SOBOLEV METRICS ON THE RIEMANNIAN MANIFOLD OF ALL RIEMANNIAN METRICS

MARTIN BAUER, PHILIPP HARMS, PETER W. MICHOR

Abstract. On the manifold $\mathcal{M}(M)$ of all Riemannian metrics on a compact manifold M one can consider the natural L^2-metric as decribed first by [10]. In this paper we consider variants of this metric which in general are of higher order. We derive the geodesic equations, we show that they are well-posed under some conditions and induce a locally diffeomorphic geodesic exponential mapping. We give a condition when Ricci flow is a gradient flow for one of this metrics.

1. Introduction

On the manifold $\mathcal{M}(M)$ of all Riemannian metrics on a compact manifold M one can consider the natural L^2-metric. It was first described by [10]. Geodesics and curvature on it were described by [13] and [14] who also described the Jacobi fields and the exponential mapping. This was extended to the space of non-degenerate bilinear structures on M in [15] and restricted to the space of almost Hermitian structures in [16]. In his thesis [7] which was published in two subsequent papers [8, 9], Brian Clarke showed that geodesic distance for the L^2-metric is a positive topological metric on $\mathcal{M}(M)$, and he determined the metric completion of $\mathcal{M}(M)$. In contrast, it was shown in [23, 22] that the natural L^2-metric on the space of immersions from a compact manifold into a Riemannian manifold has indeed vanishing geodesic distance. This also holds for the right invariant L^2-metric on diffeomorphism groups [22], and even on the Virasoro-Bott group [4] where the geodesic equation is the KdV-equation.

In this paper, guided by the results of [1, 2, 3], we investigate stronger metrics on $\mathcal{M}(M)$ than the L^2-metric. These are metrics of the following form:

$$\begin{aligned} G_g(h,k) &= \Phi(\mathrm{Vol})\int_M g^0_2(h,k)\,\mathrm{vol}(g) && \text{see 4.2}\\ \text{or } &= \int_M \Phi(\mathrm{Scal}).g^0_2(h,k)\,\mathrm{vol}(g) && \text{see 4.3}\\ \text{or } &= \int_M g^0_2((1+\Delta)^p h,k)\,\mathrm{vol}(g) && \text{see 4.4}\end{aligned}$$

where Φ is a suitable real-valued function, $\mathrm{Vol} = \int_M \mathrm{vol}(g)$ is the total volume of (M,g), Scal is the scalar curvature of (M,g), and where g^0_2 is the induced metric

Date: February 16, 2011.
2010 *Mathematics Subject Classification.* Primary 58D17, 58E30, 35A01.
All authors were supported by FWF Project 21030.

on $\binom{0}{2}$-tensors. We describe all these metrics uniformly as

$$G^P_g(h,k) = \int_M g^0_2(P_g h, k)\,\mathrm{vol}(g) = \int_M \mathrm{Tr}(g^{-1}.P_g(h).g^{-1}.k)\,\mathrm{vol}(g),$$

where $P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$ is a positive, symmetric, bijective pseudo-differential operator of order $2p, p \geq 0$, depending smoothly on the metric g. We derive the geodesic equation for the general metric and all particular cases. We show that under certain assumptions on P_g the geodesic equation is well posed and that the geodesic exponential mapping is a diffeomrophism from a neighborhood of the 0 section in the tangent bundle $T\mathcal{M}(M)$ onto a neighborhood of the diagonal in $\mathcal{M}(M) \times \mathcal{M}(M)$. The assumptions are satisfied for the metrics in 4.2 and 4.4, but not for the metric in 4.3.

Finally we derive a condition on P_g which is sufficient for the Ricci vector field to be a gradient field in the G^P-metric.

2. Notation

2.1. Metric on tensor spaces. For a Riemannian metric g, let

$$\flat = \check{g} : TM \to T^*M \quad \text{and} \quad \sharp = \check{g}^{-1} : T^*M \to TM.$$

g can be extended to the cotangent bundle $T^*M = T^0_1 M$ by setting

$$g^{-1}(\alpha,\beta) = g^0_1(\alpha,\beta) = \alpha(\beta^\sharp)$$

for $\alpha, \beta \in T^*M$, and the product metric

$$g^r_s = \bigotimes^r g \otimes \bigotimes^s g^{-1}$$

extends g to all tensor spaces $T^r_s M$. A useful formula is

$$g^0_2(h,k) = \mathrm{Tr}(g^{-1}hg^{-1}k) \quad \text{for } h,k \in T^0_2 M \text{ if } h \text{ or } k \text{ is symmetric.}$$

For a proof using orthonormal frames see [2].

2.2. Directional derivatives of functions. We will use the following ways to denote directional derivatives of functions, in particular in infinite dimensions. Given a function $F(x,y)$ for instance, we will write:

$$D_{(x,h)}F \text{ or } dF(x)(h) \text{ as shorthand for } \partial_t|_0 F(x+th,y).$$

Here (x,h) in the subscript denotes the tangent vector with foot point x and direction h. If F takes values in some linear space, we will identify this linear space and its tangent space.

2.3. Volume density. The *volume density* on M induced by the metric g is given by $\mathrm{vol}(g) = \mathrm{vol}(g) \in \Gamma(\mathrm{vol}(M))$, where $\mathrm{vol}(M)$ denotes the volume bundle. The *volume* of the manifold with respect to the metric g is given by $\mathrm{Vol} = \int_M \mathrm{vol}(g)$. The integral is well-defined since M is compact. If M is oriented we may identify the volume density with a differential form. Furthermore we have the following formula for the first variation of the volume density (see for example [1, section 3.6] for the proof):

Lemma. *The differential of the volume density*

$$\begin{cases} \Gamma(S^2_+ T^*M) & \to & \Gamma(\operatorname{vol}(M)) \\ g & \mapsto & \operatorname{vol}(g) \end{cases}$$

is given by

$$D_{(g,m)} \operatorname{vol}(g) = \frac{1}{2} \operatorname{Tr}(g^{-1}.m) \operatorname{vol}(g).$$

2.4. **Metric on tensor fields.** A metric on a space of tensor fields is defined by integrating the appropriate metric on the tensor space with respect to the volume density:

$$\widetilde{g}^r_s(h,k) = \int_M g^r_s\big(h(x),k(x)\big) \operatorname{vol}(g)(x)$$

for $h,k \in \Gamma(T^r_s M)$. According to section 2.1, if h and k are tensor fields of type $\binom{0}{2}$ and h or k is symmetric, then

$$\widetilde{g}^0_2(h,k) = \int_M \operatorname{Tr}(g^{-1}h(x)g^{-1}k(x)) \operatorname{vol}(g)(x).$$

2.5. **Covariant derivative on M.** We will use covariant derivatives on vector bundles as explained in [21, especially section 19.12]. Let X be a vector field on M. The Levi-Civita covariant derivative ∇_X on (M,g) can be extended uniquely to an operator on the space $\Gamma(T^r_s M)$ of all tensor fields on M. This covariant derivative depends on the metric g.

We define its derivative with respect to g as

$$N^r_s(m) = N^r_s(g,m) = D_{(g,m)}\big(\nabla : \Gamma(T^r_s M) \to \Gamma(T^r_{s+1}M)\big), \tag{1}$$

where m is a tangent vector to g. The operator $N^r_s(m) \in \Gamma\big(L(T^r_s M, T^r_{s+1}M)\big)$ is tensorial since

$$D_{(g,m)}\nabla(fh) = D_{(g,m)}(df \otimes h + f\nabla h) = f D_{(g,m)}\nabla h$$

holds for $f \in C^\infty(M)$ and $h \in \Gamma(T^r_s M)$. In abstract index notation one has

$$\big(N^1_0(m)\big)^{\ \ i}_{jk} = \frac{1}{2} g^{il}\big((\nabla m)_{jkl} + (\nabla m)_{kjl} - (\nabla m)_{ljk}\big), \tag{2}$$

as can be seen from the formula [5, theorem 1.174]:

$$g\big(D_{(g,m)}(\nabla_X Y), Z\big) = \frac{1}{2}\big((\nabla_X m)(Y,Z) + (\nabla_Y m)(X,Z) - (\nabla_Z m)(X,Y)\big).$$

Furthermore, $N^0_1(m) = -N^1_0(m)$ since one has for $\alpha \in \Omega^1(M)$ and $X,Y \in \mathfrak{X}(M)$:

$$(D_{(g,m)}\nabla_X \alpha)(Y) = D_{(g,m)}\big(d(\alpha(Y)).X - \alpha(\nabla_X Y)\big) = -\alpha(D_{(g,m)}\nabla_X Y).$$

Since ∇_X is a derivation on tensor products, one gets a similar property for $N^r_s(m)$:

$$\begin{aligned} &\big(N^r_s(m)\big)^{\ \ i_1 \dots i_r\, i_{r+1} \dots i_{r+s}}_{jk_1 \dots k_r\, k_{r+1} \dots k_{r+s}} = \\ &\qquad = \big(N^1_0(m)\big)^{\ \ i_1}_{jk_1} \delta^{i_2}_{k_2} \dots \delta^{i_{r+s}}_{k_{r+s}} + \dots + \delta^{i_1}_{k_1} \dots \delta^{i_{r+s-1}}_{k_{r+s-1}} \big(N^0_1(m)\big)^{\ \ i_{r+s}}_{jk_{r+s}}, \end{aligned} \tag{3}$$

where one has N^1_0 in the first r summands and N^0_1 in the last s summands.

2.6. **The adjoint of the covariant derivative.** The covariant derivative, seen as a mapping $\nabla : \Gamma(T^r_s M) \to \Gamma(T^r_{s+1} M)$ admits an adjoint $\nabla^* : \Gamma(T^r_{s+1} M) \to \Gamma(T^r_s M)$ with respect to the metric $\widetilde{g}$, i.e.: $\widetilde{g^r_{s+1}}(\nabla B, C) = \widetilde{g^r_s}(B, \nabla^* C)$. It is given by $\nabla^* B = -\operatorname{Tr}^g(\nabla B)$, where the trace contracts the first two tensor slots. This formula is proven in [2].

2.7. **Second covariant derivative.** When the covariant derivative is seen as a mapping $\nabla : \Gamma(T^r_s M) \to \Gamma(T^r_{s+1} M)$, then the *second covariant derivative* is simply $\nabla\nabla = \nabla^2 : \Gamma(T^r_s M) \to \Gamma(T^r_{s+2} M)$. For $X, Y \in \mathfrak{X}(M)$, it is given by $\nabla^2_{X,Y} = \iota_Y \iota_X \nabla^2 = \iota_Y \nabla_X \nabla = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$. Higher covariant derivatives are defined accordingly.

2.8. **Laplacian.** The *Bochner-Laplacian* is defined as $\Delta h := \nabla^* \nabla h = -\operatorname{Tr}^g(\nabla^2 h)$. It can act on all tensor fields h, and it respects the degree of the tensorfield it is acting on. Using 2.5 we get:

Lemma. *The differential of the Laplacian acting on $\binom{p}{q}$-tensors is given by:*

$$\begin{aligned} D_{(g,m)} \Delta h &= -D_{(g,m)} \operatorname{Tr}^g(\nabla^2 h) \\ &= \operatorname{Tr}(g^{-1} m g^{-1} \nabla^2 h) - \operatorname{Tr}^g\left(N^p_{q+1}(m) \nabla h\right) - \operatorname{Tr}^g\left(\nabla N^p_q(m) h\right). \end{aligned}$$

2.9. **Curvature.** The Riemann curvature tensor is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The Ricci tensor field $\operatorname{Ricci}(X,Y)$ is the trace of the map $Z \mapsto R(Z,X)Y$. The scalar curvature is $\operatorname{Scal} = \operatorname{Tr}^g(\operatorname{Ricci})$.

Lemma. *[5, theorem 1.174] The differential of the scalar curvature*

$$\begin{cases} \Gamma(S^2_+ T^* M) & \to \quad C^\infty(M), \\ \qquad g & \mapsto \quad \operatorname{Scal} \end{cases}$$

is given by

$$D_{(g,m)} \operatorname{Scal} = -\Delta(\operatorname{Tr}(g^{-1}.m)) - \nabla^*(\nabla^*(m)) - g^0_2(\operatorname{Ricci}, m).$$

3. Riemannian metrics on the manifold of Riemannian metrics

Let $P_g : \Gamma(S^2 T^* M) \to \Gamma(S^2 T^* M)$ be a positive, symmetric, bijective pseudo-differential operator of order $2p$ depending smoothly on the metric g. Then the operator P induces a metric on the manifold of Riemannian metrics, namely

$$G^P_g(h,k) = \int_M g^0_2(P_g h, k) \operatorname{vol}(g) = \int_M \operatorname{Tr}(g^{-1}.P_g h.g^{-1}.k) \operatorname{vol}(g).$$

3.1. **Geodesic equation.** Given H and K such that

$$D_{(g,m)}G^P_g(h,k) = G^P_g(K_g(h,m),k) = G^P_g(m, H_g(h,k)),$$

the geodesic equation is given by the following variant of the Christoffel symbols

$$g_{tt} = \frac{1}{2}H_g(g_t,g_t) - K_g(g_t,g_t),$$

see [24, 1, 2].

We will now compute the metric gradients H and K. The calculations at the same time show the existence of the gradients. For this aim, let $m,h,k \in T_g\mathcal{M}$. Using the formula for the variation of the volume density from section 2.3 we get

$$\begin{aligned}
G^P_g(K_g(h,m),k) &= D_{(g,m)}G^P_g(h,k) = D_{(g,m)}\int_M \operatorname{Tr}(g^{-1}.Ph.g^{-1}.k)\operatorname{vol}(g)\\
&= \int_M \operatorname{Tr}\big((D_{(g,m)}g^{-1}).Ph.g^{-1}.k\big)\operatorname{vol}(g) + \int_M \operatorname{Tr}\big(g^{-1}.(D_{(g,m)}P)h.g^{-1}.k\big)\operatorname{vol}(g)\\
&\quad + \int_M \operatorname{Tr}\big(g^{-1}.Ph.(D_{(g,m)}g^{-1}).k\big)\operatorname{vol}(g) + \int_M \operatorname{Tr}\big(g^{-1}.Ph.g^{-1}.k\big)D_{(g,m)}\operatorname{vol}(g)\\
&= \int_M \Big[-\operatorname{Tr}\big(g^{-1}.m.g^{-1}.Ph.g^{-1}.k\big) + \operatorname{Tr}\big(g^{-1}.(D_{(g,m)}P)h.g^{-1}.k\big)\\
&\quad - \operatorname{Tr}\big(g^{-1}.Ph.g^{-1}.m.g^{-1}.k\big) + \operatorname{Tr}\big(g^{-1}.Ph.g^{-1}.k\big)\frac{1}{2}\operatorname{Tr}(g^{-1}.m)\Big]\operatorname{vol}(g)\\
&= \int_M g^0_2\Big(-m.g^{-1}.Ph + (D_{(g,m)}P)h - Ph.g^{-1}.m + \frac{1}{2}\operatorname{Tr}(g^{-1}.m).Ph, k\Big)\operatorname{vol}(g).
\end{aligned}$$

Therefore the K-gradient is given by

$$K_g(h,m) = P^{-1}\Big[-m.g^{-1}.Ph + (D_{(g,m)}P)h - Ph.g^{-1}.m + \frac{1}{2}\operatorname{Tr}(g^{-1}.m).Ph\Big].$$

To calculate the H-gradient we will assume that there exists an *adjoint* in the following sense

$$\boxed{\int_M g^0_2\big((D_{(g,m)}P)h,k\big)\operatorname{vol}(g) = \int_M g^0_2\big(m,(D_{(g,.)}Ph)^*(k)\big)\operatorname{vol}(g)} \tag{1}$$

which is smooth in (g,h,k) and bilinear in (h,k). The existence of the adjoint needs to be checked for each specific operator P, usually by partial integration. Using the adjoint we can rewrite the equation above as follows:

$$\begin{aligned}
G^P_g(H_g(h,k),m) &= (D_{(g,m)}G^P_g)(h,k) = D_{(g,m)}\int_M g^0_2(Ph,k)\operatorname{vol}(g)\\
&= \int_M g^0_2\Big(-m.g^{-1}.Ph + (D_{(g,m)}P)h - Ph.g^{-1}.m + \frac{1}{2}\operatorname{Tr}(g^{-1}.m).Ph, k\Big)\operatorname{vol}(g)\\
&= \int_M g^0_2\Big(m, -Ph.g^{-1}.k\Big) + g^0_2\big(m,(D_{(g,.)}Ph)^*(k)\big) + g^0_2\Big(m, -k.g^{-1}.Ph\Big)\\
&\quad + \frac{1}{2}g^0_2\Big(m, g.\operatorname{Tr}(g^{-1}.Ph.g^{-1}.k)\Big)\operatorname{vol}(g)
\end{aligned}$$

Here we can easily read off the H-gradient:

$$H_g(h,m) = P^{-1}\big((D_{(g,.)}Ph)^*(k) - Ph.g^{-1}.k - k.g^{-1}.Ph + \tfrac{1}{2}.g.\operatorname{Tr}(g^{-1}.Ph.g^{-1}.k)\big).$$

Therefore the geodesic equation on the manifold of Riemannian metrics reads as:

$$\boxed{\begin{aligned} g_{tt} &= \frac{1}{2}H_g(g_t,g_t) - K_g(g_t,g_t) \\ &= P^{-1}\Big[(D_{(g,.)}Pg_t)^*(g_t) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.Pg_t.g^{-1}.g_t) \\ &\qquad + \frac{1}{2}g_t.g^{-1}.Pg_t + \frac{1}{2}Pg_t.g^{-1}.g_t - (D_{(g,g_t)}P)g_t - \frac{1}{2}\operatorname{Tr}(g^{-1}.g_t).Pg_t\Big] \end{aligned}}$$

We can rewrite this equation to get it in a slightly more compact form:

$$\boxed{\begin{aligned} (Pg_t)_t &= (D_{(g,g_t)}P)g_t + Pg_{tt} \\ &= (D_{(g,.)}Pg_t)^*(g_t) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.Pg_t.g^{-1}.g_t) \\ &\qquad + \frac{1}{2}g_t.g^{-1}.Pg_t + \frac{1}{2}Pg_t.g^{-1}.g_t - \frac{1}{2}\operatorname{Tr}(g^{-1}.g_t).Pg_t \end{aligned}} \tag{2}$$

3.2. **Well-posedness of some geodesic equations.** For any fixed background Riemann metric $\hat{g}$ on M and its Levi-Civita covariant derivative $\hat{\nabla}$, the *Sobolev space* $H^k(S^2T^*M)$ is the Hilbert space completion of the space $\Gamma(S^2T^*M)$ of smooth sections, in the Sobolev norm

$$\|h\|_k^2 = \sum_{j=0}^{k} \int_M \hat{g}^0_{2+j}((\hat{\nabla})^j h, (\hat{\nabla})^j h)\operatorname{vol}(\hat{g}).$$

The Sobolev space does not depend on the choices of $\hat{g}$; the resulting norms are equivalent. See [25] for more information. The following results hold:

- *Sobolev lemma.* If $k > \frac{\dim(M)}{2}$ then the identy on $\Gamma(S^2T^*M)$ extends to a injective bounded linear mapping $H^{k+p}(S^2T^*M) \to C^p(S^2T^*M)$ where $C^p(S^2T^*M)$ carries the supremum norm of all derivatives up to order p.
- *Module property of Sobolev spaces.* If $k > \frac{\dim(M)}{2}$ then pointwise evaluation $H^k(L(S^2T^*M, S^2T^*M)) \times H^k(S^2T^*M) \to H^k(S^2T^*M)$ is bounded bilinear. Likewise all other pointwise contraction operations are multilinear bounded operations. See [12], or [11, 1.3.12].

The Sobolev lemma allows us to define the Sobolev space $\mathcal{M}^k(M) := H^k(S^2_+T^*M)$ for $k > \frac{\dim(M)}{2}$.

Assumptions. *Let $P_g(h)$, $P_g^{-1}(k)$ and $(D_{(g,.)}Ph)^*(m)$ be linear pseudo-differential operators of order $2p$ in m, h and of order $-2p$ in k for some $p \geq 0$.*

As mappings in the foot point g, we assume that all mappings are non-linear, and that they are a composition of operators of the following type:

(a) Non-linear differential operators of order $l \leq 2p$, i.e.

$$A(g)(x) = A\big(x, g(x), (\hat{\nabla}g)(x), \ldots, (\hat{\nabla}^l g)(x)\big),$$

(b) Linear pseudo-differential operators of order $\leq 2p$,

such that the total (top) order of the composition is $\leq 2p$.

Since $h \mapsto P_g h$ induces a weak inner product, it is a symmetric and injective pseudodifferential operator. It is natural to assume that it is elliptic and selfadjoint. Then it is Fredholm and it has vanishing index by [25, theorem 26.2]. Thus it is invertible and $g \mapsto P_g^{-1}$ is smooth $H^k(S^2_+T^*M) \to L(H^k(S^2T^*M), H^{k+2p}(S^2T^*M))$ by the implicit function theorem on Banach spaces.

Theorem. *Let the assumptions above hold. Then for* $k > \frac{\dim(M)}{2}$*, the initial value problem for the geodesic equation* (3.1.2) *has unique local solutions in the Sobolev manifold* $\mathcal{M}^{k+2p}(M)$ *of* H^{k+2p}*-metrics. The solutions depend* C^∞ *on* t *and on the initial conditions* $g(0,\ .\ ) \in \mathcal{M}^{k+2p}(M)$ *and* $g_t(0,\ .\ ) \in H^{k+2p}(S^2T^*M)$*. The domain of existence (in* t*) is uniform in* k *and thus this also holds in* $\mathcal{M}(M)$*.*

Moreover, in each Sobolev completion $\mathcal{M}^{k+2p}(M)$*, the Riemannian exponential mapping* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in the tangent bundle, and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neighborhood of the zero section to a neighborhood of the diagonal in* $\mathcal{M}^{k+2p}(M) \times \mathcal{M}^{k+2p}(M)$*. All these neighborhoods are uniform in* $k > \frac{\dim(M)}{2}$ *and can be chosen* H^{k_0+2p}*-open, where* $k_0 > \frac{\dim(M)}{2}$*. Thus all properties of the exponential mapping continue to hold in* $\mathcal{M}(M)$*.*

This proof is an adaptation of [2, section 4.2]

Proof. We consider the geodesic equation as the flow equation of a smooth (C^∞) vector field X on the open set

$$\mathcal{M}^{k+2p} \times H^k(S^2T^*M) \subset H^{k+2p}(S^2T^*M) \times H^k(S^2T^*M).$$

We now write the geodesic equation as the flow equation of an autonomous smooth vector field $X = (X_1, X_2)$ on $\mathcal{M}^{k+2p} \times H^k$, as follows (using (3.1.2):

$$\begin{aligned} g_t &= (P_g)^{-1}h =: X_1(g,h) \\ h_t &= \big((D_{(g,.)}P_g)(P_g)^{-1}h\big)^*((P_g)^{-1}h) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.h.g^{-1}.(P_g)^{-1}h) \\ &\qquad + \frac{1}{2}(P_g)^{-1}h.g^{-1}.h + \frac{1}{2}h.g^{-1}.(P_g)^{-1}h - \frac{1}{2}\operatorname{Tr}(g^{-1}.(P_g)^{-1}h).h \qquad (1) \\ &=: X_2(g,h) \end{aligned}$$

For $(g,h) \in \mathcal{M}^{k+2p} \times H^k$ we have $(P_g)^{-1}h \in H^{k+2p}$. Thus a term by term investigation of (1), using the assumptions on the orders, shows that $X_2(g,h)$ is smooth in $(g,h) \in \mathcal{M}^{k+2p} \times H^k$ with values in H^k. Likewise $X_1(g,h)$ is smooth in $(f,h) \in \mathcal{M}^{k+2p} \times H^k$ with values in H^{k+2p}. Thus by the theory of smooth ODE's on Banach spaces, the flow Fl^k exists on $\mathcal{M}^{k+2p} \times H^k$ and is smooth in t and the initial conditions for fixed $k > \frac{\dim(M)}{2}$.

We consider C^∞ initial conditions $g_0 = g(0,\quad)$ and $h_0 = P_{g_0}g_t(0,\quad) = h(0,\quad)$ for the flow equation (1) in $\mathcal{M}(M) \times \Gamma(S^2T^*M)$. Suppose the trajectory $\mathrm{Fl}^k_t(g_0,h_0)$ of X through these intial conditions in $\mathcal{M}^{k+2p} \times H^k$ maximally exists for $t \in (-a_k, b_k)$, and the trajectory $\mathrm{Fl}^{k+1}_t(g_0,h_0)$ in $\mathcal{M}^{k+1+2p} \times H^{k+1}$ maximally exists for $t \in (-a_{k+1}, b_{k+1})$ with $a_{k+1} < a_k$ and $b_{k+1} < b_k$, say. By uniqueness of solutions we have $\mathrm{Fl}_t^{k+1}(g_0,h_0) = \mathrm{Fl}_t^k(g_0,h_0)$ for $t \in (-a_{k+1}, b_{k+1})$. We now apply the background derivative $\hat\nabla$ to both equations (5):

$$(\hat\nabla g)_t = \hat\nabla g_t = \hat\nabla X_1(g,h)$$
$$(\hat\nabla h)_t = \hat\nabla h_t = \hat\nabla X_2(g,h)$$

We claim that for $i = 1, 2$ we have

$$\hat\nabla X_i(g,h) = X_{i.1}(g,h)(\hat\nabla^{2p+1} g) + X_{i,2}(g,h)(\hat\nabla^{2p+1} h) + X_{i,3}(g,h)$$

where all $X_{i,j}(g,h)(l)$ and $X_{i,3}(g,h)$ $(i,j = 1,2)$ are smooth in all variables, of highest order $2p$ in g and h, linear and algebraic (i.e., of order 0) in l. This claim follows from the assumptions: (b) For a linear pseudo differential operator B of order q the commutator $[\nabla, B]$ is a pseudodifferential operator of order q again. (a) For a local operator we can apply the chain rule: The derivative of order $2p+1$ of g appears only linearly.

Then we write $\hat\nabla^{2p+1} g = \hat\nabla^{2p}\tilde g$ and $\hat\nabla^{2p+1} h = \hat\nabla^{2p}\tilde h$ for the highest derivatives only. The last system now becomes

$$\tilde g_t = X_{1,1}(g,h)(\hat\nabla^{2p}\tilde g) + X_{1,2}(g,h)(\hat\nabla^{2p}\tilde h) + X_{1,3}(g,h)$$
$$\tilde h_t = X_{2,1}(g,h)(\hat\nabla^{2p}\tilde g) + X_{2,2}(g,h)(\hat\nabla^{2p}\tilde h) + X_{2,3}(g,h)$$

which is inhomogeneous bounded linear in $(\tilde g, \tilde h) \in \mathcal M^{k+2p} \times H^k$ with coefficients bounded linear operators on H^{k+2p} and H^k, respectively. These coefficients are C^∞ functions of $(g,h) \in \mathcal M^{k+2p} \times H^k$ which we already know on the intervall $(-a_k, b_k)$. This equation therefore has a solution $(\tilde g(t,\ \ ), \tilde h(t,\ \ ))$ for all t for which the coefficients exists, thus for all $t \in (a_k, b_k)$. The limit $\lim_{t\nearrow b_{k+1}}(\tilde g(t,\ \ ), \tilde h(t,\ \ ))$ exists in $\mathcal M^{k+2p} \times H^k$ and by continuity it equals $(\hat\nabla g, \hat\nabla h)$ in $\mathcal M^{k+2p} \times H^k$ for some $t > b_{k+1}$. Thus the H^{k+1}-flow was not maximal and can be continued. So $(-a_{k+1}, b_{k+1}) = (-a_k, b_k)$. We can iterate this and conclude that the flow of X exists in $\bigcap_{m\ge k} \mathcal M^{m+2p} \times H^m = \mathcal M \times \Gamma$.

It remains to check the properties of the Riemannian exponential mapping $\exp^P$. It is given by $\exp^P_g(h) = c(1)$ where $c(t)$ is the geodesic emanating from value g with initial velocity h. From the form $g_{tt} = \frac12 H_g(g_t,g_t) - K_g(g_t,g_t) =: \Gamma_g(g_t,g_t)$ (see subsection 3.1), namely linearity in g_{tt} and bilinearity in g_t, and from local existence and uniqueness on each space $\mathcal M^{k+2p}(M)$ the properties claimed follow: see for example [21, 22.6 and 22.7,] for a detailed proof in terms of the spray vector field $S(g,h) = (g,h;h,\Gamma_g(h,h))$ on $T\mathcal M(M)$ which works on each $T\mathcal M^{k+2p}(M)$ without any change in notation. So we check this on the largest of this spaces $\mathcal M^{k_0}(M)$ (with the smallest k). Since the spray on $\mathcal M^{k_0}(M)$ restricts to the spray on each $\mathcal M^{k+2p}(M)$, the exponential mapping $\exp^P$ and the inverse $(\pi, \exp^P)^{-1}$ on $\mathcal M^{k_0}(M)$ restrict to the corresponding mappings on each $\mathcal M^{k+2p}(M)$. Thus the neighborhoods of existence are uniform in k. □

3.3. **Conserved Quantities.** Consider the right action of the diffeomorphism group $\mathrm{Diff}(M)$ on $\mathcal M(M)$ given by

$$(g,\phi) \mapsto \phi^* g.$$

For this action we can calculate the fundamental vector field as follows:

$$\zeta_X(g) = \mathcal{L}_X g = -2 \operatorname{Sym} \nabla(g(X)).$$

For a proof of the last equality see [5, section 1]. If the metric G^P is invariant under this action, we have the following conserved quantities (see for example [1]):

$$\begin{aligned}
\text{const} = G^P(g_t, \zeta_X(g)) &= \int_M g_2^0\big(Pg_t, -2 \operatorname{Sym} \nabla(g(X))\big) \operatorname{vol}(g) \\
&= -2 \int_M g_1^0\big(\nabla^* \operatorname{Sym} Pg_t, g(X)\big) \operatorname{vol}(g) = -2 \int_M (\nabla^* Pg_t)(X) \operatorname{vol}(g) \\
&= -2 \int_M g\big(g^{-1}\nabla^* Pg_t, X\big) \operatorname{vol}(g)
\end{aligned}$$

Since this equation holds for all vector fields X this yields

$$\boxed{(\nabla^* Pg_t) \operatorname{vol}(g) \in \Gamma(T^*M \otimes_M \operatorname{vol}(M)) \text{ is const. in time.}}$$

4. Special cases of P

4.1. **The H^0-metric.** The simplest and most natural example is the operator P of order zero given by $P(h) = h$. The induced metric is the so called L^2-metric or H^0-metric, which is well studied as mentioned in the introduction. We can easily read off the geodesic equation from the previous section:

$$\boxed{g_{tt} = \frac{1}{4}.g.g_2^0(g_t, g_t) + g_t.g^{-1}.g_t - \frac{1}{2} \operatorname{Tr}(g^{-1}.g_t).g_t}\,.$$

This coincides with the equation derived in [13] and [14]. All conditions from 3.2 are obviously satisfied. Thus the geodesic equation is well-posed.

4.2. **Conformal metrics.** Here we consider metrics of the form

$$G_g^P(h,k) = \Phi(\operatorname{Vol}) \int_M g_2^0(h,k) \operatorname{vol}(g),$$

where Φ is some positive function and $\operatorname{Vol} = \int_M \operatorname{vol}(g)$. To calculate the adjoint we will use the variational formula for the volume form from section 2.3:

$$\begin{aligned}
\int_M g_2^0\big(m, (D_{(g,.)}Ph)^*(k)\big) \operatorname{vol}(g) &= \int_M g_2^0\big((D_{(g,m)}P)h, k\big) \operatorname{vol}(g) \\
&= \Phi'.(D_{(g,m)} \operatorname{Vol}). \int_M g_2^0\big(h,k\big) \operatorname{vol}(g) \\
&= \Phi'. \int_M \operatorname{Tr}(g^{-1}.m) \operatorname{vol}(g). \int_M \operatorname{Tr}\left(g^{-1}.h.g^{-1}.k\right) \operatorname{vol}(g) \\
&= \int_M \operatorname{Tr}\left(g^{-1}.m.\Phi'. \int_M g_2^0(h,k) \operatorname{vol}(g)\right) \operatorname{vol}(g) \\
&= \int_M g_2^0\left(m, \Phi'.g. \int_M g_2^0(h,k) \operatorname{vol}(g)\right) \operatorname{vol}(g)
\end{aligned}$$

Using this formula for the adjoint, the geodesic equation reads as:

$$\boxed{\begin{aligned} g_{tt} = \frac{\Phi'}{\Phi}.g.\int_M g_2^0(g_t,g_t)\operatorname{vol}(g) + \frac{1}{4}.g.g_2^0(g_t,g_t) \\ + g_t.g^{-1}.g_t + \frac{\Phi'}{\Phi}.\int_M g_2^0(g_t,g)\operatorname{vol}(g) - \frac{1}{2}g_2^0(g_t,g).g_t \end{aligned}}$$

or

$$\boxed{\begin{aligned} (\Phi.g_t)_t = \Phi'.g.\int_M \operatorname{Tr}(g^{-1}.g_t.g^{-1}.g_t)\operatorname{vol}(g) + \frac{\Phi}{4}.g.\operatorname{Tr}(g^{-1}.g_t.g^{-1}.g_t) \\ + \Phi.g_t.g^{-1}.g_t - \frac{\Phi}{2}\operatorname{Tr}(g^{-1}.g_t).g_t \end{aligned}}$$

All conditions from theorem 3.2 are satiesfied. Thus the geodesic equation is well-posed and the geodesic exponential mapping exists and is a local diffeomorphism.

4.3. **Curvature weighted metrics.** We consider metrics weighted by scalar curvature:

$$G_g^P(h,k) = \int_M \Phi(\operatorname{Scal}).g_2^0(h,k)\operatorname{vol}(g).$$

Using the variational formula from section 2.9 we can calculate the adjoint as follows:

$$\begin{aligned} \int_M g_2^0\big(m,(D_{(g,.)}Ph)^*(k)\big)\operatorname{vol}(g) &= \int_M g_2^0\big((D_{(g,m)}P)h,k\big)\operatorname{vol}(g) \\ &= \int_M \Phi'.(D_{(g,m)}\operatorname{Scal})g_2^0\big(h,k\big)\operatorname{vol}(g) \\ &= \int_M \Phi'.\Big(-\Delta(\operatorname{Tr}(g^{-1}.m)) - \nabla^*(\nabla^*(m)) - g_2^0(\operatorname{Ricc},m)\Big)g_2^0\big(h,k\big)\operatorname{vol}(g) \\ &= \int_M \Phi'.\Big[-g_1^0\Big(\nabla\operatorname{Tr}(g^{-1}.m),\nabla g_2^0(h,k)\Big) - g_1^0\Big(\nabla^*(m),\nabla g_2^0(h,k)\Big) \\ &\qquad - g_2^0\Big(g_2^0(h,k)\operatorname{Ricc},m\Big)\Big]\operatorname{vol}(g) \\ &= \int_M \Phi'.\Big[-\operatorname{Tr}(g^{-1}.m).\nabla^*\nabla g_2^0(h,k) - g_2^0\Big(m,\nabla^2 g_2^0(h,k)\Big) \\ &\qquad - g_2^0\Big(g_2^0(h,k)\operatorname{Ricc},m\Big)\Big]\operatorname{vol}(g) \\ &= \int_M \Phi'.g_2^0\Big(m,-g.\Delta g_2^0(h,k) - \nabla^2 g_2^0(h,k) - g_2^0(h,k)\operatorname{Ricc}\Big)\operatorname{vol}(g) \end{aligned}$$

Using the formula for the geodesic equation from section 3.1 yields

$$\boxed{\begin{aligned} (\Phi.g_t)_t = \Phi'.\Big(-g.\Delta^g g_2^0(g_t,g_t) - \nabla^2 g_2^0(g_t,g_t) - g_2^0(g_t,g_t)\operatorname{Ricc}\Big) \\ + \frac{\Phi}{4}.g.\operatorname{Tr}(g^{-1}.g_t.g^{-1}.g_t) + \Phi.g_t.g^{-1}.g_t - \frac{\Phi}{2}\operatorname{Tr}(g^{-1}.g_t).g_t. \end{aligned}}$$

The conditions of theorem 3.2 are violated and therefore it is not applicable. We do not know whether the geodesic equation is well-posed.

4.4. **Sobolev metrics using the Laplacian.** We first consider the Sobolev metric of the form

$$G^P_g(h,k) = \int_M g^0_2\big((1+\Delta)^p h, k\big)\operatorname{vol}(g)$$

where Δ^g is the geometric Bochner-Laplacian described in 2.8. To calculate the adjoint we will need the following lemma:

Lemma. *The differential of the Laplacian acting on $\binom{0}{2}$-tensors admits an adjoint with repect to the metric $\widetilde{g^2_0}$, which is given by:*

$$\begin{aligned}
&\widetilde{g^2_0}\big(D_{(g,m)}\Delta h, k\big) =: \widetilde{g^0_2}\big(m, (D_{(g,.)}\Delta h)^*(k)\big)\\
&\quad = \widetilde{g^2_0}\Big(m, g^{i_1 j_1} g^{i_2 j_2}(\nabla^2 h)_{..i_1 i_2} k_{j_1 j_2} - (N^0_3(.)\nabla h)^*(g\otimes k) + (N^0_2(.)h)^*(\nabla k)\Big).
\end{aligned}$$

Here $(N^0_q(.)h)^*$ denotes the adjoint of the differential of the covariant derivative:

$$\widetilde{g^0_{q+1}}(N^0_q(m)h, k) =: \widetilde{g^0_2}\big(m, (N^0_q(.)h)^*(k)\big) = \widetilde{g^0_2}\big(m, \nabla^*(\sigma(N^0_q)(.)h)^* k\big),$$

where $h \in \Gamma(T^0_q M)$, $k \in \Gamma(T^0_{q+1}M)$ and where $\sigma(N^0_q)$ denotes the total symbol of N^0_q. It is tensorial and of the form

$$\begin{aligned}
&\sigma(N^0_q)(\widetilde{m})(h)(X_0,\dots,X_q) =\\
&\quad = -\frac{1}{2}\sum_{j=1}^{q} h\Big(X_1,\dots,X_{j-1}, \sum_{i=0}^{2}(-1)^i\big(\tau^i(\widetilde{m})(\cdot, X_0, X_j)\big)^\sharp, X_{j+1},\dots,X_q\Big),
\end{aligned}$$

where $\widetilde{m} \in \Gamma(T^*M \otimes S^2_+ T^*M)$, $h \in \Gamma(T^0_q M)$, $X_0,\dots,X_q \in \mathfrak{X}(M)$, and where τ^i is the i-th power of the cyclic permutation $\tau(\alpha\otimes\beta\otimes\gamma) = \gamma\otimes\alpha\otimes\beta$.

Proof. To prove the formula for N^0_q it suffices to show that

$$N^p_q(m)(h) = \sigma(N^p_q)(\nabla m)(h).$$

This follows from (2) and (3) in 2.5. The formula for $D_{(g,.)}\Delta$ follows from 2.8. □

We can use this lemma to calculate the adjoint of P:

$$\begin{aligned}
\int_M g^0_2\big(m, (D_{(g,.)}Ph)^*(k)\big)\operatorname{vol}(g) &= \int_M g^0_2\big((D_{(g,m)}P)h, k\big)\operatorname{vol}(g)\\
&= \sum_{i=1}^{p}\int_M g^0_2\big((1+\Delta)^{i-1}(D_{(g,m)}\Delta)(1+\Delta)^{p-i}h, k\big)\operatorname{vol}(g)\\
&= \sum_{i=1}^{p}\int_M g^0_2\big((D_{(g,m)}\Delta)(1+\Delta)^{p-i}h, (1+\Delta)^{i-1}k\big)\operatorname{vol}(g)\\
&= \sum_{i=1}^{p}\int_M g^0_2\Big(m, \big((D_{(g,.)}\Delta)(1+\Delta)^{p-i}h\big)^*(1+\Delta)^{i-1}k\Big)\operatorname{vol}(g)
\end{aligned}$$

Using the formula for the geodesic equation from section 3.1 and the formula for the adjoint of $D_{g,.}P$ yields the geodesic equation for Sobolev type metrics:

$$\boxed{\begin{aligned}&((1+\Delta)^p g_t)_t = g^{i_1 j_1} g^{i_2 j_2} (\nabla^2 (1+\Delta)^{p-i} g_t)_{..i_1 i_2} (1+\Delta)^{i-1} (g_t)_{j_1 j_2}\\ &\quad - (N_3^0(.)\nabla(1+\Delta)^{p-i} g_t)^*(g \otimes (1+\Delta)^{i-1} g_t)\\ &\quad + (N_2^0(.)(1+\Delta)^{p-i} g_t)^*(\nabla(1+\Delta)^{i-1} g_t) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.(1+\Delta)^p g_t.g^{-1}.g_t)\\ &\quad + \frac{1}{2} g_t.g^{-1}.(1+\Delta)^p g_t + \frac{1}{2}(1+\Delta)^p g_t.g^{-1}.g_t - \frac{1}{2}\operatorname{Tr}(g^{-1}.g_t).(1+\Delta)^p g_t.\end{aligned}}$$

The conditions of theorem 3.2 are valid, so the geodesic equation is well-posed.

5. The Ricci vector field

The space of metrics $\mathcal{M}(M)$ is a convex open subset in the Fréchet space $\Gamma(S^2T^*M)$. So it is contractible. A necessary and sufficient condition for Ricci curvature to be a gradient vector field with respect to the G^P-metric is that the following exterior derivative vanishes:

$$\big(dG^P(\text{Ricci},\cdot)\big)(h,k) = hG^P(\text{Ricci},k) - kG^P(\text{Ricci},h) - G^P(\text{Ricci},[h,k]) = 0.$$

It suffices to look at constant vectorfields h, k, in which case $[h,k]=0$. We have

$$\begin{aligned}&hG^P(\text{Ricci},k) - kG^P(\text{Ricci},h)\\ &= \int \Big(-\operatorname{Tr}\big(g^{-1}hg^{-1}(P\,\text{Ricci})g^{-1}k\big) + \operatorname{Tr}\big(g^{-1}kg^{-1}(P\,\text{Ricci})g^{-1}h\big)\\ &+ \operatorname{Tr}\big(g^{-1}D_{g,h}(P\,\text{Ricci})g^{-1}k\big) - \operatorname{Tr}\big(g^{-1}D_{g,k}(P\,\text{Ricci})g^{-1}h\big)\\ &- \operatorname{Tr}\big(g^{-1}(P\,\text{Ricci})g^{-1}hg^{-1}k\big) + \operatorname{Tr}\big(g^{-1}(P\,\text{Ricci})g^{-1}kg^{-1}h\big)\\ &+ \operatorname{Tr}\big(g^{-1}(P\,\text{Ricci})g^{-1}k\big)\operatorname{Tr}(g^{-1}h) - \operatorname{Tr}\big(g^{-1}(P\,\text{Ricci})g^{-1}h\big)\operatorname{Tr}(g^{-1}k)\Big)\operatorname{vol}(g).\end{aligned}$$

Some terms in this formula cancel out because for symmetric A, B, C one has

$$\operatorname{Tr}(ABC) = \operatorname{Tr}((ABC)^\top) = \operatorname{Tr}(C^\top B^\top A^\top) = \operatorname{Tr}(A^\top C^\top B^\top) = \operatorname{Tr}(ACB).$$

Therefore

$$\begin{aligned}&hG^P(\text{Ricci},k) - kG^P(\text{Ricci},h)\\ &= \int \Big(\operatorname{Tr}\big(g^{-1}D_{g,h}(P\,\text{Ricci})g^{-1}k\big) - \operatorname{Tr}\big(g^{-1}D_{g,k}(P\,\text{Ricci})g^{-1}h\big)\\ &\qquad + \operatorname{Tr}\big(g^{-1}(P\,\text{Ricci})g^{-1}k\big)\operatorname{Tr}(g^{-1}h) - \operatorname{Tr}\big(g^{-1}(P\,\text{Ricci})g^{-1}h\big)\operatorname{Tr}(g^{-1}k)\Big)\operatorname{vol}(g).\end{aligned}$$

We write $D_{g,h}(P\,\text{Ricci}) = Q(h)$ for some differential operator Q mapping symmetric two-tensors to themselves and Q^* for the adjoint of Q with respect to $\widetilde{g_2^0}$.

$$\begin{aligned}&hG^P(\text{Ricci},k) - kG^P(\text{Ricci},h)\\ &\qquad= \int \Big(g_2^0(Q(h),k) - g_2^0(Q(k),h)\\ &\qquad\qquad + g_2^0\big(P\,\text{Ricci},k\big)\operatorname{Tr}(g^{-1}h) - g_2^0\big(P\,\text{Ricci},h\big)\operatorname{Tr}(g^{-1}k)\Big)\operatorname{vol}(g)\\ &\qquad= \int g_2^0\Big(Q(h) - Q^*(h) + (P\,\text{Ricci}).\operatorname{Tr}(g^{-1}h) - g.g_2^0\big(P\,\text{Ricci},h\big),k\Big)\operatorname{vol}(g).\end{aligned}$$

We have proved:

Lemma. *The Ricci vector field* Ricci *is a gradient field for the G^P-metric iff*

$$Q(h) - Q^*(h) + (P\,\mathrm{Ricci}).\,\mathrm{Tr}(g^{-1}h) - g.g^0_2\big(P\,\mathrm{Ricci}, h\big) = 0.$$

References

[1] Martin Bauer, Philipp Harms, and Peter W. Michor. Almost local metrics on shape space of hypersurfaces in n-space, 2010. `arXiv:1001.0717`.

[2] Martin Bauer, Philipp Harms, and Peter W. Michor. Sobolev metrics on shape space of surfaces, 2010. `arXiv:1009.3616`.

[3] Martin Bauer, Philipp Harms, and Peter W. Michor. Curvature weighted metrics on shape space of hypersurfaces in n-space, 2010. `arXiv:1102.0678`.

[4] Martin Bauer, Martins Bruveris, Philipp Harms, and Peter W. Michor. The KdV-equation has vanishing geodesic distance, 2011. `arXiv:1102.0236`.

[5] Arthur L. Besse. *Einstein manifolds*. Classics in Mathematics. Springer-Verlag, Berlin, 2008.

[6] Jeff Cheeger, Mikhail Gromov, and Michael Taylor. Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. *J. Differential Geom.*, 17: 15–53, 1982.

[7] Brian Clarke. The Completion of the Manifold of Riemannian Metrics with Respect to its L^2 Metric. Doctoral Dissertation, Leipzig 2009. `arXiv:0904.0159`.

[8] Brian Clarke. The Metric Geometry of the Manifold of Riemannian Metrics over a Closed Manifold, 2009. `arXiv:0904.0174`.

[9] Brian Clarke. The Completion of the Manifold of Riemannian Metrics, 2009. `arXiv:0904.0177`.

[10] David G. Ebin. The manifold of Riemannian metrics. *Global Analysis. Proc. Symp. Pure Math. AMS*, 15, 11-40, 1970.

[11] Jürgen Eichhorn. *Global analysis on open manifolds*. Nova Science Publishers, Inc., New York, 2007.

[12] Jürgen Eichhorn and Jan Fricke. The module structure theorem for Sobolev spaces on open manifolds. *Math. Nachr.*, 194:35–47, 1998.

[13] Daniel S, Freed and David Groisser. The basic geometry of the manifold of Riemannian metrics and of its quotient by the diffeomorphism group. *Michigan Math. J.*, 36, 323–344, 1989.

[14] Olga Gil-Medrano and Peter W. Michor. The Riemannian manifold of all Riemannian metrics. *Quarterly J. Math. Oxford*, 42, 183–202, 1991.

[15] Olga Gil-Medrano, Peter W. Michor, and Martin Neuwirther. Pseudoriemannian metrics on spaces of bilinear structures. *Quarterly J. Math. Oxford*, 43, 201–221, 1992.

[16] Olga Gil-Medrano and Peter W. Michor. Geodesics on spaces of almost hermitian structures. *Israel J. Math.*, 88, 319–332, 1994.

[17] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1996.

[18] Ivan Kolář, Peter W. Michor, and Jan Slovák. *Natural operations in differential geometry*. Springer-Verlag, Berlin, 1993.

[19] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[20] A. C. G. Mennucci, A. Yezzi, and G. Sundaramoorthi. Properties of Sobolev-type metrics in the space of curves. *Interfaces Free Bound.*, 10(4):423–445, 2008.

[21] Peter W. Michor. *Topics in differential geometry*, volume 93 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[22] Peter W. Michor and David Mumford. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms. *Doc. Math.*, 10:217–245 (electronic), 2005.

[23] Peter W. Michor and David Mumford. Riemannian geometries on spaces of plane curves. *J. Eur. Math. Soc. (JEMS) 8 (2006), 1-48*, 2006. arxiv:0312384.

[24] Peter W. Michor and David Mumford. An overview of the Riemannian metrics on spaces of curves using the Hamiltonian approach. *Appl. Comput. Harmon. Anal.*, 23(1):74–113, 2007.

[25] M. A. Shubin. *Pseudodifferential operators and spectral theory*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1987.

[26] Alain Trouvé and Laurent Younes. Diffeomorphic matching problems in one dimension: Designing and minimizing matching functionals. In David Vernon, editor, *Computer Vision*, volume 1842. ECCV, 2000.

[27] Steven Verpoort. *The geometry of the second fundamental form: Curvature properties and variational aspects*. PhD thesis, Katholieke Universiteit Leuven, 2008.

[28] Laurent Younes. Computable elastic distances between shapes. *SIAM J. Appl. Math.*, 58(2):565–586 (electronic), 1998.

[29] Laurent Younes, Peter W. Michor, Jayant Shah, and David Mumford. A metric on shape space with explicit geodesics. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 19(1):25–57, 2008.

Martin Bauer, Philipp Harms, Peter W. Michor: Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria.

E-mail address: Bauer.Martin@univie.ac.at

E-mail address: Philipp.Harms@univie.ac.at

E-mail address: Peter.Michor@univie.ac.at