Math. Ann. **327** , 1 (2003), 191 - 201

DIFFERENTIABLE PERTURBATION OF UNBOUNDED OPERATORS

Andreas Kriegl, Peter W. Michor

Abstract. If $A(t)$ is a $C^{1,\alpha}$-curve of unbounded self-adjoint operators with compact resolvents and common domain of definition, then the eigenvalues can be parameterized C^1 in t. If A is C^∞ then the eigenvalues can be parameterized twice differentiably.

Theorem. *Let $t \mapsto A(t)$ for $t \in \mathbb{R}$ be a curve of unbounded self-adjoint operators in a Hilbert space with common domain of definition and with compact resolvent.*

(A) *If $A(t)$ is real analytic in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ may be parameterized real analytically in t.*

(B) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$ and if no two unequal continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors can be parameterized smoothly in t, on the whole parameter domain.*

(C) *If A is C^∞, then the eigenvalues of $A(t)$ may be parameterized twice differentiably in t.*

(D) *If $A(t)$ is $C^{1,\alpha}$ for some $\alpha > 0$ in $t \in \mathbb{R}$, then the eigenvalues of $A(t)$ may be parameterized in a C^1 way in t.*

Part (A) is due to Rellich [10] in 1940, see also [2] and [6], VII, 3.9. Part (B) has been proved in [1], 7.8, see also [8], 50.16, in 1997; there we gave also a different proof of (A). The purpose of this paper is to prove parts (C) and (D).

Both results cannot be improved to obtain a $C^{1,\beta}$-parameterization of the eigenvalues for some $\beta > 0$, by the first example below. In our proof of (D) the assumption $C^{1,\alpha}$ cannot be weakened to C^1, see the second example. For finite dimensional Hilbert spaces part (D) has been proved under the assumption of C^1 by Rellich [11], with a small inaccuracy in the auxiliary theorem on p. 48: Condition (4) must be more restrictive, otherwise the induction argument on p. 50 is not valid, since the proof on p. 52 relies on the fact that all values coincide at the point in question. A proof can also be found in [6], II, 6.8. We need a strengthened version of this result, thus our proof covers it also.

We thank T. and M. Hoffmann-Ostenhof and T. Kappeler for their interest and hints.

2000 *Mathematics Subject Classification.* Primary 26C10..

Key words and phrases. Perturbation theory, differentiable choice of eigenvalues.

PWM was supported by FWF, Projekt P 14195 MAT.

Definitions and remarks. That $A(t)$ is a real analytic, C^∞, or $C^{k,\alpha}$ curve of unbounded operators means the following: There is a dense subspace V of the Hilbert space H such that V is the domain of definition of each $A(t)$, and such that $A(t)^* = A(t)$. Moreover, we require that $t \mapsto \langle A(t)u, v\rangle$ is real analytic, C^∞, or $C^{k,\alpha}$ for each $u \in V$ and $v \in H$. This implies that $t \mapsto A(t)u$ is of the same class $\mathbb{R} \to H$ for each $u \in V$ by [8], 2.3 or [5], 2.6.2. This is true because $C^{k,\alpha}$ can be described by boundedness conditions only; and for these the uniform boundedness principle is valid. A function f is called $C^{k,\alpha}$ if it is k times differentiable and for the k-th derivative the expression $\frac{f^{(k)}(t)-f^{(k)}(s)}{|t-s|^\alpha}$ is locally bounded in $t \neq s$.

A sequence of continuous, real analytic, smooth, or twice differentiable functions λ_i is said to *parameterize the eigenvalues, if for each $z \in \mathbb{R}$ the cardinality $|\{i : \lambda_i(t) = z\}|$ equals the multiplicity of z as eigenvalue of $A(t)$.*

The proof will moreover furnish the following (stronger) versions:

(C1) *If $A(t)$ is $C^{3n,\alpha}$ in t and if the multiplicity of an eigenvalue never exceeds n, then the eigenvalues of A may be parameterized twice differentiably.*

(C2) *If the multiplicity of any eigenvalue never exceeds n, and if the resolvent $(A(t)-z)^{-1}$ is C^{3n} into $L(H,H)$ in t and z jointly, then the eigenvalues of $A(t)$ may be parameterized twice differentiably in t.*

(D1) *If the resolvent $(A(t)-z)^{-1}$ is C^1 into $L(H,H)$ in t and z jointly, then the eigenvalues of $A(t)$ may be parameterized in a C^1 way in t.*

(D2) *In the situations of* (D) *and* (D1) *the following holds: For any continuous parameterization $\lambda_i(t)$ of all eigenvalues of $A(t)$, each function λ_i has right sided derivative $\lambda_i^{(+)}(t)$ and left sided one $\lambda_i^{(-)}(t)$ at each t, and $\{\lambda_i^{(+)}(t) : \lambda_i(t) = z\}$ equals $\{\lambda_i^{(-)}(t) : \lambda_i(t) = z\}$ with correct multiplicities.*

Open problem. Construct a C^1-curve of unbounded self-adjoint operators with common domain and compact resolvent such that the eigenvalues cannot be arranged C^1.

Applications. Let M be a compact manifold and let $t \mapsto g_t$ be a smooth curve of smooth Riemannian metrics on M. Then we get the corresponding smooth curve $t \mapsto \Delta(g_t)$ of Laplace-Beltrami operators on $L^2(M)$. By theorem (C) the eigenvalues can be arranged twice differentiably.

Let Ω be a bounded region in $\mathbb{R}^n$ with smooth boundary, and let $H(t) = -\Delta + V(t)$ be a $C^{1,\alpha}$-curve of Schrödinger operators with varying potential and Dirichlet boundary conditions. Then the eigenvalues can be arranged C^1.

Example. This is an elaboration of [1], 7.4. Let $S(2)$ be the vector space of all symmetric real (2×2)-matrices. We use the general curve lemma [8], 12.2: *There exists a converging sequence of reals t_n with the following property: Let $A_n \in C^\infty(\mathbb{R}, S(2))$ be any sequence of functions which converges fast to 0, i.e., for each $k \in \mathbb{N}$ the sequence $n^k A_n$ is bounded in $C^\infty(\mathbb{R}, S(2))$. Then there exists a smooth curve $A \in C^\infty(\mathbb{R}, S(2))$ such that $A(t_n + s) = A_n(s)$ for $|s| \leq \frac{1}{n^2}$, for all n.*

We use it for

$$A_n(t) := \begin{pmatrix} \frac{1}{2^{n^2}} & \frac{t}{2^n} \\ \frac{t}{2^n} & -\frac{1}{2^{n^2}} \end{pmatrix} = \frac{1}{2^{n^2}} \begin{pmatrix} 1 & \frac{t}{s_n} \\ \frac{t}{s_n} & -1 \end{pmatrix}, \text{ where } s_n := 2^{n-n^2} \leq \frac{1}{n^2}.$$

The eigenvalues of $A_n(t)$ and their derivatives are

$$\lambda_n(t) = \pm\frac{1}{2^{n^2}}\sqrt{1+(\tfrac{t}{s_n})^2}, \quad \lambda_n'(t) = \pm\frac{2^{n^2-2n}t}{\sqrt{1+(\tfrac{t}{s_n})^2}}.$$

Then

$$\begin{aligned}\frac{\lambda'(t_n+s_n)-\lambda'(t_n)}{s_n^\alpha} &= \frac{\lambda_n'(s_n)-\lambda_n'(0)}{s_n^\alpha} = \pm\frac{2^{n^2-2n}s_n}{s_n^\alpha\sqrt{2}}\\ &= \pm\frac{2^{n(\alpha(n-1)-1)}}{\sqrt{2}} \to \infty \text{ for } \alpha>0.\end{aligned}$$

By [1], 2.1, we may always find a twice differentiable square root of a non-negative smooth function, so that the eigenvalues λ are functions which are twice differentiable but not $C^{1,\alpha}$ for any $\alpha > 0$.

Note that the normed eigenvectors cannot be chosen continuously in this example (see also example [9], §2). Namely, we have

$$A(t_n) = A_n(0) = \frac{1}{2^{n^2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad A(t_n+s_n) = A_n(s_n) = \frac{1}{2^{n^2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}.$$

Resolvent Lemma. *If A is $C^{k,\alpha}$ for some $1 \le k \le \infty$ and $\alpha > 0$, then the resolvent $(t,z) \mapsto (A(t)-z)^{-1} \in L(H,H)$ is C^k on its natural domain.*

By $C^{\infty,\alpha}$ we mean C^∞.

Proof. By definition the function $t \mapsto \langle A(t)v, u\rangle$ is of class $C^{k,\alpha}$ for each $v \in V$ and $u \in H$. Then by [4], 5 or [8], 2.3 (extended from $C^{k,1}$ to $C^{k,\alpha}$ with essentially the same proof), the curve $t \mapsto A(t)v$ is of class $C^{k,\alpha}$ into H.

For each t consider the norm $\|u\|_t^2 := \|u\|^2 + \|A(t)u\|^2$ on V. Since $A(t) = A(t)^*$ is closed, $(V, \|\ \ \|_t)$ is again a Hilbert space with inner product $\langle u,v\rangle_t := \langle u,v\rangle + \langle A(t)u, A(t)v\rangle$.

(1) *Claim. All these norms $\|\ \ \|_t$ on V are equivalent, locally uniformly in t.* We then equip V with one of the equivalent Hilbert norms, say $\|\ \ \|_0$.

Note first that $A(t) : (V, \|\ \ \|_s) \to H$ is bounded since the graph of $A(t)$ is closed in $H \times H$, contained in $V \times H$ and thus also closed in $(V, \|\ \ \|_s) \times H$. For fixed $u, v \in V$, the function $t \mapsto \langle u,v\rangle_t = \langle u,v\rangle + \langle A(t)u, A(t)v\rangle$ is $C^{k,\alpha}$ since $t \mapsto A(t)u$ is it. Thus it is also locally Lipschitz ($C^{0,1} = \mathcal{L}\text{ip}^0$). By the multilinear uniform boundedness principle ([8], 5.18) or [5], 3.7.4) the mapping $t \mapsto \langle\ \ ,\ \ \rangle_t$ is $C^{0,1}$ into the space of bounded bilinear forms on $(V, \|\ \ \|_s)$ for each fixed s. By the exponential law [5], 4.3.5 for $\mathcal{L}\text{ip}^0$ the mapping $(t,u,v) \mapsto \langle u,v\rangle_t$ is $C^{0,1}$ from $\mathbb{R} \times (V, \|\ \ \|_s) \times (V, \|\ \ \|_s) \to \mathbb{R}$ for each fixed s. Therefore and by homogeneity in (u,v) the set $\{\|u\|_t : |t| \le K, \|u\|_s \le 1\}$ is bounded by some $L_{K,s}$ in $\mathbb{R}$. Thus $\|u\|_t \le L_{K,s}\|u\|_s$ for all $|t| \le K$, i.e. all Hilbert norms $\|\ \ \|_t$ are locally uniformly equivalent, and claim (1) follows.

By [4], 5 and the linear uniform boundedness theorem we see that $t \mapsto A(t)$ is a $C^{k,\alpha}$-mapping $\mathbb{R} \to L(V,H)$, and thus is C^k in the usual sense, again by [4], 5.

Alternatively, if reference [4] is not available, one may use [8], 2.3, extended from $C^{k,1}$ to $C^{k,\alpha}$ with essentially the same proof, and note that it suffices to test with linear mappings which recognize bounded sets, by [8], 5.18. Alternatively again, one may use [5], 3.7.4 + 4.1.12, extended from $C^{k,1}$ to $C^{k,\alpha}$.

If for some $(t,z) \in \mathbb{R}\times\mathbb{C}$ the bounded operator $A(t)-z : V \to H$ is invertible, then this is true locally and $(t,z)\mapsto (A(t)-z)^{-1} : H \to V$ is C^k, by the chain rule, since inversion is smooth on the Banach space $L(V,H)$. □

Since each $A(t)$ is Hermitian with compact resolvent the *global resolvent set* $\{(t,z)\in\mathbb{R}\times\mathbb{C} : (A(t)-z) : V\to H \text{ is invertible}\}$ is open and connected. Moreover, $(A(t)-z)^{-1} : H\to H$ is a compact operator for some (equivalently any) (t,z) if and only if the inclusion $i : V \to H$ is compact, since $i = (A(t)-z)^{-1}\circ(A(t)-z) : V\to H\to H$.

Resolvent example. The resolvent lemma cannot be improved. We describe a curve $A(t)$ of self adjoint unbounded operators on ℓ^2 with compact resolvent and common domain V of definition, such that $t\mapsto\langle A(t)v,u\rangle$ is C^1 for all $v\in V$ and $u\in\ell^2$, but $t\mapsto A(t)$ is even not differentiable at 0 into $L(V,\ell^2)$.

Let $\lambda_1\in C^\infty(\mathbb{R},\mathbb{R})$ be nonnegative with compact support and $\lambda_1'(0)=0$. We consider the multiplication operator $B(t)$ on ℓ^2 given on the standard basis e_n by $B(t)e_n := (1+\frac{1}{n}\lambda_1(nt))e_n =: \lambda_n(t)e_n$ which is bounded with bounded inverse. Then the function $t\mapsto\langle B(t)x,y\rangle$ is C^1 with derivative $\langle B_1(t)x,y\rangle$, where $B_1(t)$ is given by $B_1(t)e_n=\lambda_n'(t)e_n=\lambda_1'(nt)e_n$, since for fixed t we have that

$$\mu_n(s) := \frac{\lambda_n(t+s)-\lambda_n(t)}{s} - \lambda_n'(t) = \frac{\lambda_1(nt+ns)-\lambda_1(nt)}{ns} - \lambda_1'(nt)$$

converges to 0 for $s\to 0$ pointwise in n and is bounded uniformly in n:

$$\Big|\sum_n \mu_n(s)x_ny_n\Big| \le \sum_{n=N+1}^{\infty}|\mu_n(s)x_ny_n| + \sum_{n=1}^{N}|\mu_n(s)x_ny_n| \le \sup_n|\mu_n(s)|\varepsilon + \varepsilon\|x\|\|y\|.$$

Moreover, $\langle B_1(t)x,y\rangle$ is continuous in t since $\lambda_n'(t+s)-\lambda_n'(t)=\lambda_1'(nt+ns)-\lambda_1'(nt)$ also converges to 0 for $s\to 0$ pointwise in n and is bounded uniformly in n. But $t\mapsto B(t)$ is not differentiable at 0 into $L(\ell^2,\ell^2)$ since

$$\Big\|\frac{B(t)-B(0)}{t} - B_1(0)\Big\| = \sup_n\Big|\frac{\lambda_n(t)-\lambda_n(0)}{t} - \lambda_n'(0)\Big| = \sup_n\Big|\frac{\lambda_1(nt)-\lambda_1(0)}{nt} - \lambda_1'(0)\Big|$$

is bounded away from 0, for $t\to 0$. Finally, let $C:\ell^2\to\ell^2$ be the compact invertible given by $Ce_n=\frac{1}{n}e_n$. We take $V=C(\ell^2)$, and $A(t)=B(t)\circ C^{-1}$.

Proof of the theorem. By the resolvent lemma, (D1) implies (D), and likewise (C2) implies (C) and (C1).

Proof of (D1)**.**

(2) *Claim. If $f:(a,b)\to\mathbb{R}$ is continuous and if $f(t)$ has only finitely many cluster points for $t\to b$ then the limit $\lim_{t\nearrow b} f(t)$ exists.* Otherwise, by the intermediate value theorem, we have a whole interval of cluster points.

(3) *Claim. Let z be an eigenvalue of $A(s)$ of multiplicity N. Then there exists an open box $(s-\delta, s+\delta)\times(z-\varepsilon, z+\varepsilon)$ and C^1-functions $\mu_1, \dots, \mu_N : (s-\delta, s+\delta) \to (z-\varepsilon, z+\varepsilon)$ which parameterize all eigenvalues λ with $|\lambda - z| < \varepsilon$ of $A(t)$ for $|t-s| < \delta$ with correct multiplicities.*

We choose a simple closed smooth curve γ in the resolvent set of $A(s)$ for fixed s enclosing only z among all eigenvalues of $A(s)$. Since the global resolvent set is open, no eigenvalue of $A(t)$ lies on γ, for t near s. Since

$$t \mapsto -\frac{1}{2\pi i}\int_\gamma (A(t)-z)^{-1}\,dz =: P(t,\gamma) = P(t)$$

is a C^1 curve of projections (on the direct sum of all eigenspaces corresponding to eigenvalues in the interior of γ) with finite dimensional ranges, the ranks (i.e. dimension of the ranges) must be constant: it is easy to see that the (finite) rank cannot fall locally, and it cannot increase, since the distance in $L(H,H)$ of $P(t)$ to the subset of operators of rank $\le N = \operatorname{rank}(P(s))$ is continuous in t and is either 0 or 1. So for t near s, say $t \in I := (s-\delta, s+\delta)$, there are equally many eigenvalues in the interior of γ, and we may call them $\lambda_i(t)$ for $1 \le i \le N$ (repeated with multiplicity), so that each λ_i is continuous (this is well known and follows easily from the proof of (C2)).

Then the image of $t \mapsto P(t,\gamma)$, for t near s, describes a C^1 finite dimensional vector subbundle of $\mathbb{R}\times H \to \mathbb{R}$, since its rank is constant. For each t choose an orthonormal system of eigenvectors $v_j(t)$ of $A(t)$ corresponding to these $\lambda_j(t)$. They form a (not necessarily continuous) framing of this bundle. For any t near s and any sequence $t_k \to t$ there is a subsequence again denoted by t_k such that each $v_j(t_k) \to w_j(t)$ where the $w_i(t)$ form again an orthonormal system of eigenvectors of $A(t)$ for the sum $P(t)(H)$ of the eigenspaces of the $\lambda_i(t)$ (Here we use the local triviality of the vector bundle). Now consider

$$\frac{A(t)-\lambda_i(t)}{t_k - t} v_i(t_k) + \frac{A(t_k)-A(t)}{t_k - t} v_i(t_k) - \frac{\lambda_i(t_k)-\lambda_i(t)}{t_k - t} v_i(t_k) = 0. \tag{4}$$

For $t = s$ we take the inner product of (4) with each $w_j(s)$, note that then the first summand vanishes since all $\lambda_i(s)$ agree, and let $k \to \infty$ to obtain that (for $j \ne i$) the $w_i(s)$ are a basis of eigenvectors of $P(s)A'(s)|P(s)(H)$ with eigenvalues (for $j = i$) $\lim_k \frac{\lambda_i(t_k)-\lambda_i(s)}{t_k - s}$. By (2),

$$\lim_{h\searrow 0} \frac{\lambda_i(s+h)-\lambda_i(s)}{h} = \rho_i,$$

where the ρ_i are the eigenvalues of $P(s)A'(s)|P(s)(H)$ (with correct multiplicities). So the right handed derivative $\lambda_j^{(+)}(s)$ of each λ_j exists at s. Similarly the left handed derivative $\lambda_j^{(-)}(s)$ exists, and they form the same set of numbers with the correct multiplicities. Thus there exists a permutation σ of $\{1,\dots,N\}$ such that the

$$\nu_i(t) := \begin{cases} \lambda_i(t) & \text{for } t \le s \\ \lambda_{\sigma(i)}(t) & \text{for } t \ge s \end{cases} \tag{5}$$

parameterize all eigenvalues in the box by continuous functions which are differentiable at s.

For $t \neq s$, take the inner product of (4) with $w_i(t)$ to conclude that

(6) $\lambda_i^{(+)}(t) = \langle A'(t)w_i(t), w_i(t)\rangle$ *for a unit eigenvector* $w_i(t)$ *of* $A(t)$ *with eigenvalue* $\lambda_i(t)$.

Now we show claim (3) by induction on N. Let $t_1 \in I$ be such that not all $\lambda_i(t_1)$ agree. Then $\{1, \dots, N\}$ decomposes into the subsets $\{i : \lambda_i(t_1) = w\}$. Then for i and k in different subsets $\lambda_i(t) \neq \lambda_k(t)$ for all t in an open interval I_1 containing t_1. Thus by induction on each subset (3) holds on I_1.

Next let $I_2 \subseteq I$ be an open interval containing only points t_1 as above. Let J be a maximal open subinterval on which (3) holds. Assume for contradiction that the right (say) endpoint b of J belongs to I_2, then there is a C^1-parameterization of all N eigenvalues on an open interval I_b containing b by the argument above. Let $t_2 \in J \cap I_b$. Renumbering the C^1 parameterization to the right of t_2 suitably we may extend the C^1 parameterization beyond b, a contradiction. Thus (3) holds on I_2.

Now we consider the closed set $E = \{t \in I : \lambda_1(t) = \dots = \lambda_N(t)\}$. Then $I \setminus E$ is open, thus a disjoint union of open intervals on which there exists a C^1-parameterization μ_i of all eigenvalues. Consider first the set E' of all isolated points in E. Then $E' \cup (I \setminus E)$ is again open and thus a disjoint union of open intervals, and for each point $t \in E'$ we apply in turn the following arguments: extending all μ_i's by the single value at t we get a continuous extension near t. Then by (5), we may renumber the μ_i to the right of t in such a way that they fit together differentiably at t. The derivatives are also continuous at t: They have only finitely many clusterpoints for $t_k \to t$ by applying (6) to t_k and choosing a subsequence such that the $w_i(t_k)$ converge. Now we apply the arguments surrounding (4) with the $v_j(t_k)$ replaced by $w_j(t_k)$ to conclude that (6) converges to $\rho_i(t) = \mu_i'(t)$. Thus (3) holds on $E' \cup (I \setminus E)$.

We extend each μ_i to the whole of I by taking the single continuous function on $E \setminus E'$. Let $t \in E \setminus E'$. Then for the parameterization ν_i of (5) of all eigenvalues which is differentiable at t all derivatives $\nu_i'(t)$ agree since t is a cluster point of E. Thus also $\mu_i'(t)$ exists and equals $\nu_i'(t)$. So all μ_i are differentiable on I.

To see that μ_i' is continuous at $t \in E \setminus E'$, let $t_n \to t$ be such that $\mu_i'(t_n)$ converges (to a cluster point or $\pm\infty$). Then by (6) we have $\mu_i'(t_k) = \langle A'(t_k)w_i(t_k), w_i(t_k)\rangle$ for eigenvectors $w_i(t_k)$ of $A(t_k)$ with eigenvalue $\mu_i(t_k)$. Passing to a subsequence we may assume that the $w_i(t_k)$ converge to an orthonormal basis of eigenvectors of $A(t)$, then $\langle A'(t_k)w_i(t_k), w_i(t_k)\rangle$ converges to some of the equal eigenvalues ρ_i of $P(t)A'(t)|P(t)(H)$ which also equal the $\nu_i'(t)$.

So (3) is completely proved.

(7) *Claim. Let* I *be a compact interval. Let* $t \mapsto \lambda_i(t)$ *be a differentiable eigenvalue of* $A(t)$*, defined on some subinterval of* I*. Then*

$$|\lambda_i(t_1) - \lambda_i(t_2)| \leq (1 + |\lambda_i(t_2)|)(e^{a|t_1 - t_2|} - 1)$$

holds for a positive constant a *depending only on* I.

From (6) we conclude, where $V_t = (V, \|\quad\|_t)$,

$$\begin{aligned}|\lambda_i'(t)| &\le \|A'(t)\|_{L(V_t,H)}\|w_i(t)\|_{V_t}\|w_i(t)\|_H\\ &= \|A'(t)\|_{L(V_t,H)}\sqrt{\|w_i(t)\|_H^2 + \|A(t)w_i(t)\|_H^2}\cdot 1\\ &= \|A'(t)\|_{L(V_t,H)}\sqrt{1+\lambda_i(t)^2} \le C + C|\lambda_i(t)|,\end{aligned}$$

for a constant C since all norms $\|\quad\|_t$ are locally in t uniformly equivalent, see claim (1) above. By Gronwall's lemma (see e.g. [3], (10.5.1.3)) this implies claim (7).

By the following arguments we can conclude that all eigenvalues may be parameterized in a C^1 way. Let us first number all eigenvalues of $A(0)$ (increasingly, say).

We consider families of C^1-functions $(\mu_i)_{i\in\alpha}$ indexed by ordinals α, defined on open intervals I_i containing some fixed t_0, which parameterize eigenvalues.

The set of all these sequences is partially ordered by inclusion of ordinals and then by restriction of the component functions. Obviously for each increasing chain of such sequences the union is again such a sequence. By Zorn's lemma there exists a maximal family (μ_i).

We claim that for any maximal family each component function μ_i is globally defined: If not let $b<\infty$ be the (right, say) boundary point of I_i. By claim (7) the limit $\lim_{t\nearrow b}\mu_i(t) =: z$ exists. By claim (3) there exists a box $(b-\delta, b+\delta)\times(z-\varepsilon, z+\varepsilon)$ such that all eigenvalues λ with $|\lambda - z|<\varepsilon$ of $A(t)$ for $|t-b|<\delta$ are parameterized by C^1 functions $\lambda_i : (b-\delta, b+\delta)\to(z-\varepsilon, z+\varepsilon)$ (with multiplicity). Consider the μ_j hitting this box (at the vertical boundaries only). The endpoints of the corresponding intervals I_j give a partition of $(b-\delta, b+\delta)$ into finitely many subintervals. We apply the lemma below on each subinterval and glue at the ends of the subintervals in C^1-fashion using (5) to obtain an extension of at least μ_i, so the family was not maximal.

Finally we claim that any maximal family (μ_i) parameterizes all eigenvalues of $A(t)$ with right multiplicities, for each $t\in\mathbb{R}$. If not, there is an eigenvalue z of $A(t_0)$ with $|\{i : \mu_i(t_0) = z\}|$ less than the multiplicity of z. By claim (3) and the lemma below we can then conclude again that the sequence was not maximal. □

Lemma. *Suppose that $\lambda_1,\dots,\lambda_N$ are real-valued C^1 (twice differentiable) functions defined on an interval I, and that $\mu_1,\dots,\mu_k$ for $k\le N$ are also C^1 (twice differentiable) functions on I such that $|\{j : \mu_j(t) = z\}| \le |\{i : \lambda_i(t) = z\}|$ for all $t\in I$ and $z\in\mathbb{R}$.*

Then there exist C^1 (twice differentiable) functions $\mu_{k+1},\dots,\mu_N$ on I such that for all $t\in I$ and $z\in\mathbb{R}$ we have $|\{j : 1\le j\le N, \mu_j(t) = z\}| = |\{i : \lambda_i(t) = z\}|$.

Proof. We treat the case C^1 and indicate the necessary changes in brackets for the twice differentiable case.

We use induction on N. Let us assume that the statement is true if the number of functions is less than N.

First suppose that for given $t_1\in I$ not all $\lambda_i(t_1)$ agree. Then for $i\in\{k : \lambda_1(t_1) = \lambda_k(t_1)\}\not\ni j$ we have $\lambda_i(t)\ne\lambda_j(t)$ for all t in an open interval I_1 containing t_1, and similarly for the μ_j. Thus by induction for both groups the statement holds on I_1.

Now suppose that for no point t in I we have $\lambda_1(t) = \dots = \lambda_N(t)$. Let I_1 be a maximal open subinterval of I for which the statement is true with functions μ_i^1 for $i > k$. Assume for contradiction that the right (say) endpoint b of I_1 is an interior point of I. By the first case, the statement holds for an open neighborhood I_2 of b, with functions μ_i^2 for $i > k$. Let $t_0 \in I_1 \cap I_2$. We may continue each solution μ_i^1 in $\{t \in I_1 : t \le t_0\}$ by a suitable solution $\mu_{\pi(i)}^2$ on $\{t \in I_2 : t \ge t_0\}$ for a suitable permutation π: Let $t_m \nearrow t_0$. For every m there exists a permutation π of $\{1, \dots, n\}$ such that $\mu_{\pi(i)}^2(t_m) = \mu_i^1(t_m)$ for all i. By passing to a subsequence, again denoted t_m, we may assume that the permutation does not depend on m. By passing again to a subsequence we may also assume that $(\mu_{\pi(i)}^2)'(t_m) = (\mu_i^1)'(t_m)$ (and in the twice differentiable case, again for a subsequence, that $(\mu_{\pi(i)}^2)''(t_m) = (\mu_i^1)''(t_m)$) for all i and all m. So we may paste $\mu_{\pi(i)}^2(t)$ for $t \ge t_0$ with $\mu_i^1(t)$ for $t < t_0$ to obtain a C^1 (twice differentiable) parameterization on an interval larger than I_1, a contradiction.

In the general case, we consider the closed set $E = \{t \in I : \lambda_1(t) = \dots = \lambda_N(t)\}$. Then $I \setminus E$ is open, thus a disjoint union of open intervals. By the second case the result holds on each of these open intervals. Consider first the set E' of all isolated points in E. Then $E' \cup (I \setminus E)$ is again open and thus a union of open intervals, and for each point $t \in E'$ we may renumber the μ_i to the right of t in such a way that they fit together C^1 (twice differentiable) at t. Thus the result holds on $E' \cup (I \setminus E)$.

We extend each μ_i to the whole of I by taking the single continuous function on $E \setminus E'$. Let $t \in E \setminus E'$. Then all $\lambda_i'(t) =: \lambda'(t)$ agree since t is cluster point of E (and all $\lambda_i''(t) =: \lambda''(t)$ agree by considering second order difference quotients on points in E). Thus μ_i is (twice) differentiable at t with $\mu_i'(t) = \lambda'(t)$ (and $\mu_i''(t) = \lambda''(t)$).

In the C^1 case we have still to check that μ_i' is continuous at $t \in E \setminus E'$: Let $t_n \to t$, then $\mu_i'(t_n) = \lambda'_{\sigma_n(i)}(t_n) \to \lambda'(t) = \mu_i'(t)$. □

Proof of (C2)**.** By assumption the resolvent $(A(t) - z)^{-1}$ is C^{3n} jointly in (t, z) where n may be ∞.

(8) *Claim. Let z be an eigenvalue of $A(s)$ of multiplicity $N \le n$. Then there exists an open box $(z - \varepsilon, z + \varepsilon) \times (s - \delta, s + \delta)$ and twice differentiable functions $\mu_1, \dots, \mu_N : (s - \delta, s + \delta) \to (z - \varepsilon, z + \varepsilon)$ which parameterize all eigenvalues λ with $|\lambda - z| < \varepsilon$ of $A(t)$ for $|t - s| < \delta$ with correct multiplicities.*

We choose a simple closed smooth curve γ in the resolvent set of $A(s)$ for fixed s enclosing only z among all eigenvalues of $A(s)$. As in the proof of claim (3) we see that

$$t \mapsto -\frac{1}{2\pi i} \int_\gamma (A(t) - z)^{-1}\, dz =: P(t, \gamma) = P(t)$$

is a C^{3n} curve of projections with finite dimensional ranges of constant rank.

So for t near s, there are equally many eigenvalues in the interior of γ, and we may call them $\mu_i(t)$ for $1 \le i \le N$ (repeated with multiplicity). Let us denote by $e_i(t)$ for $1 \le i \le N$ a corresponding system of eigenvectors of $A(t)$. Then by the residue theorem we have

$$\sum_{i=1}^N \mu_i(t)^p e_i(t) \langle e_i(t), \quad \rangle = -\frac{1}{2\pi i} \int_\gamma z^p (A(t) - z)^{-1}\, dz$$

which is C^{3n} in t near s, as a curve of operators in $L(H,H)$ of rank N.

(9) *Claim. Let $t \mapsto T(t) \in L(H,H)$ be a C^{3n} curve of operators of rank N in Hilbert space such that $T(0)T(0)(H) = T(0)(H)$. Then $t \mapsto \operatorname{Trace}(T(t))$ is C^{3n} near 0.*

This is claim 2 from [1], 7.8 for C^{3n} instead of C^∞. We conclude that the Newton polynomials

$$s_p(t) := \sum_{i=1}^{N} \mu_i(t)^p = -\frac{1}{2\pi i} \operatorname{Trace} \int_\gamma z^p (A(t) - z)^{-1}\, dz$$

are C^{3n} for t near s. Hence also the elementary symmetric polynomials

$$\sigma_p(t) = \sum_{i_1 < \dots < i_N} \mu_{i_1}(t) \dots \mu_{i_p}(t)$$

are C^{3n}, and thus $\{\mu_i(t) : 1 \le i \le N\}$ is the set of roots of a polynomial of degree $N \le n$ with C^{3n} coefficients. By [7] there is an arrangement of these roots such that they become twice differentiable. So claim (8) follows.

The end of the proof is now similar to the end of the proof of (D1), where one uses claim (7) (from the proof of (D1)), claim (8) instead of (3), and the lemma above. $\square$

References

[1] Alekseevsky, Dmitri; Kriegl, Andreas; Losik, Mark; Michor, Peter W., *Choosing roots of polynomials smoothly*, Israel J. Math. **105** (1998), 203-233, math.CA/9801026.

[2] Baumgärtel, Hellmut, *Endlichdimensionale analytische Störungstheorie*, Akademie-Verlag, Berlin, 1972.

[3] Dieudonné, Jean, *Foundations of modern analysis, I*, Academic Press, New York, 1960.

[4] Faure, Claude-Alain; Frölicher, Alfred, *Hölder differentiable maps and their function spaces.*, Categorical topology and its relation to analysis, algebra and combinatorics (Prague, 1988), World Sci. Publishing, Teaneck, NJ, 1989, pp. 135-142.

[5] Frölicher, Alfred; Kriegl, Andreas, *Linear spaces and differentiation theory*, Pure and Applied Mathematics, J. Wiley, Chichester, 1988.

[6] Kato, Tosio, *Perturbation theory for linear operators*, Grundlehren 132, Springer-Verlag, Berlin, 1976.

[7] Kriegl, Andreas; Losik, Mark; Michor, Peter W., *Choosing roots of polynomials smoothly, II* (2002), arXiv:math.CA/0208228.

[8] Kriegl, Andreas; Michor, Peter W., *The Convenient Setting of Global Analysis*, Surveys and Monographs, Vol. 53, AMS, Providence, 1997, http://www.ams.org/online_bks/surv53/.

[9] Rellich, Franz, *Störungstheorie der Spektralzerlegung, I*, Math. Ann. **113** (1937), 600–619.

[10] Rellich, Franz, *Störungstheorie der Spektralzerlegung, V*, Math. Ann. **118** (1940), 462–484.

[11] Rellich, Franz, *Perturbation theory of eigenvalue problems*, Lecture Notes, New York University, 1953; Gordon and Breach, New York, London, Paris, 1969.

A. Kriegl: Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria

E-mail address: Andreas.Kriegl@univie.ac.at

P. W. Michor: Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria; *and*: Erwin Schrödinger Institute of Mathematical Physics, Boltzmanngasse 9, A-1090 Wien, Austria.

E-mail address: Peter.Michor@esi.ac.at