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LIFTING SMOOTH CURVES OVER INVARIANTS FOR REPRESENTATIONS OF COMPACT LIE GROUPS, III

ANDREAS KRIEGL, MARK LOSIK, PETER W. MICHOR, AND ARMIN RAINER

Abstract. Any sufficiently often differentiable curve in the orbit space V/G of a real finite dimensional orthogonal representation $G \to O(V)$ of a finite group G admits a differentiable lift into the representation space V with locally bounded derivative. As a consequence any sufficiently often differentiable curve in the orbit space V/G can be lifted twice differentiably which is in general best possible. These results can be generalized to arbitrary polar representations. Finite reflection groups and finite rotation groups in dimensions two and three are discussed in detail.

1. Introduction

In [2] and [18] the following problem was investigated. Consider an orthogonal representation of a compact Lie group G on a real finite dimensional Euclidean vector space V. Let $\sigma_1, \dots, \sigma_n$ be a system of homogeneous generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V. Then the mapping $\sigma = (\sigma_1, \dots, \sigma_n) : V \to \mathbb{R}^n$ induces a homeomorphism between the orbit space V/G and the semialgebraic set $\sigma(V)$. Suppose a smooth curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ in the orbit space is given (smooth as curve in $\mathbb{R}^n$), does there exist a smooth lift to V, i.e., a smooth curve $\bar{c} : \mathbb{R} \to V$ with $c = \sigma \circ \bar{c}$? The answer is independent of the choice of the generators.

Note that in general no $C^{1,\alpha}$-lift for any $\alpha > 0$ exists which is shown by examples in [1], [5], and [13]. Thus twice differentiability is the best regularity condition for lifts one can expect in general.

It was shown in [2] that a real analytic curve in V/G admits a local real analytic lift to V, and that a smooth curve in V/G admits a global smooth lift, if certain genericity conditions are satisfied. In both cases the lifts may be chosen orthogonal to each orbit they meet and then they are unique up to a transformation in G, whenever the representation of G on V is polar, i.e., admits sections.

In [18] we proved that any sufficiently often differentiable curve in the orbit space V/G can be lifted to a once differentiable curve in V.

In the special case that the symmetric group S_n is acting on $\mathbb{R}^n$ by permuting the coordinates there is the following interpretation of the described lifting problem. As generators of $\mathbb{R}[\mathbb{R}^n]^{S_n}$ we may take the elementary symmetric functions

$$\sigma_j(x) = \sum_{1 \le i_1 < \cdots < i_j \le n} x_{i_1} \cdots x_{i_j} \qquad (1 \le j \le n),$$

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which constitute the coefficients (up to sign) of a monic polynomial P with roots $x_1, \dots, x_n$ via Vieta's formulas. Then a curve in the orbit space $\mathbb{R}^n/S_n = \sigma(\mathbb{R}^n)$ corresponds to a curve $P(t)$ of monic polynomials of degree n with only real roots, and a lift of $P(t)$ may be interpreted as a parameterization of the roots of $P(t)$.

This problem has been studied extensively in [1]. Moreover, the following results were proved in [17]: Any differentiable lift (parameterization of roots) of a C^{2n}-curve (of polynomials) $P : \mathbb{R} \to \mathbb{R}^n/S_n$ is actually C^1, and there always exists a twice differentiable but in general not better lift of P, if P is of class C^{3n}. Note that here the differentiability assumptions on P are not the weakest possible which is shown by the case $n = 2$, elaborated in [1] 2.1. The proof in [17] is based on the fact that the roots of a C^n-curve of polynomials $P : \mathbb{R} \to \mathbb{R}^n/S_n$ may be chosen differentiable with locally bounded derivative; this is due to Bronshtein [7] and Wakabayashi [34].

In the present paper we show the corresponding statements for arbitrary real finite dimensional orthogonal representations of finite groups. We consider representations $\rho : G \to O(V)$ with the *property* $(\mathcal{B}_k)$ that any C^k-curve in a neighborhood of 0 in the orbit space V/G admits a local differentiable lift to V with locally bounded derivative (section 3). In analogy to the polynomial case $S_n : \mathbb{R}^n$ we then show that, for representations of finite groups G with property $(\mathcal{B}_k)$, any differentiable lift of a C^{k+d}-curve is actually C^1 (section 4), and there always exists a twice differentiable lift of a C^{k+2d}-curve in the orbit space (section 5). The integer d is the maximal degree of a minimal system of homogeneous generators of the algebra of invariant polynomials $\mathbb{R}[V]^G$ (see 2.4). As a consequence we obtain in section 6 that polar representations, where the representation of the associated generalized Weyl group on some section has property $(\mathcal{B}_k)$, allow orthogonal C^1-lifts of C^{k+d}-curves and orthogonal twice differentiable lifts of C^{k+2d}-curves. In section 7 we show that property $(\mathcal{B}_k)$ is stable under subrepresentations and orthogonal direct sums. We prove in section 8, by reducing to the polynomial case, that any real finite dimensional representation $\rho : G \to O(V)$ of a finite group G has property $(\mathcal{B}_k)$, where

$$k = \max\{d, |G|/|G_{v_i}| : 1 \le i \le l\},$$

$v_i \in V_i \backslash \{0\}$ are chosen such that the cardinality of the isotropy groups G_{v_i} is maximal, and $V = V_1 \oplus \cdots \oplus V_l$ is the decomposition into irreducible subrepresentations. This establishes property $(\mathcal{B}_k)$ for polar representations with appropriate k, too. In section 9 we give a complete survey of all finite reflection groups. Section 10 is devoted to the discussion of finite rotation groups in dimensions two and three.

Still open is the question whether non-polar representations of compact Lie groups G on real finite dimensional Euclidean vector spaces V have property $(\mathcal{B}_k)$ for some $k \le \infty$.

The polynomial results have applications in the theory of partial differential equations and perturbation theory, see [19].

2. Preliminaries

2.1. **The setting.** Let G be a compact Lie group and let $\rho : G \to O(V)$ be an orthogonal representation in a real finite dimensional Euclidean vector space V with inner product $\langle \quad | \quad \rangle$. By a classical theorem of Hilbert and Nagata, the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V is finitely generated. So let $\sigma_1, \dots, \sigma_n$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ of positive degrees $d_1, \dots, d_n$. Consider the *orbit map* $\sigma = (\sigma_1, \dots, \sigma_n) : V \to \mathbb{R}^n$. Note that, if $(y_1, \dots, y_n) = \sigma(v)$ for $v \in V$, then $(t^{d_1} y_1, \dots, t^{d_n} y_n) = \sigma(tv)$ for $t \in \mathbb{R}$, and that $\sigma^{-1}(0) = \{0\}$. The image $\sigma(V)$ is a semialgebraic set in $\{y \in \mathbb{R}^n : P(y) = 0 \text{ for all } P \in I\}$ where I

is the ideal of relations between $\sigma_1, \dots, \sigma_n$. Since G is compact, σ is proper and separates orbits of G, it thus induces a homeomorphism between V/G and $\sigma(V)$.

2.2. **The problem of lifting curves.** Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a smooth curve in the orbit space; smooth as curve in $\mathbb{R}^n$. A curve $\bar{c} : \mathbb{R} \to V$ is called lift of c to V, if $c = \sigma \circ \bar{c}$ holds. *The problem of lifting smooth curves over invariants is independent of the choice of a system of homogeneous generators of* $\mathbb{R}[V]^G$ in the following sense: Suppose $\sigma_1, \dots, \sigma_n$ and $\tau_1, \dots, \tau_m$ both generate $\mathbb{R}[V]^G$. Then for all i and j we have $\sigma_i = p_i(\tau_1, \dots, \tau_m)$ and $\tau_j = q_j(\sigma_1, \dots, \sigma_n)$ for polynomials p_i and q_j. If $c^\sigma = (c_1, \dots, c_n)$ is a curve in $\sigma(V)$, then $c^\tau = (q_1(c^\sigma), \dots, q_m(c^\sigma))$ defines a curve in $\tau(V)$ of the same regularity. Any lift $\bar{c}$ to V of the curve c^σ, i.e., $c^\sigma = \sigma \circ \bar{c}$, is a lift of c^τ as well (and conversely):

$$c^\tau = (q_1(c^\sigma), \dots, q_m(c^\sigma)) = (q_1(\sigma(\bar{c})), \dots, q_m(\sigma(\bar{c}))) = (\tau_1(\bar{c}), \dots, \tau_m(\bar{c})) = \tau \circ \bar{c}.$$

2.3. **The slice theorem.** For a point $v \in V$ we denote by G_v its isotropy group and by $N_v = T_v(G.v)^\perp$ the normal subspace of the orbit $G.v$ at v. It is well known that there exists a G-invariant open neighborhood U of v which is real analytically G-isomorphic to the crossed product (or associated bundle) $G \times_{G_v} S_v = (G \times S_v)/G_v$, where S_v is a ball in N_v with center at the origin. The quotient U/G is homeomorphic to S_v/G_v. It follows that the problem of local lifting curves in V/G passing through $\sigma(v)$ reduces to the same problem for curves in N_v/G_v passing through 0. For more details see [2], [20], and [29] theorem 1.1.

A point $v \in V$ (and its orbit $G.v$ in V/G) is called *regular* if the slice representation $G_v \to O(N_v)$ is trivial. Hence a neighborhood of this point is analytically G-isomorphic to $G/G_v \times S_v \cong G.v \times S_v$. The set V_{reg} of regular points is open and dense in V, and the projection $V_{\text{reg}} \to V_{\text{reg}}/G$ is a locally trivial fiber bundle. A non regular orbit or point is called *singular*.

2.4. **The integer d.** Let $\rho : G \to O(V)$ be as in 2.1. Choose a minimal system of homogeneous generators $\sigma_1, \dots, \sigma_n$ of positive degrees $d_1, \dots, d_n$ of $\mathbb{R}[V]^G$. We associate to ρ the following number:

$$d = d(\rho) := \max\{d_1, \dots, d_n\}.$$

The integer d is well-defined and independent of the choice of a minimal system of homogeneous generators of the algebra of invariant polynomials. This follows from the fact that a system of homogeneous invariants of positive degree generates $\mathbb{R}[V]^G$ as an algebra over $\mathbb{R}$ if and only if the images of the invariants in this system generate $\mathbb{R}[V]^G_+/(\mathbb{R}[V]^G_+)^2$ as a vector space over $\mathbb{R}$, where $\mathbb{R}[V]^G_+$ is the set of invariant polynomials vanishing at 0, e.g. [11], 3.6. The grading used here is given by the degree of the polynomials. Hence a system of homogeneous algebra generators has minimal cardinality if no generator is superfluous, and the number and the degrees of the elements in a minimal system of homogeneous generators are uniquely determined.

Note that independence of d from the choice of a minimal system of homogeneous generators of $\mathbb{R}[V]^G$ also follows from the following lemma applied to the slice representation at 0.

Lemma. *Let $\rho : G \to O(V)$ be a finite dimensional representation of a compact Lie group G, let ρ' be some slice representation of ρ. Then, $d(\rho') \le d(\rho)$.*

Proof. Let $\sigma_1, \dots, \sigma_n$ be a minimal system of homogeneous generators of $\mathbb{R}[V]^G$.

For an arbitrary $v \in V$ let $\rho' : G_v \to O(N_v)$ be its slice representation, and suppose S_v is a normal slice at v. Choose a minimal system of homogeneous generators

$\tau_1, \dots, \tau_m$ of $\mathbb{R}[N_v]^{G_v}$ and assume that $\deg \tau_1 \le \dots \le \deg \tau_m = d(\rho')$. Then there exist polynomials $p_i \in \mathbb{R}[\mathbb{R}^m]$ such that

$$\sigma_i|_{S_v} = p_i(\tau_1|_{S_v}, \dots, \tau_m|_{S_v}) \qquad (1 \le i \le n).$$

On the other hand, by the slice theorem, near $v \in N_v$ we have

$$\tau_j|_{S_v} = f_j(\sigma_1|_{S_v}, \dots, \sigma_n|_{S_v}) \qquad (1 \le j \le m),$$

where f_j are real analytic functions; e.g. [30].

For contradiction assume $\deg \tau_m > d(\rho)$. Then all polynomials p_i do not depend on their last entry. Consequently, near $v \in N_v$,

$$\tau_m|_{S_v} = F(\tau_1|_{S_v}, \dots, \tau_{m-1}|_{S_v}),$$

where

$$F = f_m(p_1, \dots, p_n)$$

is real analytic. Introduce a new grading in $\mathbb{R}[\mathbb{R}^{m-1}]$ with respect to $\deg \tau_1 \le \dots \le \deg \tau_{m-1}$ and write the function F as an infinite sum of homogeneous (with respect to this grading) terms. Let $\bar{F}$ be the sum of all terms of degree $\deg \tau_m$ in this presentation of F. We obtain, near $v \in N_v$,

$$\tau_m|_{S_v} = \bar{F}(\tau_1|_{S_v}, \dots, \tau_{m-1}|_{S_v}).$$

This means τ_m is a polynomial in $\tau_1, \dots, \tau_{m-1}$ in a neighborhood of v in N_v, and, hence, everywhere. This contradicts minimality of $\tau_1, \dots, \tau_m$. □

Remark. The previous lemma allows to replace the intricate definition of the integer d given in [18] by the definition given above.

2.5. **Removing fixed points.** Let V^G be the space of G-invariant vectors in V, and let V' be its orthogonal complement in V. Then we have $V = V^G \oplus V'$, $\mathbb{R}[V]^G = \mathbb{R}[V^G] \otimes \mathbb{R}[V']^G$, and $V/G = V^G \times V'/G$.

Lemma. *Any lift $\bar{c}$ of a curve $c = (c_0, c_1)$ of class C^k $(k = 0, 1, \dots, \infty, \omega)$ in $V^G \times V'/G$ has the form $\bar{c} = (c_0, \bar{c}_1)$, where $\bar{c}_1$ is a lift of c_1 to V' of class C^k. Then the lift $\bar{c}$ is orthogonal if and only if the lift $\bar{c}_1$ is orthogonal.* □

If $V^G = \{0\}$ we may assume that $\sigma_1 : v \mapsto \langle v \mid v \rangle$ is the inner product.

2.6. **Multiplicity.** For a continuous function f defined near 0 in $\mathbb{R}$, let the *multiplicity* or *order of flatness* $m(f)$ at 0 be the supremum of all integers p such that $f(t) = t^p g(t)$ near 0 for a continuous function g. If f is C^n and $m(f) < n$, then $f(t) = t^{m(f)} g(t)$, where now g is $C^{n-m(f)}$ and $g(0) \ne 0$. Similarly, one can define multiplicity of a function at any $t \in \mathbb{R}$.

Lemma. *Let $c = (c_1, \dots, c_n)$ be a curve in $\sigma(V) \subseteq \mathbb{R}^n$ of class C^r, where $r \ge d$, and $c(0) = 0$. Then the following two conditions are equivalent:*

(1) *$c_1(t) = t^2 c_{1,1}(t)$ near 0 for a C^{r-2}-function $c_{1,1}$;*
(2) *$c_i(t) = t^{d_i} c_{i,i}(t)$ near 0 for a C^{r-d_i}-function $c_{i,i}$, for all $1 \le i \le n$.*

Proof. The proof is essentially the same as that of lemma 3.3 in [2]. □

2.7. We recall a few facts from [18]:

Lemma (a). *A curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ of class C^d admits an orthogonal C^d-lift $\bar{c}$ in a neighborhood of a regular point $c(t_0) \in V_{\text{reg}}/G$. It is unique up to a transformation from G.*

Lemma (b). *Consider a continuous curve $c : (a, b) \to X$ in a compact metric space X. Then the set A of all accumulation points of $c(t)$ as $t \searrow a$ is connected.*

Theorem. *Let $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve of class C^d. Then there exists a global differentiable lift $\bar{c} : \mathbb{R} \to V$ of c.*

2.8. Bronshtein's and Wakabayashi's result. We formulate Bronshtein's theorem [7]; compare also with Wakabayashi's version [34].

Theorem. [7] *Let*

$$P(t)(x) = x^n + \sum_{j=1}^{n} (-1)^j a_j(t) x^{n-j}$$

be a curve of monic polynomials of degree n with all roots real for all $t \in \mathbb{R}$, where $a_j \in C^n(\mathbb{R})$ for all $1 \le j \le n$. Choose a differentiable parameterization $x_1(t), \dots, x_n(t)$ of the roots of $P(t)$ (which always exists). Then, for any compact subset $K \subseteq \mathbb{R}$ there exists a constant C_K such that

$$\left| \frac{d}{dt} x_j(t) \right| \le C_K \qquad \text{for all } t \in K, 1 \le j \le n.$$

In the language of representation theory: Any C^n-curve P in the orbit space $\mathbb{R}^n/S_n$ of the standard representation of the symmetric group S_n on $\mathbb{R}^n$, by permuting the coordinates, allows a differentiable lift $x = (x_1, \dots, x_n)$ with locally bounded derivative.

3. Property ($\mathcal{B}$)

3.1. Property ($\mathcal{B}$). We shall say that an orthogonal representation $\rho : G \to O(V)$ of a compact Lie group G on a real finite dimensional Euclidean vector space V has *property* ($\mathcal{B}_k$), if:

> There exists a neighborhood $U = U(\rho)$ of 0 in $V/G = \sigma(V)$ such that each C^k-curve in U admits a local differentiable lift $\bar{c}$ to V with locally bounded derivative.

Note that property ($\mathcal{B}_k$) is independent of the choice of generators of $\mathbb{R}[V]^G$.

It is clear that, if a representation ρ has property ($\mathcal{B}_k$), then it has property ($\mathcal{B}_l$) for all $l \in \{k, k+1, \dots, \infty, \omega\}$ as well.

We shall write simply property ($\mathcal{B}$), if the degree of differentiability k is not specified.

Example. The standard representation of the symmetric group S_n on $\mathbb{R}^n$ has property ($\mathcal{B}_n$). This follows from theorem 2.8.

Proposition 3.2. *Let $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve of class C^k in the orbit space of a representation $\rho : G \to O(V)$ with property ($\mathcal{B}_k$). Then for any $t_0 \in \mathbb{R}$ there exists a local differentiable lift $\bar{c}$ of c near t_0 with locally bounded derivative.*

Proof. For each $s \in \mathbb{R}\backslash\{0\}$ let us define a C^k-curve $c_s : \mathbb{R} \to \sigma(V)$ by

$$c_s(t) = (s^{d_1} c_1(t), \dots, s^{d_n} c_n(t)).$$

There exists some $s = s(c; t_0) \in \mathbb{R}\backslash\{0\}$ such that $c_s(t) \in U$ for t near t_0, where U is the neighborhood of 0 in V/G introduced in the definition of property ($\mathcal{B}_k$). Since ρ has property ($\mathcal{B}_k$), there exists, near t_0, a local differentiable lift $\bar{c}_s$ of c_s to V with locally bounded derivative. Then, $\bar{c}(t) := s^{-1} \cdot \bar{c}_s(t)$ defines a local differentiable lift of c for t near t_0 whose derivative is locally bounded. □

Proposition 3.3. *Assume that $\rho : G \to O(V)$ is a representation of a finite group G with property ($\mathcal{B}_k$). Then any slice representation ρ' of ρ has property ($\mathcal{B}_k$) as well.*

Proof. Let $\rho' : G_v \to O(N_v)$ be an arbitrary slice representation of ρ. Consider some normal slice S_v at v for the G-action on V. Then S_v/G_v is an open neighborhood of 0 in N_v/G_v which by 2.3 is homeomorphic to $(G \times_{G_v} S_v)/G$ which in turn is an open neighborhood of $G.v$ in V/G.

Given a C^k-curve c in S_v/G_v, we may view it as a curve in $(G \times_{G_v} S_v)/G$. Since ρ has property $(\mathcal{B}_k)$ and by proposition 3.2, there exists a local differentiable lift $\bar{c}$ of c to V with locally bounded derivative. The finiteness of G implies that $N_v = V$, and hence S_v is an open neighborhood of v in V. Therefore $\bar{c}$ is a local lift of c to N_v with respect to the G_v-action. □

Lemma 3.4. *Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve in the orbit space V/G. We assume that G is finite. Let $t_0 \in \mathbb{R}$. If $\bar{c}_1$ and $\bar{c}_2$ are lifts of c which are (one-sided) differentiable at t_0 and $\bar{c}_1(t_0) = \bar{c}_2(t_0)$, then there exists some $g \in G_{\bar{c}_1(t_0)}$ such that $\bar{c}_1'(t_0) = g.\bar{c}_2'(t_0)$.*

Proof. Without loss we can assume that $t_0 = 0$.

Let $\bar{c}_1$ and $\bar{c}_2$ be lifts of $c : \mathbb{R} \to V/G$ which are (one-sided) differentiable at 0 and satisfy $\bar{c}_1(0) = \bar{c}_2(0) =: v_0$. We may suppose $V^G = \{0\}$, by lemma 2.5. We consider the following cases separately:

If $c(0) = 0$, then $\bar{c}_1(0) = \bar{c}_2(0) = 0$ and consequently, for $i = 1, 2$,

$$\sigma(\bar{c}_i'(0)) = \sigma\left(\lim_{t\to 0} \frac{\bar{c}_i(t)}{t}\right) = \lim_{t\to 0} \sigma\left(\frac{\bar{c}_i(t)}{t}\right).$$

Now, for $t \neq 0$ we have $\sigma\left(\bar{c}_i(t)/t\right) = c_{(1)}(t) \in \sigma(V)$, where

$$c_{(1)}(t) := (t^{-d_1} c_1(t), \dots, t^{-d_n} c_n(t)).$$

Since $\sigma(V)$ is closed in $\mathbb{R}^n$ (see [28]), we find

$$\sigma(\bar{c}_i'(0)) = \lim_{t\to 0} \sigma\left(\frac{\bar{c}_i(t)}{t}\right) = \lim_{t\to 0} c_{(1)}(t) \in \sigma(V),$$

i.e., σ maps $\bar{c}_1'(0)$ and $\bar{c}_2'(0)$ to the same point in $\sigma(V)$. (Note that, if only one-sided derivatives exist, then $t \to 0$ has to be replaced by $t \nearrow 0$ or $t \searrow 0$, respectively.) This shows that $\bar{c}_1'(0)$ and $\bar{c}_2'(0)$ lie in the same orbit, therefore we find some $g \in G = G_0$ with $\bar{c}_1'(0) = g.\bar{c}_2'(0)$.

If $c(0) \neq 0$: Since G is finite and therefore $N_{v_0} = V$, the ball S_{v_0} is a neighborhood of v_0 in V which contains the lifts $\bar{c}_1(t)$ and $\bar{c}_2(t)$ for t near 0. Hence, by 2.3, we may change to the slice representation $G_{v_0} \to O(N_{v_0})$. Now we may assume that c is a curve in N_{v_0}/G_{v_0} with $c(0) = 0$ and with lifts $\bar{c}_1(t)$ and $\bar{c}_2(t)$ to N_{v_0} for t near 0. So we refer to the former case. □

Note that lemma 3.4 does no longer hold, if finiteness of G is omitted:

Example. Consider the standard action of $SO(2)$ on $\mathbb{R}^2$. Then $\sigma(x_1, x_2) = x_1^2 + x_2^2$ generates $\mathbb{R}[\mathbb{R}^2]^{SO(2)}$ and $\mathbb{R}^2/SO(2) = \sigma(\mathbb{R}^2) = [0, \infty)$. We consider the curve $c(t) = t^2$ and its differentiable lifts $\bar{c}_1(t) = (t, 0)$ and $\bar{c}_2(t) = (t\cos t, t\sin t)$. We find $\bar{c}_1(2\pi) = \bar{c}_2(2\pi) = (2\pi, 0)$, but $\bar{c}_1'(2\pi) = (1, 0)$ and $\bar{c}_2'(2\pi) = (1, 2\pi)$ cannot be transformed to each other by an element of $G_{(2\pi,0)} = \{\mathrm{id}\}$.

Remark. If G is not finite, then lemma 3.4 generalizes to the following statement: *Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve in the orbit space V/G. Let $t_0 \in \mathbb{R}$. If $\bar{c}_1$ and $\bar{c}_2$ are lifts of c which are (one-sided) differentiable at t_0 and $\bar{c}_1(t_0) = \bar{c}_2(t_0) =: v_0$, then there exists some $g \in G_{v_0}$ such that $\bar{c}_1'(t_0)^\perp = g.\bar{c}_2'(t_0)^\perp$, where $\perp$ indicates the projection onto N_{v_0}.*

To see this: We consider the projection $p : G.S_{v_0} \cong G \times_{G_{v_0}} S_{v_0} \to G/G_{v_0} \cong G.v_0$ of a fiber bundle associated to the principal bundle $\pi : G \to G/G_{v_0}$, where S_{v_0} is a normal slice at v_0. Then, for t close to t_0, $\bar{c}_1$ and $\bar{c}_2$ are curves in $G.S_{v_0}$, whence $p \circ \bar{c}_i$ $(i = 1, 2)$ are curves in G/G_{v_0} which admit lifts g_i into G with $g_i(t_0) = e$, which are (one-sided) differentiable at t_0 (via the horizontal lift of the principal connection, say). Consequently, $t \mapsto g_i(t)^{-1}.\bar{c}_i(t)$ are lifts which lie in S_{v_0}, whence $\frac{d}{dt}\big|_{t=t_0} (g_i(t)^{-1}.\bar{c}_i(t)) = -g_i'(t_0).v_0 + \bar{c}_i'(t_0) \in N_{v_0}$. Thus, $\bar{c}_i'(t_0)^\perp = \frac{d}{dt}\big|_{t=t_0} (g_i(t)^{-1}.\bar{c}_i(t))$. By this observation, we may assume without loss that the lifts $\bar{c}_1$ and $\bar{c}_2$ lie in S_{v_0} for t close to t_0. Then the proof of lemma 3.4 gives the statement.

Remark 3.5. Lemma 3.4 implies that for any two differentiable lifts $\bar{c}_1$ and $\bar{c}_2$ of a curve c in V/G, where G is finite, we have $\|\bar{c}_1'(t)\| = \|\bar{c}_2'(t)\|$ for all t. So, if there exists some differentiable lift of c with locally bounded derivative, then any differentiable lift of c has this property as well.

Proposition 3.6. *Assume that $\rho : G \to O(V)$ is a representation of a finite group G with property $(\mathcal{B}_k)$. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve of class C^k. Then there exists a global differentiable lift $\bar{c}$ of c to V with locally bounded derivative.*

Proof. Proposition 3.2 provides local differentiable lifts of c with locally bounded derivative near any $t \in \mathbb{R}$.

Now let us construct from these data a global differentiable lift of c with locally bounded derivative: We glue the local differentiable lifts with locally bounded derivative differentiably. The derivative of the resulting global differentiable lift of c is then evidently locally bounded. It is sufficient to show that each local differentiable lift of c defined on an open interval I can be extended to a larger interval whenever $I \neq \mathbb{R}$.

Suppose that $\bar{c}_1 : I \to V$ is a local differentiable lift of c, and suppose the open interval I is bounded from above (say), and t_1 is its upper boundary point. Then, there exists a local differentiable lift $\bar{c}_2$ of c near t_1, and a $t_2 < t_1$ such that both $\bar{c}_1$ and $\bar{c}_2$ are defined near t_2. There is some $g \in G$ such that $\bar{c}_1(t_2) = g.\bar{c}_2(t_2)$. By lemma 3.4, we find an $h \in G_{\bar{c}_1(t_2)}$ with $\bar{c}_1'(t_2) = hg.\bar{c}_2'(t_2)$. Then $\bar{c}(t) := \bar{c}_1(t)$ for $t \le t_2$ and $\bar{c}(t) := hg.\bar{c}_2(t)$ for $t \ge t_2$ defines a differentiable lift of c on a larger interval. □

4. C^1-lifts

Proposition 4.1. *Assume that $\rho : G \to O(V)$ is a representation of a finite group G with property $(\mathcal{B}_k)$. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve of class C^{k+d}. Then for any $t_0 \in \mathbb{R}$ there exists a local differentiable lift $\bar{c}$ of c near t_0 whose derivative is continuous at t_0.*

Proof. Without loss of generality we may assume that $t_0 = 0$. We show the existence of local differentiable lifts of c whose derivatives are continuous at 0 through any $v \in \sigma^{-1}(c(0))$. By lemma 2.5 we can assume $V^G = \{0\}$.

If $c(0) \neq 0$ corresponds to a regular orbit, then unique orthogonal C^{k+d}-lifts defined near 0 exist through all $v \in \sigma^{-1}(c(0))$, by lemma 2.7(a).

If $c(0) = 0$, then c_1 must vanish of at least second order at 0, since $c_1(t) \ge 0$ for all $t \in \mathbb{R}$. That means $c_1(t) = t^2 c_{1,1}(t)$ near 0 for a C^{k+d-2}-function $c_{1,1}$. By the multiplicity lemma 2.6 we find that $c_i(t) = t^{d_i} c_{i,i}(t)$ near 0 for $1 \le i \le n$, where $c_{1,1}, c_{2,2}, \ldots, c_{n,n}$ are functions of class $C^{k+d-2}, C^{k+d-d_2}, \ldots, C^{k+d-d_n}$, respectively. We consider the following C^k-curve in $\sigma(V)$ (since $\sigma(V)$ is closed in $\mathbb{R}^n$, see [28]):

$$\begin{aligned} c_{(1)}(t) &:= (c_{1,1}(t), c_{2,2}(t), \ldots, c_{n,n}(t)) \\ &= (t^{-2} c_1(t), t^{-d_2} c_2(t), \ldots, t^{-d_n} c_n(t)). \end{aligned}$$

By property $(\mathcal{B}_k)$ and proposition 3.2, there exists a local differentiable lift $\bar{c}_{(1)}$ of $c_{(1)}$ with locally bounded derivative. Thus, $\bar{c}(t) := t \cdot \bar{c}_{(1)}(t)$ is a local differentiable lift of c near 0 with derivative $\bar{c}'(t) = \bar{c}_{(1)}(t) + t\bar{c}'_{(1)}(t)$ which is continuous at $t = 0$ with $\bar{c}'(0) = \bar{c}_{(1)}(0)$. Note that $\sigma^{-1}(0) = \{0\}$, therefore we are done in this case.

If $c(0) \neq 0$ corresponds to a singular orbit, let v be in $\sigma^{-1}(c(0))$ and consider the slice representation $G_v \to O(N_v)$. By 2.3, the lifting problem reduces to the same problem for curves in N_v/G_v now passing through 0. By proposition 3.3 we may refer to the former case. □

Theorem 4.2. *Assume that $\rho : G \to O(V)$ is a representation of a finite group G with property $(\mathcal{B}_k)$. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve of class C^{k+d}. Then any differentiable lift $\bar{c}$ of c is actually of class C^1.*

Proof. Let $\bar{c}$ be a differentiable lift of c. Let $t_0 \in \mathbb{R}$ be arbitrary. We show that $\bar{c}'$ is continuous at t_0. Let $\tilde{c}$ denote the local differentiable lift of c near t_0 with continuous derivative at t_0, provided by proposition 4.1. Consider a sequence $(t_m)_m \subseteq \mathbb{R}$ with $t_m \to t_0$. For every m there is a $g_m \in G$ such that $\bar{c}(t_m) = g_m.\tilde{c}(t_m)$. Since G is finite, we may choose a subsequence of $(t_m)_m$ again denoted by $(t_m)_m$ such that $\bar{c}(t_m) = g.\tilde{c}(t_m)$ for some fixed $g \in G$ and all m. By lemma 3.4, there exist $h_m \in G_{\bar{c}(t_m)}$ with $\bar{c}'(t_m) = h_m g.\tilde{c}'(t_m)$ for all m. Passing again to a subsequence we find a fixed $h \in G_{\bar{c}(t_m)}$ such that $\bar{c}(t_m) = h.\bar{c}(t_m) = hg.\tilde{c}(t_m)$ and $\bar{c}'(t_m) = hg.\tilde{c}'(t_m)$ for all m. Then

$$\bar{c}(t_0) = \lim_{t_m \to t_0} \bar{c}(t_m) = \lim_{t_m \to t_0} hg.\tilde{c}(t_m) = hg. \lim_{t_m \to t_0} \tilde{c}(t_m) = hg.\tilde{c}(t_0)$$

and

$$\bar{c}'(t_0) = \lim_{t_m \to t_0} \frac{\bar{c}(t_m) - \bar{c}(t_0)}{t_m - t_0} = \lim_{t_m \to t_0} \frac{hg.\tilde{c}(t_m) - hg.\tilde{c}(t_0)}{t_m - t_0} = hg.\tilde{c}'(t_0)$$

and hence

$$\lim_{t_m \to t_0} \bar{c}'(t_m) = \lim_{t_m \to t_0} hg.\tilde{c}'(t_m) = hg.\tilde{c}'(t_0) = \bar{c}'(t_0).$$

This completes the proof. □

The forgoing theorem 4.2 is false, if G is not finite:

Example. Again consider the standard action of $SO(2)$ on $\mathbb{R}^2$ with orbit map $\sigma(x_1, x_2) = x_1^2 + x_2^2$. Let us consider the curve $c(t) = t^4$ and its differentiable lift $\bar{c}(t) = \big(t^2 \cos \frac{1}{t}, t^2 \sin \frac{1}{t}\big)$. But the derivative $\bar{c}'(t) = \big(2t \cos \frac{1}{t} + \sin \frac{1}{t}, 2t \sin \frac{1}{t} - \cos \frac{1}{t}\big)$ is not continuous at $t = 0$.

Remark. The failure of theorem 4.2 in this special example really is due to the fact that $SO(2)$ is infinite, since there is the following result due to Bony [4]: Any non-negative function $f : \mathbb{R} \to \mathbb{R}$ of class C^{2m} can be represented as sum of squares $f = g^2 + h^2$ of C^m-functions g and h. This result implies that $SO(2) : \mathbb{R}^2$ has property $(\mathcal{B}_2)$, and hence any standard representation of $SO(n)$ on $\mathbb{R}^n$ $(n \geq 2)$ has property $(\mathcal{B}_2)$ as well. But see 6.3.

Note that the lifting problem for the standard representation of $SO(n)$ on $\mathbb{R}^n$ is just the problem of representing non-negative functions as sums of squares. In this regard see [3], [4], [5], [12], and [15].

5. Twice differentiable lifts

Proposition 5.1. *Assume that $\rho : G \to O(V)$ is a representation of a finite group G with property $(\mathcal{B}_k)$. Let $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve of class C^{k+2d}. Then for any $t_0 \in \mathbb{R}$ there exists a local C^1-lift $\bar{c}$ of c near t_0 which is twice differentiable at t_0.*

Proof. Without loss of generality we may assume that $t_0 = 0$. We show the existence of local C^1-lifts of c which are twice differentiable at 0 through any $v \in \sigma^{-1}(c(0))$. By lemma 2.5 we can assume $V^G = \{0\}$.

If $c(0) \neq 0$ corresponds to a regular orbit, then unique orthogonal C^{k+2d}-lifts defined near 0 exist through all $v \in \sigma^{-1}(c(0))$, by lemma 2.7(a).

If $c(0) = 0$, then as in the proof of proposition 4.1 we find that the curve

$$\begin{aligned} c_{(1)}(t) &:= (c_{1,1}(t), c_{2,2}(t), \ldots, c_{n,n}(t)) \\ &= (t^{-2}c_1(t), t^{-d_2}c_2(t), \ldots, t^{-d_n}c_n(t)) \end{aligned}$$

lies in $\sigma(V)$ and is of class C^{k+d}. By property $(\mathcal{B}_k)$ and theorem 4.2, there exists a local C^1-lift $\bar{c}_{(1)}$ of $c_{(1)}$. Thus, $\bar{c}(t) := t \cdot \bar{c}_{(1)}(t)$ is a local C^1-lift of c near 0 with derivative $\bar{c}'(t) = \bar{c}_{(1)}(t) + t\bar{c}_{(1)}'(t)$ which is differentiable at $t = 0$:

$$\lim_{t\to 0} \frac{\bar{c}'(t) - \bar{c}'(0)}{t} = \lim_{t\to 0} \frac{\bar{c}_{(1)}(t) - \bar{c}_{(1)}(0) + t\bar{c}_{(1)}'(t)}{t} = 2\bar{c}_{(1)}'(0).$$

Note that $\sigma^{-1}(0) = \{0\}$, therefore we are done in this case.

If $c(0) \neq 0$ corresponds to a singular orbit, let v be in $\sigma^{-1}(c(0))$ and consider the isotropy representation $G_v \to O(N_v)$. By 2.3, the lifting problem reduces to the same problem for curves in N_v/G_v now passing through 0. By proposition 3.3 we may refer to the former case. □

Theorem 5.2. *Assume that $\rho : G \to O(V)$ is a representation of a finite group G with property $(\mathcal{B}_k)$. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve of class C^{k+2d}. Then there exists a global twice differentiable lift $\bar{c}$ of c.*

Proof. The proof will be carried out by induction on the cardinality of G.

If $G = \{e\}$ is trivial, then $\bar{c} := c$ is a global twice differentiable lift.

So let us assume that for any finite G' with $|G'| < |G|$ and any $c : \mathbb{R} \to V/G'$ of class $C^{k+2d'}$ there exists a global twice differentiable lift $\bar{c} : \mathbb{R} \to V$ of c, where $\rho' : G' \to O(V)$ is an orthogonal representation on an arbitrary real finite dimensional Euclidean vector space V with property $(\mathcal{B}_k)$, and $d' = d(\rho')$.

We shall prove that the same is true for G. Let $c = (c_1, \ldots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be of class C^{k+2d}. We may assume that $V^G = \{0\}$, by lemma 2.5. We can write $c^{-1}(\sigma(V)\backslash\{0\}) = \bigcup_i (a_i, b_i)$, a disjoint, at most countable union, where $a_i, b_i \in \mathbb{R} \cup \{\pm\infty\}$ with $a_i < b_i$ such that each (a_i, b_i) is maximal with respect to not containing zeros of c. In particular, we have $c(a_i) = c(b_i) = 0$ for all $a_i, b_i \in \mathbb{R}$ appearing in the above presentation.

Claim: On each (a_i, b_i) there exists a twice differentiable lift $\bar{c} : (a_i, b_i) \to V\backslash\{0\}$ of the restriction $c|_{(a_i,b_i)} : (a_i, b_i) \to \sigma(V)\backslash\{0\}$. The lack of nontrivial fixed points guarantees that for all $v \in V\backslash\{0\}$ the isotropy groups G_v satisfy $|G_v| < |G|$. Therefore, by induction hypothesis, which is fulfilled by proposition 3.3 and lemma 2.4, and by 2.3, we find local twice differentiable lifts of $c|_{(a_i,b_i)}$ near any $t \in (a_i, b_i)$ and through all $v \in \sigma^{-1}(c(t))$. Suppose that $\bar{c}_1 : (a_i, b_i) \supseteq (a, b) \to V\backslash\{0\}$ is a local twice differentiable lift of $c|_{(a_i,b_i)}$ with maximal domain (a, b), where, say, $b < b_i$. Then there exists a local twice differentiable lift $\bar{c}_2$ of $c|_{(a_i,b_i)}$ near b, and there exists a $t_0 < b$ such that both $\bar{c}_1$ and $\bar{c}_2$ are defined near t_0. Let $(t_m)_m$ be a sequence with $t_m \to t_0$. By the arguments in the proof of theorem 4.2, we may pass to a subsequence and find a $g \in G$ and a $h \in G_{\bar{c}_1(t_m)}$ such that $\bar{c}_1(t_m) = g.\bar{c}_2(t_m) = hg.\bar{c}_2(t_m)$ and $\bar{c}_1'(t_m) = hg.\bar{c}_2'(t_m)$ for all m. Consequently, $\bar{c}_1(t_0) = hg.\bar{c}_2(t_0)$ and $\bar{c}_1'(t_0) = hg.\bar{c}_2'(t_0)$, and hence

$$\bar{c}_1''(t_0) = \lim_{t_m\to t_0} \frac{\bar{c}_1'(t_m) - \bar{c}_1'(t_0)}{t_m - t_0} = \lim_{t_m\to t_0} \frac{hg.\bar{c}_2'(t_m) - hg.\bar{c}_2'(t_0)}{t_m - t_0} = hg.\bar{c}_2''(t_0).$$

So $\bar{c}(t) := \bar{c}_1(t)$ for $t \le t_0$ and $\bar{c}(t) := hg.\bar{c}_2(t)$ for $t \ge t_0$ defines a twice differentiable lift of $c|_{(a_i,b_i)}$ on a larger interval than (a,b). This proves the claim.

Now let $\bar{c} : (a_i, b_i) \to V\backslash\{0\}$ be the twice differentiable lift of $c|_{(a_i,b_i)}$ constructed above. For $a_i \ne -\infty$, we put $\bar{c}(a_i) := 0$ and $\bar{c}'(a_i) := \lim_{t\searrow a_i} \frac{\bar{c}(t)}{t-a_i}$ which exists as shown in the proof of theorem 4.4 in [18]. Then $\bar{c}$ is one-sided continuous at a_i, since $\langle \bar{c}(t) \mid \bar{c}(t)\rangle = \sigma_1(\bar{c}(t)) = c_1(t)$. Let $\tilde{c}$ be a local C^1-lift of c defined near a_i which is twice differentiable at a_i, provided by proposition 5.1. Then we find

$$\lim_{t\searrow a_i} \bar{c}(t) = \bar{c}(a_i) = 0 = \tilde{c}(a_i).$$

Let $(t_m)_m \subseteq (a_i, b_i)$ be a sequence with $t_m \searrow a_i$. By the arguments in the proof of theorem 4.2, we may pass to a subsequence and find a $g \in G$ and a $h \in G_{\bar{c}(t_m)}$ such that $\bar{c}(t_m) = g.\tilde{c}(t_m) = hg.\tilde{c}(t_m)$ and $\bar{c}'(t_m) = hg.\tilde{c}'(t_m)$ for all m. Therefore we have

$$\bar{c}'(a_i) = \lim_{t_m\searrow a_i} \frac{\bar{c}(t_m)}{t_m - a_i} = \lim_{t_m\searrow a_i} \frac{hg.\tilde{c}(t_m)}{t_m - a_i} = hg.\tilde{c}'(a_i). \tag{5.1}$$

Moreover,

$$\lim_{t_m\searrow a_i} \bar{c}'(t_m) = \lim_{t_m\searrow a_i} hg.\tilde{c}'(t_m) = hg.\tilde{c}'(a_i) = \bar{c}'(a_i),$$

since $\tilde{c}$ is C^1. It follows that the set of all accumulation points of $(\bar{c}'(t))_{t\searrow a_i}$ lies in the orbit $G.\tilde{c}'(a_i)$. Since G is finite, lemma 2.7(b) implies that $\bar{c}'(t)$ converges for $t \searrow a_i$, with limit $\bar{c}'(a_i)$, because it does so along the sequence $(t_m)_m$. Otherwise put, the lift $\bar{c}$ is continuously differentiable also at the boundary point a_i of its domain.

For the sequence $(t_m)_m$ from above we can argue further

$$\frac{\bar{c}'(t_m) - \bar{c}'(a_i)}{t_m - a_i} = \frac{hg.\tilde{c}'(t_m) - hg.\tilde{c}'(a_i)}{t_m - a_i} \to hg.\tilde{c}''(a_i) \qquad \text{as } t_m \searrow a_i,$$

since the lift $\tilde{c}$ is twice differentiable at a_i. Hence the set of all accumulation points of $\left(\frac{\bar{c}'(t)-\bar{c}'(a_i)}{t-a_i}\right)_{t\searrow a_i}$ is a subset of $G_{\bar{c}'(a_i)}hg.\tilde{c}''(a_i)$: Any accumulation point of $\left(\frac{\bar{c}'(t)-\bar{c}'(a_i)}{t-a_i}\right)_{t\searrow a_i}$ corresponds to a sequence $(t_m)_m \in (a_i, b_i)$ with $t_m \searrow a_i$ such that $\frac{\bar{c}'(t_m)-\bar{c}'(a_i)}{t_m-a_i} \to \hat{h}\hat{g}.\tilde{c}''(a_i)$, where $\hat{h}$ and $\hat{g}$ are found by repeating the procedure above. ¿From the equation $\hat{h}\hat{g}.\tilde{c}'(a_i) = \bar{c}'(a_i) = hg.\tilde{c}'(a_i)$, which follows from (5.1), we can read off $(hg)^{-1}\hat{h}\hat{g} \in G_{\tilde{c}'(a_i)} = (hg)^{-1}G_{\bar{c}'(a_i)}hg$, and hence $\hat{h}\hat{g} \in G_{\bar{c}'(a_i)}hg$.

By lemma 2.7(b) we have that $\frac{\bar{c}'(t)-\bar{c}'(a_i)}{t-a_i}$ converges for $t \searrow a_i$, with limit $hg.\tilde{c}''(a_i)$, since it does so along the sequence $(t_m)_m$. That means that the one-sided second derivative of $\bar{c}$ exists at a_i. The same reasoning is true for $b_i \ne +\infty$. So we have extended our lift $\bar{c}$ twice differentiably to the closure of (a_i, b_i).

Let us now construct a global twice differentiable lift of c defined on the whole of $\mathbb{R}$. For isolated points $t_0 \in c^{-1}(0)$ the two twice differentiable lifts on the neighboring intervals can be made to match twice differentiably, by applying a fixed transformation from G to one of them: Let $\bar{c}_1$ and $\bar{c}_2$ denote the lifts left and right of t_0. Then $\bar{c}_1(t_0) = \bar{c}_2(t_0) = 0$ and, by lemma 3.4, we find some $g \in G$ such that $\bar{c}_1'(t_0) = g.\bar{c}_2'(t_0)$. Let $\tilde{c}$ be the local C^1-lift near t_0 which is twice differentiable at t_0, provided by proposition 5.1. By the same argumentation as in the previous paragraph we find $h_1, h_2 \in G$ such that

$$h_1.\tilde{c}'(t_0) = \bar{c}_1'(t_0) = g.\bar{c}_2'(t_0) = h_2.\tilde{c}'(t_0),$$

and for the one-sided second derivatives we have

$$\lim_{t\nearrow t_0} \frac{\bar{c}_1'(t) - \bar{c}_1'(t_0)}{t - t_0} = h_1.\tilde{c}''(t_0) \quad \text{and} \quad \lim_{t\searrow t_0} \frac{g.\bar{c}_2'(t) - g.\bar{c}_2'(t_0)}{t - t_0} = h_2.\tilde{c}''(t_0).$$

It follows that there is a $h := h_1 h_2^{-1} \in G_{\bar{c}_1'(t_0)}$ with $\bar{c}_1''(t_0) = hg.\bar{c}_2''(t_0)$, which shows the assertion.

Let E be the set of accumulation points of $c^{-1}(0)$. For connected components of $\mathbb{R}\backslash E$ we can proceed inductively to obtain twice differentiable lifts on them.

Let $\hat{c} : \mathbb{R} \to V$ be a global C^1-lift of c which exists by theorem 2.7 and theorem 4.2. We define the following set

$$F := \{t \in \mathbb{R} : \hat{c}(t) = \hat{c}'(t) = 0\}.$$

Note that every lift $\bar{c}$ of c has to vanish on E and is continuous there since $\langle \bar{c}(t) \mid \bar{c}(t) \rangle = \sigma_1(\bar{c}(t)) = c_1(t)$. We also claim that any lift $\bar{c}$ of c is differentiable at any point $t' \in E$ with derivative 0. Namely, the difference quotient $t \mapsto \frac{\bar{c}(t)}{t-t'}$ is a lift of the curve

$$c_{(1,t')}(t) := ((t-t')^{-d_1} c_1(t), \ldots, (t-t')^{-d_n} c_n(t))$$

in $\sigma(V)$ which vanishes at t' by the following argument: Consider the local lift $\tilde{c}$ of c near t', provided by proposition 5.1. Let $(t_m)_{m\in\mathbb{N}} \subseteq c^{-1}(0)$ be a sequence with $t' \neq t_m \to t'$, consisting exclusively of zeros of c. Such a sequence always exists since $t' \in E$. Then we have

$$\tilde{c}'(t') = \lim_{t\to t'} \frac{\tilde{c}(t) - \tilde{c}(t')}{t-t'} = \lim_{m\to\infty} \frac{\tilde{c}(t_m)}{t_m - t'} = 0.$$

Thus $c_{(1,t')}(t') = \lim_{t\to t'} \sigma(\frac{\tilde{c}(t)}{t-t'}) = \sigma(\tilde{c}'(t')) = 0$.

In particular this shows that $E \subseteq F$. If we denote by F' the accumulation points of F, then $E \subseteq F = (F\backslash F') \cup F' \subseteq c^{-1}(0)$.

Consider first some $t' \in F\backslash F'$, i.e., t' is an isolated point of F. Then again we have a local twice differentiable lift for $t \neq t'$ (left and right of t'), since near t' there are only points of $\mathbb{R}\backslash E$. Moreover, proposition 5.1 yields again a local C^1-lift near t' which is twice differentiable at t'. As above we are able to find a twice differentiable lift on the set $(\mathbb{R}\backslash E) \cup (F\backslash F')$.

Finally let $t' \in F'$, i.e., t' is an accumulation point of F. By proposition 5.1, we have again a local C^1-lift $\tilde{c}$ near t' which is twice differentiable at t'. Lemma 3.4 implies that locally near t' the set F is given by $F = \{\tilde{c}(t) = \tilde{c}'(t) = 0\}$. So we have $\tilde{c}(t') = \tilde{c}'(t') = \tilde{c}''(t') = 0$, as t' is an accumulation point of F. We extend our twice differentiable lift $\bar{c}$ on $(\mathbb{R}\backslash E) \cup (F\backslash F')$ by 0 on F' to the whole of $(\mathbb{R}\backslash E) \cup (F\backslash F') \cup F' = (\mathbb{R}\backslash E) \cup F = \mathbb{R}$. It remains to check that then $\bar{c}$ is twice differentiable at $t' \in F'$. Since $F' \subseteq E$, we obtain that $\bar{c}$ vanishes at t' and is continuous and differentiable there with derivative 0. Consider a sequence $(t_m)_m$ with $t' \neq t_m \to t'$. Passing to a subsequence, we find as above, for all m, $\bar{c}(t_m) = g.\tilde{c}(t_m)$ and $\bar{c}'(t_m) = hg.\tilde{c}'(t_m)$ for some $g \in G$ and some $h \in G_{\bar{c}(t_m)}$. Then,

$$\frac{\bar{c}'(t_m) - \bar{c}'(t')}{t_m - t'} = \frac{\bar{c}'(t_m)}{t_m - t'} = \frac{hg.\tilde{c}'(t_m)}{t_m - t'} \to hg.\tilde{c}''(t') = 0 \qquad \text{as } t_m \to t'.$$

It follows that the second derivative of $\bar{c}$ at t' exists and equals 0. This completes the proof. □

6. Polar representations

The main results of sections 4 and 5, obtained there for finite groups G, can be generalized to polar representations $G \to O(V)$.

An orthogonal representation $\rho : G \to O(V)$ of a Lie group G on a finite dimensional Euclidean vector space V is called *polar*, if there exists a linear subspace $\Sigma \subseteq V$, called a *section* or a *Cartan subspace*, which meets each orbit orthogonally. See [9], [10], and [25]. The trace of the G-action is the action of the *generalized Weyl group* $W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$ on Σ, where $N_G(\Sigma) := \{g \in G : \rho(g)(\Sigma) = \Sigma\}$

and $Z_G(\Sigma) := \{g \in G : \rho(g)(s) = s \text{ for all } s \in \Sigma\}$. The generalized Weyl group is a finite group, and is a reflection group if G is connected. If Σ' is a different section, then there is an isomorphism $W(\Sigma) \to W(\Sigma')$ induced by an inner automorphism of G.

We shall need the following generalization of Chevalley's restriction theorem, which is due to Dadok and Kac [10] and independently to Terng [33].

Theorem 6.1. *Let $\rho : G \to O(V)$ be a polar representation of a compact Lie group, with section Σ and generalized Weyl group $W(\Sigma)$. Then the algebra $\mathbb{R}[V]^G$ of G-invariant polynomials on V is isomorphic to the algebra $\mathbb{R}[\Sigma]^{W(\Sigma)}$ of $W(\Sigma)$-invariant polynomials on the section Σ, via restriction $f \mapsto f|_\Sigma$.*

As a consequence of this theorem we obtain that the orbit spaces $V/G = \sigma(V)$ and $\Sigma/W(\Sigma) = \sigma|_\Sigma(\Sigma)$ are isomorphic.

Theorem 6.2. *Let $\rho : G \to O(V)$ be a polar representation of a compact Lie group on a finite dimensional Euclidean vector space V with orbit map $\sigma : V \to \mathbb{R}^n$. Assume that $W(\Sigma) \to O(\Sigma)$ has property $(\mathcal{B}_k)$ for some section Σ. Let $c : \mathbb{R} \to \sigma(V) \subseteq \mathbb{R}^n$ be a curve in the orbit space. Then we have:*

1. *If c is of class C^{k+d}, then there exists a global orthogonal C^1-lift $\bar c : \mathbb{R} \to V$.*
2. *If c is of class C^{k+2d}, then there exists a global orthogonal twice differentiable lift $\bar c : \mathbb{R} \to V$.*

Proof. By theorem 6.1, $\sigma|_\Sigma : \Sigma \to \mathbb{R}^n$ is the orbit map for the representation $W(\Sigma) \to O(\Sigma)$, and hence the orbit spaces $V/G = \sigma(V)$ and $\Sigma/W(\Sigma) = \sigma|_\Sigma(\Sigma)$ are isomorphic.

If $c : \mathbb{R} \to \sigma(V) \cong \sigma|_\Sigma(\Sigma)$ is C^{k+d}, then by theorem 2.7 and theorem 4.2 (since $W(\Sigma)$ is finite) there exists a global C^1-lift $\bar c : \mathbb{R} \to \Sigma$, which as a curve in V is orthogonal to each G-orbit it meets, by the properties of Σ. This shows (1).

If $c : \mathbb{R} \to \sigma(V) \cong \sigma|_\Sigma(\Sigma)$ is C^{k+2d}, then statement (2) follows analogously from theorem 5.1. □

Example 6.3. The standard representation of $SO(n)$ on $\mathbb{R}^n$ is polar. Any 1-dimensional linear subspace Σ of $\mathbb{R}^n$ is a section. The associated generalized Weyl group is $W(\Sigma) = \{\pm \mathrm{id}\}$. So the representation $W(\Sigma) \to O(\Sigma)$ has property $(\mathcal{B}_2)$, since it reduces to $S_2 : \mathbb{R}^2$, the problem of finding regular roots of $x^2 - f(t) = 0$ ($f \geq 0$ and C^2), which has property $(\mathcal{B}_2)$, by theorem 2.8. Hence theorem 6.2 is applicable.

7. Stability of property $(\mathcal{B})$

Proposition 7.1. *Let $\rho : G \to O(V)$ be an orthogonal representation of a compact Lie group G on a real finite dimensional Euclidean vector space V having property $(\mathcal{B}_k)$. For any G-invariant linear subspace $W \subseteq V$ the subrepresentation $\rho' : G \to O(W)$ has property $(\mathcal{B}_k)$ as well.*

Proof. If $\sigma_1, \ldots, \sigma_n$ are generators of $\mathbb{R}[V]^G$, then their restrictions $\sigma_1|_W, \ldots, \sigma_n|_W$ generate $\mathbb{R}[W]^G$. Thus $W/G = \sigma|_W(W)$ naturally lies in $V/G = \sigma(V)$.

Let $c : \mathbb{R} \to \sigma|_W(W) \cap U$ be a C^k-curve in the orbit space $\sigma|_W(W)$, where $U = U(\rho)$ is the open neighborhood of 0 in $\sigma(V)$ from the definition of property $(\mathcal{B}_k)$ (see 3.1). We may view c as a curve in the orbit space $\sigma(V)$, and since the representation ρ has property $(\mathcal{B}_k)$, we can lift c to a local differentiable curve $\bar c$ in V with locally bounded derivative. But then $\bar c$ has obviously to lie in the G-invariant subspace W. This completes the proof. □

Proposition 7.2. *Suppose that $\rho_i : G_i \to O(V_i)$ $(1 \le i \le l)$ are orthogonal representations of compact Lie groups G_i on real finite dimensional Euclidean vector spaces V_i having property $(\mathcal{B}_{k_i})$. Then the orthogonal direct sum*

$$\rho_1 \oplus \cdots \oplus \rho_l : G_1 \times \cdots \times G_l \longrightarrow O(V_1 \oplus \cdots \oplus V_l)$$

of the representations $\rho_1, \dots, \rho_l$ has property $(\mathcal{B}_k)$, where $k = \max\{k_1, \dots, k_l\}$.

Proof. It is sufficient to consider the case $l = 2$, since the general case follows by induction.

If $\langle \quad | \quad \rangle_1$ and $\langle \quad | \quad \rangle_2$ denote the inner products on V_1 and V_2, then

$$\langle v_1 + v_2 \mid w_1 + w_2 \rangle := \langle v_1 \mid w_1 \rangle_1 + \langle v_2 \mid w_2 \rangle_2$$

defines an inner product on $V = V_1 \oplus V_2$ which makes V_1 and V_2 into orthogonal subspaces of V. The action of $G = G_1 \times G_2$ on $V_1 \oplus V_2$ is obviously again orthogonal. Moreover, we find $\mathbb{R}[V]^G = \mathbb{R}[V_1 \oplus V_2]^{G_1 \times G_2} \cong \mathbb{R}[V_1]^{G_1} \otimes \mathbb{R}[V_2]^{G_2}$ and $V/G = (V_1 \oplus V_2)/(G_1 \times G_2) \cong V_1/G_1 \times V_2/G_2$.

Now any C^k-curve c in $U_1 \times U_2 \subseteq V/G$ has the form $c = (c_1, c_2)$ for C^k-curves c_i in $U_i \subseteq V_i/G_i$, where $U_i = U(\rho_i)$ from 3.1, which allow local differentiable lifts $\bar{c}_i$ with locally bounded derivative to V_i, by assumption. This shows that $\rho = \rho_1 \oplus \rho_2$ has property $(\mathcal{B}_k)$. □

8. Finite groups G have property $(\mathcal{B})$

Theorem 8.1. *Let $\rho : G \to O(V)$ be a real finite dimensional orthogonal representation of a finite group G, and let $\sigma_1, \dots, \sigma_n$ be a minimal system of homogeneous generators of $\mathbb{R}[V]^G$. Write $V = V_1 \oplus \cdots \oplus V_l$ as orthogonal direct sum of irreducible subspaces V_i. Choose $v_i \in V_i \backslash \{0\}$ such that the cardinality of the corresponding isotropy group G_{v_i} is maximal, and put $k = \max\{d(\rho), |G|/|G_{v_i}| : 1 \le i \le l\}$. Then any curve $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ of class C^k in the orbit space admits a global differentiable lift $\bar{c}$ to V with locally bounded derivative.*

Proof. We shall reduce to the representation the polynomial case, i.e., to the standard representation of the symmetric group.

Since $d(\rho) \le k$, we can apply theorem 2.7 which provides a differentiable lift $\bar{c} : \mathbb{R} \to V$ of c.

Let i be fixed. For $g \in G$ we define a linear function

$$\begin{aligned} F_{i,g} : V &\longrightarrow \mathbb{R} \\ x &\longmapsto \langle v_i \mid g.\mathrm{pr}_{V_i}(x) \rangle = \langle v_i \mid g.x \rangle. \end{aligned}$$

Here $\mathrm{pr}_{V_i} : V \to V_i$ is the natural projection. The cardinality of distinct functions $F_{i,g}$ equals $k_i := |G|/|G_{v_i}|$.

Let $G_{v_i} \backslash G$ denote the space of right cosets of G_{v_i} in G, and introduce a numbering $G_{v_i} \backslash G = \{g_1, g_2, \dots, g_{k_i}\}$. We construct the following polynomials on V:

$$a_{i,j}(x) = \sum_{1 \le m_1 < \cdots < m_j \le k_i} F_{i,g_{m_1}}(x) \cdots F_{i,g_{m_j}}(x) \qquad 1 \le j \le k_i.$$

These polynomials $a_{i,j}$ are G-invariant by construction, and therefore expressible in the homogeneous generators $\sigma_1, \dots, \sigma_n$ of $\mathbb{R}[V]^G$, i.e., there exist polynomials $p_{i,j} \in \mathbb{R}[\mathbb{R}^n]$ such that

$$a_{i,j} = p_{i,j}(\sigma_1, \dots, \sigma_n) \qquad 1 \le j \le k_i. \tag{8.1}$$

The polynomials $a_{i,j}$, for $1 \le j \le k_i$, are elementary symmetric functions in the variables $F_{i,g}(x)$, where g runs through $G_{v_i} \backslash G = \{g_1, g_2, \dots, g_{k_i}\}$. Finally, we associate the following monic polynomial of degree k_i in one variable y:

$$P_i(x)(y) = y^{k_i} + \sum_{j=1}^{k_i}(-1)^j a_{i,j}(x) y^{k_i-j} = \prod_{j=1}^{k_i}(y - F_{i,g_j}(x)). \tag{8.2}$$

By construction, the functions $x \mapsto F_{i,g}(x)$ ($g \in G_{v_i}\backslash G$) parameterize the roots of $x \mapsto P_i(x)(y)$ which, consequently, are always real.

Now consider the functions $t \mapsto a_{i,j}(\bar{c}(t))$ ($1 \le j \le k_i$) which are of class C^k by equation (8.1). As in (8.2) we may associate a C^k-curve $t \mapsto P_i(t)(y)$ of monic polynomials of degree k_i in one variable defined by

$$P_i(t)(y) = y^{k_i} + \sum_{j=1}^{k_i}(-1)^j a_{i,j}(\bar{c}(t)) y^{k_i-j}.$$

By theorem 2.8, applied to the curve of polynomials $t \mapsto P_i(t)(y)$, the differentiable functions $t \mapsto F_{i,g}(\bar{c}(t))$ ($g \in G_{v_i}\backslash G$) which parameterize the roots of $t \mapsto P_i(t)(y)$ have locally bounded derivative.

Since V_i is irreducible, the linear span of the orbit $G.v_i$ spans V_i. If we repeat the above procedure for each $1 \le i \le l$, it follows that $\bar{c}$ is a differentiable lift of c with locally bounded derivative. This completes the proof. □

Corollary 8.2. *Any polar representation ρ of a compact Lie group G has property $(\mathcal{B}_k)$, where k is determined analogously to theorem 8.1 but for the representation $W \to O(\Sigma)$, where W is the generalized Weyl group of some section Σ. Moreover, the lifts can be chosen orthogonal.* □

Remark. The case $k = |G|$ can occur: For finite rotation groups in the plane we have $d = |G|$, and for any non-zero v the isotropy group G_v is trivial.

Remark. There are irreducible orthogonal representations of finite groups G where the inequality $d \le |G|/|G_v|$ is violated for non-zero vectors v:

Consider the rotational symmetry group T of the regular tetrahedron in $\mathbb{R}^3$. We find $d = 6$ (e.g. [8]). The isotropy group of each vertex v of the tetrahedron has 3 elements. So $|G|/|G_v| = 12/3 = 4$.

Furthermore, the same phenomenon appears for the rotational symmetry groups W and H of the cube and the regular icosahedron in $\mathbb{R}^3$, respectively. Compare with section 10.

Remark. The method of the proof of theorem 8.1 is used by L. Smith and R.E. Strong in [31] for constructing generators of invariant rings. It is related to E. Noether's [24] proof of Hilbert's finiteness theorem as recounted by H. Weyl [35].

9. Property $(\mathcal{B}_k)$ for finite reflection groups

Abusing notation we will denote finite reflection groups as well as their root systems (respectively their Coxeter graphs) with the same symbols.

Recall the characterization of finite reflection groups ([14], [16]):

If G is a finite subgroup of $O(V)$ that is generated by reflections, then V may be written as the orthogonal direct sum of G-invariant subspaces $V_0 = V^G, V_1, \dots, V_k$ with the following properties:

(a) If $G_i = \{g|_{V_i} : g \in G\}$, then G_i is a subgroup of $O(V_i)$, and G is isomorphic with $G_0 \times G_1 \times \cdots \times G_k$.
(b) G_0 consists only of the identity transformation on V_0.
(c) Each G_i ($i \ge 1$) is one of the groups

$$A_n, n \ge 1; B_n, n \ge 2; D_n, n \ge 4; I_2^n, n \ge 5, n \ne 6;$$
$$G_2; H_3; H_4; F_4; E_6; E_7; E_8.$$

We will apply theorem 8.1 to each of the irreducible finite reflection groups listed in (c). Here the inequality $d \le |G|/|G_v|$ will be satisfied for all non-zero v which can be checked directly in the table given at the end of this section.

Let $\rho : G \to O(V)$ be the standard representation of some irreducible finite reflection group G listed in (c). We consider an arbitrary slice representation $G_v \to O(N_v)$ $(v \in V)$ of ρ; note $N_v = V$. By [16], theorem 1.12, there exists a $g \in G$ such that $g.v = w$ for a w in the fundamental domain $F = \{x \in V : \langle x \mid r\rangle \ge 0 \text{ for all } r \in \Pi\}$, where Π is a system of simple roots, and G_w is generated by the simple reflections it contains. It follows that we can read off easily the information we need to determine a minimal $|G|/|G_v|$ from the Coxeter graph of G.

Easy computations yield the results collected in the table in figure 1 which gives a complete survey of the standard representations of all irreducible finite reflection groups. The integers d and $|G|$ for the listed representations can be found e.g. in [14]; in [22] also generators of the corresponding algebra of invariant polynomials are available. The integer k is the minimum of the numbers $|G|/|G_v|$ where v runs through all non-zero vectors in V. We have $d \le k$. By theorem 8.1 the representations listed in the table have property $(\mathcal{B}_k)$. This together with lemma 2.5 and proposition 7.2 treats finite reflection groups completely.

$\rho : G \to O(V)$	d	k	$\lvert G\rvert$
A_n, $n \ge 1$	$n+1$	$n+1$	$(n+1)!$
B_n, $n \ge 2$	$2n$	$2n$	$2^n n!$
D_n, $n \ge 4$	$2n-2$	$2n$	$2^{n-1}n!$
I_2^n, $n \ge 5$	n	n	$2n$
G_2	6	6	12
H_3	10	12	120
H_4	30	120	14400
F_4	12	24	1152
E_6	12	27	51840
E_7	18	56	2903040
E_8	30	240	696729600

Figure 1. Irreducible Coxeter groups and associated integers d, k, and $|G|$.

10. Property $(\mathcal{B}_k)$ for finite rotation groups

Let us denote by C_2^n the cyclic subgroup of $O(2)$ generated by the counterclockwise rotation of $\mathbb{R}^2$ through the angle $2\pi/n$. Here we have $d = |G| = n$, and for any non-zero vector $v \in \mathbb{R}^2$ its isotropy group G_v is trivial. By theorem 8.1, *all finite rotation groups C_2^n in the plane have property $(\mathcal{B}_n)$.*

Remark. This together with the result for dihedral groups I_2^n in section 9 gives a complete discussion of all finite subgroups of $O(2)$.

Next we consider finite rotation groups in 3-dimensional space.

Let P be a 2-dimensional linear subspace in $\mathbb{R}^3$. Any rotation R in $O(P)$ can be extended to a rotation in $O(3)$, by setting $Rx = x$ for all $x \in P^\perp$ and using linearity. By extending each transformation in a cyclic subgroup C_2^n of $O(P)$ in this fashion, we obtain a cyclic subgroup of rotations in $O(3)$, which will be denoted by C_3^n.

On the other hand, if S is a reflection in $O(P)$, then S may also be extended to a rotation in $O(3)$, in fact to the rotation through the angle π having the reflection line of S in P as its axis of rotation: define $Sx = -x$ for all $x \in P^\perp$ and extend by linearity. By extending each transformation in a dihedral subgroup I_2^n of $O(P)$ to a rotation in $O(3)$, the resulting set of rotations is a subgroup of $O(3)$ isomorphic with I_2^n; it shall be denoted I_3^n.

If T, W, and H denote the subgroups of rotations in $O(3)$ which leave invariant the regular tetrahedron, cube, and icosahedron each with center in the origin, then the following list provides a complete characterization of finite rotation groups in $\mathbb{R}^3$ (e.g. [14]):

$$C_3^n, n \geq 1; I_3^n, n \geq 2; T; W; H.$$

Rotation groups of type C_3^n have property $(\mathcal{B}_n)$ by construction, since the linear subspace $P^\perp$ is left pointwise invariant under the C_3^n-action, and on P it restricts to the C_2^n-action; so lemma 2.5 and the result for C_2^n give the statement.

Rotation groups of type I_3^n have property $(\mathcal{B}_{n+1})$: Note first that here $d = n + 1$. Moreover, we have the decomposition $\mathbb{R}^3 = P \oplus P^\perp$ into irreducible subrepresentations. In order to make $|G|/|G_{v_1}|$ minimal for $0 \neq v_1 \in P$ we may choose v_1 to lie on some reflection line of I_2^n in P; then $|G|/|G_{v_1}| = 2n/2 = n$. For $0 \neq v_2 \in P^\perp$ we find that $|G_{v_2}|$ is the number of rotations (including the identity) in I_2^n. So $|G|/|G_{v_2}| = 2n/n = 2$. Application of theorem 8.1 gives the assertion.

The rotational symmetry group T of the regular tetrahedron has property $(\mathcal{B}_6)$: We have $d = 6$ (e.g. [8], [27]). Further the action of T on $\mathbb{R}^3$ is irreducible. The elements of T consist of rotations through angles of $2\pi/3$ and $4\pi/3$ about each of four axes joining vertices of the tetrahedron with centers of opposite faces, rotations through the angle π about each of the three axes joining the midpoints of opposite edges, and the identity. So $|T| = 12$. The isotropy groups of non-zero vectors on axes joining vertices with centers of opposite faces have cardinality 3, those of non-zero vectors on axes joining the midpoints of opposite edges have cardinality 2, and all other isotropy groups of non-zero vectors are trivial. Hence application of theorem 8.1 gives the statement.

The rotational symmetry group W of the cube has property $(\mathcal{B}_9)$: We have $d = 9$ (e.g. [27]). The action of W on $\mathbb{R}^3$ is irreducible. The elements of W consist of rotations through angles of $\pi/2$, π, and $3\pi/2$ about each of three axes joining the centers of opposite faces, rotations through angles of $2\pi/3$ and $4\pi/3$ about each of four axes joining extreme opposite vertices, rotations through the angle π about each of six axes joining midpoints of diagonally opposite edges, and the identity. Thus $|W| = 24$. The isotropy groups of non-zero vectors on axes joining the centers of opposite faces have cardinality 4, those of non-zero vectors on axes joining extreme opposite vertices have cardinality 3, those of non-zero vectors on axes joining midpoints of diagonally opposite edges have cardinality 2, and all other isotropy groups of non-zero vectors are trivial. Apply theorem 8.1.

The rotational symmetry group H of the regular icosahedron has property $(\mathcal{B}_{15})$: We have $d = 15$ (e.g. [8]). The action of H on $\mathbb{R}^3$ is irreducible. The elements of H consist of rotations through angles of $2\pi/5$, $4\pi/5$, $6\pi/5$, and $8\pi/5$ about each of the six axes joining extreme opposite vertices, rotations through angles of $2\pi/3$ and $4\pi/3$ about each of ten axes joining centers of opposite faces, rotations through the angle π about each of fifteen axes joining midpoints of opposite edges, and the identity. Therefore $|H| = 60$. The isotropy groups of non-zero vectors on axes joining extreme opposite vertices have cardinality 5, those of non-zero vectors on axes joining centers of opposite faces have cardinality 3, those of non-zero vectors on axes joining midpoints of opposite edges have cardinality 2, and all other isotropy groups of non-zero vectors are trivial. Apply theorem 8.1.

The table in figure 2 collects the results for finite rotation groups in two and three dimensions obtained in this section. The groups in the first column of the table are meant to stay for their standard representation, d is the integer associated to representations in 2.4, and k is as in theorem 8.1.

$\rho : G \to O(V)$	d	k	$\lvert G\rvert$
C_2^n, $n \geq 1$	n	n	n
C_3^n, $n \geq 1$	n	n	n
I_3^n, $n \geq 2$	$n+1$	$n+1$	$2n$
T	6	6	12
W	9	9	24
H	15	15	60

Figure 2. Finite rotation groups in two and three dimensions and associated integers d, k, and $|G|$.

Remark. Observe that in this table we always have $d = k$, i.e., the respective representation has property $(\mathcal{B}_d)$. Since we need at least regularity C^d for a curve in the orbit space to be liftable once differentiably (theorem 2.7), we cannot expect to improve these results. Evidently this remark applies also for those representations in the table in figure 2 with $d = k$.

References

[1] D. Alekseevsky, A. Kriegl, M. Losik, P.W. Michor, *Choosing roots of polynomials smoothly*, Israel J. Math., **105** (1998), 203-233. arXiv:math.CA/9801026.

[2] D. Alekseevsky, A. Kriegl, M. Losik, P.W. Michor, *Lifting smooth curves over invariants for representations of compact Lie groups*, Transformation Groups **5** (2000), 103-110. arXiv:math.AG/0312030.

[3] J.-M. Bony, *Sur l'inégalité de Fefferman-Phong*, Seminaire: Équations aux Dérivées Partielles, 1998-1999, Exp. No. III, 16pp., École Polytech., Palaiseau, 1999.

[4] J.-M. Bony, *Sommes de carrés de fonctions dérivables*, to appear in Bull. Soc. Math. France.

[5] J.-M. Bony, F. Broglia, F. Colombini, L. Pernazza, *Nonnegative functions as squares or sums of squares*, to appear.

[6] G.E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.

[7] M.D. Bronshtein, *Smoothness of polynomials depending on parameters*, Sib. Mat. Zh. **20** (1979), 493-501 (Russian); English transl. in Siberian Math. J **20** (1980), 347-352.

[8] C.J. Cummins, J. Patera, *Polynomial icosahedral invariants*, J. Math. Phys. **29**, no. 8 (1988), 1736-1745.

[9] J. Dadok, *Polar coordinates induced by actions of compact Lie groups*, Trans. AMS **288** (1985), 125-137.

[10] J. Dadok, V. Kac, *Polar representations*, J. Algebra **92** (1985), 504-524.

[11] H. Derksen, G. Kemper, *Computational invariant theory*, Encyclopaedia of Mathematical Sciences, Springer-Verlag Berlin Heidelberg, 2002.

[12] C. Fefferman, D. Phong, *On positivity of pseudo-differential operators*, Proc. Nat. Acad. Sci. U.S.A **75**, (1978) 4673-4674.

[13] G. Glaeser, *Racine carrée d'une fonction différentiable*, Ann. Inst. Fourier (Grenoble) **13** (1963) 2, 203-210.

[14] L.C. Grove, C.T. Benson, *Finite reflection groups*, Graduate Texts in Mathematics **99**, Second Edition, Springer-Verlag New York Inc., 1985.

[15] D. Hilbert, *Über die Darstellung definiter Formen als Summe von Formquadraten*, Math. Ann. **32**, (1888) 342-350; see also Gesammelte Abhandlungen, Bd. 2 (Chelsea Publishing Company, Bronx, New York 1965) 154-161.

[16] J.E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge studies in advanced mathematics **29**, Cambridge University Press, 1990.

[17] A. Kriegl, M. Losik, P.W. Michor, *Choosing roots of polynomials smoothly, II*, Israel J. Math. **139** (2004), 183-188. arXiv:math.CA/0208228.
[18] A. Kriegl, M. Losik, P.W. Michor, A. Rainer: *Lifting smooth curves over invariants for representations of compact Lie groups, II*, J. Lie Theory **15** (2005), No. 1, 227–234. arXiv:math.RT/0402222.
[19] A. Kriegl, P.W. Michor, *Differentiable perturbation of unbounded operators*, Math. Ann. **327** (2003), 191-201. arXiv:math.FA/0204060.
[20] D. Luna, *Sur certaines opérations différentiables des groupes de Lie*, Amer. J. Math. **97** (1975), 172-181.
[21] J.N. Mather, *Differentiable invariants*, Topology **16** (1977), no. 2, 145-155.
[22] M.L. Mehta, *Basic sets of invariant polynomials for finite reflection groups*, Comm. Algebra, **16**(5) (1988), 1083-1098.
[23] D. Montgomery, C.T. Yang, *The existence of a slice*, Ann. of Math., **65** (1957), 108-116.
[24] E. Noether, *Der Endlichkeitssatz der Invarianten endlicher Gruppen*, Math. Ann. **77** (1916), 89-92.
[25] R.S. Palais, C. Terng, *A general theory of canonical forms*, Trans. AMS **300** (1987), 771-789.
[26] R.S. Palais, C. Terng, *Critical point theory and submanifold geometry*, Lecture Notes in Mathematics **1353**, Berlin, Springer-Verlag 1988.
[27] J. Patera, R.T. Sharp, P. Winternitz, *Polynomial irreducible tensors for point groups*, J. Math. Phys. **19** (1978), 2362-2376.
[28] C. Procesi, G. Schwarz, *Inequalities defining orbit spaces*, Invent. Math. **81** (1985), 539-554.
[29] G.W. Schwarz, *Lifting smooth homotopies of orbit spaces*, Publ. Math. IHES **51** (1980), 37-136.
[30] G.W. Schwarz, *Smooth functions invariant under the action of a compact Lie group*, Topology **14** (1975), 63-68.
[31] L. Smith, R.E. Strong, *On the invariant theory of finite groups: orbit polynomials and splitting principles*, J. Algebra **110** (1987), 134-157.
[32] B. Sturmfels, *Algorithms in invariant theory*, Texts and Monographs in Symbolic Computation, Springer-Verlag, Wien New York, 1993.
[33] C. Terng, *Isoparametric submanifolds and their Coxeter groups*, J. Diff. Geom. **21** (1985), 79-107.
[34] S. Wakabayashi, *Remarks on hyperbolic polynomials*, Tsukuba J. Math. **10** (1986), 17-28.
[35] H. Weyl, *The classical groups*, 2nd edition, Princeton Univ. Press, Princeton, 1946.

A. Kriegl: Institut für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria
E-mail address: Andreas.Kriegl@univie.ac.at

M. Losik: Saratov State University, ul. Astrakhanskaya, 83, 410026 Saratov, Russia
E-mail address: losikMV@info.sgu.ru

P.W. Michor: Institut für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria; and: Erwin Schrödinger Institut für Mathematische Physik, Boltzmanngasse 9, A-1090 Wien, Austria
E-mail address: Peter.Michor@esi.ac.at

A. Rainer: Institut für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria
E-mail address: armin.rainer@univie.ac.at