ in (4). Since the assertion is local on M, for the converse inclusion $\supseteq$ we may use local coordinates on $T^*Q \subset T^*M$ as in the beginning of the proof of lemma 3.3. Then $I|Q = \sum p_i \frac{\partial}{\partial p_i}$ and any horizontal form is a sum of expressions like $\Phi = f(q,p) dq^{i_1} \wedge \dots \wedge dq^{i_p} \in \Omega^p(T^*Q)$. Then $\mathcal{L}_I \Phi = l.\Phi$ means $\mathcal{L}_I f = l.f$ from which we conclude that in multi-index notation we have $f(q,p) = \sum_{|\alpha|=l} f_\alpha(q) p^\alpha$, which implies the result, since we use a partition of unity on M. □

3.3. Lemma. Collection of formulas. *In the following $X, Y \in \mathfrak{X}(M)$ are vector fields, $\varphi \in \Omega^p(M)$, $\psi \in \Omega^q(M)$, $K \in \Omega^k(M; TM)$, $L \in \Omega^l(M; TM)$, and $f \in \Omega^0(M)$. Then the following formulas hold on T^*M. We drop π^* in front of pullbacks of differential forms.*

(1) $[hX, hY] = h[X, Y]$.
(2) $[\rho\varphi, \rho\psi] = 0$, *thus also* $[h\varphi, \rho\psi] = [\rho d\psi, \rho\varphi] = 0$, *etc.*
(3) $[hX, \rho\varphi] = \rho\mathcal{L}_X\varphi$, *so also* $[hX, h\varphi] = [hX, \rho d\varphi] = h\mathcal{L}_X\varphi$.
(4) $i_{\rho\varphi}\psi = 0$ *and* $i_{\rho\varphi}\rho\psi = 0$, *so also* $\mathcal{L}_{\rho\varphi}\psi = 0$, *etc.*
(5) $\mathcal{L}_{\rho\varphi}\pi^*X = -i_X\varphi$, *so also* $\mathcal{L}_{h\varphi}\pi^*X = -i_X d\varphi$.
(6) $\mathcal{L}_{hK} f = \mathcal{L}_K f$, *so also* $\mathcal{L}_{hK}\varphi = \mathcal{L}_K\varphi$. *Similarly* $i_{hK}\varphi = i_K\varphi$.
(7) $[hL, hf] = h\mathcal{L}_L f$.
(8) $\mathcal{L}_{hK}\pi^*L = \pi^*[K, L] + (-1)^{(k-1)l} d\pi^*(i_L K)$.
(9) $d\mathcal{L}_{hK}\pi^*L = (-1)^k \mathcal{L}_{hK} d\pi^*L = d\pi^*[K, L]$.
(10) $i_{\rho\pi^*K}\psi = 0$, *so also* $\mathcal{L}_{\rho\pi^*K}\psi = 0$.
(11) $\mathcal{L}_{\rho\pi^*K}\pi^*L = -(-1)^{(k-1)l}\pi^* i_L K$.
(12) $i_{hK}\pi^*L = \pi^* i_K L$.
(13) $i_{hK} d\pi^*L = \pi^*[K, L] - (-1)^k d\pi^*(i_K L + (-1)^{(k-1)(l-1)} i_L K)$.
(14) $i_{hX}\rho\psi = -\rho i_X\psi$.
(15) $\mathcal{L}_{h\varphi}\pi^*L = -(-1)^{pl} i_L d\varphi$.
(16) $[\rho\pi^*K, h\psi] = \rho(i_K d\psi) - (-1)^k i_{hK} h\psi$.

Proof. Let us fix local coordinates $q^1, \dots, q^m$ on an open subset Q of M and induced coordinates q^i, p_j on $T^*Q \subset T^*M$, so that the Liouville form $\Theta|T^*Q = \sum p_i dq^i$ and the symplectic form is given by $\omega = -d\Theta = \sum dq^i \wedge dp_i$. We have

$$\omega(\tfrac{\partial}{\partial q^i}) = dp_i \qquad \rho(dp_i) = \tfrac{\partial}{\partial q^i}$$
$$\omega(\tfrac{\partial}{\partial p_i}) = -dq^i \qquad \rho(dq^i) = -\tfrac{\partial}{\partial p_i}$$

so that for $f \in C^\infty(M,\mathbb{R})$, $\varphi \in \Omega^p(M)$, and $X \in \mathfrak{X}(M)$ we get the following local formulas on $T^*Q \subset T^*M$:

$$\begin{aligned}
hf &= \rho(df) = -\sum \tfrac{\partial f}{\partial q^i}\tfrac{\partial}{\partial p_i}\\
h\varphi &= \rho(\sum \tfrac{\partial \varphi_{i_1\dots i_p}}{\partial q^i} dq^i \wedge dq^{i_1} \wedge \dots \wedge dq^{i_p})\\
&= \sum \tilde\varphi_{j_1\dots j_p,j}(q) dq^{j_1} \wedge \dots \wedge dq^{j_p} \otimes \tfrac{\partial}{\partial p_j}\\
hX &= -\sum \tfrac{\partial X^i}{\partial q^m} p_i \tfrac{\partial}{\partial p_m} + \sum X^i \tfrac{\partial}{\partial q^i}.
\end{aligned}$$

From this (1) and (2) follow by straightforward computation, whereas (3) follows from contemplating 2.1.(1).

(3) then can be proved as follows:

$$\begin{aligned}
[hX, \rho(f_0 df_1 \wedge \dots \wedge df_p)] &= \mathcal{L}_{hX}\left(\sum_i (-1)^{i-1} f_0 df_1 \wedge \dots \wedge \rho df_i \wedge \dots \wedge df_p\right)\\
&= \mathcal{L}_{hX} f_0 . \sum_i (-1)^{i-1} df_1 \wedge \dots \wedge hf_i \wedge \dots \wedge df_p\\
&\quad + \sum_{1\le j<i} (-1)^{i-1} f_0 . df_1 \wedge \dots \wedge \mathcal{L}_{hX} df_j \wedge \dots \wedge hf_i \wedge \dots \wedge df_p\\
&\quad + \sum_i (-1)^{i-1} f_0 . df_1 \wedge \dots \wedge \mathcal{L}_{hX} hf_i \wedge \dots \wedge df_p\\
&\quad + \sum_{1\le i<j} (-1)^{i-1} f_0 . df_1 \wedge \dots \wedge hf_i \wedge \dots \wedge \mathcal{L}_{hX} df_j \wedge \dots \wedge df_p\\
&= \rho(\mathcal{L}_X (f_0 df_1 \wedge \dots \wedge df_p)),
\end{aligned}$$

where we also use the following special cases of (3), which are immediate from the local formulas:

$$\begin{aligned}
\mathcal{L}_{hX} f &= \left(-\sum \tfrac{\partial X^k}{\partial q^m} p_k \tfrac{\partial}{\partial p_m} + \sum X^i \tfrac{\partial}{\partial q^i}\right) f = \mathcal{L}_X f\\
\mathcal{L}_{hX} df &= d\mathcal{L}_{hX} f = d\mathcal{L}_X f = \mathcal{L}_X df\\
\mathcal{L}_{hX} hf &= \left[-\sum \tfrac{\partial X^k}{\partial q^m} p_k \tfrac{\partial}{\partial p_m} + \sum X^i \tfrac{\partial}{\partial q^i}, -\sum \tfrac{\partial f}{\partial q^i}\tfrac{\partial}{\partial p_i}\right]\\
&= h\mathcal{L}_X f = \rho \mathcal{L}_X df.
\end{aligned}$$

(5) is seen as follows:

$$\begin{aligned}
\mathcal{L}(\rho(f_0 df_1 \wedge \dots \wedge df_p))\pi^* X &= i(\rho(f_0 df_1 \wedge \dots \wedge df_p)) d\pi^* X + 0\\
&= i\left(\sum_i (-1)^{i-1} f_0 df_1 \wedge \dots \widehat{df_i} \dots \wedge df_p \otimes hf_i\right) d\pi^* X\\
&= -\sum_i (-1)^{i-1} f_0 df_1 \wedge \dots \wedge i_X df_i \wedge \dots \wedge df_p = -i_X(f_0 df_1 \wedge \dots \wedge df_p)
\end{aligned}$$

where we use the special case

$$\begin{aligned} i(hf)d\pi^*X &= i\left(-\sum \tfrac{\partial f_i}{\partial q^j}\tfrac{\partial}{\partial p_j}\right)\left(\sum \tfrac{\partial X^k}{\partial q^m}p_k dq^m + \sum X^k dp_k\right) \\ &= -\mathcal{L}_X f = -i_X df. \end{aligned}$$

For the proof of the remaining formulas we assume that $K = \varphi \otimes X$ for $\varphi \in \Omega^k(M)$ with $d\varphi = 0$, and $L = \psi \otimes Y$ for $\psi \in \Omega^l(M)$ with $d\psi = 0$, where $X, Y \in \mathfrak{X}(M)$. We may do this since locally $\Omega(M; TM)$ is linearly generated by such elements. We will use the formulas of 2.1 without explicitly mentioning them. Under this assumptions we have

$$\begin{aligned} h(\varphi \otimes X) &= \rho d\pi^*(\varphi \otimes X) = -d\pi^*X \wedge \rho\varphi + \varphi \wedge hX \\ \mathcal{L}(\varphi \otimes X) &= \varphi \wedge \mathcal{L}_X. \end{aligned}$$

(6) follows from (4) via

$$\begin{aligned} \mathcal{L}_{hK} f = i_{hK} df &= i\left(-d\pi^*X \wedge \rho\varphi + \varphi \wedge hX\right) df \\ &= -d\pi^*X \wedge i_{\rho\varphi} df + \varphi \wedge i_{hX} df \\ &= 0 + \varphi \wedge i_X df = i_K df = \mathcal{L}_K f \end{aligned}$$

Then we get in turn

$$\begin{aligned} i_{hK}(f_0 df_1 \wedge \dots \wedge df_p) &= \sum_i (-1)^{i-1} f_0 df_1 \wedge \dots i_{hK} df_i \dots \wedge df_p \otimes hf_i \\ &= i_K(f_0 df_1 \wedge \dots \wedge df_p), \\ \mathcal{L}_{hK} df &= (-1)^k d\mathcal{L}_{hK} f = (-1)^k d\mathcal{L}_K f = \mathcal{L}_K df, \\ \mathcal{L}_{hK}(f_0 df_1 \wedge \dots \wedge df_p) &= \mathcal{L}_{hK} f_0 . df_1 \wedge \dots \wedge df_p) \\ &\quad + \sum_i (-1)^{i-1} f_0 df_1 \wedge \dots \mathcal{L}_{hK} df_i \dots \wedge df_p \otimes hf_i \\ &= \mathcal{L}_K(f_0 df_1 \wedge \dots \wedge df_p). \end{aligned}$$

(7) can be seen as follows, using (3), (2), and (5):

$$\begin{aligned} [hL, hf] &= [h(\psi \otimes Y), hf] = [\psi \wedge hY - d\pi^*Y \wedge \rho\psi, hf] \\ &= \psi \wedge [hY, hf] - \mathcal{L}_{hf}\psi \wedge hY - 0 \\ &\quad - d\pi^*Y \wedge [\rho\psi, hf] + \mathcal{L}_{hf} d\pi^*Y \wedge \rho\psi + 0 \\ &= \psi \wedge h\mathcal{L}_Y f - di_Y df \wedge \rho\psi = h(\psi \wedge \mathcal{L}_Y f) = h\mathcal{L}_L f \end{aligned}$$

(8) We start with the following computation, using (4), (5), and $i_{hX}\psi = i_X\psi$.

$$\begin{aligned}
\mathcal{L}_{hK}\pi^*L &= i_{hK}d\pi^*L - (-1)^{k-1}di_{hK}\pi^*L \\
&= i\,(-d\pi^*X\wedge\rho\varphi + \varphi\wedge hX)\,((-1)^l\psi\wedge d\pi^*Y) \\
&\quad + (-1)^k d\,i\,(-d\pi^*X\wedge\rho\varphi + \varphi\wedge hX)\,(\psi\wedge\pi^*Y) \\
&= -(-1)^l d\pi^*X\wedge i_{\rho\varphi}\psi\wedge d\pi^*Y - (-1)^{l+(k-2)l}d\pi^*X\wedge\psi\wedge i_{\rho\varphi}d\pi^*Y \\
&\quad + (-1)^l\varphi\wedge i_{hX}\psi\wedge d\pi^*Y + \varphi\wedge\psi\wedge i_{hX}d\pi^*Y \\
&\quad + (-1)^k d\,(-d\pi^*X\wedge i_{\rho\varphi}\psi\wedge\pi^*Y + \varphi\wedge i_{hX}\psi\wedge\pi^*Y) \\
&= 0 + (-1)^{(k-1)l}d\pi^*X\wedge\psi\wedge i_Y\varphi + (-1)^l\varphi\wedge i_X\psi\wedge d\pi^*Y \\
&\quad + \varphi\wedge\psi\wedge\pi^*[X,Y] + 0 + \varphi\wedge di_X\psi\wedge\pi^*Y - (-1)^l\varphi\wedge i_X\psi\wedge d\pi^*Y \\
&= \varphi\wedge\psi\wedge\pi^*[X,Y] + \varphi\wedge di_X\psi\wedge\pi^*Y - (-1)^{k+l}i_Y\varphi\wedge\psi\wedge d\pi^*X,
\end{aligned}$$

where we also used

$$\begin{aligned}
\mathcal{L}_{hX}\pi^*Y &= i_{hX}d\pi^*Y \\
&= i\left(-\sum \tfrac{\partial X^i}{\partial q^m}p_i\tfrac{\partial}{\partial p_m} + \sum X^i\tfrac{\partial}{\partial q^i}\right)\left(\sum \tfrac{\partial Y^k}{\partial q^n}p_k dq^n + \sum Y^k dp_k\right) \\
&= \sum\left(X^i\tfrac{\partial Y^k}{\partial q^i} - Y^i\tfrac{\partial X^k}{\partial q^i}\right)p_k = \pi^*[X,Y].
\end{aligned}$$

Then we get

$$\begin{aligned}
\pi^*[K,L] &= \pi^*[\varphi\otimes X, \psi\otimes Y], \qquad \text{use now 2.1.(2)} \\
&= \varphi\wedge\psi\wedge\pi^*[X,Y] + \varphi\wedge\mathcal{L}_X\psi\wedge\pi^*Y - \mathcal{L}_Y\varphi\wedge\psi\wedge\pi^*X + 0 + 0, \\
\mathcal{L}_{hK}\pi^*L - \pi^*[K,L] &= di_Y\varphi\wedge\psi\wedge\pi^*X - (-1)^{k+l}i_Y\varphi\wedge\psi\wedge d\pi^*X \\
&= d((-1)^{(k-1)l}\psi\wedge i_Y\varphi\wedge\pi^*X) = (-1)^{(k-1)l}d\pi^*(i_LK).
\end{aligned}$$

(9) follows from (8).
(10) $i(\rho\pi^*K)\psi = i(\pi^*X.\rho\varphi)\psi = \pi^*X.i_{\rho\varphi}\psi = 0$.
(11) We compute in turn

$$\begin{aligned}
\mathcal{L}_{\rho\pi^*K}\pi^*Y &= i(\pi^*X\wedge\rho\varphi)d\pi^*Y = \pi^*X\wedge\mathcal{L}_{\rho\varphi}\pi^*Y = -\pi^*X\wedge i_Y\varphi = -\pi^*i_YK \\
\mathcal{L}_{\rho\pi^*K}\pi^*L &= \mathcal{L}_{\rho\pi^*K}(\pi^*Y\wedge\psi) = \mathcal{L}_{\rho\pi^*K}\pi^*Y\wedge\psi + \pi^*Y\wedge\mathcal{L}_{\rho\pi^*K}\psi \\
&= -\pi^*(i_YK)\wedge\psi + 0 \\
&= -(-1)^{(k-1)l}\pi^*i_LK
\end{aligned}$$

(12) We have in turn

$$\begin{aligned}
i_{hX}\pi^*L &= i\left(-\sum\tfrac{\partial X^i}{\partial q^m}p_i\tfrac{\partial}{\partial p_m} + \sum X^i\tfrac{\partial}{\partial q^i}\right)(\psi\wedge\pi^*Y) \\
&= i_X\psi\wedge\pi^*Y = \pi^*i_XL, \\
i_{hK}\pi^*L &= i\,(-d\pi^*X\wedge\rho\varphi + \varphi\wedge hX)\,(\psi\wedge\pi^*Y) \\
&= -d\pi^*X\wedge i_{\rho\varphi}\psi\wedge\pi^*Y + \varphi\wedge i_{hX}\psi\wedge\pi^*Y \\
&= 0 + \varphi\wedge i_X\psi\wedge\pi^*Y = \pi^*i_KL.
\end{aligned}$$

(13) From (8) we get

$$\begin{aligned} i_{hK} d\pi^* L &= \mathcal{L}_{hK} \pi^* L + (-1)^{k-1} d i_{hK} \pi^* L \\ &= \pi^*[K, L] - (-1)^k d\pi^*(i_K L + (-1)^{(k-1)(l-1)} i_L K). \end{aligned}$$

(14) We just compute

$$\begin{aligned} i_{hX}\rho(f_0 df_1 \wedge \ldots \wedge df_q) &= i_{hX}\left(\sum (-1)^{j-1} f_0 df_1 \wedge \cdots \wedge hf_j \wedge \cdots \wedge df_q\right) \\ &= \sum_{k<j} (-1)^{k+j} f_0 df_1 \wedge \cdots \wedge i_{hX} df_k \wedge \cdots \wedge hf_j \wedge \cdots \wedge df_q \\ &\quad + \sum_{k>j} (-1)^{k+j-1} f_0 df_1 \wedge \cdots \wedge hf_j \wedge \cdots \wedge i_{hX} df_k \wedge \cdots \wedge df_q \\ &= -\rho i_X (f_0 df_1 \wedge \cdots \wedge df_q). \end{aligned}$$

(15) This is an easy consequence of (4) and (5), namely

$$\begin{aligned} \mathcal{L}_{h\varphi} \pi^* L = \mathcal{L}_{h\varphi}(\psi \wedge \pi^* Y) &= 0 + (-1)^{pl} \psi \wedge \mathcal{L}_{h\varphi} i^* Y \\ &= -(-1)^{pl} \psi \wedge i_Y \varphi = -(-1)^{pl} i_L d\varphi. \end{aligned}$$

(16) This can be seen by summing the following evaluations:

$$\begin{aligned} [\rho\pi^* K, h\psi] &= [\rho\pi^*(\varphi \otimes X), h\psi] = [\rho\varphi . \pi^* X, h\psi] \\ &= \pi^* X \wedge [\rho\varphi, h\psi] - (-1)^{(k-1)q} \mathcal{L}_{h\psi} \pi^* X \wedge \rho\varphi + (-1)^{k-1} d\pi^* X \wedge i_{\rho\varphi} h\psi \\ &= \rho\varphi \wedge i_X d\psi, \\ \rho(i_K d\psi) &= \rho(\varphi \wedge i_X d\psi) = \rho\varphi \wedge i_X d\psi + (-1)^k \varphi \wedge \rho i_X d\psi, \\ i_{hK} h\psi &= i(-d\pi^* X \wedge \rho\varphi + \varphi \wedge hX)(\rho d\psi) \\ &= -d\pi^* X \wedge i_{\rho\varphi} \rho d\psi + \varphi \wedge i_{hX} \rho d\psi = 0 - \varphi \wedge i_{hX} \rho d\psi. \quad \square \end{aligned}$$

3.4 The extended insertion. For $A \in \Omega^k(M; S^l TM)$ we define now the insertion operator

$$\begin{aligned} i_A : \Omega^p(M; S^m TM) &\to \Omega^{p+k-1}(M; S^{m+l-1} TM) \\ i(\varphi \otimes X_1 \vee \cdots \vee X_k)(\psi \otimes V) &= \varphi \wedge \sum_j i_{X_j} \psi \otimes X_1 \vee \ldots \widehat{X_j} \cdots \vee X_k \vee V. \end{aligned}$$

This is a graded derivation of degree $k-1$ of the graded commutative algebra $\bigoplus_{m \geq 0} \Omega^m(M, STM)$ which vanishes on the subalgebra $\Gamma(STM)$.

Lemma. More formulas. *For $A \in \Omega^k(M; S^l TM)$, where $l > 0$, and $\psi \in \Omega^q(M)$ we have on T^*M*

(1) $\mathcal{L}_{h\psi} \pi^* A = -(-1)^{qk} \pi^* i_A d\psi$.
(2) $[\rho\pi^* A, h\psi] = \rho\pi^* i_A d\psi - (-1)^k i_{hA} h\psi$.

Proof. (1) We prove this by induction on l. For $l = 1$ this is 3.3.(15). For the induction we compute as follows:

$$\begin{aligned}\mathcal{L}_{h\psi}\pi^*(X\wedge A) &= \mathcal{L}_{h\psi}\pi^*X\wedge\pi^*A + \pi^*X\wedge\mathcal{L}_{h\psi}\pi^*A\\ &= -i_X d\psi\wedge\pi^*A - (-1)^{qk}\pi^*X\wedge\pi^* i_A d\psi\\ &= -(-1)^{qk}\pi^*(A\wedge i_X d\psi + X\wedge i_A d\psi) = -(-1)^{qk}\pi^* i_{X\wedge A}d\psi.\end{aligned}$$

(2) We use again induction on l. For $l = 1$ this is 3.3.(16). The left-hand side equals:

$$\begin{aligned}&[\rho\pi^*(X\wedge A), h\psi] = [\pi^*X\wedge\rho\pi^*A, h\psi]\\ &\quad= \pi^*X\wedge[\rho\pi^*A, h\psi] - (-1)^{(k-1)q}\mathcal{L}_{h\psi}\pi^*X\wedge\rho\pi^*A + (-1)^k d\pi^*X\wedge i_{\rho\pi^*A}h\psi\\ &\quad= \pi^*X\wedge\rho\pi^* i_A d\psi - (-1)^k\pi^*X\wedge\pi^* i_{hA}h\psi\\ &\qquad + (-1)^{(k-1)q} i_X d\psi\wedge\rho\pi^*A + (-1)^k d\pi^*X\wedge i_{\rho\pi^*A}h\psi.\end{aligned}$$

For the right-hand side we get:

$$\begin{aligned}&\rho\pi^* i_{X\wedge A}d\psi - (-1)^k i_{h(X\wedge A)}h\psi = \rho\pi^*(X\wedge i_A d\psi + A\wedge i_X d\psi)\\ &\qquad - (-1)^k i(hX\wedge\pi^*A - d\pi^*X\wedge\rho\pi^*A + \pi^*X\wedge hA)h\psi\\ &\quad= \pi^*X\wedge\rho\pi^* i_A d\psi + \rho\pi^*A\wedge i_X d\psi - (-1)^k\pi^*A\wedge\rho i_X d\psi\\ &\qquad - (-1)^k\pi^*A\wedge i_{hX}h\psi + (-1)^k d\pi^*X\wedge i_{\rho\pi^*A}h\psi - (-1)^k\pi^*X\wedge i_{hA}h\psi.\end{aligned}$$

Using 3.3.(14) we see that it equals the left-hand side. □

3.5. Theorem.

(1) *The linear injective mapping*

$$h : \Gamma(\Lambda T^*M\otimes TM) = \Omega(M;TM) \xrightarrow{\pi^*} \Omega(T^*M) \xrightarrow{H} \Omega(T^*M; T(T^*M))$$

is a homomorphism for the Frölicher-Nijenhuis brackets.

(2) *The linear mapping*

$$h : \Gamma(STM) \xrightarrow{\pi^*} \Omega(T^*M) \xrightarrow{H} \Omega(T^*M; T(T^*M))$$

is a homomorphism from the symmetric Schouten bracket to the Frölicher-Nijenhuis bracket. The kernel of h is $H^0(M)$.

(3) *For differential forms $\varphi, \psi\in\Omega(M)$ we have*

$$[h\varphi, h\psi] = 0.$$

(4) *For $A\in\Omega(M;STM)$ and $\psi\in\Omega(M)$ we have*

$$[hA, h\psi] = hi_A d\psi, \qquad \textit{where}$$

$$i(\varphi\otimes X_1\vee\dots\vee X_k)\psi := \varphi\wedge\left(\sum_j i_{X_j}\psi\otimes X_1\vee\dots\widehat{X_j}\dots\vee X_k\right).$$

(5) *For $\dim M\geq 2$, in general $[h\Omega^{k_1}(M;S^{l_1}TM), h\Omega^{k_2}(M;S^{l_2}TM)]$ does not lie in the image of h, if $k_1, l_1\geq 1$ and $l_2\geq 2$ (or under the symmetric condition).*

Proof. (1) We have to show that $[hK, hL] = h[K, L]$ for $K \in \Omega^k(M; TM)$ and $L \in \Omega^l(M; TM)$ and we do this by induction on $k+l$. The case of vector fields $k+l = 0$ is well known, see 3.3.(1). Since the question is local on M and since $\Omega^{k+1}(M; TM)$ is locally linearly generated by $df \wedge K$ for $f \in \Omega^0(M)$ and $K \in \Omega^k(M; TM)$ it suffices to check that $[hK, hL] = h[K, L]$ implies $[h(df \wedge K), hL] = h[df \wedge K, L]$. We have

$$h(df \wedge K) = \rho d(df \wedge \pi^* K) = -d\pi^* K \wedge hf + df \wedge hK.$$

Using twice 2.1.(4) we get then

$$\begin{aligned}
[h(df \wedge K), hL] &= df \wedge [hK, hL] - (-1)^{(1+k)l} \mathcal{L}_{hL} df \wedge hK + 0 \\
&\quad - d\pi^* K \wedge [hf, hL] + (-1)^{(1+k)l} \mathcal{L}_{hL} d\pi^* K \wedge hf - 0 \\
&= df \wedge h[K, L] - (-1)^{(1+k)l} \mathcal{L}_L df \wedge hK \\
&\quad + d\pi^* K \wedge h\mathcal{L}_L f + (-1)^{kl} d\pi^*[L, K] \wedge hf,
\end{aligned}$$

where we used in turn induction, 3.3.(6), 3.3.(7), and 3.3.(8). On the other hand we have again by 2.1.(4)

$$\begin{aligned}
h[df \wedge K, L] &= \rho d\pi^* \Big(df \wedge [K, L] - (-1)^{(1+k)l} \mathcal{L}_L df \wedge K + 0 \Big) \\
&= -hf \wedge d\pi^*[K, L] + df \wedge h[K, L] \\
&\quad + (-1)^{kl} \rho \mathcal{L}_L df \wedge d\pi^* K + (-1)^{kl+l+1} \mathcal{L}_L df \wedge hK,
\end{aligned}$$

which equals the expression for $[h(df \wedge K), hL]$ from above.

(2) This is well known, and easy to check starting from 3.3.(1).

(3) is 3.3.(2).

(4) First we prove a partial result.

Claim. For $K \in \Omega^k(M; TM)$ and $\psi \in \Omega^q(M)$ we have $[hK, h\psi] = h\mathcal{L}_K \psi = hi_K d\psi$.

To check the claim we use induction on $q = \deg \psi$. For $q = 0$ this is 3.3.(7). Since the assertion is local on M it suffices to consider $df \wedge \psi$ for the induction step. In the following computation we use 2.1.(4), induction, 3.3.(6), and 3.3.(7):

$$\begin{aligned}
&[hK, h(df \wedge \psi)] = [hK, df \wedge h\psi - d\psi \wedge hf] \\
&\quad = (-1)^k df \wedge [hK, h\psi] + \mathcal{L}_{hK} df \wedge h\psi - 0 \\
&\qquad - (-1)^{(q+1)k} d\psi \wedge [hK, hf] - \mathcal{L}_{hK} d\psi \wedge hf + 0 \\
&\quad = (-1)^k df \wedge h\mathcal{L}_K \psi + h\mathcal{L}_K df \wedge h\psi - (-1)^{(q+1)k} d\psi \wedge h\mathcal{L}_K f - \mathcal{L}_K d\psi \wedge hf.
\end{aligned}$$

On the other hand we have by 2.1.(3)

$$\begin{aligned}
&h\mathcal{L}_K(df \wedge \psi) = h\left((-1)^k d\mathcal{L}_K f \wedge \psi + (-1)^k df \wedge \mathcal{L}_K \psi \right) \\
&\quad = \rho \left(d\mathcal{L}_K f \wedge d\psi - (-1)^k df \wedge d\mathcal{L}_K \psi \right) \\
&\quad = -h\mathcal{L}_K f \wedge d\psi + (-1)^k d\mathcal{L}_K f \wedge h\psi - (-1)^k hf \wedge d\mathcal{L}_K \psi + (-1)^k df \wedge hL_K \psi,
\end{aligned}$$

which equals the above expression. So the claim follows.

Now we can extend this result to $A \in \Omega^k(M; S^l TM)$ by induction on l. For $l = 1$ this is the claim above. For the induction we compute first the left hand side, using 2.1.(4), the claim, 3.4, induction, and 3.3.(14):

$$\begin{aligned}
[h(X.A), h\psi] &= [hX \wedge \pi^* A - d\pi^* X \wedge \rho\pi^* A + \pi^* X \wedge hA, h\psi] \\
&= \pi^* A \wedge [hX, h\psi] - (-1)^{kq} \mathcal{L}_{h\psi} \pi^* A \wedge hX + (-1)^k d\pi^* A \wedge i_{hX} h\psi \\
&\quad - d\pi^* X \wedge [\rho\pi^* A, h\psi] + (-1)^{kq} \mathcal{L}_{h\psi} d\pi^* X \wedge \rho\pi^* A - 0 \\
&\quad + \pi^* X \wedge [hA, h\psi] - (-1)^{kq} \mathcal{L}_{h\psi} \pi^* X \wedge hA + (-1)^k d\pi^* X \wedge i_{hA} h\psi \\
&= \pi^* A \wedge h i_X d\psi + \pi^* i_A d\psi \wedge hX - (-1)^k d\pi^* A \wedge \rho i_X d\psi \\
&\quad - d\pi^* X \wedge \rho i_A d\psi - (-1)^{(k-1)q} d i_X d\psi \wedge \rho\pi^* A \\
&\quad + \pi^* X \wedge h i_A d\psi + (-1)^{kq} i_X d\psi \wedge hA.
\end{aligned}$$

The right-hand side is

$$\begin{aligned}
& hi(X \wedge A)d\psi = h(A \wedge i_X d\psi + X \wedge i_A d\psi) \\
&\quad = \rho(d\pi^* A \wedge i_X d\psi + (-1)^k \pi^* A \wedge d i_X d\psi + d\pi^* X \wedge \pi^* i_A d\psi + \pi^* X \wedge d\pi^* i_A d\psi) \\
&\quad = hA \wedge i_X d\psi + (-1)^k d\pi^* A \wedge \rho i_X d\psi + (-1)^k \rho\pi^* A \wedge d i_X d\psi + \pi^* A \wedge h i_X d\psi \\
&\qquad + hX \wedge \pi^* i_A d\psi - d\pi^* X \wedge \rho\pi^* i_A d\psi + \pi^* X \wedge h\pi^* i_A d\psi),
\end{aligned}$$

which equals the left hand side.

(5) By 3.2.(3) the image of $\pi^* : \Omega(M; S^l TM)$ is the space of all horizontal forms $\Phi \in \Omega(T^*M)$ satisfying $\mathcal{L}_I \Phi = l.\Phi$. In local coordinates on M we consider then, using the bracket $\{\quad,\quad\}^1$ described in 2.3,

$$\begin{aligned}
\{\pi^*(dq^1 \otimes \frac{\partial}{\partial q^1}), \pi^*(\frac{\partial}{\partial q^1}\frac{\partial}{\partial q^2})\}^1 &= \{p_1 dq^1, p_1 p_2\}^1 = \{p_1, p_1 p_2\} dq^1 - \{q^1, p_1 p_2\} dp_1 \\
&= p_2 dp_1, \\
d\{p_1 dq^1, p_1 p_2\}^1 &= -dp_1 \wedge dp_2.
\end{aligned}$$

Thus $\{p_1 dq^1, p_1 p_2\}^1$ plus something exact can never be horizontal. □

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