A 2-COCYCLE ON A GROUP OF SYMPLECTOMORPHISMS

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Abstract. For a symplectic manifold (M, ω) with exact symplectic form we construct a 2-cocycle on the group of symplectomorphisms and indicate cases when this cocycle is not trivial.

1. Introduction

For a symplectic manifold (M, ω) such that $H^1(M, \mathbb{R}) = 0$ and the symplectic form ω is exact we indicate a formula defining a 2-cocycle on the group $\operatorname{Diff}(M, \omega)$ of symplectomorphisms with values in the trivial $\operatorname{Diff}(M, \omega)$-module $\mathbb{R}$. Let G be a connected real simple Lie group and K a maximal compact subgroup. For the symmetric Hermitian space $M = G/K$ endowed with the induced symplectic structure, we prove that the restriction of this cocycle to the group G is non-trivial. Thus this cocycle is non-trival on the whole group $\operatorname{Diff}(M, \omega)$, too. In particular, this implies that the cocycle is non-trivial for the symplectic manifold $(\mathbb{R}^2 \times M, \omega_0 + \omega_M)$, where (M, ω_M) is a non-compact symplectic manifold with exact symplectic form ω_M such that $H^1(M, \mathbb{R}) = 0$ and ω_0 is the standard symplectic form on $\mathbb{R}^2$.

For the convenience of the reader, in an appendix we consider the corresponding 2-cocycle on the Lie algebra of locally Hamiltonian and Hamiltonian vector fields and indicate when this cocycle is non-trivial.

Note that in [7] a similar 2-cocycle was constructed for the group of volume preserving diffeomorphisms on a compact n-dimensional manifold M. This cocycle takes its values in the space $H^{n-2}(M, \mathbb{R})$. Neretin in [10] constructed a 2-cocycle on the group of symplectomorphisms with compact supports.

Throughout the paper M is a connected C^∞-manifold.

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2. Preliminaries

We recall some standard facts on central extensions of groups and two-dimensional cohomology of groups (see, for example, [8], ch. 4).

Consider a group G and the field $\mathbb{R}$ as a trivial G-module. Let $C^p(G,\mathbb{R})$ be the set of maps from G^p to $\mathbb{R}$ for $p > 0$ and let $C^0(G,\mathbb{R}) = \mathbb{R}$. Define a map $D^p : C^p(G,\mathbb{R}) \to C^{p+1}(G,\mathbb{R})$ as follows: for $f \in C^p(G,\mathbb{R})$ and $g_1,\dots,g_{p+1} \in G$

$$(D^p f)(g_1,\dots,g_{p+1}) = f(g_2,\dots,g_{p+1}) + \sum_{i=1}^{p}(-1)^i f(g_1,\dots,g_i g_{i+1},\dots,g_{p+1}) + (-1)^{p+1} f(g_1,\dots,g_p). \tag{1}$$

By definition $C^*(G,\mathbb{R}) = (C^p(G,\mathbb{R}), D^p)_{p\geq 0}$ is the standard complex of nonhomogeneous cochains of the group G with values in the G-module $\mathbb{R}$ and its cohomology $H^*(G,\mathbb{R}) = (H^p(G,\mathbb{R}))_{p\geq 0}$ is the cohomology of the group G with values in the trivial G-module $\mathbb{R}$. Recall that a cochain $f \in C(G,\mathbb{R})$ is called normalized if $f(g_1,\dots,g_p) = 0$ whenever at least one of the $g_1,\dots,g_p \in G$ equals the identity e of G. It is known that the inclusion of the subcomplex of normalized cochains into $C^*(G,\mathbb{R})$ induces an isomorphism in cohomology.

Let f be a normalized 2-cocycle of G with values in a trivial G-module $\mathbb{R}$. Let $E(G,\mathbb{R}) = G\times\mathbb{R}$, with multiplication $(g_1,a_1)(g_2,a_2) = (g_1g_2, a_1g_2 + a_2 + f(g_1,g_2))$ for $a_1,a_2 \in \mathbb{R}$ and $g_1,g_2 \in G$. Then $E(G,\mathbb{R})$ is a group, and the natural projection $E(G,\mathbb{R}) = G\times\mathbb{R} \to G$ is a central extension of the group G by $\mathbb{R}$. The extension $E(G,\mathbb{R})$ is non-split iff the cocycle f is non-trivial.

If G is a topological group (finite-dimensional or infinite-dimensional Lie group) one can define a subcomplex $C^*_{\text{cont}}(G,\mathbb{R})$ ($C^*_{\text{diff}}(G,\mathbb{R})$) of the complex $C^*(G,\mathbb{R})$ (see [4], ch. 3) consisting of cochains which are continuous (smooth) functions. The cohomologies of the complexes $C^*_{\text{cont}}(G,\mathbb{R})$ and $C^*_{\text{diff}}(G,\mathbb{R})$ are isomorphic whenever G is a finite-dimensional Lie group (see [4], ch. 3 and [9]). Note that if the 2-cocycle f is continuous (differentiable), the extension $E(G,\mathbb{R})$ is a topological group (Lie group).

3. A 2-cocycle on the group of symplectomorphisms

Let (M,ω) be a non-compact symplectic manifold such that $H^1(M,\mathbb{R}) = 0$ and the symplectic form ω is exact. Let ω_1 be a 1-form on M such that $d\omega_1 = \omega$. Denote by $\mathrm{Diff}(M,\omega)$ the group of symplectomorphisms of M. We define a 2-cocycle on the group $G = \mathrm{Diff}(M,\omega)$ with values in the trivial G-module $\mathbb{R}$ as follows. Fix a point $x_0 \in M$. Then for $g_1,g_2 \in G$ we put

$$C_{x_0}(g_1,g_2) = \int_{x_0}^{g_2x_0} (g_1^*\omega_1 - \omega_1), \tag{2}$$

where the integral is taken along a smooth curve connecting the point x_0 with the point g_2x_0. Since $H^1(M,\mathbb{R}) = 0$ the 1-form $g_1^*\omega_1 - \omega_1$ is exact and the value of this integral does not depend on the choice of such a curve.

Theorem 3.1. *The function $C_{x_0} : G^2 \to \mathbb{R}$ defined by (2) is a normalized 2-cocycle on the group G with values in the trivial G-module $\mathbb{R}$. The cohomology class of C_{x_0} is independent of the choice of the point x_0 and the form ω_1.*

Proof. By (1) it is easy to check that $D^2C_{x_0} = 0$. Moreover, the 2-cocycle C_{x_0} is normalized. Since for each $g \in G$ the 1-form $g^*\omega_1 - \omega_1$ is exact, for any points $x_1, x_2 \in M$ we have $C_{x_1} - C_{x_2} = Da$, where a is a 1-cochain on G defined by $a(g) = \int_{x_1}^{x_2}(g^*\omega_1 - \omega_1)$. □

By definition, the cocycle C_{x_0} is a continuous function on $G \times G$.

Remark 3.2. Let M be a manifold such that $H^1(M,\mathbb{R}) = 0$ and let ω be an exact 2-form on M. Let $\mathrm{Diff}(M,\omega)$ be the group of diffeomorphisms of M preserving the form ω. Then the formula (2) for $g_1, g_2 \in \mathrm{Diff}(M,\omega)$ gives a 2-cocycle on the group $\mathrm{Diff}(M,\omega)$ and all statements of theorem (3.1) are true for this cocycle.

Denote by $E(\mathrm{Diff}(M,\omega))$ the central extension of the group $\mathrm{Diff}(M,\omega)$ by $\mathbb{R}$ defined by the cocycle C_{x_0}. Now we give a geometric interpretation of the extension $E(\mathrm{Diff}(M,\omega))$. We choose a form ω_1 with $d\omega_1 = \omega$ and put $\omega_2(g) = \int_{x_0}^{x}(\omega_1 - g^*\omega_1)$. Consider the trivial $\mathbb{R}$-bundle $M \times \mathbb{R}$. Clearly, the form $dt + \omega_1$ is a connection with curvature ω of this bundle. Denote by $\mathrm{Aut}(M \times \mathbb{R}, \omega)$ the group of those bundle automorphisms which respect the connection $dt + \omega_1$ and which are projectable to diffeomorphisms in $\mathrm{Diff}(M,\omega)$. It is easy to check that the group $\mathrm{Aut}(M \times \mathbb{R}, \omega)$ is isomorphic to the group $E(\mathrm{Diff}(M,\omega)) = \mathrm{Diff}(M,\omega) \times \mathbb{R}$ which acts as follows on $M \times \mathbb{R}$: $(x,t) \to (g(x), \omega_2(g)(x) + t + a)$, where $(x,t) \in M \times \mathbb{R}$ and $(g,a) \in G \times \mathbb{R}$. This gives an equivalent definition of the extension $E(\mathrm{Diff}(M,\omega))$ as a group of automorphisms of the trivial principal $\mathbb{R}$-bundle $M \times \mathbb{R}$ with connection $dt + \omega_1$.

If we replace the form ω_1 by the form $\omega_1 + df$, where f is a smooth function on M, we get an action of G on $M \times \mathbb{R}$ which is related to the initial one by the gauge transformation $(x,t) \to (x, t - f(x))$ of the bundle $M \times \mathbb{R} \to M$.

4. Examples of non-trivial 2-cocycles

The authors are not able to prove that the cocycle C_{x_0} is non-trivial for any symplectic manifold M with an exact symplectic 2-form ω. In this section we prove that for some symplectic manifolds the restrictions of this cocycle to some subgroups of $G \subset \mathrm{Diff}(M,\omega)$ turn out to be non-trivial.

4.1. **The linear symplectic space $\mathbb{R}^{2n}$ and the Heisenberg group.** Consider the space $\mathbb{R}^{2n}$ with the standard symplectic form $\omega_0 = \sum_{k=1}^{n} dx_k \wedge dx_{k+n}$ and the group $G = \mathbb{R}^{2n}$ acting on the space $\mathbb{R}^{2n}$ by translations. Applying (2) to the form

ω_0, the 1-form $\omega_1 = \frac{1}{2}\sum_{k=1}^{n}(x_{n+k}dx_k - x_k dx_{n+k})$, and the point $x_0 = 0 \in \mathbb{R}^{2n}$ we get a 2-cocycle on the group G given by

$$C_0(x, y) = \frac{1}{2}\sum_{k=1}^{n}(x_k y_{n+k} - y_k x_{n+k}),$$

where $x = (x_1, \ldots, x_{2n})$ and $y = (y_1, \ldots, y_{2n})$. The central extension of the group $\mathbb{R}^{2n}$ by $\mathbb{R}$ defined by this cocycle is the Heisenberg group. This extension is non-split since the Heisenberg group is noncommutative and thus the cocycle $C_0(x, y)$ is non-trivial.

4.2. **Symmetric Hermitian spaces and the Guichardet-Wigner cocycle.** Consider a non-compact symmetric space $M = G/K$, where G is a connected real simple Lie group and where K is a maximal compact subgroup. Then M is diffeomorphic to $\mathbb{R}^n$, where $n = \dim M$. We suppose that M admits a G-invariant complex structure, i.e., M is a symmetric Hermitian space. This condition is satisfied (up to finite covering) for the following groups: $\mathrm{SU}(p, q)$ $(p, q \geq 1)$, $\mathrm{SO}_0(2, q)$ $(q = 1$ or $q \geq 3)$, $\mathrm{Sp}(n, \mathbb{R})$ $(n \geq 1)$, $\mathrm{SO}^*(2n)$ $(n \geq 2)$, and certain real forms of E_6 and E_7.

Consider the symplectic manifold (M, ω), where the symplectic form ω is defined by the Hermitian metric on M. It is known that on each of the Lie groups mentioned above, in the complex $C_{\mathrm{diff}}(G, \mathbb{R})$ there is a non-trivial Guichardet-Wigner 2-cocycle (see [5] and [4]). By [2] this cocycle is given as follows, up to a nonzero factor:

$$(g_1, g_2) \mapsto \int_{(x_0, g_1x_0, g_1g_2x_0)} \omega, \tag{3}$$

where $g_1, g_2 \in G$, $x_0 = K \in G/K$, and the integral is taken over the oriented geodesic cone with vertex x_0 and the segment of a geodesic from g_1x_0 to $g_1g_2x_0$ as base.

We prove that the restriction of the cocycle C_{x_0} to the group G is cohomologous to the cocycle given by (3).

For the base point x_0 we define a 1-cochain γ_{x_0} on the group G as follows:

$$\gamma_{x_0}(g) = \int_{x_0}^{gx_0} \omega_1,$$

where $g \in G$ and the integral is taken along the geodesic segment from x_0 to gx_0. Consider C_{x_0} on G given by formula (2), where we choose for the curve between the points x_0 and g_2x_0 a geodesic segment from x_0 to g_2x_0. It is easy to check that on the group G the cocycle $C_{x_0} + D\gamma_{x_0}$ equals the cocycle given by (3). Thus the cocycle C_{x_0} on the group G is non-trivial in the complex $C^*_{\mathrm{diff}}(G, \mathbb{R})$.

In particular, for the group $G = \mathrm{SL}(2, \mathbb{R}) = \mathrm{SU}(1, 1)$ the symmetric space $M = G/K$ is the hyperbolic plane H^2 and ω is the area form on H^2. Instead of the group $\mathrm{SL}(2, \mathbb{R})$ we will later consider the group $\mathrm{PSL}(2, \mathbb{R})$ which acts effectively on H^2. Since $\mathrm{SL}(2, \mathbb{R})$ is a two-sheet cover of $\mathrm{PSL}(2, \mathbb{R})$, the cohomologies of these groups with values in $\mathbb{R}$ are the same. It is easy to check that the corresponding symplectic manifold (M, ω) is isomorphic to the symplectic manifold $(\mathbb{R}^2, \omega_0)$, where ω_0 is the standard symplectic form on $\mathbb{R}^2$. Unfortunately, for the groups $G \neq \mathrm{SU}(1,1)$ mentioned above we do not know whether the symplectic manifolds (M, ω) and $(\mathbb{R}^{2n}, \omega_0)$, where $\dim M = 2n$, are isomorphic or not.

The following proposition is known. We do not know a good reference for this; then we give a short proof communicated to us by Yu.A. Neretin.

Proposition 4.3. *For each symmetric Hermitian space $M = G/K$, where G is a connected simple Lie group and K is its maximal compact subgroup, the corresponding Guichardet-Wigner cocycle is non-trivial in the complex $C^*(G, \mathbb{R})$.*

Proof. Let $p : \tilde{G} \to G$ be the universal cover and let $a = C_{x_0}$ be the Guichardet-Wigner cocycle for the group G. Consider the corresponding to a 2-cocycle $\tilde{a}$ on $\tilde{G}$ induced by p. By construction, the cocycle $\tilde{a}$ is trivial, i.e., there is a smooth function b defined on $\tilde{G}$ such that for any $g, h \in \tilde{G}$ we have $\tilde{a}(g, h) = b(h) - b(gh) + b(g)$.

Assume that the cocycle a is trivial in the complex $C^*(G, \mathbb{R})$, i.e., there exists a function $f : G \to \mathbb{R}$ such that for $g, h \in \Gamma$ we have $a(g, h) = f(h) - f(gh) + f(g)$.

Then the difference $b - f \circ p$ is a homomorphism $\tilde{G} \to \mathbb{R}$. This homomorphism vanishes near the identity element of $\tilde{G}$ since the group $\tilde{G}$ is simple, and thus it vanishes on the whole of $\tilde{G}$ since $\tilde{G}$ is connected. Then the function f is smooth and the cocycle a is trivial in the complex $C^*_{\mathrm{diff}}(G, \mathbb{R})$. This contradiction proves our statement. □

5. Cases of nontriviality of the cocycle C_{x_0} for groups of symplectomorphisms

Let (M, ω_M) be a non-compact symplectic manifold such that $H^1(M, \mathbb{R}) = 0$ with an exact symplectic form ω_M.

By formula (2), the form ω_M defines a 2-cocycle C_{x_0} for the group $\mathrm{Diff}(M, \omega_M)$ with values in the trivial $\mathrm{Diff}(M, \omega_M)$-module $\mathbb{R}$. The aim of this section is to indicate cases when this cocycle is non-trivial and thus the corresponding central extension of the group $\mathrm{Diff}(M, \omega_M)$ by $\mathbb{R}$ is non-split.

Let $M = G/K$ be an Hermitian symmetric space M and let (M, ω) be the corresponding symplectic manifold which we considered in subsection 4.2.

Theorem 5.1. *For the Hermitian symmetric space $M = G/K$ and for the corresponding symplectic manifold (M, ω) the cocycle C_0 on the group $\mathrm{Diff}(M, \omega)$ is non-trivial.*

Proof. Since the group G is a subgroup of the group $\mathrm{Diff}(M, \omega)$ the statement follows from proposition 4.3. □

Recall that the symplectic manifold (H^2, ω) where ω is the area form is symplectomorphic to $(\mathbb{R}^2, \omega_0)$ where ω_0 is the standard symplectic form.

Theorem 5.2. *Let (M, ω) be a non-compact symplectic manifold such that the symplectic form ω_M is exact and let $H^1(M, \mathbb{R}) = 0$. Consider the product $\mathbb{R}^2 \times M$ of the manifold $\mathbb{R}^2$ and M as a symplectic manifold with the symplectic form $\omega = \omega_0 + \omega_M$. Then for each point $x_0 \in \mathbb{R}^2 \times M$ the cocycle C_{x_0} on the group $\mathrm{Diff}(\mathbb{R}^2 \times M, \omega)$ is non-trivial.*

Proof. Choose $\omega_{M,1} \in \Omega^1(M)$ with $d\omega_{M,1} = \omega_M$ and let $\omega_1 = x\,dy + \omega_{M,1}$. The group $\mathrm{Diff}(\mathbb{R}^2, \omega_0)$ acting on the first factor $\mathbb{R}^2$ of $\mathbb{R}^2 \times M$ is naturally included as a subgroup into the group $\mathrm{Diff}(\mathbb{R}^2 \times M, \omega)$. Thus $g^*\omega_1 - \omega_1 = g^*(x\,dy) - x\,dy$ for all g in the subgroup $\mathrm{Diff}(\mathbb{R}^2, \omega_0)$. Thus the cocycle C_{x_0} constructed from the form $dx \wedge dy + \omega_M$ on $\mathbb{R}^2 \times M$ restricts to a nontrial cocycle on the subgroup of $\mathrm{Diff}(\mathbb{R}^2, \omega_0)$ by proposition 4.3 applied to the group $\mathrm{PSL}(2, \mathbb{R})$. □

We leave to the reader to formulate the corresponding results for other symmetric Hermitian spaces G/K instead of H^2.

5.3. **Problem.** Consider an open disk M in the Euclidean plane equipped with the standard area 2-form ω. Is the 2-cocycle C_{x_0} defined by the form ω non-trivial?

6. APPENDIX

In this appendix, for a symplectic manifold (M, ω) we define a 2-cocycle on the Lie algebra $\mathrm{Vect}(M, \omega)$ of locally Hamiltonian or Hamiltonian vector fields, corresponding to the 2-cocycle C_{x_0} on the group $\mathrm{Diff}(M, \omega)$, and study conditions of its nontriviality.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ and let $\mathbb{R}$ be the trivial $\mathfrak{g}$-module. Denote by $C^p(\mathfrak{g}, \mathbb{R})$ the space of skew-symmetric p-forms on $\mathfrak{g}$ with values in $\mathbb{R}$. For $c \in C^p(\mathfrak{g}, \mathbb{R})$ and $x_1, \ldots, x_{p+1}$ put

$$(\delta^p c)(x_1, \ldots, x_{p+1}) = \sum_{i<j} (-1)^{i+j} c([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j \ldots, x_{p+1}), \tag{4}$$

where, as usual, $\hat{x}$ means that x is omitted. Then $C^*(\mathfrak{g}, \mathbb{R}) = (C^p(\mathfrak{g}, \mathbb{R}), \delta^p)_{p \geq 0}$ is the complex of standard cochains of the Lie algebra $\mathfrak{g}$ with values in the trivial $\mathfrak{g}$-module $\mathbb{R}$ and the cohomology $H^*(\mathfrak{g}, \mathbb{R})$ of this complex is the cohomology of the Lie algebra $\mathfrak{g}$ with values in the trivial $\mathfrak{g}$-module $\mathbb{R}$.

In particular, there is a bijective correspondence between $H^2(\mathfrak{g}, \mathbb{R})$ and the set of isomorphism classes of central extensions of the Lie algebra $\mathfrak{g}$ by $\mathbb{R}$.

Let (M, ω) be a symplectic manifold. Denote by $\mathrm{Vect}(M, \omega)$ the Lie algebra of locally Hamiltonian vector fields and by $\mathrm{Vect}_0(M, \omega)$ the Lie algebra of Hamiltonian vector fields on M. For a point $x_0 \in M$ and $X, Y \in \mathrm{Vect}(M, \omega)$ put $c_{x_0}(X, Y) = \omega(X, Y)(x_0)$.

Proposition 6.1. *The function $c_{x_0} : \mathfrak{g}^2 \to \mathbb{R}$ is a 2-cocycle on the Lie algebra $\mathfrak{g}$ with values in the trivial $\mathfrak{g}$-module $\mathbb{R}$. The cohomology class of c_{x_0} is independent of the choice of the point x_0.*

Proof. The proof is given by direct calculations and is based on the standard formulas $[\mathcal{L}_X, \mathbf{i}_Y] = \mathbf{i}_{[X,Y]}$ and $\mathcal{L}_X = \mathbf{i}_X d + d\mathbf{i}_X$, where $\mathbf{i}_X$ is the operator of the inner product by X and $\mathcal{L}_X$ is the Lie derivative with respect to a vector field X, (see, for example, [6], ch. 4). In particular, we have for any $x \in M$ and $X, Y \in \mathrm{Vect}(M, \omega)$ the following equality

$$c_x(X,Y) - c_{x_0}(X,Y) = -\int_{x_0}^{x} \mathbf{i}_{[X,Y]}\omega. \tag{5}$$

□

Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. We have a natural homomorphism of complexes $C^*_{\mathrm{diff}}(G, \mathbb{R}) \to C^*(\mathfrak{g}, \mathbb{R})$ (see, for example, [4], ch. 3). In particular, if $c \in C^2_{\mathrm{diff}}(G, \mathbb{R})$, the corresponding cochain $\tilde{c} \in C^2(\mathfrak{g}, \mathbb{R})$ is defined as follows:

$$\tilde{c}(X,Y) = \frac{\partial^2}{\partial t \partial s}(c(\exp tX, \exp sY) - c(\exp sY, \exp tX))_{t=0,s=0}$$

where $X, Y \in \mathfrak{g}$.

Let G be a Lie group of diffeomorphisms of M contained in the group $\mathrm{Diff}(M, \omega)$. Then for the 2-cocycle $c = C_{x_0}$ of section 3, the cocycle $\tilde{c}$ is cohomologous to the restriction of the cocycle c_{x_0} to the Lie algebra $\mathfrak{g}$ of G. Unfortunately, we cannot apply this procedure to the whole group $\mathrm{Diff}(M, \omega)$ and the Lie algebra $\mathrm{Vect}(M, \omega)$. Therefore, the problems of nontriviality of 2-cocycles C_{x_0} on the group $\mathrm{Diff}(M, \omega)$ and c_{x_0} on the Lie algebra $\mathrm{Vect}(M, \omega)$ should be solved independently.

For each $X \in \mathrm{Vect}(M, \omega)$ denote by α_X the closed 1-form such that $\alpha_X = \mathbf{i}_X \omega$. For all vector fields $X, Y \in \mathrm{Vect}(M, \omega)$ we have the following equality:

$$\omega(X,Y)\omega^n = n\alpha_X \wedge \alpha_Y \wedge \omega^{n-1} \tag{6}$$

which can be easily checked in Darboux coordinates.

Denote by X_f a Hamiltonian vector field defined by a function $f \in C^\infty(M)$. Consider the Poisson algebra $\mathrm{P}(M) = \mathrm{P}(M, \omega)$ on (M, ω), i.e., the algebra $C^\infty(M)$ endowed with the Poisson bracket $\{f, g\} = -\omega(X_f, X_g)$ for $f, g \in C^\infty(M)$.

The map $\mathrm{P}(M) \to \mathrm{Vect}_0(M, \omega)$ given by $f \to X_f$ is a homomorphism of Lie algebras which defines an extension of $\mathrm{Vect}_0(M, \omega)$ by $\mathbb{R}$. It is easy to check that this extension is isomorphic to one given by the cocycle $-c_{x_0}$.

Theorem 6.2. *For a non-compact symplectic manifold (M, ω) the cocycle c_{x_0} on the Lie algebras $\mathrm{Vect}(M, \omega)$ and $\mathrm{Vect}_0(M, \omega)$ is non-trivial.*

Proof. It suffices to prove our statement for the Lie algebra $\mathrm{Vect}_0(M, \omega)$.

First we prove that for each form $\beta \in \Omega^{2n-1}(M)$ there is a unique form $\alpha \in \Omega^1(M)$ such that $\beta = \alpha \wedge \omega^{n-1}$. Indeed, using Darboux coordinates it is easy to check that this has a unique local solution α. These are compatible and we get a global solution by gluing them.

Note that for each form $\alpha \in \Omega^1(M)$ there is a positive integer N and $2N$ functions $f_k, g_k \in C^\infty(M)$ $(k = 1, \dots, N)$ such that $\alpha = \sum_{k=1}^N f_k dg_k$ which follows easily from the existence (by dimension theory) of a finite atlas for M.

Since $H^{2n}(M, \mathbb{R}) = 0$ there is a form $\beta \in \Omega^{2n-1}(M)$ such that $\omega^n = d\beta$. Then we have $\omega^n = \sum_{k=1}^N df_k \wedge dg_k \wedge \omega^{n-1}$. By (6) and using this equality we get

$$\sum_{k=1}^{N} \{f_k, g_k\} = -n. \tag{7}$$

Assume that the extension $P(M) \to \mathrm{Vect}_0(M, \omega)$ $P(M) \to \mathrm{Vect}_0(M, \omega)$ is split. Then $P(M)$ is a direct sum of the space of constant functions on M and an ideal isomorphic to $\mathrm{Vect}_0(M, \omega)$ by $P(M) \to \mathrm{Vect}_0(M, \omega)$. Equality (7) means that these summands have nonzero intersection. This contradiction proves the statement. □

Now we consider a compact symplectic manifold (M, ω). It is known that the extension $P(M) \to \mathrm{Vect}_0(M, \omega)$ is split.

For a closed form α denote by $[\alpha]$ the cohomology class of α. Denote by L the linear map $H^p(M, \mathbb{R}) \to H^{p+2}(M, \mathbb{R})$ defined by $a \to a \smile [\omega]$, where $a \in H^p(M, \mathbb{R})$.

Theorem 6.3. *Let (M, ω) be a compact symplectic manifold. The cocycle c_{x_0} on the Lie algebra $\mathrm{Vect}(M, \omega)$ is non-trivial iff the linear map*

$$L^{n-1} : H^1(M, \mathbb{R}) \to H^{2n-1}(M, \mathbb{R})$$

is not equal zero.

Proof. We may assume that $\int_M \omega^n = 1$. Put for brevity $V = \mathrm{Vect}(M, \omega)$ and $V_0 = \mathrm{Vect}_0(M, \omega)$. Set

$$b(X, Y) = \int_M \omega(X, Y)\omega^n,$$

where $X, Y \in V$. It is easy to check that b is a 2-cocycle on V.

Multiplying both sides of equality (5) by ω^n and integrating over M we get

$$b(X, Y) - c_{x_0}(X, Y) = \int_M \left(\int_{x_0}^{x} \mathbf{i}_{[X,Y]}\omega \right) \omega^n. \tag{8}$$

Since the right hand side of (8) is a coboundary of a 1-cochain in $C^1(V, \mathbb{R})$, the cocycles c_{x_0} and b are cohomologous. By (6) we have

$$b(X, Y) = n \int_M \alpha_X \wedge \alpha_Y \wedge \omega^{n-1}, \tag{9}$$

for any $X, Y \in V$. If $X \in V_0$ the form α_X is exact, and $b(X, Y) = 0$ by (9). This proves (1).

Suppose that the cocycle b is trivial, i.e., there is a linear functional f on V such that for any $X, Y \in V$ we have $b(X, Y) = f([X, Y])$. By [1] we have $[V, V] = [V_0, V_0] = V_0$. This implies $b = 0$. So the cocycle b is trivial iff it equals zero. By (9) and the Poincaré duality this implies that $L^{n-1} = 0$ on $H^1(M, \mathbb{R})$. This proves (2). □

We know no example when $H^1(M,\mathbb{R}) \neq 0$ and the map $L^{n-1} = 0$. Moreover, if M is a compact Kählerian manifold the map $L^{n-1} : H^1(M,\mathbb{R}) \to H^{2n-1}(M,\mathbb{R})$ is an isomorphism (see, for example, [11], ch. 4). Thus in this case the cocycle c_{x_0} is non-trivial whenever $H^1(M,\mathbb{R}) \neq 0$.

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