DISSERTATION

Titel der Dissertation

Almost local metrics on shape space

Verfasser

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Introduction

In many procedures in science, engineering, imaging and medicine it is necessary to *distinguish between different geometric shapes.* Therefore it is of great importance to equip the space of all shapes with a *significant metric.* Many different representations of shapes have been developed and there are various different types of metrics on shape spaces, these include:

- Inner metrics on shape space of unparametrized immersions. These metrics are induced from metrics on parametrized immersions. See for example [37, 6, 7, 8].
- Outer metrics on various shape spaces (embedded surfaces, images, landmarks, measures and currents) that the diffeomorphism group of the ambient space is acting on. See for example [5, 9, 38, 14, 31, 24, 21, 37].
- Metamorphosis metrics. See for example [46, 23].
- The Wasserstein metric or Monge-Kantorovic metric on shape space of probability measures. See for example [2, 3, 11, 10].
- The Weil-Peterson metric on shape space of planar curves. See for example [42, 43, 28].
- Current metrics. See for example [47, 16, 17].
- (Pseudo) Metrics based on elastic deformations. See for example [19].

More references can be found in the survey papers [1, 4, 12, 29, 39, 48].

In this work we will *represent shapes as submanifolds* of a connected Riemannian manifold N which are diffeomorphic to a connected and compact manifold M. The space of all these shapes will be denoted $B_e = B_e(M, N)$ and viewed as the quotient

$$B_e = B_e(M, N) = \operatorname{Emb}(M, N)/\operatorname{Diff}(M)$$

of the open subset $\operatorname{Emb}(M, N) \subset C^\infty(M, N)$ of smooth *embeddings* of M in N, modulo the group of smooth diffeomorphisms of M (see section 1.2.11 for more details). It is natural to consider all possible *immersions* as well as embeddings, and thus introduce the larger space

$$B_i = B_i(M, N) = \operatorname{Imm}(M, N)/\operatorname{Diff}(M)$$

as the quotient of the space of C^∞ immersions by the group of diffeomorphisms of M (which is, however, no longer a manifold, but an orbifold, see section 1.2.11):

$$\begin{tikzcd} \mathrm{Emb}(M,N) \arrow[r, hook] \arrow[d] & \mathrm{Imm}(M,N) \arrow[d] \\ B_e(M,N) \arrow[r, hook] & B_i(M,N) \end{tikzcd}$$

Furthermore we will focus on the class of *inner metrics.* These are metrics that are induced by metrics on the manifold of immersions. We call these metrics inner, since they are defined intrinsically to the base manifold M.

The simplest and most natural Riemannian metric on the manifold of immersions is the L^2-metric, which is given by:

$$G^0_f(h,k) = \int_M \bar g(h,k)\,\mathrm{vol}(g),$$

where $h,k \in T_f \,\mathrm{Imm}(M,N)$ are tangent vectors at the immersion f, $\bar g$ denotes a fixed metric on the ambient space N and $g = f^*\bar g$ denotes the pullback metric on M. Unfortunately this metric induces *vanishing geodesic distance* on shape space. By vanishing geodesic distance we mean that any two points in shape space can be joined by a path of arbitrarily short length. This was first discovered by Michor and Mumford for the case of plane curves (see [36]). In [35] they proved that the vanishing geodesic distance phenomenon for the L^2-metric occurs also in the shape space $\mathrm{Imm}(M,N)/\,\mathrm{Diff}(M)$, where S^1 is replaced by a compact manifold M and Euclidean $\mathbb{R}^2$ is replaced by a Riemannian manifold N. This was the starting point of the quest for *suitable (stronger) inner metrics* on shape space.

One approach to strengthen the metric is to incorporate a *differential operator* into the definition of the metric, yielding metrics of the form:

$$G^P_f(h,k) = \int_M \bar g(P_f h,k)\,\mathrm{vol}(g),$$

where P is an equivariant differential opperator depending smoothly on the immersion. These metrics are called *Immersion-Sobolev metrics.*

The interesting special case $P = \Delta$ and $N = \mathbb{R}^2$ has been studied in [45, 55] and in [54] where an isometry to an infinite dimensional Grassmannian with the Fubini-Study metric was described. Also in the case of plane curves metrics induced by the differential operators $P = \Delta$ and $P = \Delta^2$ have been treated in [30], where estimates on the geodesic distance are proven and the metric completion of the space of curves is characterized. In [7] Immersion-Sobolev metrics have been generalized to shape space of arbitrary dimension, see also [22].

Another approach to strengthen the metric, and the approach studied in this thesis, is by *adding weights* into the definition of the metric. This yields metrics of the form

$$G^\Phi_f(h,k) = \int_M \Phi(f).\bar g(h,k)\,\mathrm{vol}(g),$$

where Φ is some positve, real-valued and $\operatorname{Diff}(M)$ invariant function depending smoothly on the immersions and possibly on $x \in M$. These metrics are called *almost local metrics*. For the case of plane curves Michor and Mumford introduced almost local metrics *weighted by curvature* (see [36, 37]). An important special case of almost local metrics is the class of *conformal metrics*, i.e. metrics of the form:

$$G_f^\Phi(h,k) = \Phi(f). \int_M \bar{g}(h,k)\operatorname{vol}(g),$$

where Φ is again some positve, real-valued and $\operatorname{Diff}(M)$ invariant function depending smoothly on the immersions but not on the point $x \in M$. For the case of plane curves these metrics have been introduced in [41] and in [51, 52, 53]. In this thesis the investigation of almost local metrics from [37] is taken up, and they are generalized to the shape space $B_i(M,N) = \operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ of surfaces of type M in N.

The contents of this thesis

Chapter 1 recalls concepts from differential geometry of surfaces in a form that is suitable for our needs. The general formalism that we shall use to compute geodesic equations and conserved quantities is explained. In the last part the derivatives of the metric, the volume form, the second fundamental form and some other curvature terms with respect to the immersion f and second derivatives for both tangent vectors horizontal are calculated.

In chapter 2 and 3 we compute the geodesic equation for almost local metrics both on the manifold of immersions and on shape space. For a flat ambient space N the sectional curvature is computed (see section 3.3). We derive conditions to ensure that the induced geodesic distance on shape space is positive (section 2.4) and for a flat ambient space $N = \mathbb{R}^n$ we compare the almost local metrics to the Fréchet Metric (section 3.6). For special choices of the weight function Φ all previously derived formulas are presented. These are the following (see section 3.4):

1. The G^0-metric or L^2-metric, where $\Phi = 1$. For this metric the geodesic distance on B_i vanishes (as shown in section 2.4.5). Sectional curvature is non-negative and has a simple expression.

2. The G^A-metric, where $\Phi = 1 + A\operatorname{Tr}(L)^2$. For the situation of plane curve this metric was treated in great detail in [36].

3. The G^B-metric, where $\Phi = 1 + B\det(L)^2$.

4. Conformal metrics, where $\Phi = \Phi(\operatorname{Vol})$. For curves these were investigated in [41, 51, 52, 53]. The full formula for sectional curvature is given only in the case that $\Phi(\operatorname{Vol}) = \operatorname{Vol}$.

5. A scale invariant almost local metric with $\Phi = \operatorname{Vol}^{\frac{1+n}{1-n}} + A\frac{\operatorname{Tr}(L)^2}{\operatorname{Vol}}$.

In Chapter 4 we study all previously defined metrics in some numerical experiments and discuss the differences in their behaviour. The experiments include:

1. Geodesics of concentric spheres.
2. Geodesics between a sphere and a translation of the sphere.
3. Geodesics between a shape and a deformation of the shape.

Chapter 1

Notations and background material

1.1 Differential geometry of surfaces and notation

In this section we will present and develop the differential geometric tools that are needed to deal with immersed surfaces. The most important point is a rigorous treatment of the covariant derivative and related concepts.

This section is based on [7, section 2]. We use the notation of [34]. Some of the definitions can also be found in [25]. A similar exposition in the same notation is [6]. This section has been written in collaboration with Philipp Harms and is up to slight modifications the same as chapter one of his PhD thesis [22].

1.1.1 Basic assumptions and convention

Assumption. *We always assume that M and N are connected manifolds of dimension*

$$1 \leq \dim(M) = m < n = \dim(N)$$

without boundary. Furthermore we will assume that M is compact.

We will work with *immersions* of M into N, i.e. smooth functions $M \to N$ with injective tangent mapping at every point. We denote the set of all such immersions by $\mathrm{Imm}(M,N)$. It is clear that only the case $\dim(M) \leq \dim(N)$ is of interest since otherwise $\mathrm{Imm}(M,N)$ would be empty. Immersions or paths of immersions are usually denoted by f. Vector fields on $\mathrm{Imm}(M,N)$ or vector fields along f will be called h, k, m, for example. Subscripts like f_t denote differentiation with respect to the indicated variable, so $f_t = \partial_t f = \partial f/\partial t$, but subscripts are also used to indicate the foot point of a tensor field.

1.1.2 Tensor bundles and tensor fields

We will deal with the *tensor bundles*

$$\begin{array}{ccc} T^r_s M & & T^r_s M \otimes f^*TN \\ \downarrow & & \downarrow \\ M & & M \end{array}$$

Here $T^r_s M$ denotes the bundle of $\binom{r}{s}$-tensors on M, i.e.

$$T^r_s M = \bigotimes^r TM \otimes \bigotimes^s T^*M,$$

and f^*TN is the pullback of the bundle TN via f, see [34, section 17.5]. A *tensor field* is a section of a tensor bundle. Generally, when E is a bundle, the space of its sections will be denoted by $\Gamma(E)$.

To clarify the notation that will be used later, some examples of tensor bundles and tensor fields are given now.

- $S^k T^*M = L^k_{\text{sym}}(TM;\mathbb{R})$ is the bundle of *symmetric* $\binom{0}{k}$*-tensors*,
- $\Lambda^k T^*M = L^k_{\text{alt}}(TM;\mathbb{R})$ is the bundle of *alternating* $\binom{0}{k}$*-tensors*,
- $\Omega^r(M) = \Gamma(\Lambda^r T^*M)$ is the space of *differential forms*,
- $\mathfrak{X}(M) = \Gamma(TM)$ is the space of *vector fields*, and
- $\Gamma(f^*TN) \cong \big\{h \in C^\infty(M,TN) : \pi_N \circ h = f\big\}$ is the space of *vector fields along f*.

1.1.3 Metric on tensor spaces

Let $\bar{g} \in \Gamma(S^2_{>0}T^*N)$ denote a fixed Riemannian metric on N. The *metric induced on M by $f \in \operatorname{Imm}(M,N)$* is the pullback metric

$$g = f^*\bar{g} \in \Gamma(S^2_{>0}T^*M), \qquad g(X,Y) = (f^*\bar{g})(X,Y) = \bar{g}(Tf.X, Tf.Y),$$

where X, Y are vector fields on M. The dependence of g on the immersion f should be kept in mind. Let

$$\flat = \check{g} : TM \to T^*M \quad \text{and} \quad \sharp = \check{g}^{-1} : T^*M \to TM.$$

g can be extended to the cotangent bundle $T^*M = T^0_1 M$ by setting

$$g^{-1}(\alpha,\beta) = g^0_1(\alpha,\beta) = \alpha(\beta^\sharp)$$

for $\alpha, \beta \in T^*M$. The product metric

$$g^r_s = \bigotimes^r g \otimes \bigotimes^s g^{-1}$$

extends g to all tensor spaces $T^r_s M$, and $g^r_s \otimes \bar{g}$ yields a metric on $T^r_s M \otimes f^*TN$.

1.1.4 Traces

The *trace* contracts pairs of vectors and co-vectors in a tensor product:

$$\mathrm{Tr}: \; T^*M \otimes TM = L(TM, TM) \to M \times \mathbb{R}$$

A special case of this is the operator i_X inserting a vector X into a co-vector or into a covariant factor of a tensor product. The inverse of the metric g can be used to define a trace

$$\mathrm{Tr}^g : T^*M \otimes T^*M \to M \times \mathbb{R}$$

contracting pairs of co-vecors. Note that Tr^g depends on the metric whereas Tr does not. The following lemma will be useful in many calculations:

Lemma.

$$g_2^0(B,C) = \mathrm{Tr}(g^{-1}Bg^{-1}C) \quad \textit{for } B, C \in T_2^0 M \textit{ if } B \textit{ or } C \textit{ is symmetric.}$$

(In the expression under the trace, B and C are seen maps $TM \to T^*M$.)

Proof. Express everything in a local coordinate system $u^1, \dots u^m$ of M.

$$\begin{aligned} g_2^0(B,C) &= g_2^0\Big(\sum_{ik} B_{ik} du^i \otimes du^k, \sum_{jl} C_{jl} du^j \otimes du^l\Big) \\ &= \sum_{ijkl} g^{ij} B_{ik} g^{kl} C_{jl} = \sum_{ijkl} g^{ji} B_{ik} g^{kl} C_{lj} = \mathrm{Tr}(g^{-1}Bg^{-1}C) \end{aligned}$$

Note that we have used only the symmetry of C. □

1.1.5 Volume density

Let $\mathrm{Vol}(M)$ be the *density bundle* over M, see [34, section 10.2]. The *volume density* on M induced by $f \in \mathrm{Imm}(M,N)$ is

$$\mathrm{vol}(g) = \mathrm{vol}(f^*\bar{g}) \in \Gamma\big(\mathrm{Vol}(M)\big).$$

The *volume* of the immersion is given by

$$\mathrm{Vol}(f) = \int_M \mathrm{vol}(f^*\bar{g}) = \int_M \mathrm{vol}(g).$$

The integral is well-defined since M is compact. If M is oriented we may identify the volume density with a differential form.

1.1.6 Metric on tensor fields

A *metric on a space of tensor fields* is defined by integrating the appropriate metric on the tensor space with respect to the volume density:

$$\widetilde{g_s^r}(B,C) = \int_M g_s^r\big(B(x), C(x)\big) \,\mathrm{vol}(g)(x)$$

for $B, C \in \Gamma(T^r_s M)$, and

$$\widetilde{g^r_s \otimes \bar{g}}(B, C) = \int_M g^r_s \otimes \bar{g}\big(B(x), C(x)\big) \operatorname{vol}(g)(x)$$

for $B, C \in \Gamma(T^r_s M \otimes f^*TN)$, $f \in \operatorname{Imm}(M, N)$. The integrals are well defined since M is compact.

1.1.7 Covariant derivative

We will use covariant derivatives on vector bundles as explained in [34, sections 19.12, 22.9]. Let $\nabla^g, \nabla^{\bar{g}}$ be the *Levi-Civita covariant derivatives* on (M, g) and $(N, \bar{g})$, respectively. For any manifold Q and vector field X on Q, one has

$$\begin{aligned}
\nabla^g_X &: C^\infty(Q, TM) \to C^\infty(Q, TM), && h \mapsto \nabla^g_X h \\
\nabla^{\bar{g}}_X &: C^\infty(Q, TN) \to C^\infty(Q, TN), && h \mapsto \nabla^{\bar{g}}_X h.
\end{aligned}$$

Usually we will simply write ∇ for all covariant derivatives. It should be kept in mind that ∇^g depends on the metric $g = f^*\bar{g}$ and therefore also on the immersion f. The following properties hold [34, section 22.9]:

1. ∇_X respects base points, i.e. $\pi \circ \nabla_X h = \pi \circ h$, where π is the projection of the tangent space onto the base manifold.
2. $\nabla_X h$ is C^∞-linear in X. So for a tangent vector $X_x \in T_x Q$, $\nabla_{X_x} h$ makes sense and equals $(\nabla_X h)(x)$.
3. $\nabla_X h$ is $\mathbb{R}$-linear in h.
4. $\nabla_X(a.h) = da(X).h + a.\nabla_X h$ for $a \in C^\infty(Q)$, the derivation property of ∇_X.
5. For any manifold $\widetilde{Q}$ and smooth mapping $q : \widetilde{Q} \to Q$ and $Y_y \in T_y\widetilde{Q}$ one has $\nabla_{Tq.Y_y} h = \nabla_{Y_y}(h \circ q)$. If $Y \in \mathfrak{X}(Q_1)$ and $X \in \mathfrak{X}(Q)$ are q-related, then $\nabla_Y(h \circ q) = (\nabla_X h) \circ q$.

The two covariant derivatives ∇^g_X and $\nabla^{\bar{g}}_X$ can be combined to yield a covariant derivative ∇_X acting on $C^\infty(Q, T^r_s M \otimes TN)$ by additionally requiring the following properties [34, section 22.12]:

6. ∇_X respects the spaces $C^\infty(Q, T^r_s M \otimes TN)$.
7. $\nabla_X(h \otimes k) = (\nabla_X h) \otimes k + A \otimes (\nabla_X k)$, a derivation with respect to the tensor product.
8. ∇_X commutes with any kind of contraction (see [34, section 8.18]). A special case of this is

$$\nabla_X\big(\alpha(Y)\big) = (\nabla_X \alpha)(Y) + \alpha(\nabla_X Y) \quad \text{for } \alpha \otimes Y : N \to T^1_1 M.$$

Property (1) is important because it implies that ∇_X respects spaces of sections of bundles. For example, for $Q = M$ and $f \in C^\infty(M, N)$, one gets

$$\nabla_X : \Gamma(T^r_s M \otimes f^*TN) \to \Gamma(T^r_s M \otimes f^*TN).$$

1.1.8 Swapping covariant derivatives

We will make repeated use of some formulas allowing to swap covariant derivatives. Let f be an immersion, h a vector field along f and X, Y vector fields on M. Since ∇ is torsion-free, one has [34, section 22.10]

$$\nabla_X Tf.Y - \nabla_Y Tf.X - Tf.[X,Y] = \mathrm{Tor}(Tf.X, Tf.Y) = 0. \tag{1}$$

Furthermore one has [34, section 24.5]

$$\nabla_X \nabla_Y h - \nabla_Y \nabla_X h - \nabla_{[X,Y]} h = R^{\bar{g}} \circ (Tf.X, Tf.Y)h, \tag{2}$$

where $R^{\bar{g}} \in \Omega^2\big(N; L(TN,TN)\big)$ is the Riemann curvature tensor of $(N, \bar{g})$.

These formulas also hold when $f : \mathbb{R} \times M \to N$ is a path of immersions, $h : \mathbb{R} \times M \to TN$ is a vector field along f and the vector fields are vector fields on $\mathbb{R} \times M$. A case of special importance is when one of the vector fields is $(\partial_t, 0_M)$ and the other $(0_\mathbb{R}, Y)$, where Y is a vector field on M. Since the Lie bracket of these vector fields vanishes, (1) and (2) yield

$$\nabla_{(\partial_t, 0_M)} Tf.(0_\mathbb{R}, Y) - \nabla_{(0_\mathbb{R}, Y)} Tf.(\partial_t, 0_M) = 0 \tag{3}$$

and

$$\nabla_{(\partial_t, 0_M)} \nabla_{(0_\mathbb{R}, Y)} h - \nabla_{(0_\mathbb{R}, Y)} \nabla_{(\partial_t, 0_M)} h = R^{\bar{g}}\big(Tf.(\partial_t, 0_M), Tf.(0_\mathbb{R}, Y)\big)h. \tag{4}$$

1.1.9 Second and higher covariant derivatives

When the covariant derivative is seen as a mapping

$$\nabla : \Gamma(T^r_s M) \to \Gamma(T^r_{s+1} M) \quad \text{or} \quad \nabla : \Gamma(T^r_s M \otimes f^*TN) \to \Gamma(T^r_{s+1} M \otimes f^*TN),$$

then the *second covariant derivative* is simply $\nabla\nabla = \nabla^2$. Since the covariant derivative commutes with contractions, ∇^2 can be expressed as

$$\nabla^2_{X,Y} := \iota_Y \iota_X \nabla^2 = \iota_Y \nabla_X \nabla = \nabla_X \nabla_Y - \nabla_{\nabla_X Y} \qquad \text{for } X, Y \in \mathfrak{X}(M).$$

Higher covariant derivates are defined as ∇^k, $k \geq 0$.

1.1.10 The adjoint of the covariant derivative

The covariant derivative

$$\nabla : \Gamma(T^r_s M) \to \Gamma(T^r_{s+1} M)$$

admits an *adjoint*

$$\nabla^* : \Gamma(T^r_{s+1} M) \to \Gamma(T^r_s M)$$

with respect to the metric $\widetilde{g}$, i.e.:

$$\widetilde{g^r_{s+1}}(\nabla B, C) = \widetilde{g^r_s}(B, \nabla^* C).$$

In the same way, ∇^* can be defined when ∇ is acting on $\Gamma(T^r_s M \otimes f^*TN)$. In either case it is given by

$$\nabla^* B = -\operatorname{Tr}^g(\nabla B),$$

where the trace is contracting the first two tensor slots of ∇B. We prove this formula now.

Proof. The result holds for decomposable tensor fields $\beta \otimes B \in \Gamma(T^r_{s+1}M)$ since

$$\begin{aligned}
\widetilde{g^r_s}\Big(\nabla^*(\beta\otimes B), C\Big) &= \widetilde{g^r_{s+1}}\Big(\beta\otimes B, \nabla C\Big) = \widetilde{g^r_s}\Big(B, \nabla_{\beta^\sharp} C\Big)\\
&= \int_M \mathcal{L}_{\beta^\sharp} g^r_s(B,C)\operatorname{vol}(g) - \int_M g^r_s(\nabla_{\beta^\sharp}B, C)\operatorname{vol}(g)\\
&= \int_M -g^r_s(B,C)\mathcal{L}_{\beta^\sharp}\operatorname{vol}(g) - \int_M g^r_s\big(\operatorname{Tr}^g(\beta\otimes\nabla B), C\big)\operatorname{vol}(g)\\
&= \widetilde{g^r_s}\Big(-\operatorname{div}(\beta^\sharp)B - \operatorname{Tr}^g(\beta\otimes\nabla B), C\Big)\\
&= \widetilde{g^r_s}\Big(-\operatorname{div}(\beta^\sharp)B + \operatorname{Tr}^g((\nabla\beta)\otimes B) - \operatorname{Tr}^g(\nabla(\beta\otimes B)), C\Big)\\
&= \widetilde{g^r_s}\Big(-\operatorname{div}(\beta^\sharp)B + \operatorname{Tr}^g(\nabla\beta)B - \operatorname{Tr}^g(\nabla(\beta\otimes B)), C\Big)\\
&= \widetilde{g^r_s}\Big(0 - \operatorname{Tr}^g(\nabla(\beta\otimes B)), C\Big)
\end{aligned}$$

Here we have used that $\nabla_X g = 0$, that ∇_X commutes with any kind of contraction and acts as a derivation on tensor products [34, section 22.12] and that $\operatorname{div}(X) = \operatorname{Tr}(\nabla X)$ for all vector fields X [34, section 25.12]. To prove the result for $\beta \otimes B \in \Gamma(T^r_{s+1}M \otimes f^*TN)$ one simply has to replace g^r_s by $g^r_s \otimes \bar{g}$. $\square$

1.1.11 Laplacian

The definition of the Laplacian used in this work is the *Bochner-Laplacian.* It can act on all tensor fields B and is defined as

$$\Delta B = \nabla^*\nabla B = -\operatorname{Tr}^g(\nabla^2 B).$$

1.1.12 Normal bundle

The *normal bundle* $\operatorname{Nor}(f)$ of an immersion f is a sub-bundle of f^*TN whose fibers consist of all vectors that are orthogonal to the image of f:

$$\operatorname{Nor}(f)_x = \big\{Y \in T_{f(x)}N : \forall X \in T_x M : \bar{g}(Y, Tf.X) = 0\big\}.$$

Any vector field h along f can be decomposed uniquely into parts *tangential* and *normal* to f as

$$h = Tf.h^\top + h^\perp,$$

where $h^\top$ is a vector field on M and $h^\perp$ is a section of the normal bundle $\operatorname{Nor}(f)$. In co-dimension one (i.e. $\dim M = n-1$) and when f is orientable, then the unit normal field ν of f can be defined. It is a section of the normal bundle in one of the above forms with constant $\bar{g}$-length one which is chosen such that

$$\big(\nu(x), T_x f.X_1, T_x f.X_2, \dots T_x f.X_{n-1}\big)$$

is a positive oriented basis in $T_{f(x)}N$ if $X_1, \dots, X_{n-1}$ is a positive oriented basis in $T_x M$. In this notation the decomposition of a vector field h along f reads as

$$h = Tf.h^\top + a.\nu.$$

The two parts are defined by the relations

$$\begin{aligned}
&a = \bar{g}(h,\nu) \in C^\infty(M)\\
&h^\top \in \mathfrak{X}(M), \text{ such that } g(h^\top, X) = \bar{g}(h, Tf(t,\cdot).X) \text{ for all } X \in \mathfrak{X}(M).
\end{aligned}$$

1.1.13 Second fundamental form and Weingarten mapping

Let X and Y be vector fields on M. Then the covariant derivative $\nabla_X Tf.Y$ splits into tangential and a normal parts as

$$\nabla_X Tf.Y = Tf.(\nabla_X Tf.Y)^\top + (\nabla_X Tf.Y)^\perp = Tf.\nabla_X Y + S(X,Y).$$

S is the *second fundamental form of* f. It is a symmetric bilinear form with values in the normal bundle of f. When Tf is seen as a section of $T^*M \otimes f^*TN$ one has $S = \nabla Tf$ since

$$S(X,Y) = \nabla_X Tf.Y - Tf.\nabla_X Y = (\nabla Tf)(X,Y).$$

Taking the trace of S yields the *vector valued mean curvature*

$$\mathrm{Tr}^g(S) \in \Gamma\big(\mathrm{Nor}(f)\big).$$

In codimension one, one can define the *scalar second fundamental form* s as

$$s(X,Y) = \bar{g}\big(S(X,Y),\nu\big).$$

Moreover, there is the *Weingarten mapping* or *shape operator* $L = g^{-1}s$. It is a g-symmetric bundle mapping defined by

$$s(X,Y) = g(LX,Y).$$

The eigenvalues of L are called *principal curvatures* and the eigenvectors *principal curvature directions*. $\mathrm{Tr}(L) = \mathrm{Tr}^g(s)$ is the *scalar mean curvature* and for surfaces in $\mathbb{R}^3$ the *Gauß-curvature* is given by $\det(L)$. The covariant derivative $\nabla_X \nu$ of the normal vector is related to L by the *Weingarten equation*

$$\nabla_X \nu = -Tf.L.X.$$

1.1.14 Directional derivatives of functions

We will use the following ways to denote directional derivatives of functions, in particular in infinite dimensions. Given a function $F(x, y)$ for instance, we will write:

$$D_{(x,h)}F \text{ as shorthand for } \partial_t|_0 F(x + th, y).$$

Here (x, h) in the subscript denotes the tangent vector with foot point x and direction h. If F takes values in some linear space, we will identify this linear space and its tangent space.

1.2 Shape space

Briefly said, by a shape we mean an *unparametrized surface*. We use the term surface regardless of whether it has dimension two or not. This section is about the infinite dimensional space of all shapes: First we give an overview of the differential calculus that is used. Then we describe some spaces of parametrized and unparametrized surfaces and how to define Riemannian metrics on them.

Assumption. *We recall the most important assumptions made in section 1.1.1: Let M and N be connected manifolds of dimension $1 \leq \dim(M) = m < n = \dim(N)$ without boundary, where M is compact in addition.*

This section is common work with Philipp Harms and can be also found in his PhD thesis [22, chapter 2].

1.2.1 Convenient calculus

The differential calculus used in this work is *convenient calculus* [27]. The overview of convenient calculus presented here is taken from [33, Appendix A]. Convenient calculus is a generalization of differential calculus beyond Banach and Fréchet spaces. For this work, the most important property of convenient calculus is that the *exponential law* holds without any restriction:

$$C^\infty(E \times F, G) \cong C^\infty\big(E, C^\infty(F, G)\big)$$

for convenient vector spaces E, F, G and a natural convenient vector space structure on $C^\infty(F, G)$. As a consequence *variational calculus* simply works: For example, a smooth curve in $C^\infty(M, N)$ can equivalently be seen as a smooth mapping $M \times \mathbb{R} \to N$. The main difficulty is that the composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology.

Let E be a *locally convex vector space.* A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous - this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, but only on its associated bornology (system of bounded sets).

E is said to be a *convenient vector space* if one of the following equivalent conditions is satisfied (called c^∞-completeness):

1. For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
2. A curve $c : \mathbb{R} \to E$ is smooth if and only if $\lambda \circ c$ is smooth for all $\lambda \in E'$, where E' is the dual consisting of all continuous linear functionals on E.
3. Any Mackey-Cauchy-sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a weak completeness requirement.

The final topology with respect to all smooth curves is called the c^∞-topology on E, which then is denoted by $c^\infty E$. For Fréchet spaces it coincides with the given locally convex topology, but on the space $\mathcal{D}$ of test functions with compact support on $\mathbb{R}$ it is strictly finer.

Let E and F be locally convex vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$. The notion of smooth mappings carries over to mappings between *convenient manifolds*, which are manifolds modelled on c^∞-open subsets of convenient vector spaces.

Theorem. *The main properties of smooth calculus are the following.*

1. *For mappings on Fréchet spaces this notion of smoothness coincides with all other reasonable definitions. Even on $\mathbb{R}^2$ this is non-trivial.*
2. *Multilinear mappings are smooth if and only if they are bounded.*
3. *If $f : E \supseteq U \to F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.*
4. *The chain rule holds.*
5. *The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the obvious injection*
$$C^\infty(U, F) \to \prod_{c \in C^\infty(\mathbb{R}, U)} C^\infty(\mathbb{R}, F) \to \prod_{c \in C^\infty(\mathbb{R}, U), \lambda \in F'} C^\infty(\mathbb{R}, \mathbb{R}).$$
6. *The exponential law holds:*
$$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$
is a linear diffeomorphism of convenient vector spaces. Note that this is the main assumption of variational calculus.
7. *A linear mapping $f : E \to C^\infty(V, G)$ is smooth (bounded) if and only if $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$. This is called the smooth uniform boundedness theorem and it is quite applicable.*

Proofs of these statements can be found in [27].

1.2.2 Manifolds of immersions and embeddings

What we sloppily called a *parametrized surface* will now be turned into a rigorous definition. Mathematically, parametrized surfaces will be modeled as immersions or embeddings of one manifold into another. We call immersions and embeddings parametrized since a change in their parametrization (i.e. applying a diffeomorphism on the domain of the function) results in a different object. We will deal with the following sets of functions:

$$\operatorname{Emb}(M,N) \subset \operatorname{Imm}_f(M,N) \subset \operatorname{Imm}(M,N) \subset C^\infty(M,N). \tag{1}$$

$C^\infty(M,N)$ is the set of smooth functions from M to N. $\operatorname{Imm}(M,N)$ is the set of all *immersions* of M into N, i.e. all functions $f \in C^\infty(M,N)$ such that $T_x f$ is injective for all $x \in M$. $\operatorname{Imm}_f(M,N)$ is the set of all *free immersions.* An immersion f is called free if the diffeomorphism group of M acts freely on it, i.e. $f \circ \varphi = f$ implies $\varphi = \operatorname{Id}_M$ for all $\varphi \in \operatorname{Diff}(M)$. $\operatorname{Emb}(M,N)$ is the set of all *embeddings* of M into N, i.e. all immersions f that are a homeomorphism onto their image.

The following lemma from [15, 1.3 and 1.4] gives sufficient conditions for an immersion to be free. In particular it implies that every embedding is free.

Lemma. *If $\varphi \in \operatorname{Diff}(M)$ has a fixed point and if $f \circ \varphi = f$ for some immersion f, then $\varphi = \operatorname{Id}_M$.*

If for an immersion f there is a point $x \in f(M)$ with only one preimage then f is free.

Since M is compact by assumption it follows that $C^\infty(M,N)$ is a *Fréchet manifold* [27, section 42.3]. All inclusions in (1) are inclusions of open subsets: $\operatorname{Imm}(M,N)$ is open in $C^\infty(M,N)$ since the condition that the differential is injective at every point is an open condition on the one-jet of f [32, section 5.1]. $\operatorname{Imm}_f(M,N)$ is open in $\operatorname{Imm}(M,N)$ by [15, theorem 1.5]. $\operatorname{Emb}(M,N)$ is open in $\operatorname{Imm}_f(M,N)$ by [27, theorem 44.1]. Therefore all function spaces in (1) are Fréchet manifolds as well.

When it is clear that M and N are the domain and target of the mappings, the abbreviations $\operatorname{Emb}, \operatorname{Imm}_f, \operatorname{Imm}$ will be used. In most cases, immersions will be used since this is the most general setting. Working with free immersions instead of immersions makes a difference in section 1.2.11, and working with embeddings instead of immersions makes a difference in section 2.4.5. The tangent and cotangent space to Imm are treated in the next section.

1.2.3 Bundles of multilinear maps over immersions

Consider the following *natural bundles of k-multilinear mappings*:

$$\begin{tikzcd} L^k(T\operatorname{Imm};\mathbb{R}) \arrow[d] & L^k(T\operatorname{Imm};T\operatorname{Imm}) \arrow[d] \\ \operatorname{Imm} & \operatorname{Imm} \end{tikzcd}$$

These bundles are isomorphic to the bundles

$$\begin{tikzcd}
L\left(\widehat{\bigotimes}^k T\,\mathrm{Imm};\mathbb{R}\right) \arrow[d] & L\left(\widehat{\bigotimes}^k T\,\mathrm{Imm};T\,\mathrm{Imm}\right) \arrow[d] \\
\mathrm{Imm} & \mathrm{Imm}
\end{tikzcd}$$

where $\widehat{\bigotimes}$ denotes the c^∞-completed bornological tensor product of locally convex vector spaces [27, section 5.7, section 4.29]. Note that $L(T\,\mathrm{Imm};T\,\mathrm{Imm})$ is not isomorphic to $T^*\,\mathrm{Imm}\ \widehat{\otimes}\ T\,\mathrm{Imm}$ since the latter bundle corresponds to multilinear mappings with finite rank.

It is worth to write down more explicitly what some of these bundles of multilinear mappings are. The *tangent space to* Imm is given by

$$\begin{aligned}
T_f\,\mathrm{Imm} &= C^\infty_f(M,TN) := \big\{h \in C^\infty(M,TN) : \pi_N \circ h = f\big\},\\
T\,\mathrm{Imm} &= C^\infty_{\mathrm{Imm}}(M,TN) := \big\{h \in C^\infty(M,TN) : \pi_N \circ h \in \mathrm{Imm}\,\big\}.
\end{aligned}$$

Thus $T_f\,\mathrm{Imm}$ is the space of vector fields along the immersion f. Now the *cotangent space to* Imm will be described. The symbol $\widehat{\otimes}_{C^\infty(M)}$ means that the tensor product is taken over the algebra $C^\infty(M)$.

$$\begin{aligned}
T^*_f\,\mathrm{Imm} &= L(T_f\,\mathrm{Imm};\mathbb{R}) = C^\infty_f(M,TN)' = C^\infty(M)'\ \widehat{\otimes}_{C^\infty(M)} C^\infty_f(M,T^*N)\\
T^*\,\mathrm{Imm} &= L(T\,\mathrm{Imm};\mathbb{R}) = C^\infty(M)'\ \widehat{\otimes}_{C^\infty(M)} C^\infty_{\mathrm{Imm}}(M,T^*N)
\end{aligned}$$

The bundle $L^2_{\mathrm{sym}}(T\,\mathrm{Imm};\mathbb{R})$ is of interest for the definition of a Riemannian metric on Imm. (The subscripts $_{\mathrm{sym}}$ and $_{\mathrm{alt}}$ indicate symmetric and alternating multilinear maps, respectively.) Letting $\otimes_S$ denotes the symmetric tensor product and $\widehat{\otimes}_S$ the c^∞-completed bornological symmetric tensor product, one has

$$\begin{aligned}
L^2_{\mathrm{sym}}(T_f\,\mathrm{Imm};\mathbb{R}) &= (T_f\,\mathrm{Imm}\ \widehat{\otimes}_S\ T_f\,\mathrm{Imm})' = \big(C^\infty_f(M,TN)\ \widehat{\otimes}_S\ C^\infty_f(M,TN)\big)'\\
&= \big(C^\infty_f(M,TN\ \otimes_S\ TN)\big)'\\
&= C^\infty(M)'\ \widehat{\otimes}_{C^\infty(M)} C^\infty_f(M,T^*N\ \otimes_S\ T^*N)\\
L^2_{\mathrm{sym}}(T\,\mathrm{Imm};\mathbb{R}) &= C^\infty(M)'\ \widehat{\otimes}_{C^\infty(M)} C^\infty_{\mathrm{Imm}}(M,T^*N\ \otimes_S\ T^*N)
\end{aligned}$$

1.2.4 The diffeomorphism group

This result is taken from [27, section 43.1] with slight simplifications due to the compactness of M.

Theorem. *For a smooth compact manifold M the group* $\mathrm{Diff}(M)$ *of all smooth diffeomorphisms of M is an open submanifold of $C^\infty(M,M)$. Composition and inversion are smooth. The Lie algebra of the smooth infinite dimensional Lie group* $\mathrm{Diff}(M)$ *is the convenient vector space $\mathfrak{X}(M)$ of all smooth vector fields on M, equipped with the negative of the usual Lie bracket.* $\mathrm{Diff}(M)$ *is a regular Lie group in the sense that the right evolution*

$$\mathrm{evol}^r : C^\infty\big(\mathbb{R},\mathfrak{X}(M)\big) \to \mathrm{Diff}(M)$$

as defined in [27, section 38.4] exists and is smooth. The exponential mapping

$$\exp : \mathfrak{X}(M) \to \mathrm{Diff}(M)$$

is the flow mapping to time 1*, and it is smooth.*

The diffeomorphism group $\mathrm{Diff}(M)$ acts smoothly on $C^\infty(M,N)$ and its subspaces $\mathrm{Imm}, \mathrm{Imm}_f$ and Emb by composition from the right. The action is given by the mapping

$$\mathrm{Imm}(M,N) \times \mathrm{Diff}(M) \to \mathrm{Imm}(M,N), \qquad (f,\varphi) \mapsto r(f,\varphi) = r^\varphi(f) = f \circ \varphi.$$

The tangent prolongation of this group action is given by the mapping

$$T\,\mathrm{Imm}(M,N) \times \mathrm{Diff}(M) \to T\,\mathrm{Imm}(M,N), \qquad (h,\varphi) \mapsto Tr^\varphi(h) = h \circ \varphi.$$

1.2.5 Riemannian metrics on immersions

A Riemannian metric G on Imm is a section of the bundle

$$L^2_{\mathrm{sym}}(T\,\mathrm{Imm};\mathbb{R})$$

such that at every $f \in \mathrm{Imm}$, G_f is a symmetric positive definite bilinear mapping

$$G_f : T_f\,\mathrm{Imm} \times T_f\,\mathrm{Imm} \to \mathbb{R}.$$

Each metric is *weak* in the sense that G_f, seen as a mapping

$$G_f : T_f\,\mathrm{Imm} \to T_f^*\,\mathrm{Imm}$$

is injective. (But it can never be surjective.)

Assumption. *We will always assume that the metric G is compatible with the action of* $\mathrm{Diff}(M)$ *on* $\mathrm{Imm}(M,N)$ *in the sense that the group action is given by isometries.*

This means that $G = (r^\varphi)^* G$ for all $\varphi \in \mathrm{Diff}(M)$, where r^φ denotes the right action of φ on Imm that was described in section 1.2.4. This condition can be spelled out in more details using the definition of r^φ as follows:

$$G_f(h,k) = \big((r^\varphi)^*G\big)(h,k) = G_{r^\varphi(f)}\big(Tr^\varphi(h), Tr^\varphi(k)\big) = G_{f\circ\varphi}(h\circ\varphi, k\circ\varphi).$$

1.2.6 Covariant derivative $\nabla^{\bar g}$ on immersions

The covariant derivative $\nabla^{\bar g}$ defined in section 1.1.7 induces a *covariant derivative over immersions* as follows. Let Q be a smooth manifold. Then we identify

$$h \in C^\infty\big(Q, T\,\mathrm{Imm}(M,N)\big) \qquad \text{and} \qquad X \in \mathfrak{X}(Q)$$

with

$$h^\wedge \in C^\infty(Q \times M, TN) \qquad \text{and} \qquad (X, 0_M) \in \mathfrak{X}(Q \times M).$$

As described in section 1.1.7 one has the covariant derivative

$$\nabla^{\bar{g}}_{(X,0_M)} h^\wedge \in C^\infty\big(Q \times M, TN\big).$$

Thus one can define

$$\nabla_X h = \Big(\nabla^{\bar{g}}_{(X,0_M)} h^\wedge\Big)^\vee \in C^\infty\big(Q, T\operatorname{Imm}(M,N)\big).$$

This covariant derivative is torsion-free by section 1.1.8, formula (1). It respects the metric $\bar{g}$ but in general does not respect G.

It is helpful to point out some special cases of how this construction can be used. The case $Q = \mathbb{R}$ will be important to formulate the geodesic equation. The expression that we will be interested in is $\nabla_{\partial_t} f_t$, which is well-defined when $f : \mathbb{R} \to \operatorname{Imm}$ is a path of immersions and $f_t : \mathbb{R} \to T\operatorname{Imm}$ is its velocity.

Another case of interest is $Q = \operatorname{Imm}$. Let $h, k, m \in \mathfrak{X}(\operatorname{Imm})$. Then the covariant derivative $\nabla_m h$ is well-defined and tensorial in m. Requiring ∇_m to respect the grading of the spaces of multilinear maps, to act as a derivation on products and to commute with compositions of multilinear maps, one obtains as in section 1.1.7 a covariant derivative ∇_m acting on all mappings into the natural bundles of multilinear mappings over Imm. In particular, $\nabla_m P$ and $\nabla_m G$ are well-defined for

$$P \in \Gamma\big(L(T\operatorname{Imm}; T\operatorname{Imm})\big), \quad G \in \Gamma\big(L^2_{\text{sym}}(T\operatorname{Imm}; \mathbb{R})\big)$$

by the usual formulas

$$\begin{aligned}(\nabla_m P)(h) &= \nabla_m\big(P(h)\big) - P(\nabla_m h),\\ (\nabla_m G)(h,k) &= \nabla_m\big(G(h,k)\big) - G(\nabla_m h, k) - G(h, \nabla_m k).\end{aligned}$$

1.2.7 Metric gradients

The *metric gradients* $H, K \in \Gamma\big(L^2(T\operatorname{Imm}; T\operatorname{Imm})\big)$ are uniquely defined by the equation

$$(\nabla_m G)(h,k) = G\big(K(h,m), k\big) = G\big(m, H(h,k)\big),$$

where h, k, m are vector fields on Imm and the covariant derivative of the metric tensor G is defined as in the previous section. (This is a generalization of the definition used in [37] that allows for a curved ambient space $N \neq \mathbb{R}^n$.)

Existence of H, K has to proven case by case for each metric G, usually by partial integration. We will prove the existence of H, K for various almost local metrics in section 2.2 and in section 3.1.

Assumption. *Nevertheless we will assume for now that the metric gradients H, K exist.*

1.2.8 Geodesic equation on immersions

Theorem. *Given H, K as defined in the previous section and ∇ as defined in section 1.2.6, the geodesic equation reads as*

$$\nabla_{\partial_t} f_t = \frac{1}{2} H_f(f_t, f_t) - K_f(f_t, f_t),$$

This is the same result as in [37, section 2.4], but in a more general setting.

Proof. Let $f : (-\varepsilon, \varepsilon) \times [0,1] \times M \to N$ be a one-parameter family of curves of immersions with fixed endpoints. The variational parameter will be denoted by $s \in (-\varepsilon, \varepsilon)$ and the time-parameter by $t \in [0,1]$. In the following calculation, let G_f denote G composed with f, i.e.

$$G_f : \mathbb{R} \to \mathrm{Imm} \to L^2_{\mathrm{sym}}(T\,\mathrm{Imm}; \mathbb{R}).$$

Remember that the covariant derivative on Imm that has been introduced in section 1.2.6 is torsion-free so that one has

$$\nabla_{\partial_t} f_s - \nabla_{\partial_s} f_t = Tf.[\partial_t, \partial_s] + \mathrm{Tor}(f_t, f_s) = 0.$$

Thus the first variation of the energy of the curves is

$$\begin{aligned}
\partial_s \frac{1}{2} \int_0^1 G_f(f_t, f_t) dt &= \frac{1}{2} \int_0^1 (\nabla_{\partial_s} G_f)(f_t, f_t) + \int_0^1 G_f(\nabla_{\partial_s} f_t, f_t) dt \\
&= \frac{1}{2} \int_0^1 (\nabla_{f_s} G)(f_t, f_t) + \int_0^1 G_f(\nabla_{\partial_t} f_s, f_t) dt \\
&= \frac{1}{2} \int_0^1 (\nabla_{f_s} G)(f_t, f_t) dt + \int_0^1 \partial_t \; G_f(f_s, f_t) dt \\
&\qquad - \int_0^1 (\nabla_{f_t} G)(f_s, f_t) dt - \int_0^1 G_f(f_s, \nabla_{\partial_t} f_t) dt \\
&= \int_0^1 G\Big(f_s, \frac{1}{2} H(f_t, f_t) + 0 - K(f_t, f_t) - \nabla_{\partial_t} f_t\Big) dt.
\end{aligned}$$

If $f(0, \cdot, \cdot)$ is energy-minimizing, then one has at $s = 0$ that

$$\frac{1}{2} H(f_t, f_t) - K(f_t, f_t) - \nabla_{\partial_t} f_t = 0. \qquad \square$$

1.2.9 Geodesic equation for the momentum on immersions

In the previous section we have derived the geodesic equation for the velocity f_t. In many applications it is more convenient to formulate the geodesic equation as an equation for the momentum $G(f_t, \cdot) \in T^*_f \mathrm{Imm}$. $G(f_t, \cdot)$ is an element of the *smooth cotangent bundle*, also called *smooth dual*, which is given by

$$G(T\,\mathrm{Imm}) := \coprod_{f \in \mathrm{Imm}} \{G_f(h, \cdot) : h \in T_f \mathrm{Imm}\} \subset T^* \mathrm{Imm}\,.$$

It is strictly smaller than $T^* \operatorname{Imm}$ since at every $f \in \operatorname{Imm}$ the metric $G_f : T_f \operatorname{Imm} \to T_f^* \operatorname{Imm}$ is injective but not surjective. It is called smooth since it does not contain distributional sections of f^*TN, whereas $T_f^* \operatorname{Imm}$ does.

Theorem. *The geodesic equation for the momentum $p \in T^* \operatorname{Imm}$ is given by*

$$\begin{cases} p = G(f_t, \cdot) \\ \nabla_{\partial_t} p = \frac{1}{2} G_f\big(H(f_t, f_t), \cdot\big) \end{cases}$$

where H is the metric gradient defined in section 1.2.7 and ∇ is the covariant derivative action on mappings into $T^* \operatorname{Imm}$ as defined in section 1.2.6.

Proof. Let G_f denote G composed with the path $f : \mathbb{R} \to \operatorname{Imm}$, i.e.

$$G_f : \mathbb{R} \to \operatorname{Imm} \to L^2_{\mathrm{sym}}(T \operatorname{Imm}; \mathbb{R}).$$

Then one has

$$\begin{aligned} \nabla_{\partial_t} p &= \nabla_{\partial_t}\big(G_f(f_t, \cdot)\big) = (\nabla_{\partial_t} G_f)(f_t, \cdot) + G_f(\nabla_{\partial_t} f_t, \cdot) \\ &= (\nabla_{f_t} G)(f_t, \cdot) + G_f\Big(\frac{1}{2} H(f_t, f_t) - K(f_t, f_t), \cdot\Big) \\ &= G_f\big(K(f_t, f_t), \cdot\big) + G_f\Big(\frac{1}{2} H(f_t, f_t) - K(f_t, f_t), \cdot\Big) \qquad \square \end{aligned}$$

This equation is equivalent to *Hamilton's equation* restricted to the smooth cotangent bundle:

$$\begin{cases} p = G(f_t, \cdot) \\ p_t = (\operatorname{grad}^\omega E)(p). \end{cases}$$

Here ω denotes the restriction of the *canonical symplectic form* on $T^* \operatorname{Imm}$ to the smooth cotangent bundle and E is the *Hamiltonian*

$$E : G(T \operatorname{Imm}) \to \mathbb{R}, \quad E(p) = G^{-1}(p, p)$$

which is only defined on the smooth cotangent bundle.

1.2.10 Conserved momenta

We will describe how a group acting on Imm by isometries defines a *momentum mapping* that is conserved along geodesics in Imm. This section is very similar to [6, section 4]. A more detailed treatment and proofs can be found in [37].

Let us consider an infinite dimensional regular Lie group with Lie algebra $\mathfrak{g}$ and a right action $g \mapsto r^g$ of this group on Imm. Let Imm be endowed with a Riemannian metric G. The basic assumption (assumption 1.2.5) is that the action is by isometries:

$$G = (r^g)^* G, \quad \text{i.e.} \quad G_f(h, k) = G_{r^g(f)}\big(T_f(r^g) h, T_f(r^g) k\big).$$

Denote by $\mathfrak{X}(\operatorname{Imm})$ the set of vector fields on Imm. Then we can specify the group action by the fundamental vector field mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(\operatorname{Imm})$, which will be a bounded Lie algebra homomorphism. The fundamental vector field $\zeta_X, X \in \mathfrak{g}$ is the infinitesimal action in the sense:

$$\zeta_X(f) = \partial_t|_0 r^{\exp(tX)}(f).$$

The key to the Hamiltonian approach is to write the infinitesimal action as a Hamiltonian vector field, i.e. as the ω-gradient of some function. This function will be called the *momentum map.* ω is a two-form on $T\,\mathrm{Imm}$,

$$\omega \in \Gamma\big(L^2_{\mathrm{alt}}(TT\,\mathrm{Imm};\mathbb{R})\big)$$

that is obtained as the pullback of the canonical symplectic form on $T^*\,\mathrm{Imm}$ via the metric

$$G : T\,\mathrm{Imm} \to T^*\,\mathrm{Imm}\,.$$

The ω-gradient is defined by the relation

$$\operatorname{grad}^\omega f \in \mathfrak{X}(T\,\mathrm{Imm}), \qquad \omega(\operatorname{grad}^\omega f, \cdot) = df,$$

where f is a smooth function on $T\,\mathrm{Imm}$. Not all functions have an ω-gradient because

$$\omega : TT\,\mathrm{Imm} \to T^*T\,\mathrm{Imm}$$

is injective, but not surjective. We will denote the set of functions that have a smooth ω-gradient by

$$C^\infty_\omega(T\,\mathrm{Imm},\mathbb{R}) \subset C^\infty(T\,\mathrm{Imm},\mathbb{R}).$$

We define the momentum map as

$$j : \mathfrak{g} \to C^\infty(T\,\mathrm{Imm},\mathbb{R}), \quad j_X(h_f) = G_f\big(\zeta_X(f), h_f\big)$$

and verify that it has the desired properties: Assuming that the metric gradients H, K exist (assumption 1.2.7), one can prove that

$$j_X \in C^\infty_\omega(T\,\mathrm{Imm},\mathbb{R}) \quad \text{and} \quad \operatorname{grad}^\omega(j_X) = \zeta_X.$$

Thus the momentum map fits into the following commutative diagram of Lie algebras:

$$\begin{tikzcd}
H^0(T\,\mathrm{Imm}) \arrow[r, "i"] & C^\infty_\omega(T\,\mathrm{Imm},\mathbb{R}) \arrow[r, "\operatorname{grad}^\omega"] & \mathfrak{X}(T\,\mathrm{Imm},\omega) \arrow[r, "\omega"] & H^1(T\,\mathrm{Imm}) \\
 & & \mathfrak{g} \arrow[ul, "j"] \arrow[u, "\zeta^{T\,\mathrm{Imm}}"'] &
\end{tikzcd}$$

Here $\mathfrak{X}(T\,\mathrm{Imm},\omega)$ is the space of vector fields on $T\,\mathrm{Imm}$ whose flow leaves ω fixed. All arrows in this diagram are homomorphism of Lie algebras. The sequence at the top is exact when it is extended by zeros on the left and right end.

By *Emmy Noether's theorem*, the momentum mapping is constant along any geodesic $f : \mathbb{R} \to \mathrm{Imm}$. Thus for any $X \in \mathfrak{g}$ we have that

$$j_X(f_t) = G_f\big(\zeta_X(f), f_t\big) \quad \text{is constant in } t.$$

We will now consider several group actions on Imm and calculate the corresponding conserved momenta.

- Consider the smooth right action of the group $\operatorname{Diff}(M)$ on $\operatorname{Imm}(M,N)$ given by composition from the right:
$$f \mapsto f \circ \varphi \quad \text{for} \quad \varphi \in \operatorname{Diff}(M).$$
This action is isometric by assumption, see section 1.2.5. For $X \in \mathfrak{X}(M)$ the fundamental vector field is given by
$$\zeta_X(f) = \partial_t|_0(f \circ \operatorname{Fl}_t^X) = Tf \circ X$$
where Fl_t^X denotes the flow of X. The *reparametrization momentum*, for any vector field X on M is thus $G_f(Tf \circ X, h_f)$. Assuming that the metric is reparametrization invariant, it follows that along any geodesic $f(t,\cdot)$, the expression $G_f(Tf \circ X, f_t)$ is constant for all X.

For a flat ambient space $N = \mathbb{R}^n$ we can consider in addition the following group actions:

- The left action of the Euclidean motion group $\mathbb{R}^n \rtimes SO(n)$ on $\operatorname{Imm}(M,\mathbb{R}^n)$ given by
$$f \mapsto A + Bf \quad \text{for} \quad (A,B) \in \mathbb{R}^n \times SO(n).$$
The fundamental vector field mapping is
$$\zeta_{(A,X)}(f) = A + Xf \quad \text{for} \quad (A,X) \in \mathbb{R}^n \times \mathfrak{so}(n).$$
The *linear momentum* is thus $G_f(A,h), A \in \mathbb{R}^n$ and if the metric is translation invariant, $G_f(A,f_t)$ will be constant along geodesics for every $A \in \mathbb{R}^n$. The *angular momentum* is similarly $G_f(X.f,h), X \in \mathfrak{so}(n)$ and if the metric is rotation invariant, then $G_f(X.f,f_t)$ will be constant along geodesics for each $X \in \mathfrak{so}(n)$.
- The action of the scaling group of $\mathbb{R}$ given by $f \mapsto e^r f$, with fundamental vector field $\zeta_a(f) = a.f$. If the metric is scale invariant, then the *scaling momentum* $G_f(f,f_t)$ will be constant along geodesics.

1.2.11 Shape space

$\operatorname{Diff}(M)$ acts smoothly on $C^\infty(M,N)$ and its subsets $\operatorname{Imm}, \operatorname{Imm}_f$ and Emb by composition from the right. *Shape space* is defined as the orbit space with respect to this action. That means that in shape space, two mappings that differ only in their parametrization will be regarded the same.

Theorem. *Let M be compact and of dimension $\leq n$. Then* $\operatorname{Imm}_f(M,N)$ *is the total space of a smooth principal fiber bundle with structure group* $\operatorname{Diff}(M)$, *whose base manifold is a Hausdorf smooth Fréchet manifold denoted by*
$$B_{i,f}(M,N) = \operatorname{Imm}_f(M,N)/\operatorname{Diff}(M).$$
The same result holds for the open subset $\operatorname{Emb}(M,N) \subset \operatorname{Imm}_f(M,N)$. *The corresponding base space is denoted by*
$$B_e(M,N) = \operatorname{Emb}(M,N)/\operatorname{Diff}(M).$$

However, the space

$$B_i(M,N) = \mathrm{Imm}(M,N)/\mathrm{Diff}(M)$$

is not a smooth manifold, but has singularities of orbifold type: Locally, it looks like a finite dimensional orbifold times an infinite dimensional Fréchet space.

The proofs for free and non-free immersions can be found in [15] and the one for embeddings in [27, section 44.1].

As with immersions and embeddings, we will sometimes write $B_{i,f}, B_i, B_e$ when it is clear that M and N are the domain and target of the mappings.

1.2.12 Riemannian submersions and geodesics

The concept of a Riemannian submersion will allow us to induce a Riemannian metric on shape space. We will now explain in general terms what a Riemannian submersion is and how horizontal geodesics in the top space correspond nicely to geodesics in the quotient space. The definitions and results of this section are taken from [34, section 26].

Let $\pi : E \to B$ be a submersion of smooth manifolds, that is, $T\pi : TE \to TB$ is surjective. Then

$$V = V(\pi) := \ker(T\pi) \subset TE$$

is called the *vertical subbundle*. If E carries a Riemannian metric G, then we can go on to define the *horizontal subbundle* as the G-orthogonal complement of V:

$$\mathrm{Hor} = \mathrm{Hor}(\pi, G) := V(\pi)^{\perp} \subset TE.$$

Now any vector $X \in TE$ can be decomposed uniquely in vertical and horizontal components as

$$X = X^{\mathrm{ver}} + X^{\mathrm{hor}}.$$

This definition extends to the cotangent bundle as follows: An element of T^*E is called horizontal when it annihilates all vertical vectors, and vertical when it annihilates all horizontal vectors.

In the setting described so far, the mapping

$$T_x\pi|_{\mathrm{Hor}_x} : \mathrm{Hor}_x \to T_{\pi(x)}B$$

is an isomorphism of vector spaces for all $x \in E$. If both (E, G_E) and (B, G_B) are Riemannian manifolds and if this mapping is an isometry for all $x \in E$, then we will call π a *Riemannian submersion.*

Theorem. *Consider a Riemannian submersion $\pi : E \to B$, and let $c : [0,1] \to E$ be a geodesic in E.*

1. *If $c'(t)$ is horizontal at one t, then it is horizontal at all t.*
2. *If $c'(t)$ is horizontal then $\pi \circ c$ is a geodesic in B.*

3. *If every curve in B can be lifted to a horizontal curve in E, then there is a one-to-one correspondence between curves in B and horizontal curves in E. This implies that instead of solving the geodesic equation on B one can equivalently solve the equation for horizontal geodesics in E.*

See [34, section 26] for the proof.

1.2.13 Riemannian metrics on shape space

Now the previous chapter is applied to the submersion $\pi : \mathrm{Imm} \to B_i$:

Theorem. *Given a* $\mathrm{Diff}(M)$*-invariant Riemannian metric on* Imm*, there is a unique Riemannian metric on the quotient space B_i such that the quotient map $\pi : \mathrm{Imm} \to B_i$ is a Riemannian submersion.*

One also gets a description of the tangent space to shape space: When $f \in \mathrm{Imm}$, then $T_{\pi(f)}B_i$ is isometric to the horizontal bundle at f. The horizontal bundle depends on the definition of the metric. For the almost local metrics, it consists of vector fields along f that are everywhere normal to f, see section 2.3.1.

Assumption. *In the following we will always assue that a* $\mathrm{Diff}(M)$*-invariant Riemannian metric on the manifold of immersions is given, and that shape space is endowed with the unique Riemannian metric turning the projection into a Riemannian submersion.*

1.2.14 Geodesic equation on shape space

We will apply theorem 1.2.12 to the Riemannian submersion $\pi : \mathrm{Imm} \to B_i$.

Theorem. *Assuming that every curve in B_i can be lifted to a horizontal curve in* Imm*, the geodesic equation on shape space is equivalent to*

$$\begin{cases} f_t = f_t^{\mathrm{hor}} \in \mathrm{Hor} \\ (\nabla_{\partial_t} f_t)^{\mathrm{hor}} = \Big(\frac{1}{2}H(f_t,f_t) - K(f_t,f_t)\Big)^{\mathrm{hor}}, \end{cases} \tag{1}$$

where f is a horizontal curve in Imm*, where H, K are the metric gradients defined in section 1.2.7 and where ∇ is the covariant derivative defined in section 1.2.6.*

Proof. Theorem 1.2.12 states that the geodesic equation on shape space is equivalent to the horizontal geodesic equation on Imm which is given by

$$\begin{cases} f_t = f_t^{\mathrm{hor}} \\ \nabla_{\partial_t} f_t = \frac{1}{2}H_f(f_t,f_t) - K_f(f_t,f_t) \end{cases} \tag{2}$$

Clearly (2) implies (1). To prove the converse it remains to show that

$$(\nabla_{\partial_t} f_t)^{\mathrm{vert}} = \Big(\frac{1}{2}H(f_t,f_t) - K(f_t,f_t)\Big)^{\mathrm{vert}}.$$

As the following proof shows, this is a consequence of the conservation of the momentum along f and of the invariance of the metric under $\mathrm{Diff}(M)$.

Recall the infinitesimal action of $\mathrm{Diff}(M)$ on $\mathrm{Imm}(M,N)$. For any $X \in \mathfrak{X}(M)$ it is given by the fundamental vector field

$$\zeta_X \in \mathfrak{X}(\mathrm{Imm}), \qquad \zeta_X(f) = \partial_s|_0 r\big(f, \exp(sX)\big) = \partial_s|_0\big(f \circ Fl^X_t\big) = Tf.X.$$

Here r is the right action of $\mathrm{Diff}(M)$ on $\mathrm{Imm}(M,N)$ defined in section 1.2.4. When $f : \mathbb{R} \to \mathrm{Imm}$ is a curve of immersions, one obtains a two-parameter family of immersions

$$g : \mathbb{R}\times\mathbb{R} \to \mathrm{Imm}, \qquad g(s,t) = r\big(f(t), \exp(sX)\big)$$

that satisfies

$$\begin{aligned}\nabla_{\partial_t} Tg.\partial_s &= \nabla_{\partial_s} Tg.\partial_t + Tg.[\partial_t,\partial_s] + \mathrm{Tor}(Tg.\partial_t, Tg.\partial_s)\\ &= \nabla_{\partial_s} T\big(r^{\exp(sX)}\big) f_t + 0 + 0\end{aligned}$$

since ∇ is torsion-free. This implies

$$\nabla_{\partial_t}\zeta_X(f) = \nabla_{\partial_t} Tg.\partial_s|_0 = \nabla_{\partial_s|_0} T\big(r^{\exp(sX)}\big) f_t.$$

$\zeta_X(f)$ is vertical and f_t is horizontal by assumption. Thus the momentum mapping $G_f\big(\zeta_X(f), f_t\big)$ is constant and equals zero. Its derivative is

$$\begin{aligned}
0 &= \partial_t\Big(G_f\big(\zeta_X(f), f_t\big)\Big)\\
&= (\nabla_{\partial_t} G_f)\big(\zeta_X(f), f_t\big) + G_f\big(\nabla_{\partial_t}\zeta_X(f), f_t\big) + G_f\big(\zeta_X(f), \nabla_{\partial_t} f_t\big)\\
&= (\nabla_{f_t} G)\big(\zeta_X(f), f_t\big) + G_f\Big(\nabla_{\partial_s|_0} T\big(r^{\exp(sX)}\big) f_t, f_t\Big) + G_f\big(\zeta_X(f), \nabla_{\partial_t} f_t\big)\\
&= G_f\big(K_f(f_t,f_t) + \nabla_{\partial_t} f_t, \zeta_X(f)\big)\\
&\quad + G_{r^{\exp(sX)}f}\Big(\nabla_{\partial_s} T\big(r^{\exp(sX)}\big) f_t, T\big(r^{\exp(sX)}\big) f_t\Big)\Big|_{s=0}\\
&= G_f\big(K_f(f_t,f_t) + \nabla_{\partial_t} f_t, \zeta_X(f)\big)\\
&\quad + \frac{1}{2}\partial_s|_0\bigg(G_{r^{\exp(s.X)}f}\Big(T\big(r^{\exp(sX)}\big) f_t, T\big(r^{\exp(s.X)}\big) f_t\Big)\bigg)\\
&\quad - \frac{1}{2}\,\big(\nabla_{\partial_s} G_{r^{\exp(sX)}f}\big)\Big(T\big(r^{\exp(sX)}\big) f_t, T\big(r^{\exp(sX)}\big) f_t\Big)\Big|_{s=0}\\
&= G_f\big(K_f(f_t,f_t) + \nabla_{\partial_t} f_t, \zeta_X(f)\big)\\
&\quad + \frac{1}{2}\partial_s|_0\big(G_f(f_t,f_t)\big) - \frac{1}{2}\big(\nabla_{\zeta_X(f)} G\big)(f_t,f_t)\\
&= G_f\Big(K_f(f_t,f_t) + \nabla_{\partial_t} f_t + 0 - \frac{1}{2}H_f(f_t,f_t), \zeta_X(f)\Big)
\end{aligned}$$

Any vertical tangent vector to f is of the form $\zeta_X(f)$ for some $X \in \mathfrak{X}(M)$. Therefore

$$0 = \Big(\nabla_{\partial_t} f_t - \frac{1}{2}H_f(f_t,f_t) + K_f(f_t,f_t)\Big)^{\mathrm{vert}}. \qquad \square$$

It will be shown in section 2.3.1 that curves in B_i can be lifted to horizontal curves in Imm for the class of almost local metrics. Thus all assumptions and conclusions of the theorem hold.

1.2.15 Geodesic equation on shape space in terms of the momentum

As in the previous section we apply theorem 1.2.12 to the Riemannian submersion $\pi : \mathrm{Imm} \to B_i$. This yields:

Theorem. *Assuming that every curve in B_i can be lifted to a horizontal curve in* Imm*, the geodesic equation for the momentum on shape space is equivalent to*

$$\begin{cases} p = G(f_t, \cdot) \in \mathrm{Hor} \\ \big(\nabla_{\partial_t} G(f_t, \cdot)\big)^{\mathrm{hor}} = \dfrac{1}{2} G\big(H(f_t, f_t), \cdot\big)^{\mathrm{hor}}, \end{cases}$$

Here f is a curve in Imm*, H is the metric gradient defined in section 1.2.7, and ∇ is the covariant derivative defined in section 1.2.6. f is horizontal because p is horizontal.*

The proof of this theorem is a consequence of the previous section and of section 1.2.9.

1.2.16 Inner versus outer metrics

There are two similar yet different approaches on how to define a Riemannian metric on shape space.

The metrics on shape space presented in this work are induced by metrics on $\mathrm{Imm}(M, N)$. One might call them *inner metrics* since they are defined intrinsically to M. Intuitively, these metrics can be seen as describing a deformable material that the shape itself is made of.

In contrast to these metrics, there are also metrics that are induced from metrics on $\mathrm{Diff}(N)$ by the same construction of Riemannian submersions. (The widely used LDDMM algorithm is based on such a metric.) The differential operator governing these metrics is defined on all of N, even outside of the shape. When the shape is deformed, the surrounding ambient space is deformed with it. Intuitively, such metrics can be seen as describing some deformable material that the ambient space is made of. Therefore one might call them *outer metrics*.

The following diagram illustrates both approaches. Metrics are defined on one of the top spaces and induced on the corresponding space below by the construction of Riemannian submersions.

$$\begin{tikzcd}
\mathrm{Diff}(N) \arrow[d] & \\
\mathrm{Emb}(M,N) \arrow[r, hook] \arrow[d] & \mathrm{Imm}(M,N) \arrow[d] \\
B_e(M,N) \arrow[r, hook] & B_i(M,N)
\end{tikzcd}$$

1.3 Formulas for first variations

Recall that many operators like

$$g = f^*\bar{g}, \quad S = S^f, \quad \mathrm{vol}(g), \quad \nabla = \nabla^g, \quad \Delta = \Delta^g, \quad \dots$$

implicitly depend on the immersion f. We want to calculate their derivative with respect to f, which we call *the first variation.* We will use this formulas to calculate the metric gradients that are needed for the geodesic equation.

This section is based on [6, 8], with modifications due to possibly curved ambient space N and higher codimension $n-m$. Some of the formulas can also be found in [7, 13, 35, 49]. Some of the variation formulas are equal to [22, section 3].

1.3.1 Paths of immersions

All of the concepts introduced in section 1.1 can be recast for a path of immersions instead of a fixed immersion. This allows to study variations immersions. So let $f : \mathbb{R} \to \mathrm{Imm}(M,N)$ be a path of immersions. By convenient calculus [27], f can equivalently be seen as $f : \mathbb{R} \times M \to N$ such that $f(t,\cdot)$ is an immersion for each t. We can replace bundles over M by bundles over $\mathbb{R} \times M$:

$$\begin{tikzcd}
\mathrm{pr}_2^* T^r_s M \arrow[d] & \mathrm{pr}_2^* T^r_s M \otimes f^*TN \arrow[d] & \mathrm{Nor}(f) \arrow[d] \\
\mathbb{R} \times M & \mathbb{R} \times M & \mathbb{R} \times M
\end{tikzcd}$$

Here pr_2 denotes the projection $\mathrm{pr}_2 : \mathbb{R} \times M \to M$. The covariant derivative $\nabla_Z h$ is now defined for vector fields Z on $\mathbb{R} \times M$ and sections h of the above bundles. The vector fields $(\partial_t, 0_M)$ and $(0_\mathbb{R}, X)$, where X is a vector field on M, are of special importance. Let

$$\mathrm{ins}_t : M \to \mathbb{R} \times M, \qquad x \mapsto (t,x).$$

Then by [34, 22.9.6] one has for vector fields X, Y on M

$$\nabla_X Tf(t,\cdot).Y = \nabla_X T(f \circ \mathrm{ins}_t) \circ Y = \nabla_X Tf \circ T\,\mathrm{ins}_t \circ Y$$

$$
\begin{aligned}
&= \nabla_X Tf \circ (0_{\mathbb{R}}, Y) \circ \mathrm{ins}_t = \nabla_{T\,\mathrm{ins}_t \circ X} Tf \circ (0_{\mathbb{R}}, Y) \\
&= \big(\nabla_{(0_{\mathbb{R}}, X)} Tf \circ (0_{\mathbb{R}}, Y)\big) \circ \mathrm{ins}_t\,.
\end{aligned}
$$

This shows that one can recover the static situation at t by using vector fields on $\mathbb{R} \times M$ with vanishing $\mathbb{R}$-component and evaluating at t.

1.3.2 Setting for first variations

In all of this chapter, let f be an immersion and $f_t \in T_f \,\mathrm{Imm}$ a tangent vector to f. The reason for calling the tangent vector f_t is that in calculations it will often be the derivative of a curve of immersions through f. Using the same symbol f for the fixed immersion and for the path of immersions through it, one has in fact that

$$D_{(f,f_t)} F = \partial_t F(f(t)).$$

For the sake of brevity we will write ∂_t instead of $(\partial_t, 0_M)$ and X instead of $(0_{\mathbb{R}}, X)$, where X is a vector field on M.

1.3.3 Tangential variation of equivariant tensor fields

Let the smooth mapping $F : \mathrm{Imm}(M, N) \to \Gamma(T^r_s M)$ take values in some space of tensor fields over M, or more generally in any natural bundle over M, see [26].

Lemma. *If F is equivariant with respect to pullbacks by diffeomorphisms of M, i.e.*

$$F(f) = (\varphi^* F)(f) = \varphi^* \Big(F\big((\varphi^{-1})^* f\big)\Big)$$

for all $\varphi \in \mathrm{Diff}(M)$ and $f \in \mathrm{Imm}(M, N)$, then the tangential variation of F is its Lie-derivative:

$$
\begin{aligned}
D_{(f, Tf.f_t^\top)} F &= \partial_t|_0 F\Big(f \circ Fl_t^{f_t^\top}\Big) = \partial_t|_0 F\Big((Fl_t^{f_t^\top})^* f\Big) \\
&= \partial_t|_0 \Big(Fl_t^{f_t^\top}\Big)^* (F(f)) = \mathcal{L}_{f_t^\top}(F(f)).
\end{aligned}
$$

This allows us to calculate the tangential variation of the pullback metric and the volume density, because these tensor fields are natural with respect to pullbacks by diffeomorphisms.

1.3.4 Variation of the metric

Lemma. *The differential of the pullback metric*

$$
\begin{cases}
\mathrm{Imm} & \to \quad \Gamma(S^2_{>0} T^* M), \\
f & \mapsto \quad g = f^* \bar{g}
\end{cases}
$$

is given by

$$D_{(f,f_t)} g = 2 \,\mathrm{Sym}\, \bar{g}(\nabla f_t, Tf) = -2 \bar{g}(f_t^\perp, S) + 2 \,\mathrm{Sym}\, \nabla (f_t^\top)^\flat$$

$$= -2\bar{g}(f_t^\perp, S) + \mathcal{L}_{f_t^\top} g.$$

In codimension one this formula specializes to

$$D_{(f,f_t)} g = -2\bar{g}(f_t, \nu).s + \mathcal{L}_{f_t^\top}(g).$$

Here Sym denotes the symmetric part of the tensor field C of type $\binom{0}{2}$ given by

$$\big(\operatorname{Sym}(C)\big)(X,Y) := \frac{1}{2}\big(C(X,Y) + C(Y,X)\big).$$

Proof. Let $f : \mathbb{R} \times M \to N$ be a path of immersions. Swapping covariant derivatives as in section 1.1.8 formula (3) one gets

$$\begin{aligned}\partial_t\big(g(X,Y)\big) &= \partial_t\big(\bar{g}(Tf.X, Tf.Y)\big) = \bar{g}(\nabla_{\partial_t} Tf.X, Tf.Y) + \bar{g}(Tf.X, \nabla_{\partial_t} Tf.Y)\\ &= \bar{g}(\nabla_X f_t, Tf.Y) + \bar{g}(Tf.X, \nabla_Y f_t) = \big(2\operatorname{Sym}\bar{g}(\nabla f_t, Tf)\big)(X,Y).\end{aligned}$$

Splitting f_t into its normal and tangential part yields

$$\begin{aligned}2\operatorname{Sym}\bar{g}(\nabla f_t, Tf) &= 2\operatorname{Sym}\bar{g}(\nabla f_t^\perp + \nabla Tf.f_t^\top, Tf)\\ &= -2\operatorname{Sym}\bar{g}(f_t^\perp, \nabla Tf) + 2\operatorname{Sym} g(\nabla f_t^\top, \cdot)\\ &= -2\bar{g}(f_t^\perp, S) + 2\operatorname{Sym}\nabla(f_t^\top)^\flat.\end{aligned}$$

Finally the relation

$$D_{(f, Tf.f_t^\top)} g = 2\operatorname{Sym}\nabla(f_t^\top)^\flat = \mathcal{L}_{f_t^\top} g$$

follows either from the equivariance of g with respect to pullbacks by diffeomorphisms (see 1.3.3) or directly from

$$\begin{aligned}(\mathcal{L}_X g)(Y,Z) &= \mathcal{L}_X\big(g(Y,Z)\big) - g(\mathcal{L}_X Y, Z) - g(Y, \mathcal{L}_X Z)\\ &= \nabla_X\big(g(Y,Z)\big) - g(\nabla_X Y - \nabla_Y X, Z) - g(Y, \nabla_X Z - \nabla_Z X)\\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = (\nabla_Y X)^\flat(Z) + (\nabla_Z X)^\flat(Y)\\ &= (\nabla_Y X^\flat)(Z) + (\nabla_Z X^\flat)(Y) = 2\operatorname{Sym}\big(\nabla(X^\flat)\big)(Y,Z).\end{aligned}$$ □

1.3.5 Variation of the inverse of the metric

Lemma. *The differential of the inverse of the pullback metric*

$$\begin{cases} \operatorname{Imm} & \to \quad \Gamma(L(T^*M, TM)),\\ f & \mapsto \quad g^{-1} = (f^*\bar{g})^{-1}\end{cases}$$

is given by

$$D_{(f,f_t)} g^{-1} = D_{(f,f_t)} (f^*\bar{g})^{-1} = 2\bar{g}(f_t^\perp, g^{-1}Sg^{-1}) + \mathcal{L}_{f_t^\top}(g^{-1})$$

In codimension one this formula specializes to

$$D_{(f,f_t)} g^{-1} = -2\bar{g}(f_t, \nu).L.g^{-1} + \mathcal{L}_{f_t^\top}(g^{-1}).$$

Proof.

$$\begin{aligned}\partial_t g^{-1} &= -g^{-1}(\partial_t g) g^{-1} = -g^{-1}\big(-2\bar{g}(f_t^\perp, S) + \mathcal{L}_{f_t^\top} g\big) g^{-1}\\ &= 2g^{-1}\bar{g}(f_t^\perp, S) g^{-1} - g^{-1}(\mathcal{L}_{f_t^\top} g) g^{-1} = 2\bar{g}(f_t^\perp, g^{-1}Sg^{-1}) + \mathcal{L}_{f_t^\top}(g^{-1})\end{aligned}$$ □

1.3.6 Variation of the volume density

Lemma. *The differential of the volume density*

$$\begin{cases} \mathrm{Imm} & \to & \mathrm{Vol}(M), \\ f & \mapsto & \mathrm{vol}(g) = \mathrm{vol}(f^*\bar{g}) \end{cases}$$

is given by

$$D_{(f,f_t)}\,\mathrm{vol}(g) = \mathrm{Tr}^g\big(\bar{g}(\nabla f_t, Tf)\big)\,\mathrm{vol}(g) = \Big(\mathrm{div}^g(f_t^\top) - \bar{g}\big(f_t^\perp, \mathrm{Tr}^g(S)\big)\Big)\,\mathrm{vol}(g).$$

In codimension one this formula reads as

$$D_{(f,f_t)}\,\mathrm{vol}(g) = \Big(\mathrm{div}^g(f_t^\top) - \bar{g}(f_t^\perp, \nu).\,\mathrm{Tr}(L)\Big)\,\mathrm{vol}(g).$$

Proof. Let $g(t) \in \Gamma(S^2_{>0}T^*M)$ be any curve of Riemannian metrics. Then

$$\partial_t\,\mathrm{vol}(g) = \frac{1}{2}\,\mathrm{Tr}(g^{-1}.\partial_t g)\,\mathrm{vol}(g).$$

This follows from the formula for $\mathrm{vol}(g)$ in a local oriented chart $(u^1, \ldots u^n)$ on M:

$$\begin{aligned}
\partial_t\,\mathrm{vol}(g) &= \partial_t\sqrt{\det((g_{ij})_{ij})}\;du^1\wedge\cdots\wedge du^{n-1} \\
&= \frac{1}{2\sqrt{\det((g_{ij})_{ij})}}\,\mathrm{Tr}(\mathrm{adj}(g)\partial_t g)\;du^1\wedge\cdots\wedge du^{n-1} \\
&= \frac{1}{2\sqrt{\det((g_{ij})_{ij})}}\,\mathrm{Tr}(\det((g_{ij})_{ij})g^{-1}\partial_t g)\;du^1\wedge\cdots\wedge du^{n-1} \\
&= \frac{1}{2}\,\mathrm{Tr}(g^{-1}.\partial_t g)\,\mathrm{vol}(g)
\end{aligned}$$

Now we can set $g = f^*\bar{g}$ and plug in the formula

$$\partial_t g = \partial_t(f^*\bar{g}) = 2\,\mathrm{Sym}\,\bar{g}(\nabla f_t, Tf)$$

from 1.3.4. This immediately proves the first formula:

$$\partial_t\,\mathrm{vol}(g) = \frac{1}{2}\,\mathrm{Tr}\big(g^{-1}.2\,\mathrm{Sym}\,\bar{g}(\nabla f_t, Tf)\big) = \mathrm{Tr}^g\big(\bar{g}(\nabla f_t, Tf)\big).$$

Expanding this further yields the second formula:

$$\begin{aligned}
\partial_t\,\mathrm{vol}(g) &= \mathrm{Tr}^g\Big(\nabla\bar{g}(f_t, Tf) - \bar{g}(f_t, \nabla Tf)\Big) \\
&= \mathrm{Tr}^g\Big(\nabla\bar{g}(f_t, Tf) - \bar{g}(f_t, S)\Big) = -\nabla^*\bar{g}(f_t, Tf) - \bar{g}\big(f_t, \mathrm{Tr}^g(S)\big) \\
&= -\nabla^*\big((f_t^\top)^\flat\big) - \bar{g}\big(f_t^\perp, \mathrm{Tr}^g(S)\big) = \mathrm{div}(f_t^\top) - \bar{g}\big(f_t^\perp, \mathrm{Tr}^g(S)\big).
\end{aligned}$$

Here we have used

$$\nabla Tf = S \quad \text{and} \quad \mathrm{div}(f_t^\top) = \mathrm{Tr}(\nabla f_t^\top) = \mathrm{Tr}^g\big((\nabla f_t^\top)^\flat\big) = -\nabla^*\big((f_t^\top)^\flat\big).$$

Note that by 1.3.3, the formula for the tangential variation would have followed also from the equivariance of the volume form with respect to pullbacks by diffeomorphisms. □

1.3.7 Variation of the volume

Lemma. *The differential of the total Volume*

$$\begin{cases} \mathrm{Imm} & \to & \mathbb{R}, \\ f & \mapsto & \mathrm{Vol}(f) = \int_M \mathrm{vol}(f^*\bar{g}) \end{cases}$$

is given by

$$D_{(f,f_t)} \mathrm{Vol}(f) = D_{(f,f_t)} \int_M \mathrm{vol}(g) = -\int_M \bar{g}\big(f_t^\perp, \mathrm{Tr}^g(S)\big) \,\mathrm{vol}(g).$$

In codimension one this formula reads as

$$D_{(f,f_t)} \mathrm{vol}(g) = -\int_M \bar{g}(f_t^\perp, \nu). \,\mathrm{Tr}(L) \,\mathrm{vol}(g).$$

Proof. This follows from 1.3.6. The integral over the divergence term vanishes by the Theorem of Stokes. □

1.3.8 Variation of the vector valued second fundamental form

Lemma. *The differential of the vector valued second fundamental form*

$$\begin{cases} \mathrm{Imm} & \to & \Gamma(S^2T^*M \otimes \mathrm{Nor}(f)), \\ f & \mapsto & S \end{cases}$$

is given by

$$D_{(f,f_t)}S = \nabla^2 f_t + R^{\bar{g}}(f_t, Tf)Tf = \nabla^2 f_t^\perp + R^{\bar{g}}(f_t^\perp, Tf)Tf + \mathcal{L}_{f_t^\top}(S).$$

Recall that ∇^2 stands for the bilinear mapping $(X, Y) \mapsto \nabla^2_{X,Y}$ defined in 1.1.9.

Proof. By definition $S(X, Y) = \nabla_X(Tf.Y) - Tf.\nabla_X Y$. Therefore

$$\begin{aligned} \partial_t S^f(X, Y) &= \nabla_{\partial_t} \nabla_X(Tf.Y) - \nabla_{\partial_t} Tf.\nabla_X Y \\ &= \nabla_X \nabla_Y Tf.\partial_t + R^{\bar{g}}(Tf\partial_t, TfX)TfY - \nabla_{\nabla_X Y} Tf.\partial_t \\ &= \nabla_X \nabla_Y f_t - \nabla_{\nabla_X Y} f_t + R^{\bar{g}}(f_t, TfX)TfY \\ &= \nabla^2 f_t + R^{\bar{g}}(f_t, Tf)Tf, \end{aligned}$$

where we interchanged covariant derivatives as in 1.1.8.(3) and 1.1.8.(4). By 1.3.3, the formula for the tangential variation follows from the equivariance of the second fundamental form with respect to pullbacks by diffeomorphisms:

$$\begin{aligned} S^{f\circ\Phi}(X, Y) &= \nabla^{\bar{g}}\big(T(f \circ \Phi).Y\big) - T(f \circ \Phi)\nabla_X^{(f\circ\Phi)^*\bar{g}} Y \\ &= \nabla^{\bar{g}}_X\big(Tf.T\Phi.Y\big) - Tf.T\Phi.\nabla_X^{(f\circ\Phi)^*\bar{g}} Y \end{aligned}$$

$$\begin{aligned}
&= \nabla^{\bar g}_X\big(Tf.\Phi_*Y\circ\Phi\big) - Tf(\nabla^{f^*\bar g}_{\Phi_*X}\Phi_*Y)\circ\Phi \\
&= \nabla^{\bar g}_{T\Phi.X}\big(Tf.\Phi_*Y\big) - Tf(\nabla^{f^*\bar g}_{\Phi_*X}\Phi_*Y)\circ\Phi \\
&= \Big(\nabla^{\bar g}_{\Phi_*X}Tf.\Phi_*Y\Big)\circ\Phi - Tf(\nabla^{f^*\bar g}_{\Phi_*X}\Phi_*Y)\circ\Phi = S^f(\Phi_*X,\Phi_*Y)\circ\Phi
\end{aligned}$$

In the above calculations we used property (5) from section 1.1.7 and the naturality of the covariant derivative. □

1.3.9 Variation of the scalar valued second fundamental form (codimension one)

Lemma. *The differential of the scalar second fundamental form*

$$\begin{cases} \mathrm{Imm} & \to & \Gamma(S^2T^*M), \\ f & \mapsto & s^f \end{cases}$$

is given by

$$\begin{aligned}
D_{(f,f_t)}s &= \bar g(\nabla^2 f_t,\nu) + \bar g(R^{\bar g}(f_t,Tf)Tf,\nu) \\
&= \nabla^2\bar g(f_t,\nu) - \bar g(f_t,\nu).g\circ(L\otimes L) + \mathcal{L}_{f_t^\top}.s + \bar g(R^{\bar g}(f_t^\perp,Tf)Tf,\nu)
\end{aligned}$$

Proof. By definition $s(X,Y) = \bar g\big(S(X,Y),\nu\big) = \bar g\big(\nabla_X(Tf.Y),\nu\big)$. Using the formula for the first variation of the vector valued second fundamental form (1.3.8) yields

$$\begin{aligned}
\partial_t s(X,Y) &= \bar g\big(\partial_t S(X,Y),\nu\big) + \bar g\big(S(X,Y),\partial_t\nu\big) \\
&= \bar g\big(\nabla^2_{X,Y}f_t,\nu\big) + \bar g\big(R^{\bar g}(f_t,TfX)TfY,\nu\big) + 0 \\
&= \bar g\Big(\nabla^2_{X,Y}f_t^\perp + R^{\bar g}(f_t^\perp,TfX)TfY + \mathcal{L}_{f_t^\top}S(X,Y),\nu\Big),
\end{aligned}$$

where the term $\bar g\big(S(X,Y),\partial_t\nu\big)$ vanishes since $\partial_t\nu$ is tangential (see 1.3.14). To get the second formula we calculate:

$$\begin{aligned}
\partial_t s(X,Y) &= \bar g\Big(\nabla^2_{X,Y}\big(\bar g(f_t,\nu).\nu\big),\nu\Big) + \bar g\big(\mathcal{L}_{f_t^\top}(s(X,Y).\nu),\nu\big) \\
&\quad + \bar g\big(R^{\bar g}(f_t,TfX)TfY,\nu\big) \\
&= \nabla^2_{X,Y}\bar g(f_t,\nu) + 0 + \bar g(f_t,\nu).\bar g\big(\nabla^2_{X,Y}\nu,\nu\big) + \mathcal{L}_{f_t^\top}s(X,Y) + 0 \\
&\quad + \bar g\big(R^{\bar g}(f_t,TfX)TfY,\nu\big) \\
&= \nabla^2_{X,Y}\bar g(f_t,\nu) - \bar g(f_t,\nu).\bar g\big(\nabla_X\nu,\nabla_Y\nu\big) + 0 + \mathcal{L}_{f_t^\top}s(X,Y) \\
&\quad + \bar g\big(R^{\bar g}(f_t,TfX)TfY,\nu\big) \\
&= \nabla^2_{X,Y}\bar g(f_t,\nu) - \bar g(f_t,\nu).g\big(LX,LY\big) + 0 + \mathcal{L}_{f_t^\top}s(X,Y) \\
&\quad + \bar g\big(R^{\bar g}(f_t,TfX)TfY,\nu\big).
\end{aligned}$$

□

1.3.10 Variation of the vector valued mean curvature

Lemma. *The differential of the vector valued mean curvature*

$$\begin{cases} \mathrm{Imm} & \to & \Gamma(\mathrm{Nor}(f)), \\ f & \mapsto & \mathrm{Tr}^g(S) \end{cases}$$

is given by

$$D_{(f,f_t)} \operatorname{Tr}^g(S) = \operatorname{Tr}\big(2\bar g(f_t^\perp, g^{-1}.S.g^{-1}).S\big) - \Delta(f_t^\perp) \\ + \operatorname{Tr}^g(R^{\bar g}(f_t^\perp, Tf)Tf) + \mathcal{L}_{f_t^\top} \operatorname{Tr}^g(S)$$

Proof.

$$\begin{aligned}
\partial_t \operatorname{Tr}^g(S) &= \partial_t \operatorname{Tr}(g^{-1}.S) = \operatorname{Tr}\big((\partial_t g^{-1}).S\big) + \operatorname{Tr}\big(g^{-1}.\partial_t S\big) \\
&= \operatorname{Tr}\big(2\bar g(f_t^\perp, g^{-1}Sg^{-1}).S\big) + \operatorname{Tr}\big(\mathcal{L}_{f_t^\top}(g^{-1}).S\big) + \operatorname{Tr}\big(g^{-1}.\nabla^2 f_t^\perp\big) \\
&\quad + \operatorname{Tr}\big(g^{-1}R^{\bar g}(f_t^\perp, Tf)Tf\big) + \operatorname{Tr}\big(g^{-1}.(\mathcal{L}_{f_t^\top} S)\big) \\
&= \operatorname{Tr}\big(2\bar g(f_t^\perp, g^{-1}Sg^{-1}).S\big) + \mathcal{L}_{f_t^\top} \operatorname{Tr}\big(g^{-1}.S\big) - \operatorname{Tr}\big(g^{-1}.(\mathcal{L}_{f_t^\top} S)\big) \\
&\quad - \Delta f_t^\perp + \operatorname{Tr}\big(g^{-1}R^{\bar g}(f_t^\perp, Tf)Tf\big) + \operatorname{Tr}\big(g^{-1}.(\mathcal{L}_{f_t^\top} S)\big) \\
&= \operatorname{Tr}\big(2\bar g(f_t^\perp, g^{-1}.S.g^{-1}).S\big) - \Delta(f_t^\perp) \\
&\quad + \operatorname{Tr}\big(g^{-1}R^{\bar g}(f_t^\perp, Tf)Tf\big) + \mathcal{L}_{f_t^\top} \operatorname{Tr}^g(S)
\end{aligned}$$

Note that by 1.3.3, the formula for the tangential variation would have followed also from the equivariance of the vector valued mean curvature with respect to pullbacks by diffeomorphisms. ☐

1.3.11 Variation of the scalar Weingarten map (codimension one)

Lemma. *The differential of the scalar Weingarten map*

$$\begin{cases} \operatorname{Imm} & \to & \Gamma(\operatorname{End}(TM)), \\ f & \mapsto & L^f \end{cases}$$

is given by

$$D_{(f,f_t)}L = g^{-1}.\nabla^2\big(\bar g(f_t,\nu)\big) + \bar g(f_t,\nu)L^2 + g^{-1}.\bar g\big(R^{\bar g}(f_t^\perp, Tf)Tf\big) + \mathcal{L}_{f_t^\top}(L).$$

Proof. From $L = g^{-1}.s$ follows

$$\begin{aligned}
\partial_t L &= g^{-1}.\partial_t s + \partial_t(g^{-1}).s \\
&= g^{-1}.\Big(\nabla^2(\bar g(f_t,\nu)) - \bar g(f_t,\nu)g.L^2 + \bar g\big(R^{\bar g}(f_t^\perp, Tf)Tf\big) + \mathcal{L}_{f_t^\top}(s)\Big) \\
&\quad + \Big(2\bar g(f_t,\nu)Lg^{-1} + \mathcal{L}_{f_t^\top}(g^{-1})\Big).s \\
&= g^{-1}.\nabla^2\big(\bar g(f_t,\nu)\big) + \bar g(f_t,\nu).L^2 + g^{-1}.\bar g\big(R^{\bar g}(f_t^\perp, Tf)Tf\big) + \mathcal{L}_{f_t^\top}(L).
\end{aligned}$$ ☐

1.3.12 Variation of the scalar mean curvature (codimension one)

Lemma. *The differential of the scalar mean curvature*

$$\begin{cases} \operatorname{Imm} & \to & C^\infty(M), \\ f & \mapsto & \operatorname{Tr}(L^f) \end{cases}$$

is given by

$$\begin{aligned} D_{(f,f_t)} \operatorname{Tr}(L) &= -\Delta\big(\bar{g}(f_t,\nu)\big) + \bar{g}(f_t,\nu).\operatorname{Tr}\left(L^2\right) \\ &\quad + \operatorname{Tr}^g\left(\bar{g}\big(R^{\bar{g}}(f_t^\perp, Tf)Tf\big)\right) + d\big(\operatorname{Tr}(L)\big)(f_t^\top). \end{aligned}$$

Proof. This statement follows from the linearity of the trace operator and from the previous equation for $D_{(f,f_t)}L$. □

1.3.13 Variation of the Gaußcurvature (codimension one)

Lemma. *The differential of the Gaußcurvature*

$$\begin{cases} \mathrm{Imm} & \to & C^\infty(M), \\ f & \mapsto & \det(L) \end{cases}$$

is given by

$$\begin{aligned} D_{(f,f_t)} \det(L) &= \operatorname{Tr}(L).\det(L).\bar{g}(f_t,\nu) + \operatorname{Tr}^g\left(g.\operatorname{C}(L).\nabla^2\big(\bar{g}(f_t,\nu)\big)\right) \\ &\quad + \operatorname{Tr}^g\left(g.C(L).\bar{g}\big(R^{\bar{g}}(f_t^\perp, Tf)Tf\big)\right) + d\det(L)(f_t^\top) \end{aligned}$$

where $C(L)$ is the classical adjoint of L uniquely determined by

$$\operatorname{C}(L).L = L.\operatorname{C}(L) = \det(L).I.$$

Proof. For the normal part we have

$$\begin{aligned} \partial_t \det(L) &= \operatorname{Tr}\big(\operatorname{C}(L).\partial_t L\big) \\ &= \operatorname{Tr}\Big(\operatorname{C}(L).\Big(g^{-1}\nabla^2\big(\bar{g}(f_t,\nu)\big) + \bar{g}(f_t,\nu).L^2 + g^{-1}.\bar{g}(R^{\bar{g}}(\partial_t, Tf)Tf)\Big)\Big) \\ &= \operatorname{Tr}\Big(\operatorname{C}(L).g^{-1}.\nabla^2\bar{g}(f_t,\nu) + \bar{g}(f_t,\nu).\det(L).L \\ &\qquad + C(L)g^{-1}.\bar{g}(R^{\bar{g}}(\partial_t, Tf)Tf)\Big)\Big) \\ &= \operatorname{Tr}(L).\det(L).\bar{g}(f_t,\nu) + \operatorname{Tr}^g\left(g.\operatorname{C}(L).\nabla^2\big(\bar{g}(f_t,\nu)\big)\right) \\ &\qquad + \operatorname{Tr}^g\left(g.C(L).\bar{g}\big(R^{\bar{g}}(f_t^\perp, Tf)Tf\big)\right) \end{aligned}$$

For the tangential part, we use the observation of section 1.3.3. □

1.3.14 Variation of the normal vector field (codimension one)

Lemma. *The normal vector field is a smooth map $\nu : \mathbb{R}\times M \to TN$. Therefore, as explained in section 1.1.7, we can take its covariant derivative along vector fields on $\mathbb{R}\times M$. Identifying ∂_t with the vector field $(\partial_t, 0_M)$ on $\mathbb{R}\times M$, we get*

$$\nabla_{\partial_t}\nu = -Tf.\Big(Lf_t^\top + \operatorname{grad}^g\left(\bar{g}(f_t,\nu)\right)\Big).$$

Proof. $\nabla_{\partial_t}\nu$ is tangential because $\bar g(\nabla_{\partial_t}\nu,\nu) = \frac{1}{2}\partial_t \bar g(\nu,\nu) = 0$. Therefore one can write $\nabla_{\partial_t}\nu = Tf.(\nabla_{\partial_t}\nu)^\top$. Then for all $X \in \mathfrak{X}(M)$ we have

$$g((\nabla_{\partial_t}\nu)^\top, X) = \bar g\big(\nabla_{\partial_t}\nu, Tf.X\big) = 0 - \bar g\big(\nu, \nabla_{\partial_t}Tf.X\big) = -\bar g(\nu, \nabla_X Tf.\partial_t),$$

where in the last step we swapped X and ∂_t as in section 1.1.8 formula (3). Splitting into normal and tangential parts yield:

$$\begin{aligned} g((\nabla_{\partial_t}\nu)^\top, X) &= -\bar g(\nu, \nabla_X f_t) = -\bar g\Big(\nu, \nabla_X(Tf.f_t^\top + \bar g(f_t,\nu).\nu)\Big) \\ &= -\bar g\Big(\nu, \nabla_X(Tf.f_t^\top + \bar g(f_t,\nu).\nu)\Big) \\ &= -s(X, f_t^\top) - \nabla_X\big(\bar g(f_t,\nu)\big) - 0 \\ &= -g\Big(Lf_t^\top + \operatorname{grad}^g \bar g(f_t,\nu), X\Big) \end{aligned}$$

□

1.3.15 Variation of the covariant derivative

In this section, let $\nabla = \nabla^g = \nabla^{f^*\bar g}$ be the Levi-Civita covariant derivative acting on vector fields on M. Since any two covariant derivatives on M differ by a tensor field, the first variation of $\nabla^{f^*\bar g}$ is tensorial. It is given by the tensor field $D_{(f,f_t)}\nabla^{f^*\bar g} \in \Gamma(T^1_2 M)$.

The tensor field $D_{(f,f_t)}\nabla^{f^*\bar g}$ is determined by the following relation holding for vector fields X, Y, Z on M:

$$g\big((D_{(f,f_t)}\nabla)(X,Y), Z\big) = \frac{1}{2}(\nabla D_{(f,f_t)} g)\big(X\otimes Y\otimes Z + Y\otimes X\otimes Z - Z\otimes X\otimes Y\big)$$

Proof. The defining formula for the covariant derivative is

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\Big[&Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &- g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y])\Big]. \end{aligned}$$

Taking the derivative $D_{(f,f_t)}$ yields

$$\begin{aligned} &(D_{(f,f_t)}g)(\nabla_X Y, Z) + g\big((D_{(f,f_t)}\nabla)(X,Y), Z\big) \\ &= \frac{1}{2}\Big[X\big((D_{(f,f_t)}g)(Y,Z)\big) + Y\big((D_{(f,f_t)}g)(Z,X)\big) - Z\big((D_{(f,f_t)}g)(X,Y)\big) \\ &\qquad - (D_{(f,f_t)}g)(X,[Y,Z]) + (D_{(f,f_t)}g)(Y,[Z,X]) + (D_{(f,f_t)}g)(Z,[X,Y])\Big]. \end{aligned}$$

Then the result follows by replacing all Lie brackets in the above formula by covariant derivatives using $[X,Y] = \nabla_X Y - \nabla_Y X$ and by expanding all terms of the form $X\big((D_{(f,f_t}g)(Y,Z)\big)$ using

$$\begin{aligned} &X\big((D_{(f,f_t)}g)(Y,Z)\big) = \\ &\qquad (\nabla_X D_{(f,f_t)}g)(Y,Z) + (D_{(f,f_t)}g)(\nabla_X Y, Z) + (D_{(f,f_t)}g)(Y, \nabla_X Z). \end{aligned}$$

□

1.4 Formulas for second variations

In section 3.3 we will calculate the second derivative of the metric G in a chart. Therefore we will need second variation formulas of the volume form, the pullback metric and the mean curvature.

This section is taken from [6].

1.4.1 Setting for second variations

All formulas for second derivatives will be used in section 3.3.2, where we work in codimension one and with flat ambient space $N = \mathbb{R}^n$, i.e. $R^{\bar{g}}(X, Y)Z = 0$. There we consider a curve of immersions

$$f(t, x) = \exp_{f_0(x)}(t.a(x).\nu^{f_0}(x)) = f_0(x) + t.a(x).\nu^{f_0}(x),$$

for a fixed immersion f_0. This curve of immersions has the property that at $t = 0$ its first derivative and the covariant derivative of the first derivative are both horizontal, i.e.,

$$f|_{t=0} = f_0, \quad \partial_t|_0 f = a.\nu^{f_0}, \quad \text{and} \quad \nabla_{\partial_t} Tf.\partial_t|_{t=0} = 0. \tag{1}$$

Assumption. *In all calculations of second variations we will assume that we have codimension one, a flat ambient space N and that the above properties hold.*

1.4.2 Second variation of the metric

Lemma. *The second derivative of the pullback metric*

$$\left\{ \begin{array}{ccc} \text{Imm} & \to & \Gamma(S^2_{>0} T^*M), \\ f & \mapsto & g = f^*\bar{g} \end{array} \right.$$

along a curve of immersions f satisfying property (1) *from section 1.4.1 is given by*

$$\partial_t^2|_0 f^*\bar{g} = 2(da \otimes da) + 2a^2 g_0 \circ (L^{f_0} \otimes L^{f_0}).$$

Proof. Since $\nabla_{\partial_t} Tf.\partial_t|_0 = 0$, we have

$$\begin{aligned} \partial_t^2|_0 g(X, Y) &= \partial_t^2|_0 \bar{g}(Tf.X, Tf.Y) \\ &= \partial_t|_0 \bar{g}(\nabla_{\partial_t} Tf.X, Tf.Y) + \partial_t|_0 \bar{g}(Tf.X, \nabla_{\partial_t} Tf.Y) \\ &= 2\bar{g}(\nabla_{\partial_t} Tf.X|_0, \nabla_{\partial_t} Tf.Y|_0) + 0 + 0 = 2\bar{g}(\nabla_X Tf.\partial_t, \nabla_Y Tf.\partial_t) \end{aligned}$$

Using $Tf.\partial_t = a.\nu^{f_0}$ we get

$$\begin{aligned} \partial_t^2|_0 (g(X, Y)) &= 2da(X).da(Y) + 2a^2 \bar{g}(\nabla_X \nu^{f_0}, \nabla_Y \nu^{f_0}) \\ &= 2(da \otimes da)(X, Y) + 2a^2.g_0(L^{f_0} X, L^{f_0} Y) \end{aligned}$$ □

1.4.3 Second variation of the inverse metric

Lemma. *The second derivative of the inverse of the pullback metric*

$$\begin{cases} \operatorname{Imm} & \to & \Gamma(L(T^*M, TM)), \\ f & \mapsto & g^{-1} = (f^*\bar{g})^{-1} \end{cases}$$

along a curve of immersions f satisfying property (1) *from section 1.4.1 is given by*

$$\partial_t^2|_0 (f^*\bar{g})^{-1} = 6a^2 (L^{f_0})^2 . g_0^{-1} - 2 g_0^{-1} (da \otimes da) g_0^{-1}.$$

Proof. We look at $g = f^*\bar{g}$ as a bundle map from TM to T^*M. Then

$$\begin{aligned} \partial_t^2|_0 (g^{-1}) &= \partial_t|_0 \big(- g^{-1}.\partial_t g. g^{-1}\big) = 2 g_0^{-1}.\partial_t|_0 g. g_0^{-1}.\partial_t|_0 g. g_0^{-1} - g_0^{-1}.\partial_t^2|_0 g. g_0^{-1} \\ &= 2(-2aL^{f_0})^2 . g_0^{-1} - g_0^{-1}.\big(2(da \otimes da) + 2a^2 g_0 \circ (L^{f_0} \otimes L^{f_0})\big).g_0^{-1} \\ &= 8a^2 (L^{f_0})^2 . g_0^{-1} - 2 g_0^{-1} (da \otimes da) g_0^{-1} - 2a^2 (L^{f_0})^2 . g_0^{-1} \end{aligned}$$

□

1.4.4 Second variation of the volume form

Lemma. *The second derivative of the volume form*

$$\begin{cases} \operatorname{Imm} & \to & \Omega^{n-1}(M), \\ f & \mapsto & \operatorname{vol}(g) = \operatorname{vol}(f^*\bar{g}) \end{cases}$$

along a curve of immersions f satisfying property (1) *from section 1.4.1 is given by*

$$\partial_t^2|_0 \operatorname{vol}(g) = \Big[a^2 \operatorname{Tr}(L^{f_0})^2 - a^2 \operatorname{Tr}\big((L^{f_0})^2\big) + \|da\|_{g^{-1}}^2 \Big] \operatorname{vol}(g_0),$$

Proof. In section 1.3.6 we showed that for any curve of Riemannian metrics $g(t) \in \Gamma(S^2_{>0} T^*M)$, we have

$$\partial_t \operatorname{vol}(g) = \frac{1}{2} \operatorname{Tr}\big(g^{-1}.\partial_t g\big) \operatorname{vol}(g).$$

Therefore

$$\begin{aligned} \partial_t^2 \operatorname{vol}(g) &= \partial_t \frac{1}{2} \operatorname{Tr}\big(g^{-1}.\partial_t g\big) \operatorname{vol}(g) = \frac{1}{2} \operatorname{Tr}\big(\partial_t(g^{-1}).\partial_t g\big) \operatorname{vol}(g) \\ &\quad + \frac{1}{2} \operatorname{Tr}\big(g^{-1}.\partial_t^2 g\big) \operatorname{vol}(g) + \frac{1}{2} \operatorname{Tr}\big(g^{-1}.\partial_t g\big) \partial_t \operatorname{vol}(g) \end{aligned}$$

Evaluating at $t = 0$ and setting $g(t) = f^*\bar{g}$ we get

$$\begin{aligned} \partial_t^2|_0 \operatorname{vol}(g) =& \frac{1}{2} \operatorname{Tr}\big((2aL^{f_0} g_0^{-1}).(-2a.s^{f_0})\big) \operatorname{vol}(g_0) \\ &+ \frac{1}{2} \operatorname{Tr}\big(g_0^{-1}.2(da \otimes da)\big) \operatorname{vol}(g_0) \\ &+ \frac{1}{2} \operatorname{Tr}\big(g_0^{-1}.2a^2 g_0.(L^{f_0})^2\big) \operatorname{vol}(g_0) \\ &+ \frac{1}{2} \operatorname{Tr}\big(g_0^{-1}.(-2a.s^{f_0})\big)\big(-\operatorname{Tr}(L^{f_0}).a\big) \operatorname{vol}(g_0) \\ =& \Big[a^2 \operatorname{Tr}(L^{f_0})^2 - a^2 \operatorname{Tr}\big((L^{f_0})^2\big) + \|da\|_{g^{-1}}^2 \Big] \operatorname{vol}(g_0) \end{aligned}$$

□

1.4.5 Second variation of the second fundamental form

The second derivative of the second fundamental form

$$\begin{cases} \mathrm{Imm} & \to & \Gamma(S^2T^*M), \\ f & \mapsto & s^f \end{cases}$$

along a curve of immersions f satisfying Property (1.4.1.1) is given by

$$\begin{aligned} \partial_t^2|_0 s &= 2(da\otimes da)\big(\mathrm{Id}\otimes L^{f_0} + L^{f_0}\otimes \mathrm{Id}\big) - \|da\|^2_{g_0^{-1}}.s^{f_0} \\ &\quad + 2.a(\nabla_{\mathrm{grad}^{g_0}(a)} s^{f_0}). \end{aligned}$$

Proof. From section 1.3.10 we have

$$\partial_t s(X,Y) = \bar g(\overline{\nabla}^2_{X,Y} Tf.\partial_t, \nu) = \bar g(\overline{\nabla}^2_{X,Y} f_t, \nu).$$

Using $\overline{\nabla}_{\partial_t} f_t = 0$ we get

$$\begin{aligned} \partial_t^2 s(X,Y) &= \bar g(\overline{\nabla}^2_{X,Y} f_t, \overline{\nabla}_{\partial_t}\nu) + \bar g(\overline{\nabla}_{\partial_t}\overline{\nabla}_X\overline{\nabla}_Y f_t - \overline{\nabla}_{\partial_t}\overline{\nabla}_{\nabla_X Y} f_t, \nu) \\ &= \bar g(\overline{\nabla}^2_{X,Y} f_t, \overline{\nabla}_{\partial_t}\nu) + 0 - \bar g(\overline{\nabla}_{\nabla_X Y}\overline{\nabla}_{\partial_t} f_t + \overline{\nabla}_{[\partial_t,\nabla_X Y]} f_t, \nu) \\ &= \bar g(\overline{\nabla}^2_{X,Y} f_t, \overline{\nabla}_{\partial_t}\nu) + 0 - \bar g(\overline{\nabla}_{[\partial_t,\nabla_X Y]} f_t, \nu) \\ &= \bar g(\overline{\nabla}^2_{X,Y} f_t, \overline{\nabla}_{\partial_t}\nu) - \bar g(\overline{\nabla}_{(D_{(f,f_t)}\nabla)(X,Y)} f_t, \nu) \end{aligned}$$

In the last step we used

$$\begin{aligned} [\partial_t, \nabla_X^{f^*\bar g} Y] &= [(\partial_t, 0_M), (0_{\mathbb R}, \nabla_X^{f^*\bar g} Y)] \\ &= \big(0_{\mathbb R}, (D_{(f,f_t)}\nabla)(X,Y)\big) = (D_{(f,f_t)}\nabla)(X,Y). \end{aligned}$$

Evaluating at $t=0$ yields:

$$\begin{aligned} &\partial_t^2|_0 s(X,Y) = \\ &\quad = \bar g(\overline{\nabla}^2_{X,Y}(a.\nu^{f_0}), -Tf_0.\,\mathrm{grad}^{g_0} a) - \bar g(\overline{\nabla}_{(D_{(f_0,a.\nu^{f_0})}\nabla)(X,Y)}(a.\nu^{f_0}), \nu^{f_0}) \\ &\quad = 0 + \bar g(da(X).\overline{\nabla}_Y \nu^{f_0} + da(Y).\overline{\nabla}_X \nu^{f_0}, -Tf_0.\,\mathrm{grad}^{g_0} a) \\ &\qquad + \bar g(a.\overline{\nabla}^2_{X,Y}(\nu^{f_0}), -Tf_0.\,\mathrm{grad}^{g_0} a) - da\big((D_{(f_0,a.\nu^{f_0})}\nabla)(X,Y)\big) + 0. \end{aligned}$$

We will treat the three terms separately. The first one gives, using $\overline{\nabla}_Z\nu = -Tf.L.Z$:

$$\begin{aligned} &\bar g(da(X).\overline{\nabla}_Y \nu^{f_0} + da(Y).\overline{\nabla}_X \nu^{f_0}, -Tf_0.\,\mathrm{grad}^{g_0} a) = \\ &\quad = g_0(da(X)L^{f_0}Y + da(Y)L^{f_0}X, \mathrm{grad}^{g_0} a) \\ &\quad = da(X).da(L^{f_0}Y) + da(Y).da(L^{f_0}X). \end{aligned}$$

For the second term we get:

$$\begin{aligned} &\bar g(a.\overline{\nabla}^2_{X,Y}(\nu^{f_0}), -Tf_0.\,\mathrm{grad}^{g_0} a) = -a\bar g(\overline{\nabla}_X\overline{\nabla}_Y \nu^{f_0} - \overline{\nabla}_{\nabla_X Y}\nu^{f_0}, Tf_0.\,\mathrm{grad}^{g_0} a) \\ &\quad = -a\bar g(-\overline{\nabla}_X(Tf_0 L^{f_0} Y) + Tf_0 L^{f_0}\nabla_X Y, Tf_0.\,\mathrm{grad}^{g_0} a) \end{aligned}$$

$$
\begin{aligned}
&= -a\bar{g}(-(\overline{\nabla}Tf_0)(X, L^{f_0}Y) - Tf_0\nabla_X(L^{f_0}Y) + Tf_0L^{f_0}\nabla_XY, Tf_0.\operatorname{grad}^{g_0} a) \\
&= 0 + a\bar{g}(Tf_0(\nabla_X L^{f_0})(Y), Tf_0.\operatorname{grad}^{g_0} a) = ag_0\big((\nabla_X L^{f_0})(Y), \operatorname{grad}^{g_0} a\big) \\
&= a\nabla_X\big(g_0(L^{f_0}Y, \operatorname{grad}^{g_0} a)\big) - ag_0(L^{f_0}\nabla_XY, \operatorname{grad}^{g_0} a) \\
&\qquad - ag_0(L^{f_0}Y, \nabla_X \operatorname{grad}^{g_0} a) \\
&= a\nabla_X\big(s^{f_0}(Y, \operatorname{grad}^{g_0} a)\big) - as^{f_0}(\nabla_XY, \operatorname{grad}^{g_0} a) - as^{f_0}(Y, \nabla_X \operatorname{grad}^{g_0} a) \\
&= a(\nabla_X s)(Y, \operatorname{grad}^{g_0} a)
\end{aligned}
$$

$\overline{\nabla}^2_{X,Y}\nu$ is symmetric in X, Y because the ambient space $\mathbb{R}^n$ is flat. Therefore the last formula and the symmetry of s imply that

$$
a(\nabla_X s)(Y, \operatorname{grad}^{g_0} a) = a(\nabla_Y s)(X, \operatorname{grad}^{g_0} a) = a(\nabla_{\operatorname{grad}^{g_0} a} s)(X, Y).
$$

The third term yields, using formula 1.3.15:

$$
\begin{aligned}
&- g_0\big((D_{(f,a.\nu^{f_0})}\nabla)(X,Y), \operatorname{grad}^{g_0}(a)\big) = \\
&\quad = -\frac{1}{2}(\nabla(-2a.s^{f_0}))(X, Y, \operatorname{grad}^{g_0}(a)) - \frac{1}{2}(\nabla(-2a.s^{f_0}))(Y, X, \operatorname{grad}^{g_0}(a)) \\
&\qquad + \frac{1}{2}(\nabla(-2a.s^{f_0}))(\operatorname{grad}^{g_0}(a), X, Y) \\
&\quad = da(X).s^{f_0}(Y, \operatorname{grad}^{g_0}(a)) + a.(\nabla_X s^{f_0})(Y, \operatorname{grad}^{g_0}(a)) \\
&\qquad + da(Y).s^{f_0}(X, \operatorname{grad}^{g_0}(a)) + a.(\nabla_Y s^{f_0})(X, \operatorname{grad}^{g_0}(a)) \\
&\qquad - da(\operatorname{grad}^{g_0}(a)).s^{f_0}(X, Y) - a.(\nabla_{\operatorname{grad}^{g_0}(a)} s^{f_0})(X, Y) \\
&\quad = da(X).da(L^{f_0}Y) + da(Y).da(L^{f_0}X) \\
&\qquad - \|da\|^2_{g^{-1}}.s^{f_0}(X, Y) + a.(\nabla_{\operatorname{grad}^{g_0}(a)} s^{f_0})(X, Y).
\end{aligned}
$$

□

1.4.6 Second variation of the mean curvature

The second derivative of the mean curvature

$$
\begin{cases} \operatorname{Imm} & \to \quad C^\infty(M), \\ \quad f & \mapsto \quad \operatorname{Tr}(L^f) \end{cases}
$$

along a curve of immersions f satisfying Property (1.4.1.1) is given by

$$
\begin{aligned}
\partial_t^2|_0 \operatorname{Tr}(L) &= 2a^2 \operatorname{Tr}\big((L^{f_0})^3\big) + 4a \operatorname{Tr}\big(L^{f_0}g_0^{-1}.\nabla^2 a\big) + 2\operatorname{Tr}(g^{-1}(da \otimes da)L^{f_0}) \\
&\quad - \|da\|^2_{g_0^{-1}} \operatorname{Tr}(L^{f_0}) + 2a \operatorname{Tr}^{g_0}\big(\nabla_{\operatorname{grad}^{g_0} a} s^{f_0}\big)
\end{aligned}
$$

Proof. From $\operatorname{Tr}(L) = \operatorname{Tr}(g^{-1}.s)$ we get

$$
\partial_t^2 \operatorname{Tr}(L) = \operatorname{Tr}\big(\partial_t^2(g^{-1}).s\big) + 2\operatorname{Tr}\big(\partial_t(g^{-1}).\partial_t s\big) + \operatorname{Tr}\big(g^{-1}.\partial_t^2 s\big)
$$

Evaluating at $t = 0$ we get

$$
\partial_t^2|_0 \operatorname{Tr}(L) = \operatorname{Tr}\big(6a^2(L^{f_0})^2.g_0^{-1}.s^{f_0}\big) + \operatorname{Tr}\big(-2g_0^{-1}.(da \otimes da).g_0^{-1}.s^{f_0}\big)
$$

$$
\begin{aligned}
&+ 2\operatorname{Tr}\left(2aL^{f_0}g_0^{-1}.\nabla^2 a\right) + 2\operatorname{Tr}\left(2aL^{f_0}g_0^{-1}.(-ag_0(L^{f_0})^2)\right)\\
&+ 2.\operatorname{Tr}\left(g_0^{-1}.((da\otimes da\circ L^{f_0}) + (da\circ L^{f_0}\otimes da)\right)\\
&- \|da\|^2_{g_0^{-1}}\operatorname{Tr}(L^{f_0}) + 2a\operatorname{Tr}^{g_0}\left(\nabla_{\mathrm{grad}^{g_0} a}s^{f_0}\right)\\
={}& 2a^2\operatorname{Tr}\left((L^{f_0})^3\right) - 2\operatorname{Tr}\left(g_0^{-1}.(da\otimes da).L^{f_0}\right)\\
&+ 4a\operatorname{Tr}\left(L^{f_0}g_0^{-1}.\nabla^2 a\right) + 4\operatorname{Tr}\left(g_0^{-1}.(da\otimes da).(L^{f_0})\right)\\
&- \|da\|^2_{g_0^{-1}}\operatorname{Tr}(L^{f_0}) + 2a\operatorname{Tr}^{g_0}\left(\nabla_{\mathrm{grad}^{g_0} a}s^{f_0}\right) \qquad \square
\end{aligned}
$$

Chapter 2

Surfaces

2.1 Almost local metrics

The G^0 metric is the simplest metric on $\mathrm{Imm}(M,N)$. It is given by the following formula:

$$G^0_f(h,k) = \int_M \bar{g}(h,k)\,\mathrm{vol}(g).$$

This metric is well studied, see for example [35]. Unfortunately it induces vanishing geodesic distance on shape space, see [35, 36] or section 2.4.5. There are different ways to strengthen the metric. In [7] we incorporated a differential operator in the definition of the metric, whereas in this work we will study metrics of the form

$$G^\Phi_f(h,k) = \int_M \Phi(f).\bar{g}(h(x),k(x))\,\mathrm{vol}(g)(x),$$

where Φ is a $\mathrm{Diff}(M)$-invariant function depending on the immersion f and possibly on x. These metrics are called *almost local* metrics. This definition includes as an important special case *conformal versions* of the G^0 metric, i.e. metrics of the form

$$G^\Phi_f(h,k) = \Phi(f)\int_M \bar{g}(h,k).\,\mathrm{vol}(g),$$

where Φ is again some $\mathrm{Diff}(M)$-invariant function depending on the immersion f but not on x. Conformal metrics have been studied in [41, 30].

Assumption. *In this chapter we will consider functions Φ depending on the volume and the mean curvature, i.e.*

$$\Phi = \Phi\big(\,\mathrm{Vol}, \|\mathrm{Tr}^g(S)(x)\|^2_{\bar{g}}\,\big).$$

We will calculate the geodesic equation on both, Imm and B_i. In section 2.4 we will state some conditions for Φ ensuring that the induced geodesic distance on shape space is non-vanishing.

For this class of weight functions, some work has already been done by [37] for the special case of immersions of the unit circle in the plane. The special case of hypersurfaces in n-space has been studied in [6], see also chapter 3.

2.2 The geodesic equation on the manifold of immersions

We use the method of section 1.2.8 and section 1.2.9 to calculate the geodesic equation. So we need to compute the metric gradients. The calculation at the same time shows the existence of the gradients. For vector fields m, h, k on Imm one has

$$\begin{aligned}
(\nabla^{\bar g}_m G^\Phi)(h,k) &= D_{(f,m)} \int_M \Phi.\bar g(h,k)\operatorname{vol}(g) - \int_M \Phi.\bar g\big(\nabla^{\bar g}_m h, k\big)\operatorname{vol}(g)\\
&\qquad - \int_M \Phi.\bar g(h, \nabla^{\bar g}_m k)\operatorname{vol}(g)\\
&= \int_M (D_{(f,m)}\Phi)\bar g(h,k)\operatorname{vol}(g) + \int_M \Phi\bar g\big(\nabla^{\bar g}_m h,k\big)\operatorname{vol}(g)\\
&\qquad + \int_M \Phi\bar g(h,\nabla^{\bar g}_m k)\operatorname{vol}(g) + \int_M \Phi\bar g(h,k) D_{(f,m)}\operatorname{vol}(g)\\
&\qquad - \int_M \Phi\bar g\big(\nabla^{\bar g}_m h,k\big)\operatorname{vol}(g) - \int_M \Phi\bar g(h,\nabla^{\bar g}_m k)\operatorname{vol}(g)\\
&= \int_M (D_{(f,m)}\Phi)\bar g(h,k)\operatorname{vol}(g) + \int_M \Phi\bar g(h,k) D_{(f,m)}\operatorname{vol}(g)\\
&= \int_M (\partial_1\Phi).(D_{(f,m)}\operatorname{Vol}).\bar g(h,k)\operatorname{vol}(g)\\
&\qquad + \int_M (\partial_2\Phi).D_{(f,m)}\left\|\operatorname{Tr}^g(S)\right\|^2_{\bar g}.\bar g(h,k)\operatorname{vol}(g)\\
&\qquad + \int_M \Phi.\bar g(h,k).(D_{(f,m)}\operatorname{vol}(g)).
\end{aligned}$$

To read off the K-gradient of the metric, we write this expression as

$$\begin{aligned}
\int_M \Phi.\bar g\bigg(\Big[\frac{\partial_1\Phi}{\Phi}(D_{(f,m)}\operatorname{Vol}) + 2.\frac{\partial_2\Phi}{\Phi}\bar g\Big(D_{(f,m)}\operatorname{Tr}^g(S), \operatorname{Tr}^g(S)\Big)&\\
+ \frac{D_{(f,m)}\operatorname{vol}(g)}{\operatorname{vol}(g)}\Big]h,k\bigg)\operatorname{vol}(g)&
\end{aligned}$$

Therefore, using the formulas from section 1.3 we can calculate the K gradient:

$$\boxed{\begin{aligned}
&K_f(m,h) = \\
&= \Big[\frac{\partial_1\Phi}{\Phi}(D_{(f,m)}\operatorname{Vol}) + 2.\frac{\partial_2\Phi}{\Phi}\bar g\Big(D_{(f,m)}\operatorname{Tr}^g(S), \operatorname{Tr}^g(S)\Big) + \frac{D_{(f,m)}\operatorname{vol}(g)}{\operatorname{vol}(g)}\Big]h \\
&= \Big[\frac{\partial_1\Phi}{\Phi}\Big(\int_M -\bar g\big(m^\perp, \operatorname{Tr}^g(S)\big)\operatorname{vol}(g)\Big) \\
&\qquad + 2.\frac{\partial_2\Phi}{\Phi}\Big(\bar g\Big(2\operatorname{Tr}\big(\bar g(m^\perp, g^{-1}.S.g^{-1}).S\big), \operatorname{Tr}^g(S)\Big) - \bar g\big(\Delta(m^\perp), \operatorname{Tr}^g(S)\big) \\
&\qquad\qquad + \bar g(\operatorname{Tr}^g(R^{\bar g}(m^\perp, Tf).Tf), \operatorname{Tr}^g(S)) + \bar g\big(\mathcal{L}_{m^\top}\operatorname{Tr}^g(S), \operatorname{Tr}^g(S)\big)\Big) \\
&\qquad + \operatorname{div}^g(m^\top) - \bar g\big(m^\perp, \operatorname{Tr}^g(S)\big)\Big]h.
\end{aligned}}$$

To calculate the H-gradient, we treat the four summands of $D_{(f,m)}G_f^\Phi(h,k)$ separately. The first summand is

$$\begin{aligned}
&\int_M (\partial_1\Phi)(D_{(f,m)}\operatorname{Vol}(x))\bar g(h(x),k(x))\operatorname{vol}(g)(x) \\
&\quad = -\int_{x\in M}(\partial_1\Phi)\int_{y\in M}\bar g\big(m^\perp(y), \operatorname{Tr}^g(S)(y)\big)\operatorname{vol}(g)(y)\bar g(h(x),k(x))\operatorname{vol}(g)(x) \\
&\quad = \int_{y\in M}\bar g\Big(m^\perp(y), -\operatorname{Tr}^g(S)(y)\int_{x\in M}(\partial_1\Phi).\bar g(h(x),k(x))\operatorname{vol}(g)(x)\Big)\operatorname{vol}(g)(y) \\
&\quad = G_f^\Phi\left(m, -\frac{1}{\Phi}\operatorname{Tr}^g(S)\int_M(\partial_1\Phi).\bar g(h,k)\operatorname{vol}(g)\right).
\end{aligned}$$

In the calculation of the second term we will make use of the selfadjointness of the Laplacian, i.e. for any tensor fields $B, C \in T_s^r(M)$ we have

$$\int_M g_s^r(\Delta B, C)\operatorname{vol}(g) = \int_M g_s^r(\nabla^*\nabla B, C)\operatorname{vol}(g) = \int_M g_s^r(B, \Delta C)\operatorname{vol}(g),$$

of the following Leibnitz rule for the derivative of the mean curvature:

$$\begin{aligned}
&\bar g\Big(\mathcal{L}_{m^\top}\operatorname{Tr}^g(S), \operatorname{Tr}^g(S)\Big) = \\
&\qquad = \frac{1}{2}\Big(\bar g\big(\mathcal{L}_{m^\top}\operatorname{Tr}^g(S), \operatorname{Tr}^g(S)\big) + \bar g\big(\operatorname{Tr}^g(S), \mathcal{L}_{m^\top}\operatorname{Tr}^g(S)\big)\Big) \\
&\qquad = \frac{1}{2}\mathcal{L}_{m^\top}\bar g\big(\operatorname{Tr}^g(S), \operatorname{Tr}^g(S)\big) = \frac{1}{2}d\left\|\operatorname{Tr}^g(S)\right\|_{\bar g}^2(m^\top),
\end{aligned}$$

and of a symmetry property of the curvature tensor (see [34, 24.4.4]):

$$\bar g(R^{\bar g}(X,Y)Z,U) = -\bar g(R^{\bar g}(Y,X)Z,U) = -\bar g(R^{\bar g}(Z,U)Y,X).$$

Thus the second summand is given by

$$\begin{aligned}
&\int_M(\partial_2\Phi).D_{(f,m)}\left\|\operatorname{Tr}^g(S)\right\|_{\bar g}^2.\bar g(h,k)\operatorname{vol}(g) = \\
&\qquad = 2\int_M(\partial_2\Phi).\bar g\Big(2\operatorname{Tr}\big(\bar g(m^\perp, g^{-1}.S.g^{-1}).S\big), \operatorname{Tr}^g(S)\Big).\bar g(h,k)\operatorname{vol}(g)
\end{aligned}$$

$$
\begin{aligned}
&- 2\int_M (\partial_2\Phi).(\bar g\otimes g_0^0)\big(\Delta(m^\perp),\operatorname{Tr}^g(S)\big).\bar g(h,k)\operatorname{vol}(g)\\
&+ 2\int_M (\partial_2\Phi)\bar g\Big(\operatorname{Tr}^g(R^{\bar g}(m^\perp,Tf)Tf),\operatorname{Tr}^g(S)\Big)\bar g(h,k)\operatorname{vol}(g)\\
&+ 2.\int_M (\partial_2\Phi)\frac{1}{2}d\left\|\operatorname{Tr}^g(S)\right\|^2_{\bar g}(m^\top).\bar g(h,k)\operatorname{vol}(g)\\
= {}& 4\int_M (\partial_2\Phi).\operatorname{Tr}\Big(\bar g(m^\perp,g^{-1}.S.g^{-1}).\bar g\big(S,\operatorname{Tr}^g(S)\big)\Big)\bar g(h,k)\operatorname{vol}(g)\\
&- 2\int_M (\partial_2\Phi).(\bar g\otimes g_0^0)\Big(\Delta(m^\perp),\operatorname{Tr}^g(S)\Big)\bar g(h,k)\operatorname{vol}(g)\\
&+ 2\int_M (\partial_2\Phi)\operatorname{Tr}^g\Big(\bar g\big(R^{\bar g}(m^\perp,Tf)Tf,\operatorname{Tr}^g(S)\big)\Big)\bar g(h,k)\operatorname{vol}(g)\\
&+ \int_M (\partial_2\Phi).g\Big(\operatorname{grad}^g\left\|\operatorname{Tr}^g(S)\right\|^2_{\bar g},m^\top\Big).\bar g(h,k)\operatorname{vol}(g)\\
= {}& 4\int_M (\partial_2\Phi).\bar g\bigg(m^\perp,\operatorname{Tr}\Big(g^{-1}.S.g^{-1}.\bar g\big(S,\operatorname{Tr}^g(S)\big).\bar g(h,k)\Big)\bigg)\operatorname{vol}(g)\\
&- 2\int_M (\bar g\otimes g_0^0)\Big(m^\perp,\Delta\big((\partial_2\Phi).\operatorname{Tr}^g(S).\bar g(h,k)\big)\Big)\operatorname{vol}(g)\\
&- 2\int_M (\partial_2\Phi)\operatorname{Tr}^g\Big(\bar g\big(R^{\bar g}(Tf,\operatorname{Tr}^g(S))Tf,m^\perp\big)\Big)\bar g(h,k)\operatorname{vol}(g)\\
&+ \int_M (\partial_2\Phi).\bar g\Big(Tf.\operatorname{grad}^g\left\|\operatorname{Tr}^g(S)\right\|^2_{\bar g},Tf.m^\top\Big).\bar g(h,k)\operatorname{vol}(g)\\
= {}& G_f^\Phi\bigg(m^\perp,4.\frac{\partial_2\Phi}{\Phi}\operatorname{Tr}\Big(g^{-1}.S.g^{-1}.\bar g\big(S,\operatorname{Tr}^g(S)\big).\bar g(h,k)\Big)\bigg)\\
&- G_f^\Phi\bigg(m^\perp,\frac{2}{\Phi}\Delta\Big((\partial_2\Phi).\operatorname{Tr}^g(S).\bar g(h,k)\Big)\bigg)\\
&- 2\int_M \bar g\Big(\bar g(h,k)(\partial_2\Phi)\operatorname{Tr}^g\big(R^{\bar g}(Tf,\operatorname{Tr}^g(S))Tf\big),m^\perp\Big)\operatorname{vol}(g)\\
&+ \int_M (\partial_2\Phi).\bar g\Big(m,Tf.\bar g(h,k).\operatorname{grad}^g\left\|\operatorname{Tr}^g(S)\right\|^2_{\bar g}\Big)\operatorname{vol}(g)\\
= {}& G_f^\Phi\bigg(m,4.\frac{\partial_2\Phi}{\Phi}\operatorname{Tr}\Big(g^{-1}.S.g^{-1}.\bar g\big(S,\operatorname{Tr}^g(S)\big).\bar g(h,k)\Big)\bigg)\\
&- G_f^\Phi\bigg(m,\frac{2}{\Phi}\Big(\Delta\Big((\partial_2\Phi).\operatorname{Tr}^g(S).\bar g(h,k)\Big)\Big)^\perp\bigg)\\
&- G_f^\Phi\bigg(m,2.\frac{\partial_2\Phi}{\Phi}\bar g(h,k)\operatorname{Tr}^g\Big(R^{\bar g}(Tf,\operatorname{Tr}^g(S))Tf\Big)^\perp\bigg)\\
&+ G_f^\Phi\bigg(m,\frac{\partial_2\Phi}{\Phi}Tf.\bar g(h,k).\operatorname{grad}^g\left\|\operatorname{Tr}^g(S)\right\|^2_{\bar g}\bigg)
\end{aligned}
$$

In the calculation of the last term of the $H_f(m,h)$ gradient, we will make use of the following formula, which is valid for $\phi\in C^\infty(M)$ and $X\in\mathfrak{X}(M)$:

$$
\begin{aligned}
0 &= \int_M \operatorname{div}(\phi.X).\operatorname{vol}(g) = \int_M \mathcal{L}_{\phi.X}\operatorname{vol}(g)\\
&= \int_M (d\circ i_{\phi.X}+i_{\phi.X}\circ d)\operatorname{vol}(g) = \int_M d(\phi.i_X\operatorname{vol}(g))
\end{aligned}
$$

$$
\begin{aligned}
&= \int_M d\phi \wedge i_X \operatorname{vol}(g) + \int_M \phi \wedge d(i_X \operatorname{vol}(g)) \\
&= \int_M \big(- i_X(d\phi \wedge \operatorname{vol}(g)) + i_X \circ d\phi \wedge \operatorname{vol}(g)\big) + \int_M \phi.\mathcal{L}_X \operatorname{vol}(g) \\
&= \int_M d\phi(X) \operatorname{vol}(g) + \int_M \phi. \operatorname{div}(X) \operatorname{vol}(g).
\end{aligned}
$$

Therefore we can calculate the last summand, which is given by

$$
\begin{aligned}
&\int_M \Phi.\bar{g}(h,k)(D_{(f,m)} \operatorname{vol}(g)) = \\
&\quad = \int_M \Phi.\bar{g}(h,k)\Big(\operatorname{div}^g(m^\top) - \bar{g}\big(m^\perp, \operatorname{Tr}^g(S)\big)\Big) \operatorname{vol}(g) \\
&\quad = \int_M -\Big(d\big(\Phi.\bar{g}(h,k)\big)(m^\top) + \Phi.\bar{g}\big(m^\perp, \bar{g}(h,k). \operatorname{Tr}^g(S)\big)\Big) \operatorname{vol}(g) \\
&\quad = \int_M -\bar{g}\Big(Tf. \operatorname{grad}^g\big(\Phi.\bar{g}(h,k)\big), Tf.m^\top\Big) \operatorname{vol}(g) - G_f^\Phi\Big(m^\perp, \bar{g}(h,k) \operatorname{Tr}^g(S)\Big) \\
&\quad = G_f^\Phi\Big(m, -\frac{1}{\Phi} Tf. \operatorname{grad}^g\big(\Phi.\bar{g}(h,k)\big) - \bar{g}(h,k) \operatorname{Tr}^g(S)\Big)
\end{aligned}
$$

Summing up all the terms the H-gradient is given by

$$
\boxed{
\begin{aligned}
H_f(h,k) = &-\frac{1}{\Phi} \operatorname{Tr}^g(S) \int_M (\partial_1 \Phi).\bar{g}(h,k) \operatorname{vol}(g) \\
&+ 4.\frac{\partial_2 \Phi}{\Phi} \operatorname{Tr}\Big(g^{-1}.S.g^{-1}.\bar{g}\big(S, \operatorname{Tr}^g(S)\big).\bar{g}(h,k)\Big) \\
&- \frac{2}{\Phi}\Big(\Delta\Big((\partial_2\Phi). \operatorname{Tr}^g(S).\bar{g}(h,k)\Big)\Big)^\perp \\
&- 2.\frac{\partial_2 \Phi}{\Phi} \bar{g}(h,k) \operatorname{Tr}^g\Big(R^{\bar{g}}\big(Tf, \operatorname{Tr}^g(S)\big)Tf\Big)^\perp \\
&+ \frac{\partial_2 \Phi}{\Phi} Tf.\bar{g}(h,k). \operatorname{grad}^g \|\operatorname{Tr}^g(S)\|_{\bar{g}}^2 \\
&- \frac{1}{\Phi} Tf. \operatorname{grad}^g\big(\Phi.\bar{g}(h,k)\big) - \bar{g}(h,k) \operatorname{Tr}^g(S)
\end{aligned}
}
$$

Using the formulas from section 1.2.8 and section 1.2.9 leads:

Theorem. *The geodesic equation for an almost local metric G^Φ depending on* *volume and mean curvature on* $\operatorname{Imm}(M,N)$ *is given by*

$$
\boxed{
\begin{aligned}
\nabla_{\partial_t} f_t &= \frac{1}{2} H_f(f_t,f_t) - K_f(f_t,f_t)\\
&= -\frac{1}{2.\Phi}\operatorname{Tr}^g(S)\int_M (\partial_1\Phi).\,\|f_t\|^2_{\bar g}\operatorname{vol}(g)\\
&\quad + 2.\frac{\partial_2\Phi}{\Phi}\operatorname{Tr}\Big(g^{-1}.S.g^{-1}.\bar g\big(S,\operatorname{Tr}^g(S)\big).\,\|f_t\|^2_{\bar g}\Big)\\
&\quad - \frac{1}{\Phi}\Big(\Delta\Big((\partial_2\Phi).\operatorname{Tr}^g(S).\,\|f_t\|^2_{\bar g}\Big)\Big)^\perp\\
&\quad - \frac{\partial_2\Phi}{\Phi}\|f_t\|^2_{\bar g}\operatorname{Tr}^g\Big(R^{\bar g}\big(Tf,\operatorname{Tr}^g(S)\big)Tf\Big)^\perp\\
&\quad + \frac{\partial_2\Phi}{2.\Phi}Tf.\,\|f_t\|^2_{\bar g}.\operatorname{grad}^g\|\operatorname{Tr}^g(S)\|^2_{\bar g} - \frac{1}{2.\Phi}Tf.\operatorname{grad}^g\big(\Phi.\,\|f_t\|^2_{\bar g}\big)\\
&\quad - \frac{1}{2}\|f_t\|^2_{\bar g}\operatorname{Tr}^g(S) - \bigg[\frac{\partial_1\Phi}{\Phi}\Big(\int_M -\bar g\big(f_t^\perp,\operatorname{Tr}^g(S)\big)\operatorname{vol}(g)\Big)\\
&\quad + 2.\frac{\partial_2\Phi}{\Phi}\bigg(\bar g\Big(2\operatorname{Tr}\big(\bar g(f_t^\perp,g^{-1}.S.g^{-1}).S\big),\operatorname{Tr}^g(S)\Big)\\
&\qquad - \bar g\big(\Delta(f_t^\perp),\operatorname{Tr}^g(S)\big) + \bar g\Big(\operatorname{Tr}^g\big(R^{\bar g}(f_t^\perp,Tf)Tf\big),\operatorname{Tr}^g(S)\Big)\\
&\qquad + \bar g\big(\mathcal{L}_{f_t^\top}\operatorname{Tr}^g(S),\operatorname{Tr}^g(S)\big)\bigg)\\
&\quad + \operatorname{div}^g(f_t^\top) - \bar g\big(f_t^\perp,\operatorname{Tr}^g(S)\big)\bigg]f_t
\end{aligned}
}
$$

According to section 1.2.9 we can rewrite this equation as an equation for the momentum p:

$$
\boxed{
\begin{aligned}
p &= \Phi(\operatorname{Vol},\operatorname{Tr}(L))f_t\otimes\operatorname{vol}(g)\\
\nabla_{\partial_t}p &= -\frac{1}{2}\operatorname{Tr}^g(S)\int_M(\partial_1\Phi).\,\|f_t\|^2_{\bar g}\operatorname{vol}(g)\otimes\operatorname{vol}(g)\\
&\quad + 2.(\partial_2\Phi)\operatorname{Tr}\Big(g^{-1}.S.g^{-1}.\bar g\big(S,\operatorname{Tr}^g(S)\big).\,\|f_t\|^2_{\bar g}\Big)\otimes\operatorname{vol}(g)\\
&\quad - \Big(\Delta\Big((\partial_2\Phi).\operatorname{Tr}^g(S).\,\|f_t\|^2_{\bar g}\Big)\Big)^\perp\otimes\operatorname{vol}(g)\\
&\quad - (\partial_2\Phi)\,\|f_t\|^2_{\bar g}\operatorname{Tr}^g\Big(R^{\bar g}\big(Tf,\operatorname{Tr}^g(S)\big)Tf\Big)^\perp\otimes\operatorname{vol}(g)\\
&\quad + \frac{\partial_2\Phi}{2}Tf.\,\|f_t\|^2_{\bar g}.\operatorname{grad}^g\|\operatorname{Tr}^g(S)\|^2_{\bar g}\otimes\operatorname{vol}(g)\\
&\quad - \frac{1}{2}Tf.\operatorname{grad}^g\big(\Phi.\,\|f_t\|^2_{\bar g}\big)\otimes\operatorname{vol}(g) - \frac{\Phi}{2}\|f_t\|^2_{\bar g}\operatorname{Tr}^g(S)\otimes\operatorname{vol}(g)
\end{aligned}
}
$$

2.2.1 Momentum mappings

The metric G^Φ is invariant under the action of the reparametrization group $\operatorname{Diff}(M)$. According to section 1.2.10 the momentum mapping for this group action is constant along any geodesic in $\mathrm{Imm}(M,N)$:

$$\boxed{\begin{array}{rr} \forall X \in \mathfrak{X}(M): \displaystyle\int_M \Phi\big(\mathrm{Vol(f)}, \|\mathrm{Tr}^g(S)\|^2_{\bar g}\big)\bar g(Tf.X, f_t)\,\mathrm{vol}(g) & \text{rep. mom.} \\ \text{or } \Phi\big(\mathrm{Vol(f)}, \|\mathrm{Tr}^g(S)\|^2_{\bar g}\big) g(f_t^\top)\,\mathrm{vol}(g) \in \Gamma(T^*M \otimes_M \mathrm{vol}(M)) & \text{rep. mom.} \end{array}}$$

For a flat ambient space $N = \mathbb{R}^n$ the metric G^Φ is in addition invariant under the action of the Euclidean motion group $\mathbb{R}^n \rtimes \mathrm{SO}(n)$. This yields the following conserved quantities:

$$\boxed{\begin{array}{rr} \displaystyle\int_M \Phi\big(\mathrm{Vol(f)}, \|\mathrm{Tr}^g(S)\|^2_{\bar g}\big) f_t\,\mathrm{vol}(g) & \text{lin. mom.} \\ \forall X \in \mathfrak{so}(n): \displaystyle\int_M \Phi\big(\mathrm{Vol(f)}, \|\mathrm{Tr}^g(S)\|^2_{\bar g}\big)\bar g(X.f, f_t)\,\mathrm{vol}(g) & \text{ang. mom.} \\ \text{or } \displaystyle\int_M \Phi\big(\mathrm{Vol(f)}, \|\mathrm{Tr}^g(S)\|^2_{\bar g}\big)(f \wedge f_t)\,\mathrm{vol}(g) \in \textstyle\bigwedge^2\mathbb{R}^n \cong \mathfrak{so}(n)^* & \text{ang. mom.} \end{array}}$$

2.3 The geodesic equation on shape space

2.3.1 The horizontal bundle

Since $\mathrm{vol}(f^*\bar g)$ and $\mathrm{Tr}^g(S)$ react equivariantly to the action of the group $\mathrm{Diff}(M)$, every G^Φ-metric is $\mathrm{Diff}(M)$-invariant. As described in Section 1.2.13 it induces a Riemannian metric on B_i (off the singularities) such that the projection $\pi : \mathrm{Imm} \to B_i$ is a Riemannian submersion. The restriction to almost local metrics is very beneficial, namely:

Lemma 1. *For an almost local metric G^Φ the horizontal bundle at the point f equals the set of sections of the normal bundle (see section 1.1.12) along f.*

Proof. By definition, a tangent vector h to $f \in \mathrm{Imm}(M,N)$ is horizontal if and only if it is G^Φ-perpendicular to the $\mathrm{Diff}(M)$-orbits. This is the case if and only if $\bar g(h(x), T_x f.X_x) = 0$ at every point $x \in M$. □

According to section 1.2.14 the calculation of the geodesic equation can be done on the horizontal bundle instead of on B_i assuming that every path in B_i corresponds to exactly one horizontal path in Imm. The following lemma shows that this assumptions is satisfied.

Lemma 2. *For any smooth path f in $\mathrm{Imm}(M,N)$ there exists a smooth path φ in $\mathrm{Diff}(M)$ with $\varphi(t,\ ) = \mathrm{Id}_M$ depending smoothly on f such that the path $f(t,\varphi(t,x))$ is horizontal, i.e. $\partial_t f(t,\varphi(t,x))$ lies in the horizontal bundle.*

Proof. The proof is taken from [35, Section 2.5]. The basic idea is to write the path φ as the integral curve of a time dependent vector field. This method is called the Moser-Trick.

In the following we will write $f \circ \varphi$ for the map $f(t, \varphi(t,x))$, etc. We look for φ as the integral curve to the time dependent vector field $\xi(t,x)$ on M, given by

$$\partial_t \varphi = \xi \circ \varphi$$

We want the following expression to vanish for all $x \in M$ and $X_x \in T_x M$:

$$\begin{aligned} \bar{g}(\partial_t(f \circ \varphi)(x), T(f \circ \varphi).X_x) &= \bar{g}((\partial_t f)(\varphi(x)) + Tf.(\partial_t \varphi)(x), Tf \circ T\varphi.X) \\ &= \bar{g}((\partial_t f)(\varphi(x)) + Tf.\xi(\varphi(x)), Tf \circ T\varphi.X) \end{aligned}$$

Since $T\varphi$ is surjective, $T\varphi.X$ exhausts the tangent space $T_{\varphi(x)}M$, and we have

$$(\partial_t f)(\varphi(x)) + Tf.\xi(\varphi(x)) \quad \perp \quad f.$$

This holds for all $x \in M$, and by the surjectivity of φ, we also have

$$(\partial_t f)(x) + Tf.\xi(x) \quad \perp \quad f$$

at all $x \in M$. This determines the non-autonomous vector field $\xi = -(f_t)^\top$ uniquely. □

2.3.2 The geodesic equation on shape space

As described in section 1.2.14 and 2.3.1 geodesics in B_i correspond to horizontal geodesics in Imm. A horizontal geodesic f in Imm has $f_t = f_t^\perp$. The geodesic equation is then given by

$$\nabla_{\partial_t} f_t = \frac{1}{2} H(f_t^\perp, f_t^\perp) - K(f_t^\perp, f_t^\perp),$$

see section 1.2.14. This equation splits into a normal and a tangential part. The normal part is given by

$$(\nabla_{\partial_t} f_t)^\perp = \Big(\frac{1}{2} H(f_t^\perp, f_t^\perp) - K(f_t^\perp, f_t^\perp)\Big)^\perp.$$

From section 2.2, where we calculated the geodesic equation on Imm we can read off the tangential part of this equation:

$$\begin{aligned} (\nabla_{\partial_t} f_t)^\top &= \Big(\frac{1}{2} H(f_t^\perp, f_t^\perp) - K(f_t^\perp, f_t^\perp)\Big)^\top \\ &= +\frac{\partial_2 \Phi}{2.\Phi} \left\| f_t^\perp \right\|_{\bar{g}}^2 . \operatorname{grad}^g \|\operatorname{Tr}^g(S)\|_{\bar{g}}^2 - \frac{1}{2.\Phi} \operatorname{grad}^g \left(\Phi. \left\| f_t^\perp \right\|_{\bar{g}}^2\right) \\ &= +\frac{\partial_2 \Phi}{2.\Phi} \left\| f_t^\perp \right\|_{\bar{g}}^2 . \operatorname{grad}^g \|\operatorname{Tr}^g(S)\|_{\bar{g}}^2 \\ &\quad - \frac{1}{2} \operatorname{grad}^g \left(\left\| f_t^\perp \right\|_{\bar{g}}^2 \right) - \frac{\left\| f_t^\perp \right\|_{\bar{g}}^2}{2.\Phi} (\partial_2 \Phi). \operatorname{grad}^g \left(\|\operatorname{Tr}^g(S)\|_{\bar{g}}^2 \right) \\ &= -\frac{1}{2} \operatorname{grad}^g \left(\left\| f_t^\perp \right\|_{\bar{g}}^2 \right), \end{aligned}$$

where we used the the following Leibnitz rule for the gradient:

$$g(\operatorname{grad}^g(f_1.f_2), X) = g(f_1 \operatorname{grad}^g f_2 + f_2 \operatorname{grad}^g f_1, X).$$

In section 1.2.14 we proved that this equation is satisfied automatically. We will nevertheless check this by hand. The following calculation holds for all vector fields $X \in \mathfrak{X}(M)$:

$$\begin{aligned} g\big((\nabla_{\partial_t} f_t)^\top, X\big) &= \bar{g}\big(Tf.(\nabla_{\partial_t} f_t)^\top, Tf.X\big) = \bar{g}(\nabla_{\partial_t} f_t, Tf.X) \\ &= \nabla_{\partial_t} \bar{g}(f_t^\perp, Tf.X) - \bar{g}(f_t, \nabla_{\partial_t} Tf.X) \\ &= 0 - \bar{g}(f_t, \nabla_X f_t) = -\frac{1}{2} d(\bar{g}(f_t, f_t))(X) \\ &= -\frac{1}{2} g(\operatorname{grad}^g(\|f_t\|_{\bar{g}}^2), X) = -\frac{1}{2} g(\operatorname{grad}^g(\|f_t^\perp\|_{\bar{g}}^2), X). \end{aligned}$$

Therefore we have:

$$(\nabla_{\partial_t} f_t)^\top = -\frac{1}{2} \operatorname{grad}^g(\|f_t^\perp\|_{\bar{g}}^2).$$

According to section 1.2.15 we can rewrite the geodesic equation as an equation for the momentum. This yields:

Theorem. *On the smooth cotangent bundle the horizontal geodesic equation for the momentum of an almost local metric G^Φ is given by:*

$$\boxed{\begin{aligned} f_t &= f_t^\perp \in \operatorname{Nor}(f) \\ p &= \Phi.f_t \otimes \operatorname{vol}(g) \\ \nabla_{\partial_t} p &= -\frac{1}{2} \operatorname{Tr}^g(S) \int_M (\partial_1 \Phi).\, \|f_t^\perp\|_{\bar{g}}^2 \operatorname{vol}(g) \otimes \operatorname{vol}(g) \\ &\quad + 2.(\partial_2\Phi) \operatorname{Tr}\Big(g^{-1}.S.g^{-1}.\bar{g}\big(S, \operatorname{Tr}^g(S)\big).\, \|f_t^\perp\|_{\bar{g}}^2\Big) \otimes \operatorname{vol}(g) \\ &\quad - \Big(\Delta\Big((\partial_2\Phi).\operatorname{Tr}^g(S).\, \|f_t^\perp\|_{\bar{g}}^2\Big)\Big)^\perp \otimes \operatorname{vol}(g) \\ &\quad - (\partial_2\Phi)\, \|f_t^\perp\|_{\bar{g}} . \operatorname{Tr}^g\Big(R^{\bar{g}}(Tf, \operatorname{Tr}^g(S))Tf\Big)^\perp \otimes \operatorname{vol}(g) \\ &\quad - \frac{1}{2}.\Phi\, \|f_t\|_{\bar{g}}^2 \operatorname{Tr}^g(S) \otimes \operatorname{vol}(g) \end{aligned}}$$

2.4 Geodesic distance on shape space

We will state some conditions on Φ ensuring that the almost local metric G^Φ induces non-vanishing geodesic distance on B_i. The proofs are based on a comparison between the G^Φ-length of a path and its area swept out. The main result is in section 2.4.5. This section is based on [6, section 7], with slight modifications due to a possibly curved ambient space.Some of the ideas can also be found in [22, 7, 35]

Geodesic distance on B_i is given by

$$\operatorname{dist}_{B_i}^{G^\Phi}(F_0, F_1) = \inf_F L_{B_i}^{G^\Phi}(F),$$

where the infimum is taken over all $F : [0,1] \to B_i$ with $F(0) = F_0$ and $F(1) = F_1$. $L_{B_i}^{G^\Phi}$ is the length of paths in B_i given by

$$L_{B_i}^{G^\Phi}(F) = \int_0^1 \sqrt{G_F^\Phi(F_t, F_t)}dt \quad \text{for } F : [0,1] \to B_i.$$

Letting $\pi : \mathrm{Imm} \to B_i$ denote the projection, we have

$$L^{G^\Phi}_{B_i}(\pi \circ f) = L^{G^\Phi}_{\mathrm{Imm}}(f) = \int_0^1 \sqrt{G^\Phi_f(f_t, f_t)}dt \quad \text{for horizontal } f : [0,1] \to \mathrm{Imm}.$$

By non-vanishing geodesic distance on B_i we mean that $\mathrm{dist}^{G^\Phi}_{B_i}$ separates points.

2.4.1 Area swept out

For a path of immersions f seen as a mapping $f : [0,1] \times M \to N$ one has

$$(\text{area swept out by } f) = \int_{[0,1]\times M} \mathrm{vol}(f(\cdot,\cdot)^*\bar{g}) = \int_0^1 \int_M \left\| f_t^\perp \right\| \mathrm{vol}(g) dt.$$

2.4.2 First area swept out bound

Lemma. *For an almost local metric G^Φ satisfying*

$$\Phi \geq C_1 \qquad \text{for } C_1 > 0.$$

and a horizontal path $f : [0,1] \to \mathrm{Imm}$, we have the area swept out bound

$$\sqrt{C_1}\ (\text{area swept out by } f) \leq \max_t \sqrt{\mathrm{Vol}\big(f(t)\big)}.L^{G^\Phi}_{\mathrm{Imm}}(f)$$

The proof is an adaptation of the one given in [35, section 3.4] for the G^A-metric.

Proof.

$$\begin{aligned}
L^{G^\Phi}_{\mathrm{Imm}}(f) &= \int_0^1 \sqrt{G^\Phi_f(f_t, f_t)}dt \\
&= \int_0^1 \Big(\int_M \Phi \left\| f_t \right\|^2 \mathrm{vol}(g) \Big)^{\frac{1}{2}} dt \geq \sqrt{C_1} \int_0^1 \Big(\int_M \left\| f_t \right\|^2 \mathrm{vol}(g) \Big)^{\frac{1}{2}} dt \\
&\geq \sqrt{C_1} \int_0^1 \Big(\int_M \mathrm{vol}(g) \Big)^{-\frac{1}{2}} \int_M 1. \left\| f_t \right\| \mathrm{vol}(g) dt \\
&\geq \sqrt{C_1} \min_t \Big(\int_M \mathrm{vol}(g) \Big)^{-\frac{1}{2}} \int_0^1 \int_M 1. \left\| f_t \right\| \mathrm{vol}(g) dt \\
&= \sqrt{C_1} \Big(\max_t \int_M \mathrm{vol}(g) \Big)^{-\frac{1}{2}} \int_0^1 \int_M 1. \left\| f_t \right\| \mathrm{vol}(g) dt
\end{aligned}$$

□

2.4.3 Lipschitz continuity of $\sqrt{\mathrm{Vol}}$

Lemma. *For an almost local metric G^Φ, the condition*

$$\Phi \geq C_2 \left\| \mathrm{Tr}(L) \right\|^2_{\bar{g}}$$

implies the Lipschitz continuity of the map

$$\sqrt{\mathrm{Vol}} : (B_i, \mathrm{dist}_{G^\Phi}^{B_i}) \to \mathbb{R}_{\geq 0}$$

by the inequality holding for fF_1 and F_2 in B_i:

$$\sqrt{\mathrm{Vol}(F_1)} - \sqrt{\mathrm{Vol}(F_2)} \leq \frac{1}{2\sqrt{C_2}} \mathrm{dist}_{G^\Phi}^{B_i}(F_1, F_2),$$

The proof is an adaptation of the one given in [35, section 3.3] for the G^A-metric.

Proof. Let $f : [0,1] \to \mathrm{Imm}$ be a path, and let f_t denote its derivative. Using the formula from section 1.3.7 for the variation of the volume we get

$$\begin{aligned}
\partial_t \mathrm{Vol}(f) &= -\int_M \bar{g}(f_t, \mathrm{Tr}^g(S)) \,\mathrm{vol}(g) \leq \left| \int_M \bar{g}(f_t, \mathrm{Tr}^g(S)) \,\mathrm{vol}(g) \right| \\
&\leq \Big(\int_M 1^2 \,\mathrm{vol}(g) \Big)^{\frac{1}{2}} \Big(\int_M \bar{g}(f_t, \mathrm{Tr}^g(S))^2 \,\mathrm{vol}(g) \Big)^{\frac{1}{2}} \\
&\leq \sqrt{\mathrm{Vol}(f)} \Big(\int_M \|f_t\|_{\bar{g}}^2 \, \|\mathrm{Tr}^g(S)\|_{\bar{g}}^2 \,\mathrm{vol}(g) \Big)^{\frac{1}{2}} \\
&\leq \sqrt{\mathrm{Vol}(f)} \Big(\int_M \frac{\Phi}{C_2} \, \|f_t\|_{\bar{g}}^2 \,\mathrm{vol}(g) \Big)^{\frac{1}{2}} \leq \frac{1}{\sqrt{C_2}} \sqrt{\mathrm{Vol}(f)} \sqrt{G_f^\Phi(f_t, f_t)}.
\end{aligned}$$

Thus

$$\partial_t \sqrt{\mathrm{Vol}(f)} = \frac{\partial_t \mathrm{Vol}(f)}{2\sqrt{\mathrm{Vol}(f)}} \leq \frac{1}{2\sqrt{C_2}} \sqrt{G_f^\Phi(f_t, f_t)}.$$

By integration we get

$$\begin{aligned}
\sqrt{\mathrm{Vol}(f_1)} - \sqrt{\mathrm{Vol}(f_0)} &= \int_0^1 \partial_t \sqrt{\mathrm{Vol}(f)} dt \\
&\leq \int_0^1 \frac{1}{2\sqrt{C_2}} \sqrt{G_f^\Phi(f_t, f_t)} = \frac{1}{2\sqrt{C_2}} \; L_{\mathrm{Imm}}^{G^\Phi}(f).
\end{aligned}$$

Now take the infimum over all paths $f : [0,1] \to \mathrm{Imm}$ with $\pi(f(0)) = F_0$ and $\pi(f(1)) = F_1$. $\square$

2.4.4 Second area swept out bound

Lemma. *For an almost local metric G^Φ satisfying*

$$\Phi \geq C \,\mathrm{Vol} \qquad \textit{with } C > 0$$

and a horizontal path $f : [0,1] \to \mathrm{Imm}$, we get the area swept out bound

$$\sqrt{C} \; (\textit{area swept out by } f) \leq L_{\mathrm{Imm}}^{G^\Phi}(f),$$

The proof is adapted from proofs for the case of planar curves that can be found in [37, section 3.7], [41, Lemma 3.2], [53, proposition 1] and [52, theorem 7.5].

Proof.

$$\begin{aligned}
L^{G^\Phi}_{\mathrm{Imm}}(f) &= \int_0^1 \sqrt{G^\Phi_f(f_t,f_t)}dt = \int_0^1 \Big(\int_M \Phi\,\|f_t\|^2\,\mathrm{vol}(g)\Big)^{\frac12} dt \\
&\geq \sqrt{C}\int_0^1 \sqrt{\mathrm{Vol}(f)}\Big(\int_M \|f_t\|^2\,\mathrm{vol}(g)\Big)^{\frac12} dt \\
&\geq \sqrt{C}\int_0^1 \int_M 1.\,\|f_t\|\,\mathrm{vol}(g)dt \\
&= \sqrt{C}\int_{[0,1]\times M} \mathrm{vol}(f(\cdot,\cdot)^*\bar g)dt = \sqrt{C}\ (\text{area swept out by } f). \qquad \square
\end{aligned}$$

2.4.5 Geodesic distance

Theorem. *At least on B_e, the almost local metric G^Φ induces non-vanishing geodesic distance if one of the two following conditions holds:*

(1) $\Phi \geq C_1 + C_2\,\mathrm{Tr}(L)^2$ *for* $C_1, C_2 > 0$.

(2) $\Phi \geq C\,\mathrm{Vol}$ *for* $C > 0$.

On the other hand, the almost local metric G^Φ induces vanishing geodesic distance on shape space if

(3) $\Phi \leq C_3\,\mathrm{Vol}^{-k}$ *for* $k \geq 0$.

Proof. The first part of this theorem is a consequence of the previous estimates. The following proof of the vanishing geodesic distance result for $\Phi = 1$ is taken from [35], the figure illustrating the construction is from [36].

Take a path $f(t,x)$ in $\mathrm{Imm}(M,N)$ from f_0 to f_1 and make it horizontal by the same method that was used in 2.3.1. Horizontality for the H^0-metric simply means $\bar g(f_t, Tf) = 0$. This forces a reparametrization on f_1.

Let $\alpha : M \to [0,1]$ be a surjective Morse function whose singular values are all contained in the set $\{\frac{k}{2N} : 0 \leq k \leq 2N\}$ for some integer N. We shall use integers n below and we shall use only multiples of N.

Then the level sets $M_r := \{x \in M : \alpha(x) = r\}$ are of Lebesque measure 0. We shall also need the slices $M_{r_1,r_2} := \{x \in M : r_1 \leq \alpha(x) \leq r_2\}$. Since M is compact there exists a constant C such that the following estimate holds uniformly in t:

$$\int_{M_{r_1,r_2}} \mathrm{vol}(f(t,\quad)^*\bar g) \leq C(r_2 - r_1)\int_M \mathrm{vol}(f(t,\quad)^*\bar g)$$

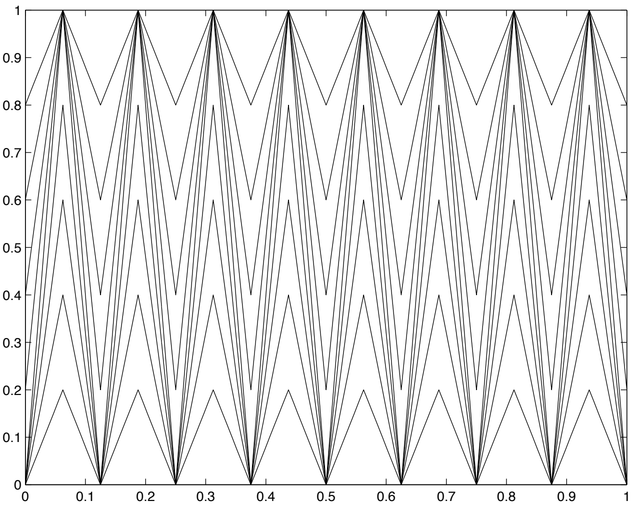

Figure 2.1: Plot of the function φ. Each zig-zagged line corresponds to $\varphi(t, \cdot)$ for some fixed values of t, namely $t = \frac{1}{10}, \frac{2}{10}, \dots, \frac{9}{10}$.

Let $\tilde{f}(t,x) = f(\varphi(t, \alpha(x)), x)$ where $\varphi : [0,1] \times [0,1] \to [0,1]$ is given as in [36], 3.10 by

$$\varphi(t,\alpha) = \begin{cases} 2t(2n\alpha - 2k), & 0 \le t \le 1/2,\ \frac{2k}{2n} \le \alpha \le \frac{2k+1}{2n} \\ 2t(2k+2-2n\alpha), & 0 \le t \le 1/2,\ \frac{2k+1}{2n} \le \alpha \le \frac{2k+2}{2n} \\ 2t - 1 + 2(1-t)(2n\alpha - 2k), & 1/2 \le t \le 1,\ \frac{2k}{2n} \le \alpha \le \frac{2k+1}{2n} \\ 2t - 1 + 2(1-t)(2k+2-2n\alpha), & 1/2 \le t \le 1,\ \frac{2k+1}{2n} \le \alpha \le \frac{2k+2}{2n}. \end{cases}$$

See figure 2.1 for an illustration of the construction.

Then we get $T\tilde{f} = \varphi_\alpha . d\alpha . f_t + Tf$ and $\tilde{f}_t = \varphi_t . f_t$ where

$$\varphi_\alpha = \begin{cases} +4nt \\ -4nt \\ +4n(1-t) \\ -4n(1-t) \end{cases}, \qquad \varphi_t = \begin{cases} 4n\alpha - 4k \\ 4k + 4 - 4n\alpha \\ 2 - 4n\alpha + 4k \\ -(2 - 4n\alpha + 4k) \end{cases}.$$

We use horizontality $\bar{g}(f_t, Tf) = 0$ to determine $\tilde{f}_t^\perp = \tilde{f}_t + T\tilde{f}(X)$ where $X \in TM$ satisfies $0 = \bar{g}(\tilde{f}_t + T\tilde{f}(X), T\tilde{f}(\xi))$ for all $\xi \in TM$. We also use

$$d\alpha(\xi) = g(\operatorname{grad}^g \alpha, \xi) = \bar{g}\big(Tf(\operatorname{grad}^g \alpha), Tf(\xi)\big)$$

and get

$$\begin{aligned} 0 &= \bar{g}\big(\tilde{f}_t + T\tilde{f}(X), T\tilde{f}(\xi)\big) \\ &= g\Big(\varphi_t f_t + \varphi_\alpha d\alpha(X) f_t + Tf(X), \varphi_\alpha d\alpha(\xi) f_t + Tf(\xi)\Big) \\ &= \varphi_t . \varphi_\alpha . g(\operatorname{grad}^g \alpha, \xi) \|f_t\|^2 + \\ &\quad + \varphi_\alpha^2 . g(\operatorname{grad}^g \alpha, X) . g(\operatorname{grad}^g \alpha, \xi) \|f_t\|^2 + \bar{g}(Tf(X), Tf(\xi)) \end{aligned}$$

$$= (\varphi_t.\varphi_\alpha + \varphi_\alpha^2.g(\operatorname{grad}^g \alpha, X))\|f_t\|^2 g(\operatorname{grad}^g \alpha, \xi) + g(X,\xi)$$

This implies that $X = \lambda \operatorname{grad}^g \alpha$ for a function λ and in fact we get

$$\tilde{f}_t^\perp = \frac{\varphi_t}{1+\varphi_\alpha^2\|d\alpha\|_g^2\|f_t\|^2} f_t - \frac{\varphi_t\varphi_\alpha\|f_t\|_g^2}{1+\varphi_\alpha^2\|d\alpha\|_{f^*\bar g}^2\|f_t\|^2} Tf(\operatorname{grad}^g \alpha)$$

and

$$\|\tilde{f}_t\|^2 = \frac{\varphi_t^2\|f_t\|^2}{1+\varphi_\alpha^2\|d\alpha\|_{f^*\bar g}^2\|f_t\|^2}$$

From $T\tilde f = \varphi_\alpha.d\alpha.f_t + Tf$ and $\bar g(f_t, Tf) = 0$ we get for the volume form

$$\operatorname{vol}(\tilde f^*\bar g) = \sqrt{1+\varphi_\alpha^2\,\|d\alpha\|_g^2\|f_t\|^2}\ \operatorname{vol}(g).$$

For the horizontal length we get

$$L^{\mathrm{hor}}(\tilde f) = \int_0^1 \Big(\int_M \|\tilde f_t^\perp\|^2 \operatorname{vol}(\tilde f^*\bar g)\Big)^{\frac12} dt =$$

$$= \int_0^1 \Big(\int_M \frac{\varphi_t^2\|f_t\|^2}{\sqrt{1+\varphi_\alpha^2\|d\alpha\|_g^2\|f_t\|^2}} \operatorname{vol}(g)\Big)^{\frac12} dt =$$

$$= \int_0^{\frac12} \Bigg(\sum_{k=0}^{n-1}\Big(\int_{M_{\frac{2k}{2n},\frac{2k+1}{2n}}} \frac{(4n\alpha-4k)^2\|f_t\|^2}{\sqrt{1+(4nt)^2\|d\alpha\|_g^2\|f_t\|^2}} \operatorname{vol}(g) +$$

$$+ \int_{M_{\frac{2k+1}{2n},\frac{2k+2}{2n}}} \frac{(4k+4-4n\alpha)^2\|f_t\|^2}{\sqrt{1+(4nt)^2\|d\alpha\|_g^2\|f_t\|^2}} \operatorname{vol}(g)\Big)\Bigg)^{\frac12} dt +$$

$$+ \int_{\frac12}^1 \Bigg(\sum_{k=0}^{n-1}\Big(\int_{M_{\frac{2k}{2n},\frac{2k+1}{2n}}} \frac{(2-4n\alpha+4k)^2\|f_t\|^2}{\sqrt{1+(4n(1-t))^2\|d\alpha\|_g^2\|f_t\|^2}} \operatorname{vol}(g) +$$

$$+ \int_{M_{\frac{2k+1}{2n},\frac{2k+2}{2n}}} \frac{(2-4n\alpha+4k)^2\|f_t\|^2}{\sqrt{1+(4n(1-t))^2\|d\alpha\|_g^2\|f_t\|^2}} \operatorname{vol}(g)\Big)\Bigg)^{\frac12} dt$$

Let $\varepsilon > 0$. The function $(t,x) \mapsto \|f_t(\varphi(t,\alpha(x)),x)\|^2$ is uniformly bounded. On $M_{\frac{2k}{2n},\frac{2k+1}{2n}}$ the function $4n\alpha - 4k$ has values in $[0,2]$. Choose disjoint geodesic balls centered at the finitely many singular values of the Morse function α of total g-volume $< \varepsilon$. Restricted to the union M_{sing} of these balls the integral above is $O(1)\varepsilon$. So we have to estimate the integrals on the complement $\tilde M = M \setminus M_{\mathrm{sing}}$ where the function $\|d\alpha\|_g$ is uniformly bounded from below by $\eta > 0$.

Let us estimate one of the sums above. We use the fact that the singular points of the Morse function α lie all on the boundaries of the sets $\tilde M_{\frac{2k}{2n},\frac{2k+1}{2n}}$ so that we can transform the integrals as follows:

$$\sum_{k=0}^{n-1}\int_{\tilde M_{\frac{2k}{2n},\frac{2k+1}{2n}}} \frac{(4n\alpha-4k)^2\|f_t\|^2}{\sqrt{1+(4nt)^2\|d\alpha\|_g^2\|f_t\|^2}} \operatorname{vol}(g) =$$

$$= \sum_{k=0}^{n-1} \int_{\frac{2k}{2n}}^{\frac{2k+1}{2n}} \int_{\tilde{M}_r} \frac{(4nr-4k)^2 \|f_t\|^2}{\sqrt{1+(4nt)^2 \|d\alpha\|_g^2 \|f_t\|^2}} \frac{\mathrm{vol}(i_r^* f^* \bar{g})}{\|d\alpha\|_g} \, dr$$

We estimate this sum of integrals: Consider first the set of all $(t, r, x) \in M_r$ such that $|f_t(\varphi(t, r), x)| < \varepsilon$. There we estimate by

$$O(1).n.16n^2.\varepsilon^2.(r^3/3)|_{r=0}^{r=1/2n} = O(\varepsilon).$$

On the complementary set where $|f_t(\varphi(t, r), x)| \geq \varepsilon$ we estimate by

$$O(1).n.16n^2.\frac{1}{4nt\eta^2\varepsilon}(r^3/3)|_{r=0}^{r=1/2n} = O(\frac{1}{nt\eta^2\varepsilon})$$

which goes to 0 if n is large enough. The other sums of integrals can be estimated similarly, thus $L^{\mathrm{hor}}(\tilde{f})$ goes to 0 for $n \to \infty$. It is clear that one can approximate φ by a smooth function whithout changing the estimates essentially. $\square$

Chapter 3

Hypersurfaces in n-space

In chapter 2 we studied metrics weighted by volume and mean curvature. In codimension one it is natural to incorporate $\det(L)$ weights in the definition of the metric. This yields almost local metrics with

$$\Phi = \Phi\big(\operatorname{Vol}, \operatorname{Tr}(L), \det(L)\big).$$

For such metrics we will derive the geodesic equation on the manifold of immersions and on shape space (section 3.1 and section 3.2). In section 3.3 we will calculate the sectional curvature of an almost local metrics weighted by mean curvature and Volume and in section 3.4 all the previously derived formulas are presented for special cases of Φ. We will study the totally geodesic subset of concentric spheres (section 3.5) and finally in section 3.6 we will compare various almost local metrics to the Fréchet metric.

Assumption. *In this chapter we will study hypersurfaces in n-space, i.e. we have*

$$\dim(M) = n-1 \quad \textit{and} \quad N = \mathbb{R}^n.$$

This chapter is based on [6, 8, 7].

3.1 The geodesic equation for the momentum on immersions

Assumption. *In this section and in section 3.2 we will assume that the weight function Φ depends on Volume, mean curvature and* $\det(L)$*.*

Let $m \in T_f \operatorname{Imm}(M, \mathbb{R}^n)$ with $m = \bar{g}(m,\nu).\nu + Tf.m^\top = a.\nu + Tf.m^\top$. We calculate:

$$\begin{aligned}(\nabla^{\bar{g}}_m G^\Phi)(h,k) &= D_{(f,m)} G^\Phi_f(h,k)\\ &= \int_M (\partial_1 \Phi)(D_{(f,m)} \operatorname{Vol}) \bar{g}(h,k) \operatorname{vol}(g)\end{aligned}$$

$$
\begin{aligned}
&+\int_M (\partial_2\Phi)(D_{(f,m)}\operatorname{Tr}(L))\bar{g}(h,k)\operatorname{vol}(g)\\
&+\int_M (\partial_3\Phi)(D_{(f,m)}\det(L))\bar{g}(h,k)\operatorname{vol}(g)\\
&+\int_M \Phi\bar{g}(h,k)(D_{(f,m)}\operatorname{vol}(g)).
\end{aligned}
$$

As in section 2.2, we can easily read off the K-gradient:

$$
\boxed{
\begin{aligned}
K_f(m,h) = \Big[&\frac{\partial_1\Phi}{\Phi}\Big(\int_M -\operatorname{Tr}(L).a\operatorname{vol}(g)\Big)\\
&+\frac{\partial_2\Phi}{\Phi}\big(-\Delta a + a\operatorname{Tr}(L^2) + d\operatorname{Tr}(L)(m^\top)\big)\\
&+\frac{\partial_3\Phi}{\Phi}\Big(\operatorname{Tr}(L).\det(L).a + g_2^0\big(g.\operatorname{C}(L),\nabla^2(a)\big) + d\det(L)(m^\top)\Big)\\
&+\operatorname{div}^g(m^\top) - \operatorname{Tr}(L).a\Big]h.
\end{aligned}
}
$$

To calculate the H-gradient, we treat the four summands of $D_{(f,m)}G_f^\Phi(h,k)$ separately. To the first and last term we can apply the same analysis as in 2.2. The second summand is given by

$$
\begin{aligned}
&\int_M (\partial_2\Phi)(D_{(f,m)}\operatorname{Tr}(L))\ \bar{g}(h,k)\operatorname{vol}(g)\\
&\quad= \int_M (\partial_2\Phi)\big(-\Delta a + a\operatorname{Tr}(L^2) + d\operatorname{Tr}(L)(m^\top)\big)\bar{g}(h,k)\operatorname{vol}(g)\\
&\quad= \int_M -a.\Delta\big((\partial_2\Phi)\bar{g}(h,k)\big)\operatorname{vol}(g) + \int_M a.(\partial_2\Phi)\operatorname{Tr}(L^2).\bar{g}(h,k)\operatorname{vol}(g)\\
&\qquad+\int_M (\partial_2\Phi)g\big(\operatorname{grad}^g(\operatorname{Tr}(L)),m^\top\big)\bar{g}(h,k)\operatorname{vol}(g)\\
&\quad= G_f^\Phi\Big(m,-\frac{1}{\Phi}\Delta\big((\partial_2\Phi)\bar{g}(h,k)\big).\nu\Big) + G_f^\Phi\Big(m,\frac{1}{\Phi}(\partial_2\Phi)\operatorname{Tr}(L^2)\bar{g}(h,k).\nu\Big)\\
&\qquad+G_f^\Phi\Big(m,\frac{1}{\Phi}(\partial_2\Phi)\bar{g}(h,k)Tf.\operatorname{grad}^g(\operatorname{Tr}(L))\Big)
\end{aligned}
$$

The third summand is

$$
\begin{aligned}
&\int_M (\partial_3\Phi)(D_{(f,m)}\det(L))\bar{g}(h,k)\operatorname{vol}(g) =\\
&\quad= \int_M (\partial_3\Phi)\operatorname{Tr}(L).\det(L).a.\bar{g}(h,k)\operatorname{vol}(g)\\
&\qquad+\int_M (\partial_3\Phi)d\det(L)(m^\top)\bar{g}(h,k)\operatorname{vol}(g)\\
&\qquad+\int_M g_2^0\big((\partial_3\Phi).\bar{g}(h,k).g.\operatorname{C}(L),\nabla^2(a)\big)\operatorname{vol}(g)\\
&\quad= \int_M (\partial_3\Phi)\operatorname{Tr}(L).\det(L).a.\bar{g}(h,k)\operatorname{vol}(g)\\
&\qquad+\int_M (\partial_3\Phi)g\big(\operatorname{grad}^g(\det(L)),m^\top\big)\bar{g}(h,k)\operatorname{vol}(g)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_M \nabla^*\nabla^*\Big((\partial_3\Phi).g.\,\mathrm{C}(L)\bar g(h,k)\Big).a\,\mathrm{vol}(g)\\
&= G_f^\Phi\Big(m, \frac{\partial_3\Phi}{\Phi}\,\mathrm{Tr}(L).\det(L).\bar g(h,k).\nu\Big)\\
&\quad + G_f^\Phi\Big(m, \frac{\partial_3\Phi}{\Phi}\bar g(h,k).Tf.\,\mathrm{grad}^g(\det(L))\Big)\\
&\quad + G_f^\Phi\Big(m, \frac{1}{\Phi}\nabla^*\nabla^*\big((\partial_3\Phi).g.\,\mathrm{C}(L)\bar g(h,k)\big).\nu\Big).
\end{aligned}
$$

Summing up all the terms the H-gradient is given by

$$
\boxed{
\begin{aligned}
H_f(h,k) = \Big[&-\frac{1}{\Phi}\,\mathrm{Tr}(L)\int_M(\partial_1\Phi)\bar g(h,k)\,\mathrm{vol}(g) - \frac{1}{\Phi}\Delta\big((\partial_2\Phi)\bar g(h,k)\big)\\
&+\frac{\partial_2\Phi}{\Phi}\,\mathrm{Tr}(L^2)\bar g(h,k) + \frac{\partial_3\Phi}{\Phi}.\,\mathrm{Tr}(L).\det(L).\bar g(h,k)\\
&+\frac{1}{\Phi}\nabla^*\nabla^*\big((\partial_3\Phi).g.\,\mathrm{C}(L)\bar g(h,k)\big) - \bar g(h,k)\,\mathrm{Tr}(L)\Big]\nu^f\\
&+\frac{1}{\Phi}Tf.\Big[(\partial_2\Phi)\bar g(h,k)\,\mathrm{grad}^g(\mathrm{Tr}(L))\\
&+(\partial_3\Phi)\bar g(h,k)\,\mathrm{grad}^g(\det(L)) - \mathrm{grad}^g(\Phi\bar g(h,k))\Big]
\end{aligned}
}
$$

Using the formula from section 1.2.9

Theorem. *The geodesic equation for the momentum of an almost local metric* G^Φ *on* Imm *is given by*

$$
\boxed{
\begin{aligned}
p &= \Phi(\mathrm{Vol},\mathrm{Tr}(L),\det(L)).f_t\otimes\mathrm{vol}(g) = \Phi.(a.\nu + Tf.f_t^\top)\otimes\mathrm{vol}(g),\\
p_t &= \frac{1}{2}\Big[-\mathrm{Tr}(L)\int_M\partial_1\Phi\,\|f_t\|^2\,\mathrm{vol}(g) - \Delta\big((\partial_2\Phi)\,\|f_t\|^2\big)\\
&\qquad + (\partial_2\Phi)\,\mathrm{Tr}(L^2)\,\|f_t\|^2 + (\partial_3\Phi).\,\mathrm{Tr}(L).\det(L).\,\|f_t\|^2\\
&\qquad + \nabla^*\nabla^*\big((\partial_3\Phi).g.\,\mathrm{C}(L)\,\|f_t\|^2\big) - \Phi\,\|f_t\|^2\,\mathrm{Tr}(L)\Big]\nu\otimes\mathrm{vol}(g)\\
&\quad + \frac{1}{2}Tf.\Big[(\partial_2\Phi)\,\|f_t\|^2\,\mathrm{grad}^g(\mathrm{Tr}(L))\\
&\qquad + (\partial_3\Phi)\,\|f_t\|^2\,\mathrm{grad}^g(\det(L)) - \mathrm{grad}^g(\Phi\,\|f_t\|^2)\Big]\otimes\mathrm{vol}(g)
\end{aligned}
}
$$

3.2 The geodesic equation on shape space

For an almost local metric in codimension one a horizontal geodesic f in Imm has $f_t = a.\nu$ with $a \in C^\infty(\mathbb{R}\times M)$. In the following we will calculate the horizontal geodesic equation in terms of the velocity. This will allow us to rewrite the horizontal geodesic equation as an equation for the function a only.

The horizontal geodesic equation is given by

$$
f_{tt} = \underbrace{a_t.\nu}_{\text{normal}} + \underbrace{a.\nu_t}_{\text{tang.}} = \frac{1}{2}H(a.\nu,a.\nu) - K(a.\nu,a.\nu),
$$

see section 1.2.14.

In 1.2.14 it was shown the tangential part of the geodesic equation is satisfied automatically. Again we will check this by hand. Using the formulas from section 2.2, we can easily read off the tangential part of the geodesic equation

$$\begin{aligned} a.\nu_t &= \frac{1}{2\Phi} Tf.\Big[(\partial_2\Phi)a^2 \operatorname{grad}^g(\operatorname{Tr}(L)) + (\partial_3\Phi)a^2 \operatorname{grad}^g(\det(L)) - \operatorname{grad}^g(\Phi a^2)\Big] \\ &= \frac{1}{2\Phi} Tf.\Big[a^2 \operatorname{grad}^g(\Phi) - \Phi.\operatorname{grad}^g(a^2) - a^2.\operatorname{grad}^g(\Phi)\Big] \\ &= -\frac{1}{2\Phi}\Phi Tf.\operatorname{grad}^g(a^2) = -Tf.a.\operatorname{grad}^g(a). \end{aligned}$$

By the variational formula for ν in section 1.3.14 this equation is satisfied automatically.

The normal part is given by

$$\begin{aligned} a_t &= \bar g\big(\frac{1}{2}H(a.\nu, a.\nu) - K(a.\nu, a.\nu), \nu\big) \\ &= \frac{1}{\Phi}\big[\frac{1}{2}\Phi a^2 \operatorname{Tr}(L^f) - \frac{1}{2}\operatorname{Tr}(L^f)\int_M (\partial_1\Phi)a^2 \operatorname{vol}(f^*\bar g) - \frac{1}{2}a^2\Delta(\partial_2\Phi) \\ &\quad + 2a\operatorname{Tr}^g(d(\partial_2\Phi)\otimes da) + (\partial_2\Phi)\operatorname{Tr}^g(da\otimes da) \\ &\quad + (\partial_1\Phi)a\int_M \operatorname{Tr}(L^f).a\operatorname{vol}(f^*\bar g) - \frac{1}{2}(\partial_2\Phi)\operatorname{Tr}((L^f)^2)a^2\big]. \end{aligned}$$

We can rewrite this equation by expanding Laplacians of products as follows:

$$\Delta(a_1a_2) = (\Delta a_1)a_2 - 2\operatorname{Tr}^g(da_1\otimes da_2) + a_1(\Delta a_2).$$

Theorem. *The horizontal geodesic equation of an almost local metric G^Φ on immersions is given by*

$$\boxed{\begin{aligned} f_t &= a.\nu, \\ a_t &= \frac{1}{\Phi}\Big[\frac{\Phi}{2}a^2\operatorname{Tr}(L) - \frac{1}{2}\operatorname{Tr}(L)\int_M (\partial_1\Phi)a^2\operatorname{vol}(g) - \frac{1}{2}a^2\Delta(\partial_2\Phi) \\ &\quad + 2ag^{-1}(d(\partial_2\Phi), da) + (\partial_2\Phi)\,\|da\|^2_{g^{-1}} \\ &\quad + (\partial_1\Phi)a\int_M \operatorname{Tr}(L).a\operatorname{vol}(g) - \frac{1}{2}(\partial_2\Phi)\operatorname{Tr}(L^2)a^2 \\ &\quad + \frac{1}{2}\nabla^*\nabla^*\big((\partial_3\Phi).g.\operatorname{C}(L)a^2\big) - (\partial_3\Phi)g_2^0\Big(g.\operatorname{C}(L),\nabla^2(a)\Big).a \\ &\quad - \frac{\partial_3\Phi}{2}.\operatorname{Tr}(L).\det(L).a^2\Big] \end{aligned}}$$

This equation is in accordance to the equation in section 2.3.2. For the case of curves immersed in $\mathbb{R}^2$, this formula specializes to the formula given in [37, section 3.4]. (When verifying this, remember that $\Delta = -D_s^2$ and $\operatorname{Tr}(L) = \det(L)$ in the notation of [37].)

3.3 Sectional curvature on shape space

The following part is taken from [6].

Assumption. *In all of this section we will assume that the weight function Φ depends on Volume and mean curvature only.*

To compute the sectional curvature we will use the following formula, which is valid in a chart:

$$\begin{aligned}
&R_0(a_1, a_2, a_1, a_2) = G_0^\Phi(R_0(a_1, a_2)a_1, a_2) = \\
&\quad \frac{1}{2}d^2G_0^\Phi(a_1, a_1)(a_2, a_2) - d^2G_0^\Phi(a_1, a_2)(a_1, a_2) + \frac{1}{2}d^2G_0^\Phi(a_2, a_2)(a_1, a_1) \\
&\quad + G_0^\Phi(\Gamma_0(a_1, a_1), \Gamma_0(a_2, a_2)) - G_0^\Phi(\Gamma_0(a_1, a_2), \Gamma_0(a_1, a_2)).
\end{aligned}$$

Sectional curvature is given by

$$R_0(a_1, a_2, a_2, a_1) = -R_0(a_1, a_2, a_1, a_2).$$

Therefore we have to calculate the metric in a chart and calculate its second derivative.

3.3.1 The almost local metric G^Φ in a chart

In the following section we will follow the method of [35]. First we will construct a local chart for B_i. Let $f_0 : M \to \mathbb{R}^n$ be a fixed immersion, which will be the center of our chart. Consider the mapping

$$\begin{aligned}
\psi = \psi_{f_0} &: C^\infty(M, (-\epsilon, \epsilon)) \to \mathrm{Imm}(M, \mathbb{R}^n) \\
\psi(a)(x) &= \exp^{\bar{g}}_{f_0(x)}(a(x).\nu^{f_0}(x)) = f_0(x) + a(x).\nu^{f_0}(x),
\end{aligned}$$

where ϵ is so small that $\psi(a)$ is an immersion for each a.

Denote by π the projection from $\mathrm{Imm}(M, \mathbb{R}^n)$ to $B_i(M, \mathbb{R}^n)$. The inverse on its image of $\pi \circ \psi : C^\infty(M, (-\epsilon, \epsilon)) \to B_i(M, \mathbb{R}^n)$ is then a smooth chart on $B_i(M, \mathbb{R}^n)$. We want to calculate the induced metric in this chart, i.e.

$$((\pi \circ \psi)^* G^\Phi)_a(b_1, b_2)$$

for any $a \in C^\infty(M, (-\epsilon, \epsilon))$ and $b_1, b_2 \in C^\infty(M)$. We shall fix the function a and work with the ray of points $t.a$ in this chart. Everything will revolve around the map:

$$f(t, x) = \psi(t.a)(x) = f_0(x) + t.a(x).\nu^{f_0}(x)$$

We shall use a fixed chart (u, U) on M with $\partial_i = \frac{\partial}{\partial u^i}$. Then in this chart, the pullback metric is given by

$$g|_U = \sum_{i,j}^{n-1} g_{ij} du^i \otimes du^j = \sum_{i,j}^{n-1} \bar{g}\big(\partial_i f, \partial_j f\big) du^i \otimes du^j,$$

the volume density by

$$\mathrm{vol}(g) = \sqrt{\det(\bar{g}\,(\partial_i f, \partial_j f))}|du^1 \wedge \cdots \wedge du^{n-1}|,$$

the second fundamental form by

$$s_{ij} = s(\partial_i, \partial_j) = \bar{g}(\nabla_{\partial_i} Tf.\partial_j, \nu) = \bar{g}(\frac{\partial^2 f}{\partial_i \partial_j}, \nu^{f_0}),$$

and the mean curvature by $\mathrm{Tr}(L) = \sum_{i,j} g^{ij} s_{ij}$. To calculate the metric G^Φ in this chart we have to understand how

$$T_{t.a}\psi.b_1 = b_1(x).\nu^{f_0}(x)$$

splits into a tangential and horizontal part with respect to the immersion $f(t,\ \ )$. The tangential part locally has the form

$$Tf.(T_{(t.a)}\psi.(b_1))^\top = \sum_{i=1}^{n-1} c^i \partial_i f(t,x),$$

where the coefficients c^i are given by

$$c^i = \sum_{j=1}^{n-1} g^{ij} \bar{g}\left(b_1(x)\nu^{f_0}(x), \partial_j f(t,x)\right).$$

Thus the horizontal part is

$$(T_{t.a}\psi.b_1).\nu = (T_{t.a}\psi.b_1) - Tf.(T_{(t.a)}\psi.(b_1))^\top = b_1(x)\nu^{f_0}(x) - \sum_{i=1}^{n-1} c^i \partial_i f(t,x).$$

Lemma. *The expression of G^Φ in the chart $(\pi \circ \psi)^{-1}$ is:*

$$\begin{aligned}
\big((\pi\circ\psi_{f_0})^* G^\Phi\big)_{(t.a)}(b_1,b_2) &= G^\Phi_{\pi(\psi(t.a))}\big(T_{t.a}(\pi\circ\psi).b_1, T_{t.a}(\pi\circ\psi)b_2\big)\\
&= G^\Phi_{\psi(t.a)}\big((T_{t.a}\psi.b_1)^\perp.\nu, (T_{t.a}\psi.b_2)^\perp.\nu\big)\\
&= \int_M \Phi\bar{g}\left((T_{t.a}\psi.b_1)^\perp.\nu, (T_{t.a}\psi.b_2)\right)\mathrm{vol}(g)\\
&= \int_M \Phi\left(b_1.b_2 - \sum_{i=1}^{n-1} c^i \bar{g}(\partial_i f(t,x), b_2(x).\nu^{f_0}(x))\right)\mathrm{vol}(g)
\end{aligned}$$

3.3.2 Second derivative of the G^Φ-metric in the chart

We will calculate

$$\partial_t^2|_0((\pi\circ\psi_{f_0})^* G^\Phi)_{(t.a)}(b_1,b_2).$$

We will use the following arguments repeatedly:

$$\partial_t|_0\partial_j f = \partial_j\partial_t|_0 f = \partial_j(a.\nu^{f_0}) = (\partial_j a)\nu^{f_0} + a\,\underbrace{(\partial_j\nu^{f_0})}_{\text{tang.}},$$

$$\bar{g}\left(b_1(x)\nu^{f_0}(x), \partial_j f(t,x)\right)|_{t=0} = 0,$$

and consequently $c_i|_{t=0} = 0$.

$$\begin{aligned}
\partial_t|_0 c_i &= \sum_{j=1}^{n-1} \partial_t|_0(g^{ij}).0 + \sum_{j=1}^{n-1} g^{ij}\partial_t|_0 \bar{g}\left(b_1\nu^{f_0}, \partial_j f\right) \\
&= \sum_{j=1}^{n-1} g^{ij}\bar{g}\left(b_1\nu^{f_0}, \partial_t|_0\partial_j f\right) = \sum_{j=1}^{n-1} g^{ij}\bar{g}\left(b_1\nu^{f_0}, \partial_j(a.\nu^{f_0})\right) = \sum_{j=1}^{n-1} g^{ij} b_1 \partial_j a.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\big(b_1.b_2 - \sum_{i=1}^{n-1} c^i \bar{g}(\partial_i f, b_2.\nu^{f_0})\big)\big|_{t=0} = b_1.b_2 \\
&\partial_t|_0\big(b_1.b_2 - \sum_{i=1}^{n-1} c^i \bar{g}(\partial_i f, b_2.\nu^{f_0})\big) = \\
&\quad = -\sum_{i=1}^{n-1}(\partial_t|_0 c^i).0 - \sum_{i=1}^{n-1} 0.\partial_t|_0\bar{g}(\partial_i f, b_2.\nu^{f_0}) = 0 \\
&\partial_t^2|_0\big(b_1.b_2 - \sum_{i=1}^{n-1} c^i \bar{g}(\partial_i f, b_2.\nu^{f_0})\big) = \\
&\quad = -\sum_{i=1}^{n-1}(\partial_t^2|_0 c^i).0 - 2\sum_{i=1}^{n-1}(\partial_t|_0 c^i)\partial_t|_0\bar{g}(\partial_i f, b_2.\nu^{f_0}) - \sum_{i=1}^{n-1} 0.\partial_t^2|_0\bar{g}(\partial_i f, b_2.\nu^{f_0}) \\
&\quad = -2\sum_{i=1}^{n-1}(\partial_t|_0 c^i)\bar{g}(\partial_i(a.\nu^{f_0}), b_2.\nu^{f_0}) = -2\sum_{i=1}^{n-1}(\partial_t|_0 c^i)(\partial_i a) b_2 \\
&\quad = -2 b_1 b_2 \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} g^{ij}\partial_j a.\partial_i a = -2 b_1 b_2 \left\|da\right\|_{g^{-1}}^2 .
\end{aligned}$$

The derivatives of Φ are

$$\begin{aligned}
&\partial_t|_0\big(\Phi \circ (\mathrm{Vol}, \mathrm{Tr}(L))\big) = (\partial_1\Phi).(\partial_t|_0 \mathrm{Vol}) + (\partial_2\Phi).(\partial_t|_0 \mathrm{Tr}(L)) \\
&\partial_t^2|_0\big(\Phi \circ (\mathrm{Vol}, \mathrm{Tr}(L))\big) = (\partial_1\partial_1\Phi).(\partial_t|_0 \mathrm{Vol})^2 + (\partial_2\partial_2\Phi).(\partial_t|_0 \mathrm{Tr}(L))^2 \\
&\qquad + 2(\partial_1\partial_2\Phi).(\partial_t|_0 \mathrm{Vol}).(\partial_t|_0 \mathrm{Tr}(L)) + (\partial_1\Phi)(\partial_t^2|_0 \mathrm{Vol}) + (\partial_2\Phi)(\partial_t^2|_0 \mathrm{Tr}(L)).
\end{aligned}$$

Lemma. *The second derivative of the G^Φ-metric in the chart $(\pi \circ \psi)^{-1}$ is given by:*

$$\begin{aligned}
\partial_t^2|_0\big((\pi \circ \psi_{f_0})^* G^\Phi\big)_{(t.a)}(b_1, b_2) &= \Big(d^2\big((\pi \circ \psi_{f_0})^* G^\Phi\big)(0)(a,a)\Big)(b_1, b_2) \\
&= \int_M \ldots\ b_1.b_2 \,\mathrm{vol}(g)
\end{aligned} \tag{1}$$

over the following expression

$$\begin{aligned}
\ldots = \Phi\Big(\frac{\partial_t^2|_0 \mathrm{vol}}{\mathrm{vol}} - 2\left\|da\right\|_{g^{-1}}^2\Big) &+ (\partial_1\Phi).\Big((\partial_t^2|_0 \mathrm{Vol}) + 2(\partial_t|_0 \mathrm{Vol})\frac{\partial_t|_0 \mathrm{vol}}{\mathrm{vol}}\Big) \\
&+ (\partial_2\Phi).\Big((\partial_t^2|_0 \mathrm{Tr}(L)) + 2(\partial_t|_0 \mathrm{Tr}(L))\frac{\partial_t|_0 \mathrm{vol}}{\mathrm{vol}}\Big) + (\partial_1\partial_1\Phi).(\partial_t|_0 \mathrm{Vol})^2 \\
&+ 2(\partial_1\partial_2\Phi).(\partial_t|_0 \mathrm{Vol})(\partial_t|_0 \mathrm{Tr}(L)) + (\partial_2\partial_2\Phi).(\partial_t|_0 \mathrm{Tr}(L))^2.
\end{aligned}$$

3.3.3 Sectional curvature on shape space

To understand the structure in the formulas for the sectional curvature tensor, we will use some facts from linear algebra.

Lemma 1. *Let $V = C^\infty(M)$, and let P and Q be bilinear and symmetric maps $V \times V \to V$. Then*

$$\boxplus(P,Q)(a_1 \wedge a_2, b_1 \wedge b_2) := \frac{1}{2}\big(P(a_1,b_1)Q(a_2,b_2) - P(a_1,b_2)Q(a_2,b_1) \\ + P(a_2,b_2)Q(a_1,b_1) - P(a_2,b_1)Q(a_1,b_2)\big)$$

defines a symmetric, bilinear map $(V \wedge V) \otimes (V \wedge V) \to V$.

Also $\boxplus(P,Q) = \boxplus(Q,P)$. The symbol $\boxplus$ stands for the Young tableau encoding the symmetries, see [20]. We have

$$\boxplus(P,Q)(a_1 \wedge a_2, a_1 \wedge a_2) \\ = \frac{1}{2}P(a_1,a_1)Q(a_2,a_2) - P(a_1,a_2)Q(a_2,a_1) + \frac{1}{2}P(a_2,a_2)Q(a_1,a_1).$$

P is called positive semidefinite if for all $x \in M$ and $a \in C^\infty(M)$, $P(a,a)(x) \geq 0$. P is called negative semidefinite if $-P$ is positive semidefinite. We will write $P \geq 0, P \leq 0, P \lesseqgtr 0$ if P is positive semidefinite, negative semidefinite or indefinite.

Lemma 2. *If P and Q are positive semidefinite bilinear and symmetric maps $V \times V \to V$, then also $\boxplus(P,Q)$ is a positive semidefinite symmetric, bilinear map.*

Proof. To shorten notation, we will write for instance P_{12} instead of $P(a_1,a_2)$. The Cauchy inequality applied to P and Q gives us

$$P_{12}Q_{12} \leq \sqrt{P_{11}P_{22}Q_{11}Q_{22}},$$

and therefore we have

$$\begin{aligned}\boxplus(P,Q)(a_1 \wedge a_2, a_1 \wedge a_2) &= \frac{1}{2}P_{11}Q_{11} - P_{12}Q_{12} + \frac{1}{2}P_{22}Q_{22} \\ &\geq \frac{1}{2}P_{11}Q_{22} - \sqrt{P_{11}P_{22}Q_{11}Q_{22}} + \frac{1}{2}P_{22}Q_{11} \\ &= \frac{1}{2}\Big(\sqrt{P_{11}Q_{22}} - \sqrt{P_{22}Q_{11}}\Big)^2 \geq 0. \qquad \square\end{aligned}$$

Let $\lambda, \mu : V \to V$. Then the map $\lambda \otimes \mu : V \otimes V \to V$ is given by

$$(\lambda \otimes \mu)(a \otimes b) = \lambda(a).\mu(b),$$

where the multiplication is in $V = C^\infty(M)$. Denote by $\lambda \vee \mu$ the symmetrization of the tensor product given by

$$\lambda \vee \mu = \tfrac{1}{2}(\lambda \otimes \mu + \mu \otimes \lambda).$$

We will make use of the following simplifications:

Lemma 3. *Let $\lambda, \beta, \mu, \nu, : V \to V$. Then the bilinear symetric map*

$$\boxplus(\lambda \vee \beta, \mu \vee \nu)$$

satisfies the following properties:

(S1) $\boxplus(\lambda \vee \mu, \lambda \vee \nu)(a_1 \wedge a_2, a_1 \wedge a_2) = -\frac{1}{4}(\lambda \otimes \mu)(a_1 \wedge a_2).(\lambda \otimes \nu)(a_1 \wedge a_2)$,

(S2) $\boxplus(\lambda \vee \mu, \lambda \otimes \lambda) = 0$,

(S3) $\boxplus(\lambda \otimes \lambda, \mu \vee \nu)(a_1 \wedge a_2, a_1 \wedge a_2) = \frac{1}{2}(\lambda \otimes \mu)(a_1 \wedge a_2).(\lambda \otimes \nu)(a_1 \wedge a_2)$.

Proof. For the proof of simplification (S1) we calculate:

$$\begin{aligned}
&\boxplus(\lambda \vee \mu, \lambda \vee \nu)(a_1 \wedge a_2, a_1 \wedge a_2) \\
&\quad = \frac{1}{2}(\lambda \otimes \mu \otimes \lambda \otimes \nu)\Big[a_1 \otimes a_1 \otimes a_2 \otimes a_2 + \quad a_2 \otimes a_2 \otimes a_1 \otimes a_1 \\
&\qquad\qquad -\frac{1}{2}a_1 \otimes a_2 \otimes a_1 \otimes a_2 - \frac{1}{2}a_1 \otimes a_2 \otimes a_2 \otimes a_1 \\
&\qquad\qquad -\frac{1}{2}a_2 \otimes a_1 \otimes a_1 \otimes a_2 - \frac{1}{2}a_2 \otimes a_1 \otimes a_2 \otimes a_1\Big]
\end{aligned}$$

Using the symmetries of the quasilinear mapping $\lambda \otimes \mu \otimes \lambda \otimes \mu$, we can swap the first and third position in the tensor product of the two summands in the first line. Then the expression inside the square brackets equals $-\frac{1}{2}(a_1 \wedge a_2) \otimes (a_1 \wedge a_2)$.

Since $\lambda \otimes \lambda$ vanishes when applied to elements of $V \wedge V$, simplification (S2) is a direct consequence of (S1).

For the proof of simplification (S3) we calculate:

$$\begin{aligned}
&\boxplus(\lambda \otimes \lambda, \mu \vee \nu)(a_1 \wedge a_2, a_1 \wedge a_2) \\
&\quad = \frac{1}{2}(\lambda \otimes \lambda \otimes \mu \otimes \nu)\Big[a_1 \otimes a_1 \otimes a_2 \otimes a_2 + a_2 \otimes a_2 \otimes a_1 \otimes a_1 \\
&\qquad\qquad -a_1 \otimes a_2 \otimes a_1 \otimes a_2 - a_1 \otimes a_2 \otimes a_2 \otimes a_1\Big]
\end{aligned}$$

Using symmetries as above, we can replace third summand $a_1 \otimes a_2 \otimes a_1 \otimes a_2$ by $a_2 \otimes a_1 \otimes a_2 \otimes a_1$, because the first two tensor components of $\lambda \otimes \lambda \otimes \mu \otimes \nu$ are equal. Then, swapping the second and third position in all tensor products, we get

$$\begin{aligned}
&\boxplus(\lambda \otimes \lambda, \mu \otimes \nu)(a_1 \wedge a_2, a_1 \wedge a_2) \\
&\quad = \frac{1}{2}(\lambda \otimes \mu \otimes \lambda \otimes \nu)\Big[a_1 \otimes a_2 \otimes a_1 \otimes a_2 + a_2 \otimes a_1 \otimes a_2 \otimes a_1 \\
&\qquad\qquad -a_2 \otimes a_1 \otimes a_1 \otimes a_2 - a_1 \otimes a_2 \otimes a_2 \otimes a_1\Big]
\end{aligned}$$

The expression inside the square brackets equals $(a_1 \wedge a_2) \otimes (a_1 \wedge a_2)$. □

For orthonormal a_1, a_2 sectional curvature is the negative of the curvature tensor $R_0(a_1, a_2, a_1, a_2)$. We will use the following *formula for the curvature* *tensor*, which is valid in a chart:

$$\begin{aligned}
&R_0(a_1,a_2,a_1,a_2) = G_0^\Phi(R_0(a_1,a_2)a_1,a_2) = \\
(1)\qquad &\frac{1}{2}d^2G_0^\Phi(a_1,a_1)(a_2,a_2) - d^2G_0^\Phi(a_1,a_2)(a_1,a_2) + \frac{1}{2}d^2G_0^\Phi(a_2,a_2)(a_1,a_1) \\
&\quad + G_0^\Phi(\Gamma_0(a_1,a_1),\Gamma_0(a_2,a_2)) - G_0^\Phi(\Gamma_0(a_1,a_2),\Gamma_0(a_1,a_2)).
\end{aligned}$$

Looking at Formula (1) from section 3.3.2 we can express the second derivative of the metric G^Φ in the chart as

$$\begin{aligned}
&\Big(d^2(\pi\circ\psi_{f_0})^*G^\Phi(0)(a_1,a_2)\Big)(b_1,b_2) \\
&= \int_M \Big(\Phi.P_1(a_1,a_2) + (\partial_1\Phi)P_2(a_1,a_2) + (\partial_2\Phi)P_3(a_1,a_2) + (\partial_1\partial_1\Phi)P_4(a_1,a_2) \\
&\qquad + (\partial_1\partial_2\Phi)P_5(a_1,a_2) + (\partial_2\partial_2\Phi)P_6(a_1,a_2)\Big)P(b_1,b_2)\operatorname{vol}(g),
\end{aligned}$$

where $P(b_1,b_2) = b_1.b_2$, so $P = \mathrm{id}\otimes\mathrm{id}$, and where the P_i are obtained by symmetrizing the terms in Formula (1) from section 3.3.2.

For the rest of this section, we do not note the pullback via the chart anymore, writing G_0^Φ instead of $\big((\pi\circ\psi_{f_0})^*G^\Phi\big)(0)$, for example. To further shorten our notation, we write L instead of L^{f_0} and g instead of g_0. The following terms are calculated using the variational formulas from section 1.3.

$$\begin{aligned}
P_1(a,a) &= \frac{\partial_t^2|_0\operatorname{vol}}{\operatorname{vol}} - 2\,\|da\|^2_{g^{-1}} = a^2\big(\operatorname{Tr}(L)^2 - \operatorname{Tr}(L^2)\big) - \|da\|^2_{g^{-1}} \\
P_2(a,a) &= (\partial_t^2|_0\operatorname{Vol}) + 2(\partial_t|_0\operatorname{Vol})\frac{\partial_t|_0\operatorname{vol}}{\operatorname{vol}} \\
&= \int_M a^2\big(\operatorname{Tr}(L)^2 - \operatorname{Tr}(L^2)\big) + \int_M \|da\|^2_{g^{-1}}\operatorname{vol}(g) \\
&\quad + 2\operatorname{Tr}(L).a\int_M \operatorname{Tr}(L).a\operatorname{vol}(g) \\
P_3(a,a) &= (\partial_t^2|_0\operatorname{Tr}(L)) + 2(\partial_t|_0\operatorname{Tr}(L))\frac{\partial_t|_0\operatorname{vol}}{\operatorname{vol}} \\
&= 2a^2\operatorname{Tr}(L^3) + 4a\operatorname{Tr}\big(L.g^{-1}.\nabla^2 a\big) + 2\operatorname{Tr}(g^{-1}(da\otimes da)L) \\
&\quad - \|da\|^2_{g^{-1}}\operatorname{Tr}(L) + 2a\operatorname{Tr}^g\big(\nabla_{\operatorname{grad}a}s\big) \\
&\quad + 2\big(-\Delta a + a\operatorname{Tr}(L^2)\big)(-\operatorname{Tr}(L).a) \\
&= 2a^2\operatorname{Tr}(L^3) + 4a\operatorname{Tr}\big(L.g^{-1}.\nabla^2 a\big) + 2\operatorname{Tr}(g^{-1}(da\otimes da)L) \\
&\quad - \|da\|^2_{g^{-1}}\operatorname{Tr}(L) + 2a\operatorname{Tr}^g\big(\nabla_{\operatorname{grad}a}s\big) \\
&\quad + 2\operatorname{Tr}(L)a\Delta a - 2\operatorname{Tr}(L)\operatorname{Tr}(L^2).a^2 \\
P_4(a,a) &= (\partial_t|_0\operatorname{Vol})^2 = \Big(\int_M \operatorname{Tr}(L).a\operatorname{vol}(g)\Big)^2 \\
P_5(a,a) &= 2(\partial_t|_0\operatorname{Vol})(\partial_t|_0\operatorname{Tr}(L)) = 2\int_M -\operatorname{Tr}(L).a\operatorname{vol}(g)\big(-\Delta a + a\operatorname{Tr}(L^2)\big) \\
&= 2\Delta a\int_M \operatorname{Tr}(L).a\operatorname{vol}(g) - 2\operatorname{Tr}(L^2)a\int_M \operatorname{Tr}(L).a\operatorname{vol}(g)
\end{aligned}$$

$$\begin{aligned} P_6(a,a) &= (\partial_t|_0 \operatorname{Tr}(L))^2 = \big(-\Delta a + a \operatorname{Tr}(L^2)\big)^2 \\ &= (\Delta a)^2 - 2a\Delta a \operatorname{Tr}(L^2) + a^2 \operatorname{Tr}(L^2)^2 \end{aligned}$$

Then the *first part of the curvature tensor* is given by

$$\begin{aligned} &\frac{1}{2} d^2 G_0^\Phi(a_1,a_1)(a_2,a_2) - d^2 G_0^\Phi(a_1,a_2)(a_1,a_2) + \frac{1}{2} d^2 G_0^\Phi(a_2,a_2)(a_1,a_1) \\ &\quad = \int_M \big(\Phi . \boxplus(P_1,P) + (\partial_1 \Phi) \boxplus(P_2,P) + (\partial_2\Phi)\boxplus(P_3,P) \\ &\qquad + (\partial_1\partial_1\Phi)\boxplus(P_4,P) + (\partial_1\partial_2\Phi)\boxplus(P_5,P) + (\partial_2\partial_2\Phi)\boxplus(P_6,P)\big) \operatorname{vol}(g) \\ &\qquad \cdot (a_1 \wedge a_2, a_1 \wedge a_2). \end{aligned}$$

Note that P is positive definite, so that $\boxplus(P_i,P)$ is positive semidefinite if P_i is positive semidefinite. We can always assume that Φ is positive because otherwise G^Φ would not be a Riemannian metric.

$\boxed{P_1 P}$ $\qquad P_1 = P_1^1 + P_1^2,$

with

$$\begin{aligned} P_1^1 &= (\operatorname{Tr}(L)^2 - \operatorname{Tr}(L^2)) \operatorname{id} \otimes \operatorname{id} \\ P_1^2 &= -\operatorname{Tr}^g(d \otimes d) \end{aligned}$$

Applying simplification (S3) to $\boxplus(P_1^1,P)$ and $\boxplus(P_1^2,P)$, we get

$$\boxplus(P_1^1,P) = \frac{1}{2}(\operatorname{Tr}(L)^2 - \operatorname{Tr}(L^2))(\operatorname{id}\otimes\operatorname{id})^2 = 0$$

on $(V \wedge V) \otimes (V \wedge V)$ and

$$\begin{aligned} &\boxplus(P_1^2,P) = -\frac{1}{2}\operatorname{Tr}^g\big((\operatorname{id}\otimes d)^2\big), \\ &\boxplus(P_1^2,P)(a_1\wedge a_2, a_1\wedge a_2) = -\frac{1}{2}\,\|a_1 da_2 - a_2 da_1\|^2_{g^{-1}} \leq 0. \end{aligned}$$

Therefore we have

$$\int_M \Phi . \boxplus(P_1,P)(a_1\wedge a_2, a_1\wedge a_2) \operatorname{vol}(g) \leq 0.$$

$\boxed{P_2 P}$ $\qquad P_2 = P_2^1 + P_2^2 + P_2^3$

with

$$\begin{aligned} P_2^1 &= \int_M (\operatorname{id}\otimes\operatorname{id})(\operatorname{Tr}(L)^2 - \operatorname{Tr}(L^2)) \operatorname{vol}(g) \\ P_2^2 &= 2\operatorname{Tr}(L)(\operatorname{id} \vee \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g)) \end{aligned}$$

$$P_2^3 = \int_M \operatorname{Tr}^g(d \otimes d) \operatorname{vol}(g)$$

P_2^1 is indefinite. Applying Simplification (S2) we get $\boxplus(P_2^2, P) = 0$. P_2^3 and therefore also $\boxplus(P_2^3, P)$ is positive semidefinite. Therefore

$$\int_M (\partial_1\Phi) \boxplus(P_2^1, P)(a_1 \wedge a_2, a_1 \wedge a_2) \operatorname{vol}(g) \lesseqgtr 0,$$
$$\int_M (\partial_1\Phi) \boxplus(P_2^3, P)(a_1 \wedge a_2, a_1 \wedge a_2) \operatorname{vol}(g) \geq 0$$

$\boxed{P_3 P}$ $$P_3 = P_3^1 + P_3^2 + P_3^3,$$

with

$$\begin{aligned} P_3^1 &= 2 \operatorname{id} \vee \Big(\operatorname{Tr}(L^3) \operatorname{id} + 2 \operatorname{Tr}(L g^{-1} \nabla^2(\operatorname{id})) + \operatorname{Tr}^g(\nabla_{\operatorname{grad} id} s) \\ &\qquad + \operatorname{Tr}(L)\Delta(\operatorname{id}) - \operatorname{Tr}(L) \operatorname{Tr}(L)^2 \operatorname{id} \Big) \\ P_3^2 &= 2 \operatorname{Tr}\left(g^{-1}(d \otimes d) L\right) \\ P_3^3 &= -\operatorname{Tr}^g(d \otimes d) \operatorname{Tr}(L) \end{aligned}$$

Applying Simplification (S2) we get that $\boxplus(P_3^1, P)$ vanishes. Furthermore,

$$\begin{aligned} \boxplus(P_3^2, P)(a_1 \wedge a_2, a_1 \wedge a_2) &= a_1^2 \operatorname{Tr}(g^{-1}(da_2 \otimes da_2).L) \\ &\quad - 2a_1 a_2 \operatorname{Tr}(g^{-1}(da_1 \otimes da_2).L) \\ &\quad + a_2^2 \operatorname{Tr}(g^{-1}(da_1 \otimes da_1).L) \\ &= g_2^0\big((a_1 da_2 - a_2 da_1) \otimes (a_1 da_2 - a_2 da_1), s)\big) \lesseqgtr 0 \\ \boxplus(P_3^3, P)(a_1 \wedge a_2, a_1 \wedge a_2) &= -\frac{1}{2} \left\| a_1 da_2 - a_2 da_1 \right\|_{g^{-1}}^2 \operatorname{Tr}(L) \lesseqgtr 0 \end{aligned}$$

$\boxed{P_4 P}$ $$P_4 = \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g) \otimes \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g)$$

Applying Simplification (S3) we get

$$\boxplus(P_4, P) = \frac{1}{2} \Big(\operatorname{id} \otimes \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g) \Big)^2.$$

Therefore, if $\partial_1\partial_1\Phi \geq 0$

$$\int_M \partial_1\partial_1\Phi \boxplus(P_4, P)(a_1 \wedge a_2, a_1 \wedge a_2) \operatorname{vol}(g) \geq 0,$$

$\boxed{P_5 P}$ $$P_5 = P_5^1 + P_5^2$$

with

$$\begin{aligned} P_5^1 &= 2 \Big(\Delta \vee \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g) \Big) \\ P_5^2 &= -2 \operatorname{Tr}(L^2) \Big(\operatorname{id} \vee \int_M \operatorname{Tr}(L) a_2 \operatorname{vol}(g) \Big) \end{aligned}$$

Applying Simplification (S3) we get that $\boxplus(P_5^1, P)$ is the indefinite form given by

$$\boxplus(P_5^1, P) = \big(\operatorname{id} \otimes \Delta\big) \otimes \big(\operatorname{id} \otimes \int_M \operatorname{Tr}(L) \operatorname{id} \operatorname{vol}(g)\big)$$

Simplification (S2) gives $\boxplus(P_5^2, P) = 0$. Therefore

$$\int_M (\partial_1\partial_2\Phi) \boxplus(P_5^1, P)(a_1 \wedge a_2, a_1 \wedge a_2) \operatorname{vol}(g) \lesseqgtr 0.$$

$\boxed{P_6 P}$ $\quad P_6 = P_6^1 + P_6^2$

with

$$\begin{aligned}
P_6^1 &= \Delta \otimes \Delta \\
P_6^2 &= \operatorname{Tr}(L^2)^2 \operatorname{id} \otimes \operatorname{id} \\
P_6^3 &= -2 \operatorname{Tr}(L^2) \operatorname{id} \vee \Delta
\end{aligned}$$

Applying Simplification (S2) we get that $\boxplus(P_6^2, P)$ and $\boxplus(P_6^3, P)$ vanish. Simplification (S3) gives

$$\boxplus(P_6^1, P) = \frac{1}{2}(\operatorname{id} \otimes \Delta)^2$$

We get

$$\int_M (\partial_2\partial_2\Phi) \boxplus(P_6, P)(a_1 \wedge a_2, a_1 \wedge a_2) \operatorname{vol}(g) \geq 0$$

if $\partial_2\partial_2\Phi \geq 0$.

Now we come to the *second part of the curvature tensor* $R_0(a_1, a_2, a_1, a_2)$, which is given by

$$G_0(\Gamma_0(a_1, a_1), \Gamma_0(a_2, a_2)) - G_0(\Gamma_0(a_1, a_2), \Gamma_0(a_1, a_2)).$$

From the geodesic equation calculated in section 3.2, which is given by

$$\begin{aligned}
a_t = \Gamma_0(a, a) = \frac{1}{\Phi}\Big[&\frac{1}{2}\Phi a^2 \operatorname{Tr}(L) - \frac{1}{2}\operatorname{Tr}(L)\int_M (\partial_1\Phi) a^2 \operatorname{vol}(g) - \frac{1}{2}a^2 \Delta(\partial_2\Phi) \\
&+ 2a \operatorname{Tr}^g(d(\partial_2\Phi) \otimes da) + (\partial_2\Phi)\operatorname{Tr}^g(da \otimes da) \\
&+ (\partial_1\Phi) a \int_M \operatorname{Tr}(L).a \operatorname{vol}(g) - \frac{1}{2}(\partial_2\Phi)\operatorname{Tr}(L^2)a^2\Big],
\end{aligned}$$

we can extract the Christoffel symbol by symmetrization and get

$$\Gamma_0(a_1, a_2) = \frac{1}{\Phi}\sum_{i=1}^{5} Q_i(a_1, a_2),$$

where $Q_1, \dots, Q_5$ are the symmetrizations of the summands in the geodesic equation. Q_i are given by

$$\begin{aligned}
Q_1 &= \frac{1}{2}\Big(\Phi \operatorname{Tr}(L) - \Delta(\partial_2\Phi) - (\partial_2\Phi)\operatorname{Tr}(L^2)\Big)\operatorname{id}\otimes\operatorname{id},\\
Q_2 &= -\frac{1}{2}\operatorname{Tr}(L)\int_M (\partial_1\Phi)\operatorname{id}\otimes\operatorname{id}\operatorname{vol}(g),\\
Q_3 &= 2\operatorname{id}\vee\operatorname{Tr}^g(d(\partial_2\Phi)\otimes d)\\
Q_4 &= (\partial_1\Phi)\operatorname{id}\vee\int_M \operatorname{Tr}(L)\operatorname{id}\operatorname{vol}(g),\\
Q_5 &= (\partial_2\Phi)\operatorname{Tr}^g(d\otimes d).
\end{aligned}$$

Then

$$\begin{aligned}
&G_0(\Gamma_0(a_1,a_1),\Gamma_0(a_2,a_2)) - G_0(\Gamma_0(a_1,a_2),\Gamma_0(a_1,a_2))\\
&\quad= \int_M \frac{1}{\Phi}\sum_i \boxplus(Q_i,Q_i)(a_1\wedge a_2, a_1\wedge a_2)\operatorname{vol}(g)\\
&\qquad+ \int_M \frac{2}{\Phi}\sum_{i<j}\boxplus(Q_i,Q_j)(a_1\wedge a_2, a_1\wedge a_2)\operatorname{vol}(g).
\end{aligned}$$

The contribution of the following terms to $R_0(a_1,a_2,a_1,a_2)$ is $\int_M \frac{1}{\Phi}\dots\operatorname{vol}(g)$ over the terms listed.

$\boxed{Q_1Q_1}$ $\quad \boxplus(Q_1,Q_1) = 0$

according to Simplification (S2).

$\boxed{Q_2Q_2}$ $\quad \boxplus(Q_2,Q_2)(a_1\wedge a_2, a_1\wedge a_2) = \dfrac{\operatorname{Tr}(L)^2}{4}\Big[$

$$\int_M (\partial_1\Phi)a_1^2\operatorname{vol}(g).\int_M (\partial_1\Phi)a_2^2\operatorname{vol}(g) - \Big(\int_M (\partial_1\Phi)a_1a_2\operatorname{vol}(g)\Big)^2\Big]$$

which is positive by the Cauchy-Schwarz inequality, assuming that $\partial_1\Phi \geq 0$.

$\boxed{Q_3Q_3}$ $\quad \boxplus(Q_3,Q_3)(a_1\wedge a_2, a_1\wedge a_2) =$

$$\begin{aligned}
&= -\Big(\big(\operatorname{id}\otimes\operatorname{Tr}^g(d(\partial_2\Phi)\otimes d)\big)(a_1\wedge a_2)\Big)^2\\
&= -g^{-1}\big(d(\partial_2\Phi), a_1da_2 - a_2da_1\big)^2 \leq 0
\end{aligned}$$

according to Simplification (S1).

$\boxed{Q_4Q_4}$ $\quad \boxplus(Q_4,Q_4) = -\dfrac{1}{4}(\partial_1\Phi)^2(\operatorname{id}\otimes\displaystyle\int_M \operatorname{Tr}(L)\operatorname{id}\operatorname{vol}(g))^2 \leq 0$

according to Simplification (S1).

$\boxed{Q_5Q_5}$ $\quad \boxplus(Q_5,Q_5) = (\partial_2\Phi)^2\big(\,\|da_1\|^2_{g^{-1}}\,\|da_2\|^2_{g^{-1}} - g^{-1}(da_1,da_2)^2\big)$

$$= (\partial_2\Phi)^2 \, \|da_1 \wedge da_2\|^2_{g^0_2} \geq 0$$

by the Cauchy-Schwarz inequality.

The contribution of the following terms to $R_0(a_1, a_2, a_1, a_2)$ is $\int_M \frac{2}{\Phi} \ldots \mathrm{vol}(g)$ over the terms listed.

$\boxed{Q_1 Q_2}$ $$\boxplus(Q_1, Q_2) = -\frac{1}{4}\big(\Phi \operatorname{Tr}(L)^2 - \operatorname{Tr}(L)\Delta(\partial_2\Phi) - \operatorname{Tr}(L)\operatorname{Tr}(L^2)(\partial_2\Phi)\big). \\ . \boxplus\Big(\mathrm{id} \otimes \mathrm{id}, \int_M (\partial_1\Phi)\, \mathrm{id} \otimes \mathrm{id} \, \mathrm{vol}(g)\Big),$$

where the second factor is ≥ 0 assuming that $\partial_1\Phi \geq 0$.

$\boxed{Q_1 Q_3}$ $\boxplus(Q_1, Q_3) = 0$

according to Simplification (S2).

$\boxed{Q_1 Q_4}$ $\boxplus(Q_1, Q_4) = 0$

according to Simplification (S2).

$\boxed{Q_1 Q_5}$ $$\boxplus(Q_1, Q_5) = \frac{1}{4}\big(\Phi \operatorname{Tr}(L)(\partial_2\Phi) - (\partial_2\Phi)\Delta(\partial_2\Phi) - \operatorname{Tr}(L^2)(\partial_2\Phi)^2\big). \\ . \, \|a_1 da_2 - a_2 da_1\|^2_{g^{-1}}$$

$\boxed{Q_2 Q_3}$ $\boxplus(Q_2, Q_3) \lesseqgtr 0$

$\boxed{Q_2 Q_4}$ $$\boxplus(Q_2, Q_4) = -\frac{1}{2}(\partial_1\Phi)\operatorname{Tr}(L) \cdot \\ \cdot \boxplus\Big(\int_M (\partial_1\Phi)\, \mathrm{id} \otimes \mathrm{id}\, \mathrm{vol}(g), \mathrm{id} \vee \int_M \operatorname{Tr}(L)\, \mathrm{id}\, \mathrm{vol}(g)\Big)$$

This form is indefinite, but we have

$$\int_M \frac{2}{\Phi} \boxplus(Q_2, Q_4)\, \mathrm{vol}(g) = -\boxplus(\tilde{Q}_2, \tilde{Q}_4),$$

with the positive semidefinite form

$$\tilde{Q}_2 = \int_M (\partial_1\Phi)\, \mathrm{id} \otimes \mathrm{id}\, \mathrm{vol}(g),$$

and the form

$$\tilde{Q}_4 = \int_M \operatorname{Tr}(L)\frac{1}{\Phi}(\partial_1\Phi)\, \mathrm{id}\, \mathrm{vol}(g) \vee \int_M \operatorname{Tr}(L)\, \mathrm{id}\, \mathrm{vol}(g),$$

which is positive semidefinite if $\frac{\partial_1\Phi}{\Phi}$ is a non-negative constant.

$\boxed{Q_2 Q_5}$ $\boxplus(Q_2, Q_5) = -\frac{1}{2}(\partial_2\Phi)\operatorname{Tr}(L) \cdot$

$$\cdot \boxplus \Big(\int_M (\partial_1 \Phi)\, \mathrm{id} \otimes \mathrm{id}\, \mathrm{vol}(g), \mathrm{Tr}^g(d \otimes d) \Big) \lesseqgtr 0,$$

because of the factor $(\partial_2\Phi)\,\mathrm{Tr}(L)$. But the factor

$$\boxplus \Big(\int_M (\partial_1 \Phi)\, \mathrm{id} \otimes \mathrm{id}\, \mathrm{vol}(g), \mathrm{Tr}^g(d \otimes d) \Big)$$

is positive definite.

$\boxed{Q_3Q_4}$ $\boxplus(Q_3, Q_4) \lesseqgtr 0$

$\boxed{Q_3Q_5}$ $\boxplus(Q_3, Q_5)(a_1 \wedge a_2, a_1 \wedge a_2) =$

$$= (\partial_2\Phi)\Big(a_1 g^{-1}(d(\partial_2\Phi), da_1)\, \|da_2\|^2_{g^{-1}} - \big(a_1 g^{-1}(d(\partial_2\Phi), da_2) + a_2 g^{-1}(d(\partial_2\Phi), da_1)\big) g^{-1}(da_1, da_2) + a_2 g^{-1}(d(\partial_2\Phi), da_2)\, \|da_1\|^2_{g^{-1}} \Big)$$
$$= (\partial_2\Phi) g_2^0\big(d(\partial_2\Phi) \otimes (a_1 da_2 - a_2 da_1), da_1 \wedge da_2 \big) \lesseqgtr 0$$

$\boxed{Q_4Q_5}$ $\boxplus(Q_4, Q_5) \lesseqgtr 0$

We are now able to compile a list of all negative, positive and indefinite terms of $R_0(a_1, a_2, a_1, a_2)$. Remember that negative terms of $R_0(a_1, a_2, a_1, a_2)$ make a positive contribution to sectional curvature. Positive sectional curvature is connected to the vanishing of geodesic distance because the space wraps up on itself in tighter and tighter ways.

$\boxed{P_4P}$ $\boxed{P_6P}$ $\boxed{Q_2Q_2}$ $\boxed{Q_5Q_5}$ are positive, assuming $\partial_1\Phi, \partial_1\partial_1\Phi, \partial_2\partial_2\Phi \geq 0$.

$\boxed{P_1P}$ $\boxed{Q_3Q_3}$ $\boxed{Q_4Q_4}$ $\boxed{Q_1Q_2}$ are the negative, assuming that $\partial_1\Phi \geq 0$.

$\boxed{Q_2Q_4}$ is negative assuming that $\frac{\partial_1\Phi}{\Phi}$ is a non-negative constant, and indefinite otherwise.

$\boxed{Q_2Q_5}$ is negative assuming that $\mathrm{Tr}(L)(\partial_2\Phi)$ is positive, and indefinite otherwise.

$\boxed{P_2P}$ $\boxed{P_3P}$ $\boxed{P_5P}$ $\boxed{Q_1Q_5}$ $\boxed{Q_2Q_3}$ $\boxed{Q_3Q_4}$ $\boxed{Q_3Q_5}$ $\boxed{Q_4Q_5}$ are indefinite.

3.4 Special cases of almost local metrics

3.4.1 The G^0-metric

The G^0-metric is the special case of a G^Φ-metric with $\Phi \equiv 1$. Thus its geodesic equation can be read off from section 3.1. It reads as

$$\boxed{\begin{aligned} & f_t = a.\nu + Tf.f_t^\top \\ & f_{tt} = -\frac{1}{2}\left(\|f_t\|^2 \,\mathrm{Tr}(L).\nu + Tf.\,\mathrm{grad}^g(\|f_t\|^2) \right) + \big(\mathrm{Tr}(L).a - \mathrm{div}^g(h f_t^\top) \big)\,.f_t. \end{aligned}}$$

We have three conserved quantities, namely:

$$\boxed{\begin{array}{lr} g(f_t^\top)\operatorname{vol}(g) \in \Gamma(T^*M \otimes_M \operatorname{vol}(M)) & \text{reparametrization momentum} \\ \displaystyle\int_M f_t \operatorname{vol}(g) & \text{linear momentum} \\ \displaystyle\int_M (f \wedge f_t)\operatorname{vol}(g) \in \textstyle\bigwedge^2 \mathbb{R}^n \cong \mathfrak{so}(n)^* & \text{angular momentum} \end{array}}$$

The geodesic equation on $B_i(M, \mathbb{R}^n)$ is well studied. We can read it off from section 3.2:

$$\boxed{f_t = a.\nu, \qquad a_t = \frac{\operatorname{Tr}(L).a^2}{2}.}$$

Sectional curvature is given by

$$\boxed{R_0(a_1, a_2, a_2, a_1) = \frac{1}{2}\int_M \|a_1 da_2 - a_2 da_1\|^2_{g^{-1}} \operatorname{vol}(g) \geq 0.}$$

This formula is in accordance with [35, section 4.5] since we have codimension one and a flat ambient space, so that only term(6) remains, and for the case of plain curves, it is in accordance with [37, section 3.5].

The G^0-metric induces vanishing geodesic distance, see section 2.4.5.

3.4.2 The G^A-metric

For a constant $A > 0$, the G^A-metric is defined as

$$G^A_f(h,k) = \int_M (1 + A\operatorname{Tr}(L)^2)\bar{g}(h,k)\operatorname{vol}(g).$$

This metric has been introduced by [36, 35, 37]. It corresponds to an almost local metric G^Φ with $\Phi(x, y, z) = (1 + Ay^2)$, thus its geodesic equation on $\operatorname{Imm}(M, \mathbb{R}^n)$ is given by (see section 3.1):

$$\boxed{\begin{aligned} & f_t = a.\nu + Tf.f_t^\top, \\ & f_{tt} = \frac{1}{2}\Big[- \frac{\Delta\big((2A\operatorname{Tr}(L))\,\|f_t\|^2\big)}{1 + A\operatorname{Tr}(L)^2} + \|f_t\|^2 . \operatorname{Tr}(L)\Big(\frac{2A\operatorname{Tr}(L^2)}{1 + A\operatorname{Tr}(L)^2} - 1\Big)\Big]\nu \\ & \qquad + \frac{Tf.\Big[(2A\operatorname{Tr}(L))\,\|f_t\|^2 \operatorname{grad}^g(\operatorname{Tr}(L)) - \operatorname{grad}^g((1 + A\operatorname{Tr}(L)^2)\,\|f_t\|^2)\Big]}{2(1 + A\operatorname{Tr}(L)^2)} \\ & \qquad - \Big[\frac{(2A\operatorname{Tr}(L))}{1 + A\operatorname{Tr}(L)^2}\big(- \Delta a + a\operatorname{Tr}(L^2) + d\operatorname{Tr}(L)(f_t^\top)\big) \\ & \qquad + \operatorname{div}^g(f_t^\top) - \operatorname{Tr}(L).a\Big] f_t. \end{aligned}}$$

The conserved quantities have the form

$$\boxed{\begin{array}{rr} (1+A\operatorname{Tr}(L)^2)\,g(f_t^\top)\operatorname{vol}(g)\in\Gamma(T^*M\otimes_M\operatorname{vol}(M)) & \text{reparam. momentum}\\ \displaystyle\int_M(1+A\operatorname{Tr}(L)^2)f_t\operatorname{vol}(g) & \text{linear momentum}\\ \displaystyle\int_M(1+A\operatorname{Tr}(L)^2)(f\wedge f_t)\operatorname{vol}(g)\in{\textstyle\bigwedge}^2\mathbb{R}^n\cong\mathfrak{so}(n)^* & \text{angular momentum}\end{array}}$$

The horizontal geodesic equation for the G^A metric reduces to

$$\boxed{\begin{aligned} &f_t=a.\nu\\ &a_t=\frac{1}{2}a^2\operatorname{Tr}(L)+\frac{-a^2A\Delta(\operatorname{Tr}(L))+4Aag^{-1}(d\operatorname{Tr}(L),da)}{(1+A\operatorname{Tr}(L)^2)}\\ &\qquad+\frac{2A\operatorname{Tr}(L)\,\|da\|^2_{g^{-1}}-A\operatorname{Tr}(L)\operatorname{Tr}(L^2)a^2}{(1+A\operatorname{Tr}(L)^2)}\end{aligned}}$$

For the case of curves immersed in $\mathbb{R}^2$, this formula specializes to the formula given in [36, section 4.2]. (When verifying this, remember that $\Delta=-D_s^2$ in the notation of [36].)

The curvature tensor $R_0(a_1,a_2,a_1,a_2)$ is the sum of:

$\boxed{P_1P}$ $\boxed{Q_3Q_3}$ negative terms,

$\boxed{P_6P}$ $\boxed{Q_5Q_5}$ positive terms, and

$\boxed{P_3P}$ $\boxed{Q_1Q_5}$ $\boxed{Q_3Q_5}$ indefinite terms.

$$\boxed{\begin{aligned} R_0(a_1,a_2,a_1,a_2)&=\int_M A(a_1\Delta a_2-a_2\Delta a_1)^2\operatorname{vol}(g)\\ &+\int_M 2A\operatorname{Tr}(L)g_2^0\big((a_1da_2-a_2da_1)\otimes(a_1da_2-a_2da_1),s\big)\operatorname{vol}(g)\\ &+\int_M\frac{1}{1+A\operatorname{Tr}(L)^2}\Big[-4A^2g^{-1}\big(d\operatorname{Tr}(L),a_1da_2-a_2da_1\big)^2\\ &-\Big(\frac{1}{2}\big(1+A\operatorname{Tr}(L)^2\big)^2+2A^2\operatorname{Tr}(L)\Delta(\operatorname{Tr}(L))+2A^2\operatorname{Tr}(L^2)\operatorname{Tr}(L)^2\Big)\cdot\\ &\quad\cdot\|a_1da_2-a_2da_1\|^2_{g^{-1}}+(2A^2\operatorname{Tr}(L)^2)\,\|da_1\wedge da_2\|^2_{g_0^2}\\ &+(8A^2\operatorname{Tr}(L))g_2^0\big(d\operatorname{Tr}(L)\otimes(a_1da_2-a_2da_1),da_1\wedge da_2\big)\Big]\operatorname{vol}(g)\end{aligned}}$$

We want to express the curvature in terms of the basic skew symmetric forms. Therefore, mimicking the notation of [36, 37] we define

$$W_2=a_1da_2-a_2da_1,\quad W_{22}=a_1\Delta a_2-a_2\Delta a_1,\quad W_{12}=da_1\wedge da_2.$$

Then the above equation reads as:

$$\boxed{\begin{aligned}
R_0(a_1,a_2,a_1,a_2) &= \int_M AW_{22}^2 \operatorname{vol}(g) + \int_M 2A\operatorname{Tr}(L) g_2^0\big(W_2\otimes W_2, s\big)\operatorname{vol}(g)\\
&+ \int_M \frac{1}{1+A\operatorname{Tr}(L)^2}\bigg[-4A^2 g^{-1}\big(d\operatorname{Tr}(L),W_2\big)^2\\
&- \Big(\frac{1}{2}\big(1+A\operatorname{Tr}(L)^2\big)^2 + 2A^2\operatorname{Tr}(L)\Delta(\operatorname{Tr}(L)) + 2A^2\operatorname{Tr}(L^2)\operatorname{Tr}(L)^2\Big)\,\|W_2\|_{g^{-1}}^2\\
&+ (2A^2\operatorname{Tr}(L)^2)\,\|W_{12}\|_{g_0^2}^2 + (8A^2\operatorname{Tr}(L))g_2^0\big(d\operatorname{Tr}(L)\otimes W_2, W_{12}\big)\bigg]\operatorname{vol}(g)
\end{aligned}}$$

For the case of plain curves, this formula specializes to the formula given in [37, section 3.6].

The G^A-metric satisfies condition (1) from section 2.4, thus it induces non-vanishing geodesic distance.

3.4.3 The G^B-metric

For a constant $B > 0$, the G^B-metric is defined as

$$G_f^A(h,k) = \int_M (1+B\det(L)^2)\bar{g}(h,k)\operatorname{vol}(g).$$

This metric is another generalization of the plane curves G^A-metric. It corresponds to an almost local metric G^Φ with $\Phi(x,y,z) = (1+Bz^2)$, thus its geodesic equation on $\operatorname{Imm}(M,\mathbb{R}^n)$ is given by (see section 3.1):

$$\boxed{\begin{aligned}
f_t &= a.\nu + Tf.f_t^\top,\\
f_{tt} &= \frac{1}{2}\Big[\frac{2B\det(L)}{1+B\det(L)^2}.\operatorname{Tr}(L).\det(L).\,\|f_t\|^2\\
&\quad + \frac{1}{1+B\det(L)^2}\nabla^*\nabla^*\big((2B\det(L)).g.\operatorname{C}(L)\,\|f_t\|^2\big) - \|f_t\|^2\operatorname{Tr}(L)\Big]\nu\\
&\quad + \frac{1}{2(1+B\det(L)^2)}Tf.\Big[(2B\det(L))\,\|f_t\|^2\operatorname{grad}^g(\det(L))\\
&\qquad - \operatorname{grad}^g((1+B\det(L)^2)\,\|f_t\|^2)\Big]\\
&\quad - \bigg[\frac{2B\det(L)}{1+B\det(L)^2}\Big(\operatorname{Tr}(L).\det(L).a + g_2^0\big(g.\operatorname{C}(L),\nabla^2(a)\big)\\
&\qquad + d\det(L)(f_t^\top)\Big) + \operatorname{div}^g(f_t^\top) - \operatorname{Tr}(L).a\bigg]f_t.
\end{aligned}}$$

The conserved quantities have the form

$$\boxed{\begin{aligned} &(1+B\det(L)^2)\,g(f_t^\top)\operatorname{vol}(g)\in\Gamma(T^*M\otimes_M\operatorname{vol}(M)) && \text{reparam. momentum}\\ &\int_M(1+B\det(L)^2)f_t\operatorname{vol}(g) && \text{linear momentum}\\ &\int_M(1+B\det(L)^2)(f\wedge f_t)\operatorname{vol}(g)\in{\textstyle\bigwedge^2}\mathbb{R}^n\cong\mathfrak{so}(n)^* && \text{angular momentum}\end{aligned}}$$

The horizontal geodesic equation for the G^B metric reduces to

$$\boxed{\begin{aligned} f_t &= a.\nu,\\ a_t &= \frac{1}{2}\Big[\frac{2B\det(L)}{1+B\det(L)^2}.\operatorname{Tr}(L).\det(L).a^2\\ &\qquad+\frac{1}{1+B\det(L)^2}\nabla^*\nabla^*\big((2B\det(L)).g.\operatorname{C}(L)a^2\big)-a^2\operatorname{Tr}(L)\Big]\\ &\quad-\left[\frac{2B\det(L)}{1+B\det(L)^2}\Big(\operatorname{Tr}(L).\det(L).a+g_2^0\big(g.\operatorname{C}(L),\nabla^2(a)\big)\Big)-\operatorname{Tr}(L).a\right]a.\end{aligned}}$$

3.4.4 Conformal metrics

The conformal metrics correspond to almost local metrics G^Φ with $\Phi=\Phi(\operatorname{Vol})$. For the case of planar curves these metrics have been treated in [51, 52, 53, 41]. [41] provides very interesting estimates on geodesic distance induced by metrics with $\Phi(\operatorname{Vol})=\operatorname{Vol}$ and e^{Vol}. The geodesic equation on $\operatorname{Imm}(M,\mathbb{R}^n)$ is given by:

$$\boxed{\begin{aligned} f_t &= h = a.\nu+Tf.h^\top,\\ h_t &= -\frac{1}{2}\Big[\frac{\Phi'}{\Phi}\left(\int_M\|h\|^2\operatorname{vol}(g)\right)\operatorname{Tr}(L).\nu\\ &\qquad+\|h\|^2\operatorname{Tr}(L).\nu+Tf.\operatorname{grad}^g(\|h\|^2)\Big]\\ &\qquad+\left[\frac{\Phi'}{\Phi}\left(\int_M\operatorname{Tr}(L).a\operatorname{vol}(g)\right)+\operatorname{Tr}(L).a-\operatorname{div}^g(h^\top)\right].h\end{aligned}}$$

The conserved quantities are given by

$$\boxed{\begin{aligned} &\Phi(\operatorname{Vol})g(f_t^\top)\operatorname{vol}(g)\in\Gamma(T^*M\otimes_M\operatorname{vol}(M)) && \text{reparam. momentum}\\ &\Phi(\operatorname{Vol})\int_M f_t\operatorname{vol}(g) && \text{linear momentum}\\ &\Phi(\operatorname{Vol})\int_M(f\wedge f_t)\operatorname{vol}(g)\in{\textstyle\bigwedge^2}\mathbb{R}^n\cong\mathfrak{so}(n)^* && \text{angular momentum}\end{aligned}}$$

The horizontal part of the geodesic equation is given by

$$a_t=\bar{g}\Big(\frac{1}{2}H(a.\nu,a.\nu)-K(a.\nu,a.\nu),\nu\Big)$$

$$= -\frac{\Phi'}{2\Phi}\left(\int_M a^2 \operatorname{vol}(g)\right)\operatorname{Tr}(L) + \frac{1}{2}a^2\operatorname{Tr}(L) + \frac{\Phi'}{\Phi}\left(\int_M a.\operatorname{Tr}(L)\operatorname{vol}(g)\right)a.$$

To simplify this equation let $b(t) = \Phi(\mathrm{Vol}).a(t)$. We get

$$\begin{aligned}
b_t &= \Phi'.(D_{(f,a.\nu)}\operatorname{Vol}).a + \Phi.a_t \\
&= -\Phi'.a.\int_M \operatorname{Tr}(L).a.\operatorname{vol}(g) + \Phi\frac{1}{2}a^2.\operatorname{Tr}(L) \\
&\quad - \frac{1}{2}\Phi'\left(\int_M a^2\operatorname{vol}(g)\right).\operatorname{Tr}(L) + \Phi'.a.\int_M \operatorname{Tr}(L).a\operatorname{vol}(g) \\
&= -\frac{1}{2}\Phi'\int_M a^2\operatorname{vol}(g).\operatorname{Tr}(L) + \frac{1}{2}\Phi a^2.\operatorname{Tr}(L).
\end{aligned}$$

Thus the geodesic equation of the conformal metric G^Φ on B_i is

$$\boxed{\begin{aligned}
f_t &= \frac{b(t)}{\Phi(\mathrm{Vol})}\nu \\
b_t &= \frac{\operatorname{Tr}(L)}{2\Phi(\mathrm{Vol})}\left(b^2 - \frac{\Phi'(\mathrm{Vol})}{\Phi(\mathrm{Vol})}\int_M b^2\operatorname{vol}(g)\right).
\end{aligned}}$$

For the case of curves immersed in $\mathbb{R}^2$, this formula specializes to the formula given in [37, section 3.7].

Assuming that Φ' and Φ'' are non-negative, the curvature tensor consists of the following summands.

$\boxed{P_4P}$ $\boxed{Q_2Q_2}$ are the positive summands.

$\boxed{P_1P}$ $\boxed{Q_4Q_4}$ $\boxed{Q_1Q_2}$ are the negative summands.

$\boxed{Q_2Q_4}$ is indefinite, but assuming that $\frac{\Phi'}{\Phi}$ is a non-negative constant, it is negative. Solving the ODE $\frac{\Phi'}{\Phi} = C > 0$ leeds to $\Phi(\mathrm{Vol}) = e^{C.\mathrm{Vol}}$. In the case of curves, conformal metrics of this type have been studied by [30] and [41].

$\boxed{P_2P}$ is indefinite.

Since the formula for sectional curvature with general $\Phi = \Phi(\mathrm{Vol})$ is still too long, we will only print the formula for $\Phi(\mathrm{Vol}) = \mathrm{Vol}$. To shorten notation we will write $\overline{a}$ for the integral over $a \in C^\infty(M)$, i.e.

$$\overline{a} = \int_M a\operatorname{vol}(g).$$

Then the sectional curvature reads as:

$$\boxed{\begin{aligned}&R_0(a_1,a_2,a_1,a_2) = -\frac{1}{2}\operatorname{Vol}\int_M \|a_1 da_2 - a_2 da_1\|^2_{g^{-1}} \operatorname{vol}(g)\\
&\quad + \frac{1}{4\operatorname{Vol}}\overline{\operatorname{Tr}(L)^2}\Big(\overline{a_1^2}.\overline{a_2^2} - \overline{a_1.a_2}^2\Big)\\
&\quad + \frac{1}{4}\Big(\overline{a_1^2}.\overline{\operatorname{Tr}(L)^2 a_2^2} - 2\overline{a_1.a_2}.\overline{\operatorname{Tr}(L)^2 a_1.a_2} + \overline{a_2^2}.\overline{\operatorname{Tr}(L)^2 a_1^2}\Big)\\
&\quad - \frac{3}{4\operatorname{Vol}}\Big(\overline{a_1^2}.\overline{\operatorname{Tr}(L)a_2}^2 - 2\overline{a_1.a_2}.\overline{\operatorname{Tr}(L)a_1}.\overline{\operatorname{Tr}(L)a_2} + \overline{a_2^2}.\overline{\operatorname{Tr}(L)a_1}^2\Big)\\
&\quad + \frac{1}{2}\Big(\overline{a_1^2}.\overline{\operatorname{Tr}^g((da_2)^2)} - 2\overline{a_1.a_2}.\overline{\operatorname{Tr}^g(da_1.da_2)} + \overline{a_2^2}\overline{\operatorname{Tr}^g((da_1)^2)}\Big)\\
&\quad - \frac{1}{2}\Big(\overline{a_1^2}.\overline{a_2^2.\operatorname{Tr}(L^2)} - 2.\overline{a_1.a_2}.\overline{a_1.a_2.\operatorname{Tr}(L^2)} + \overline{a_2^2}.\overline{a_1^2.\operatorname{Tr}(L^2)}\Big).\end{aligned}}$$

For the case of curves immersed in $\mathbb{R}^2$, this formula is in accordance with the formula given in [37, section 3.7].

From Condition (2) in section 2.4 we read off that the conformal metrics induce non-vanishing geodesic distance if $\Phi(\operatorname{Vol}) \geq C.\operatorname{Vol}$ for some constant $C > 0$.

3.4.5 A scale invariant metric

For a constant $A > 0$ we define the metric

$$G_f^{SI}(h,k) = \int_M \Big(\operatorname{Vol}^{\frac{1+n}{1-n}} + A\frac{\operatorname{Tr}(L)^2}{\operatorname{Vol}}\Big)\bar{g}(h,k)\operatorname{vol}(g).$$

Scale invariance means that this metric does not change when f, h, k are replaced by $\lambda f, \lambda h, \lambda k$ for $\lambda > 0$. To see that G^{SI} is scale invariant, we calculate as in [37] how the scaling factor λ changes the metric, volume form, volume and mean curvature. We fix an oriented chart $(u^1, \ldots, u^{n-1})$ on M. Then

$$\begin{aligned}(\lambda f)^*\bar{g}(\partial_i,\partial_j) &= \bar{g}(T(\lambda f).\partial_i, T(\lambda f).\partial_j) = \lambda^2.f^*\bar{g}(\partial_i,\partial_j)\\
\operatorname{vol}((\lambda.f)^*\bar{g}) &= \sqrt{\det(\lambda^2(f^*\bar{g})|_U)}\, du^1\wedge\ldots\wedge d^{u-1} = \lambda^{n-1}\operatorname{vol}(f^*\bar{g})\\
\operatorname{Tr}(L((\lambda f)^*\bar{g}))) &= ((\lambda f)^*\bar{g})^{ij}\bar{g}(\frac{\partial^2(\lambda f)}{\partial_i\partial_j}, \nu^{\lambda.f})\\
&= \frac{\lambda}{\lambda^2}(f^*\bar{g})^{ij}\bar{g}(\frac{\partial^2 f}{\partial_i\partial_j}, \nu^f) = \frac{1}{\lambda}\operatorname{Tr}(L(f)).\end{aligned}$$

The scale invariance of the metric G^{SI} follows. Thus along geodesics we have an additional conserved quantity (see section 1.2.10), namely:

$$\boxed{\int_M \Big(\operatorname{Vol}^{\frac{1+n}{1-n}} + A\frac{\operatorname{Tr}(L)^2}{\operatorname{Vol}}\Big)\bar{g}(f,f_t)\operatorname{vol}(g) \qquad \text{scaling momentum}}$$

From 3.2 we can read off the geodesic equation for G^{SI} on B_i:

$$\boxed{\begin{aligned}
&f_t = a.\nu,\\
&a_t = \frac{1}{2}a^2\,\mathrm{Tr}(L) + \frac{1}{\mathrm{Vol}^{\frac{1+n}{1-n}} + A\frac{\mathrm{Tr}(L)^2}{\mathrm{Vol}}}\\
&\qquad \Big[-\frac{1}{2}\,\mathrm{Tr}(L)\int_M \Big(\tfrac{1+n}{1-n}\,\mathrm{Vol}^{\frac{2n}{1-n}} - A\frac{\mathrm{Tr}(L)^2}{\mathrm{Vol}^2}\Big)a^2\,\mathrm{vol}(g) - A\frac{\Delta(\mathrm{Tr}(L)).a^2}{\mathrm{Vol}}\\
&\qquad + \frac{4A.a}{\mathrm{Vol}}g^{-1}(d\,\mathrm{Tr}(L),da) + \frac{2A\,\mathrm{Tr}(L)}{\mathrm{Vol}}\,\|da\|^2_{g^{-1}}\\
&\qquad + \Big(\tfrac{1+n}{1-n}\,\mathrm{Vol}^{\frac{2n}{1-n}} - A\frac{\mathrm{Tr}(L)^2}{\mathrm{Vol}^2}\Big)a\int_M \mathrm{Tr}(L).a\,\mathrm{vol}(g) - A\frac{\mathrm{Tr}(L^2)\,\mathrm{Tr}(L)}{\mathrm{Vol}}a^2\Big].
\end{aligned}}$$

For the case of curves immersed in $\mathbb{R}^2$, this formula specializes to the formula given in [37, section 3.8]. (When verifying this, remember that $\Delta = -D_s^2$ in the notation of [37].)

The metric G^{SI} induces non-vanishing geodesic distance. This follows from the fact that log(Vol) is Lipschitz, see [37, section 3.8].

3.5 The set of concentric spheres

For an almost local metric, the set of spheres with common center $x \in \mathbb{R}^n$ is a totally geodesic subspace of B_i. The reason is that it is the fixed point set of a group of isometries acting on B_i, namely the group of rotations of $\mathbb{R}^n$ around x. (We also have to assume uniqueness of solutions to the geodesic equation.) For the G^A metric and plane curves the set of concentric spheres has been studied in [36] and for Sobolev type metrics they have been studied in [7, 22] Some work for the G^0-metric has also been done by [40].

Theorem. *Within a set of concentric spheres, any sphere is uniquely described by its radius r. Thus the geodesic equation within a set of concentric spheres reduces to an ordinary differential equation for the radius. It is given by:*

$$\boxed{r_{tt} = -r_t^2\frac{n-1}{\Phi}\Big[\frac{1}{2r}\Phi + \frac{\partial_1\Phi}{2}\frac{n\pi^{\frac{n}{2}}r^{n-2}}{\Gamma(1+\frac{n}{2})} + \frac{1}{2r^2}(\partial_2\Phi) + \frac{(-1)^n}{2.r^n}(\partial_3\Phi)\Big].}$$

The space of concentric spheres is geodesically complete with respect to a G^Φ metric iff

$$\int_0^{r_1} r^{\frac{n-1}{2}}\sqrt{\Phi\Big(\frac{n\pi^{\frac{n}{2}}r^{n-1}}{\Gamma(1+\frac{n}{2})}, -(n-1)/r, 1/(-r)^{2n-2}\Big)}dr = \infty \qquad r_1 > 0$$

and

$$\int_{r_0}^{\infty} r^{\frac{n-1}{2}}\sqrt{\Phi\Big(\frac{n\pi^{\frac{n}{2}}r^{n-1}}{\Gamma(1+\frac{n}{2})}, -(n-1)/r, 1/(-r)^{2n-2}\Big)}dr = \infty \qquad r_0 > 0.$$

For the metrics studied in this work, this yields:

$$
\begin{aligned}
\Phi &= \mathrm{Vol}^k = \frac{n^k \pi^{\frac{kn}{2}}}{\Gamma(1+\frac{n}{2})^k} r^{k(n-1)} : && \textit{incomplete} \\
\Phi &= e^{\mathrm{Vol}} = e^{\frac{n\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}} e^{r^{n-1}} : && \textit{incomplete} \\
\Phi &= 1 + A\,\mathrm{Tr}(L)^{2k} = 1 + A\tfrac{(n-1)^{2k}}{r^{2k}} : && \textit{complete iff } k \geq \frac{n+1}{2} \\
\Phi &= 1 + B\det(L)^{2l} = 1 + B\frac{1}{r^{2l(n-1)}} : && \textit{complete iff } l \geq \frac{1}{2} + \frac{1}{n-1} \\
\Phi &= \mathrm{Vol}^{\frac{1+n}{1-n}} + A\frac{\mathrm{Tr}(L)^2}{\mathrm{Vol}} = \frac{C(n)}{r^{n+1}} : && \textit{complete.}
\end{aligned}
$$

Proof. The differential equation for the radius can be read off the geodesic equation in section 2.3, when it is taken into account that all functions are constant on each sphere, and that

$$
\mathrm{Vol} = \frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(1+\frac{n}{2})}, \quad L = -\tfrac{1}{r}\,\mathrm{Id}_{TM}, \quad \mathrm{Tr}(L^k) = (-1)^k \tfrac{n-1}{r^k}, \quad \det(L) = \tfrac{1}{(-r)^{n-1}}.
$$

To determine whether the space of concentric spheres is complete, we calculate the length of a path f connecting a sphere with radius r_0 to a sphere with radius r_1:

$$
\begin{aligned}
L_{B_i}^{G^\Phi}(f) &= \int_0^1 \sqrt{G_f^\Phi(f_t^\perp, f_t^\perp)}dt = \\
&\int_0^1 \sqrt{\int_M \Phi\big(\mathrm{Vol}, \mathrm{Tr}(L), \det(L)\big) r_t^2 \,\mathrm{vol}(g)}dt \\
&= \int_0^1 |r_t| \sqrt{\Phi\Big(\frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(1+\frac{n}{2})}, -(n-1)/r, 1/(-r)^{2n-2}\Big) \frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(1+\frac{n}{2})}}dt \\
&= \frac{\sqrt{n}\pi^{\frac{n}{4}}}{\sqrt{\Gamma(1+\frac{n}{2})}} \int_{r_0}^{r_1} r^{\frac{n-1}{2}} \sqrt{\Phi\Big(\frac{n\pi^{\frac{n}{2}} r^{n-1}}{\Gamma(1+\frac{n}{2})}, -(n-1)/r, 1/(-r)^{2n-2}\Big)}dr. \square
\end{aligned}
$$

3.6 The Fréchet distance

The vector space structure of $\mathbb{R}^n$ allows us to define a Fréchet metric on shape space $B_i(M, \mathbb{R}^n)$. In 3.6.1 it is shown how this metric is related to a L^∞ Finsler metric, and in 3.6.2 the Fréchet metric is compared to almost local metrics.

3.6.1 Fréchet distance and Finsler metric

The Fréchet distance on shape space $B_i(M, \mathbb{R}^n)$ is defined as

$$
\mathrm{dist}_\infty^{B_i}(F_0, F_1) = \inf_{f_0, f_1} \|f_0 - f_1\|_{L^\infty},
$$

where the infimum is taken over all f_0, f_1 with $\pi(f_0) = F_0, \pi(f_1) = F_1$. As before, π denotes the projection $\pi : \mathrm{Imm} \to B_i$. Fixing f_0 and f_1, one has

$$\mathrm{dist}_\infty^{B_i}\big(\pi(f_0), \pi(f_1)\big) = \inf_\varphi \|f_0 \circ \varphi - f_1\|_{L^\infty},$$

where the infimum is taken over all $\varphi \in \mathrm{Diff}(M)$. The Fréchet distance is related to the Finsler metric

$$G^\infty : T\,\mathrm{Imm}(M, \mathbb{R}^n) \to \mathbb{R}, \qquad h \mapsto \left\|h^\perp\right\|_{L^\infty}.$$

Lemma. *The path length distance induced by the Finsler metric G^∞ provides an upper bound for the Fréchet distance:*

$$\mathrm{dist}_\infty^{B_i}(F_0, F_1) \leq \mathrm{dist}_{G^\infty}^{B_i}(F_0, F_1) = \inf_f \int_0^1 \|f_t\|_{G^\infty}\, dt,$$

where the infimum is taken over all paths

$$f : [0,1] \to \mathrm{Imm}(M, \mathbb{R}^n) \quad \textit{with} \quad \pi(f(0)) = F_0, \quad \pi(f(1)) = F_1.$$

Proof. Since any path f can be reparametrized such that f_t is normal to f, one has

$$\inf_f \int_0^1 \left\|f_t^\perp\right\|_{L^\infty} dt = \inf_f \int_0^1 \|f_t\|_{L^\infty}\, dt,$$

where the infimum is taken over the same class of paths f as described above. Therefore

$$\begin{aligned}\mathrm{dist}_\infty^{B_i}(F_0, F_1) &= \inf_f \|f(1) - f(0)\|_{L^\infty} = \inf_f \left\|\int_0^1 f_t dt\right\|_{L^\infty} \leq \inf_f \int_0^1 \|f_t\|_{L^\infty}\, dt \\ &= \inf_f \int_0^1 \left\|f_t^\perp\right\|_{L^\infty} dt = \mathrm{dist}_{G^\infty}^{B_i}(F_0, F_1).\end{aligned}$$

□

It is claimed in [30, theorem 13] that $d_\infty = \mathrm{dist}_{G^\infty}$. However, the proof given there only works on the vector space $C^\infty(M, \mathbb{R}^n)$ and not on $B_i(M, \mathbb{R}^n)$. The reason is that convex combinations of immersions are used in the proof, but that the space of immersions is not convex.

3.6.2 Almost local versus Fréchet distance on shape space

Theorem. *On the shape space $B_i(M, \mathbb{R}^n)$ the G^Φ distance can not be bounded from below by the Fréchet distance if one of the following conditions holds:*

(1) $\Phi \leq C_1 + C_2 \operatorname{Tr}(L)^{2k}$ *for* $C_1, C_2 > 0$ *and* $k < (\dim(M) + 2)/2$,

(2) $\Phi \leq C\, \mathrm{Vol}^k$ *for* $C > 0$,

(3) $\Phi \leq C\, e^{Vol}$ *for* $C > 0$,

(4) $\Phi \leq C_1 + C_2 \operatorname{del}(L)^{2l}$ *for* $C_1, C_2 > 0$ *and* $l < \dfrac{1}{2} + \dfrac{1}{\dim(M)}$.

Indeed, then the identity map

$$\mathrm{Id} : (B_i(M, \mathbb{R}^n), d_{G^\Phi}) \to (B_i(M, \mathbb{R}^n), d_\infty)$$

is not continuous.

Note that this result also holds in higher codimension $n - m$.

Proof. Let f_0 be a fixed immersion of M into $\mathbb{R}^n$, and let f_1 be a translation of f_0 by a vector h of length ℓ. We will show that the H^p-distance between $\pi(f_0)$ and $\pi(f_1)$ is bounded by a constant $2L$ that does not depend on ℓ. It follows that the H^p-distance can not be bounded from below by the Fréchet distance, and this proves the claim.

For small r_0, we calculate the G^Φ-length of the following path of immersions: First scale f_0 by a factor r_0, then translate it by h, and then scale it again until it has reached f_1. The following calculation shows that under one of the above assumption the immersion f_0 can be scaled down to zero in finite G^Φ-path length L.

$$\begin{aligned}
&L^{G^\Phi}_{\mathrm{Imm}}(r.f_0) = \\
&= \int_0^1 \sqrt{\int_M \Phi\big(\operatorname{Vol}(r.f_0), \operatorname{Tr}(L^{r.f_0}), \det(L^{r.f_0})\big)\bar{g}\big(r_t.f_0, r_t.f_0\big) \operatorname{vol}\big((r.f_0)^*\bar{g}\big)}\, dt \\
&= \int_0^1 \sqrt{\int_M r_t^2.\Phi\big(r^m \operatorname{Vol}(f_0), \frac{1}{r}\operatorname{Tr}(L^{f_0}), \frac{1}{r^m}\det(L^{f_0})\big)\bar{g}\big(f_0, f_0\big) r^m \operatorname{vol}\big(f_0^*\bar{g}\big)}\, dt \\
&= \int_1^0 \sqrt{\int_M \Phi\big(r^m \operatorname{Vol}(f_0), \frac{1}{r}\operatorname{Tr}(L^{f_0}), \frac{1}{r^m}\det(L^{f_0})\big)\bar{g}\big(f_0, f_0\big) r^m \operatorname{vol}\big(f_0^*\bar{g}\big)}\, dr
\end{aligned}$$

The last integral converges for all of the above assumptions. Scaling down to $r_0 > 0$ needs even less effort. So we see that the length of the shrinking and growing part of the path is bounded by $2L$.

The energy needed for a pure translation of the scaled immersion by distance ℓ is given by ($f = t.h$, with $\bar{g}(h, h) = \ell^2$):

$$\begin{aligned}
L^{G^\Phi}_{\mathrm{Imm}}(f) &= \int_0^1 \int_M \Phi.\bar{g}(f_t, f_t) \operatorname{vol}(g) dt \\
&= \int_0^1 \int_M \Phi.t^2\ell^2 \operatorname{vol}(g) dt = \ell^2 \int_M \Phi \operatorname{vol}(g) \\
&= \begin{cases} O(r^{(m-2k)}), & \text{if } \Phi \text{ satisfies (1)} \\ O(r^{m(k+1)}), & \text{if } \Phi \text{ satisfies (2)} \\ O(e^{r.m}.r^m), & \text{if } \Phi \text{ satisfies (3)} \end{cases}
\end{aligned}$$

This length tends to zero as r tends to zero. Therefore

$$\operatorname{dist}^{G^\Phi}_{B_i}\big(\pi(f_0), \pi(f_1)\big) \le \operatorname{dist}^{G^P}_{\mathrm{Imm}}(f_0, f_1) \le 2L. \qquad \square$$

Chapter 4

Numerical results

4.1 Discretizing the horizontal path energy

We want to solve the boundary value problem for geodesics in shape space of surfaces in $\mathbb{R}^3$ with respect to several almost local metrics, more specifically with respect to G^Φ-metrics with

$$\Phi = \mathrm{Vol}^k, \quad \Phi = e^{\mathrm{Vol}}, \quad \Phi = 1 + A\,\mathrm{Tr}(L)^{2k} + B\det(L)^{2l}$$

and the scale-invariant metric

$$\Phi = \mathrm{Vol}^{\frac{1+3}{1-3}} + A\frac{\mathrm{Tr}(L)^2}{\mathrm{Vol}}.$$

In order to solve this infinite-dimensional problem numerically, we will reduce it to a finite-dimensional problem by approximating an immersed surface by a triangular mesh. This chapter is based on [6, 8].

One approach to solve the boundary value problem is by the method of geodesic shooting. This method is based on iteratively solving the initial value problem for geodesics while suitably adapting the initial conditions.

Another approach, and the approach we will follow, is to minimize horizontal path energy

$$E^{\mathrm{hor}}(f) = \int_0^1 \int_M \Phi(\mathrm{Vol}, \mathrm{Tr}(L), \det(L))(f_t^\perp)^2 \,\mathrm{vol}(g)$$

over the set of paths f of immersions with fixed endpoints. Note that by definition, the horizontal path energy does not depend on reparametrizations of the surface. Nevertheless we want the triangular mesh to stay regular. This can be achieved by adding a penalty functional to the horizontal path energy.

4.1.1 Discrete path energy

To discretize the horizontal path energy

$$E^{\mathrm{hor}}(f) = \int_0^1 \int_M \Phi(\mathrm{Vol}, \mathrm{Tr}(L), \det(L))(f_t^{\perp})^2 \,\mathrm{vol}(g),$$

one has to find a discrete version of all the involved terms, notably the gauss and mean curvature. We will follow [44] to do this. Let V, E, F denote the vertices, edges and faces of the triangular mesh, and let $\mathrm{star}(p)$ be the set of faces surrounding vertex p. Then the discrete mean curvature at vertex p can be defined as

$$\mathrm{Tr}(L)(p) = \frac{\|\text{vector mean curvature}\|}{\|\text{vector area}\|} = \frac{\|\nabla_p(\text{surface area})\|}{\|\nabla_p(\text{enclosed volume})\|}.$$

Here ∇_p stands for a discrete gradient, and

$$(\text{vector mean curvature})_p = \nabla_p(\text{surface area}) = \sum_{(p,p_i)\in E} (\cot\alpha_i + \cot\beta_i)(p - p_i)$$

is the vector mean curvature defined by the cotangent formula. In this formula, α_i and β_i are the angles opposite the edge (p, p_i) in the two adjacent triangles. For the numerical simulation it is advantageous to express this formula in terms of scalar and cross products instead of the cotangents. Furthermore,

$$(\text{vector area})_p = \nabla_p(\text{enclosed volume}) = \sum_{f\in\mathrm{Star}(p)} \nu(f).(\text{surface area of } f)$$

is the vector area at vertex p.

The discrete Gauss curvature can be defined as

$$\det(L)(p) = \frac{\overline{\Theta}(p)}{\text{Area of star}(p)}.$$

Here $\overline{\Theta}(p)$ stands for the angular deflection at p, defined by

$$\overline{\Theta}(p) = 2\pi - \sum_{i=1}^{\#(\mathrm{star}(p))} \theta_i,$$

where θ_i denotes the internal angle of the i-th corner of vertex p and $\#(\mathrm{star}(p)$ the number of faces adjacent to vertex p. We discretize the time by

$$0 = t_1 < \ldots < t_{N+1} = 1.$$

Then the $(N-1)(\#V)$ free variables representing the path of immersions f are

$$f(t_i, p), \qquad \text{with } 2 \le i \le N,\ p \in V.$$

$f(0,p)$ and $f(1,p)$ are not free variables, since they define the fixed boundary shapes. f_t can be approximated by either forward increments

$$f_t^{fw}(t_i, p) = \frac{f(t_{i+1}, p) - f(t_i, p)}{t_{i+1} - t_i}$$

or backward increments

$$f_t^{bw}(t_i, p) = \frac{f(t_i, p) - f(t_{i-1}, p)}{t_i - t_{i-1}}.$$

We use a combination of both to make path energy symmetric. (Instead of this we could have also used the central difference quotient. However minimizing an energy functional depending on central differences favors oscillations, since they are not felt by the central differences.) Using the discrete definitions of normal vector and increments we can calculate $f_t^\perp$ at every vertex p and are now able to write down the discrete horizontal path energy:

$$\boxed{\begin{aligned} G_f^\Phi(h,k) &= \sum_{p\in V}\sum_{F\ni p} \Phi\Big(\operatorname{Vol}, \operatorname{Tr}(L)(p), \det(L)(p)\Big) \\ &\qquad .\bar{g}\big(h(p), \nu(F)\big).\bar{g}\big(k(p), \nu(F)\big)\frac{\operatorname{area}(\operatorname{star}^f(p))}{3} \\ E^{\text{hor}}(f) &= \sum_{i=1}^N \frac{t_{i+1}-t_i}{2} \\ &\qquad .\Big(G^\Phi_{f(t_i)}\big(f_t^{fw}(t_i), f_t^{fw}(t_i)\big) + G^\Phi_{f(t_{i+1})}\big(f_t^{bw}(t_{i+1}), f_t^{fw}(t_{i+1})\big)\Big). \end{aligned}}$$

This is not the only way to discretize the energy functional. There are several ways to distribute the discrete energy on faces, vertices and edges. Depending on how this was done, the minimizer converged faster, slower or even not at all. However if the minimizer converged to a smooth solution, the results were qualitatively the same. This increased our belief in the discretization. However we do not guarantee the accuracy of the simulations in this section.

This energy functional does not depend on the parametrization of the surface at each instant of time. So we are free to choose a suitable parametrization. We do this by adding to the energy functional a term penalizing irregular meshes. So instead of minimizing horizontal path energy, we minimize the sum of horizontal path energy and a penalty term. The penalty term measures the deviation of angles from the "perfect angle" 2π divided by the number of surrounding triangles, i.e.

$$\sum_{t=2}^N \sum_{p\in V} \sum_{(p,q,r)\in\Delta} \Big|\sphericalangle(pq, pr) - (\text{perfect angle})\Big|^k, \quad k \in \mathbb{N}.$$

4.1.2 Numerical implementation

Discrete path energy depends on a very high number of real variables, namely three times the number of vertices times one less than the number of time steps. In the numerical experiments that we have done, this were between 5.000 and 50.000 variables. To solve this problem we used the nonlinear solver IPOPT (Interior Point OPTimizer [50]). IPOPT uses a filter based line search method to compute the minimum. In this process it needs the gradient and the Hessian of the energy. IPOPT was invoked by AMPL (A Modeling Language for Mathematical Programming [18]). The advantage of using AMPL is that it is able to automatically calculate the gradient and Hessian. The user only has to write a model and data file for AMPL in a quite readable notation (see appendix 4.4 for the model file). The data file containing the definition of the combinatorics of the triangle mesh was automatically generated by the computer algebra system Mathematica. As an example, some discretizations of the sphere that we used can be seen in figure 4.1.

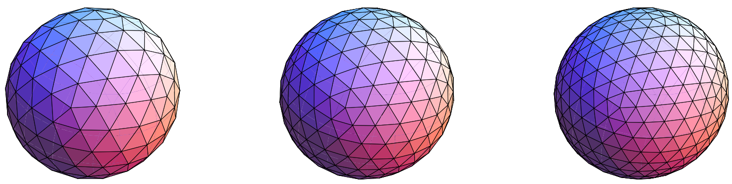

Figure 4.1: Triangulations of a sphere with 320, 500 and 720 triangles, respectively.

4.2 Scaling a sphere

In section 3.5 we studied the set of concentric spheres in n dimensions. In dimension three the geodesic equation for the radius simplifies to

$$\boxed{r_{tt} = -r_t^2 \frac{1}{\Phi}\big[\frac{1}{r}\Phi + \partial_1 \Phi 4r^2\pi + \frac{1}{r^2}(\partial_2\Phi) + \frac{1}{r^2}(\partial_3\Phi)\big].}$$

This equation is in accordance with the numerical results obtained by minimizing the discrete path energy defined in section 4.1.1. As will be seen, the numerics show that the shortest path connecting two concentric spheres in fact consists of spheres with the same center, and that the above differential equation is (at least qualitatively) satisfied. Furthermore, in our experiments the optimal paths obtained were independent of the initial path used as a starting value for the optimization.

In all numerical experiments of this section we used 50 timesteps and a triangulation with 320 triangles.

For conformal metrics of the type $\Phi = \mathrm{Vol}^k$ and $\Phi = e^{\mathrm{Vol}}$, the differential equation for the radius is:

$$\begin{aligned} \Phi = \mathrm{Vol}^k : \qquad & r_{tt} = -r_t^2 \frac{k+1}{r}, \\ \Phi = e^{\mathrm{Vol}} : \qquad & r_{tt} = -r_t^2 \Big(\frac{1}{r} + 4r\pi\Big). \end{aligned}$$

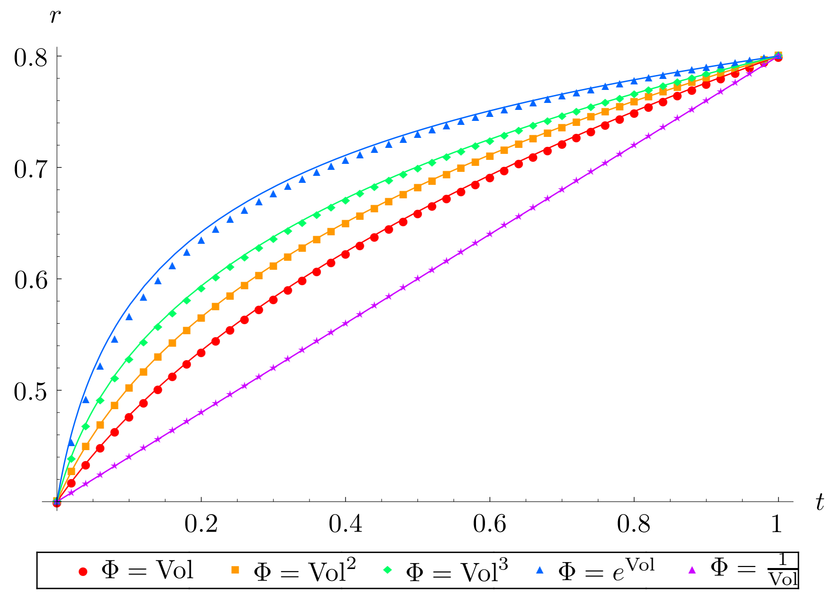

Figure 4.2: Geodesics between concentric spheres of radius 0.4 to 0.8 for several conformal metrics. Solid lines are the exact solutions.

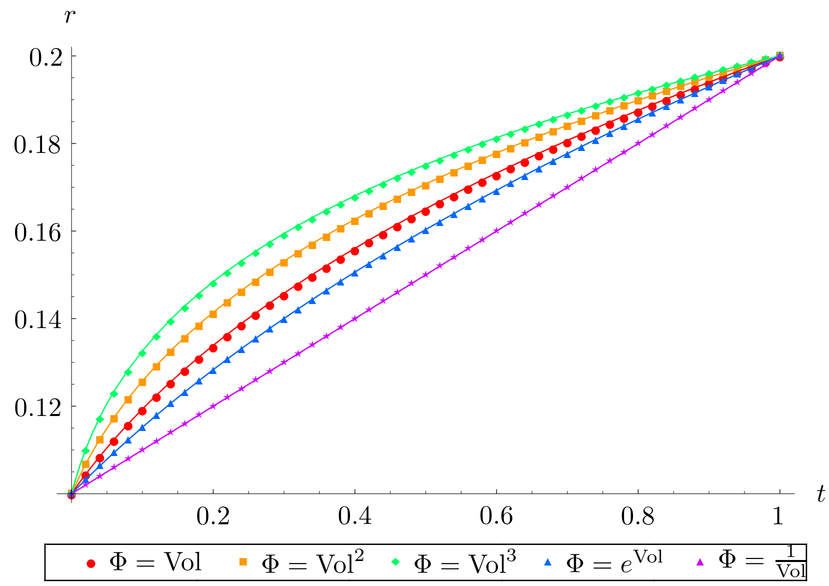

Figure 4.3: Geodesics between concentric spheres of radius 0.1 to 0.2 for several conformal metrics. Solid lines are the exact solutions.

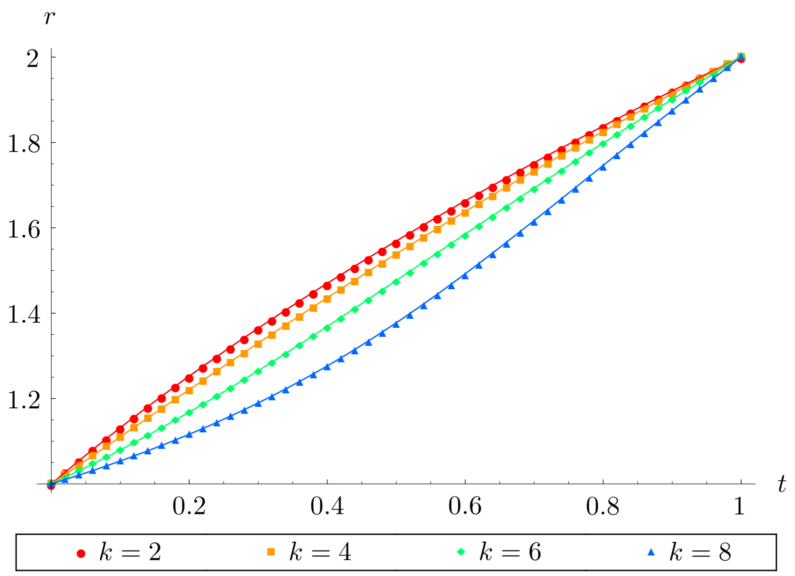

Figure 4.4: Geodesics between concentric spheres for $\Phi = 1 + 0.1\,\mathrm{Tr}(L)^k$, and varying k. Solid lines are the exact solutions.

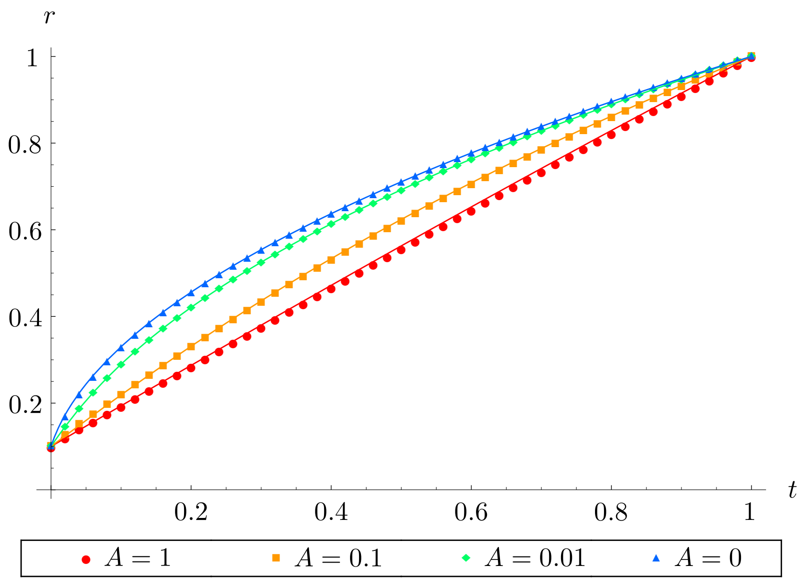

Figure 4.5: Geodesics between concentric spheres for $\Phi = 1 + A\,\mathrm{Tr}(L)^2$ and varying A. Solid lines are the exact solutions.

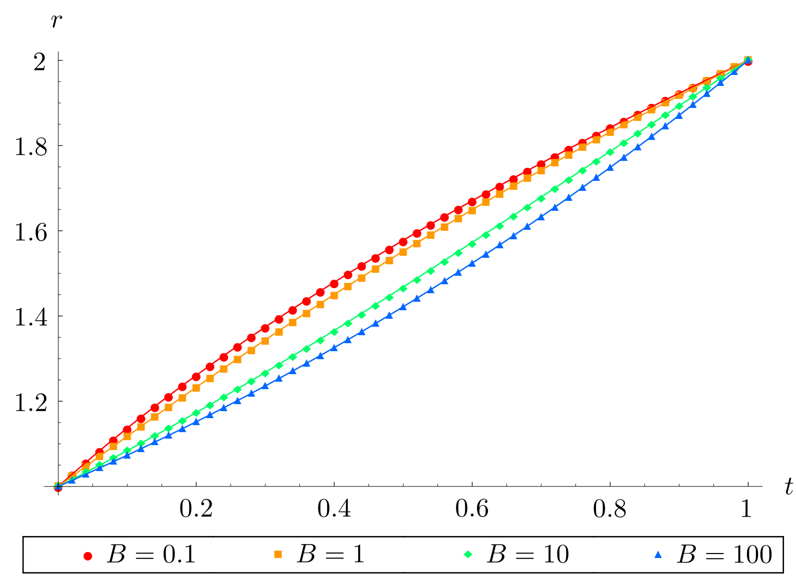

Figure 4.6: Geodesics between concentric spheres for Gauss curvature weighted metrics with $\Phi = 1 + B\det(L)^2$.

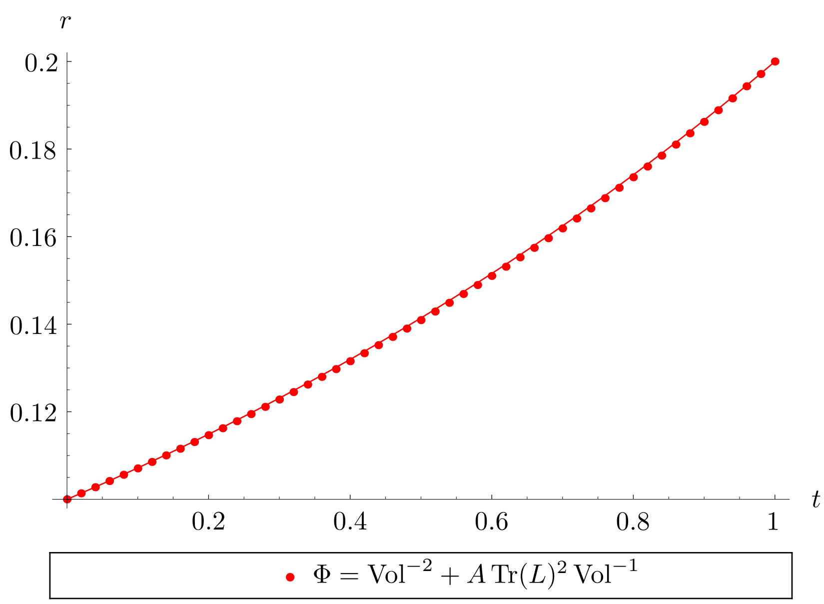

Figure 4.7: Geodesics between concentric spheres for the scale-invariant metric.

Note that the equation for $\Phi = \mathrm{Vol}^{-1}$ is $r_{tt} = 0$. These equations have explicit analytic solutions given by

$$\begin{aligned} \Phi &= \mathrm{Vol}^k : & r &= C_1\big((k+2)t - C_2\big)^{\frac{1}{k+2}} \\ \Phi &= e^{\mathrm{Vol}} : & r &= \frac{1}{2\pi}\sqrt{\log\big(C_1 t + C_2\big)}. \end{aligned}$$

A comparison of the numerical results with the exact analytic solutions can be seen in figure 4.2 and 4.3. The solid lines are the exact solutions. For the numerical solutions, 50 time steps and a triangulation with 320 triangles (see figure 4.1) were used. Note that for big radii as in figure 4.2, the solution for $\Phi = e^{\mathrm{Vol}}$ has a very steep ascent, is more curved and lies above the solutions for $\Phi = \mathrm{Vol}, \mathrm{Vol}^2, \mathrm{Vol}^3$. For small radii, it lies below these solutions, as can be seen in figure 4.3. Note also that when the ascent gets too steep, the discrete solution is somewhat inexact as in figure 4.2.

For mean curvature weighted metrics, the differential equation for the radius is:

$$\Phi = 1 + A\,\mathrm{Tr}(L)^{2k} : \qquad r_{tt} = -r_t^2\Big(\frac{1}{r} - \frac{2kA2^{2k-1}}{r^{2k+1} + A2^{2k}r}\Big).$$

The numerics for these metrics are shown in figure 4.4 and figure 4.5. Note that we got convergence to a path consisting of concentric spheres even for the G^0-metric ($A = 0$), even though we know from the theory that this is not the shortest path. In fact, there are no shortest paths for the G^0 metric since it has vanishing geodesic distance [35].

The geodesic equation for $\Phi = 1 + B\det(L)^{2l}$ on a set of concentric spheres in B_i reads as):

$$\Phi = 1 + B\det(L)^{2l} \qquad r_{tt} = -r_t^2\left(\frac{1}{r} - \frac{2l.B}{r^{4l+1} + B.r}\right).$$

Note that $\det(L) = \big(\mathrm{Tr}(L)/(2)\big)^2$. Therefore this equation is equal to the equation for metrics weighted by mean curvature with suitably adapted coefficients. A comparison of the numerical results with the exact analytic solutions can be seen in figure 4.6.

For the scale-invariant metric, the differential equation is given by:

$$\Phi = \mathrm{Vol}^{-2} + A\frac{\mathrm{Tr}(L)^2}{\mathrm{Vol}} : \qquad r_{tt} = \frac{r_t^2}{r}.$$

This equation has an explicit analytical solution:

$$\Phi = \mathrm{Vol}^{-2} + A\frac{\mathrm{Tr}(L)^2}{\mathrm{Vol}} : \qquad r = C_1 e^{C_2 t}.$$

Note that this equation and therefore its solution is independent of A. Again, this is confirmed by the numerics, see figure 4.7.

4.3 Translation of a sphere

In this section we will study geodesics between a sphere and a translated sphere for various almost local metrics of the type $\Phi = \mathrm{Vol}^k$, $\Phi = e^{\mathrm{Vol}}$ and $\Phi = 1 + A\,\mathrm{Tr}(L)^{2k} + B\det(L)^{2l}$.

Depending on the distance (relative to the radius) of the two translated spheres, different behaviors can be observed.

High distance:

- *Shrink and grow:* For some metrics it is possible to shrink a sphere in finite time to zero. For these metrics long translation go via a shrinking and growing part. Studied metrics of this behavior are: $\Phi = \mathrm{Vol}^k$, $\Phi = e^{\mathrm{Vol}}$ and $\Phi = 1 + A\,\mathrm{Tr}(L)^2$. This phenomenon is studied in more detail in section 4.3.1, see also figure 4.9.
- *Moving an optimal middle shape:* For some of the metrics translation of a sphere with a certain optimal radius is a geodesic. For these metrics geodesics for long translations scale the sphere to the optimal radius and translate the sphere with the optimal radius. Metrics with this behavior include $\Phi = 1+A\,\mathrm{Tr}(L)^{2k}$ for $k > 1$ and $\Phi = 1+B\det(L)^{2l}$. This behavior is studied in section 4.3.2.

Low distance:

- Geodesics of pure translation. ($\Phi = 1 + A\,\mathrm{Tr}(L)^{2k}$ for $k > 1$ and $\Phi = 1 + B\det(L)^{2l}$, c.f. figure 4.12)
- Geodesics that pass through an ellipsoid, where the longer principal axis is in the direction of the translation (Conformal metrics, c.f. figure 4.8).
- Geodesics that pass through an ellipsoid, where the principal axis is in the direction of the translation is shorter ($\Phi = 1 + A\,\mathrm{Tr}(L)^{2k}$ for $k > 1$ and $\Phi = 1 + B\det(L)^{2l}$, c.f. figure 4.12).
- Geodesics that pass through an cigar shaped figure ($\Phi = 1+A\,\mathrm{Tr}(L)^2$, c.f. figure 4.11.)

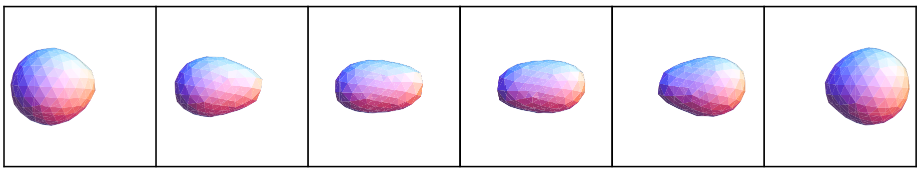

Figure 4.8: Geodesic between two unit spheres translated by distance 1.5 for $\Phi = \mathrm{Vol}$. 20 timesteps and a triangulation with 500 triangles were used. Time progresses from left to right. Boundary shapes $t = 0$ and $t = 1$ are not included.

4.3.1 Shrink and grow

In section 3.5 we showed that it is possible to shrink a sphere to zero in finite time for some of the metrics, namely conformal metrics with $\Phi = \mathrm{Vol}^k$ or $\Phi = e^{\mathrm{Vol}}$ and for the G^A metric. For these metrics geodesics of long translation will go via a shrinking and growing part, and almost all of the translation will be done with the shrunken version of the shape. We now want to get an bound about the ratio of distance and radius of the boundary spheres where these behavior cannot occur. To do this, we compare the energy needed for a pure translation

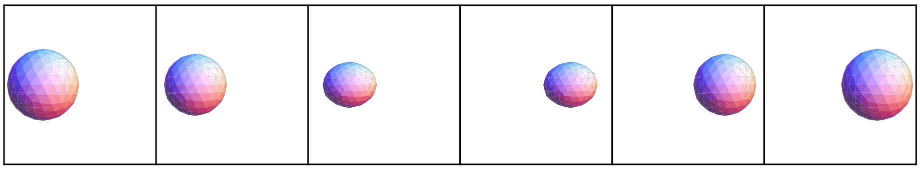

Figure 4.9: Geodesic between two unit spheres translated by distance 2 for $\Phi = e^{\mathrm{Vol}}$. 20 timesteps and a triangulation with 500 triangles were used. Time progresses from left to right. Boundary shapes $t = 0$ and $t = 1$ are not included.

with the energy needed to first shrink the sphere to almost zero, move it, and then blow it up again.

The energy needed for a pure translation of a sphere with radius r by distance ℓ in the direction of a unit vector e_1 is given by

$$\begin{aligned}
E &= \int_0^1 \int_{S^2} \Phi(\mathrm{Vol}, \mathrm{Tr}(L))\bar{g}(\ell.e_1, \nu)^2 \,\mathrm{vol}(g)dt \\
&= \Phi(4r^2\pi, -\frac{2}{r}) \int_0^\pi \int_0^{2\pi} \bar{g}\left(\ell.e_1, \begin{pmatrix} \cos\varphi\sin\theta \\ \sin\varphi\sin\theta \\ \cos\theta \end{pmatrix}\right)^2 r^2 \sin\theta \; d\varphi d\theta \\
&= \Phi(4r^2\pi, -\frac{2}{r}) \int_0^\pi \int_0^{2\pi} \ell^2.(\cos\varphi\sin\theta)^2 r^2 \sin\theta d\varphi d\theta = \Phi(4r^2\pi, -\frac{2}{r}).\frac{4\pi}{3}\ell^2.r^2
\end{aligned}$$

Any other unit vector can be chosen instead of e_1, yielding the same result.

We will now calculate the energy needed for shrinking the sphere, moving it, and blowing it up again. The energy needed for translating a sphere of radius almost zero can be neglected. Shrinking and blowing up is done using the solutions to the geodesic equation for the radius from the last section, where one has to adapt the constants to the boundary conditions. For the shrinking part, we have $r(0) = r$ and $r(\frac{1}{2}) = 0$, and for the growing part we have $r(\frac{1}{2}) = 0, r(1) = r$, see figure 4.10 (left).

The energy of the path is

$$\begin{aligned}
\Phi = \mathrm{Vol}^k: \qquad & E = \int_0^1 \mathrm{Vol}^k \int_{S^2} r_t^2 \,\mathrm{vol}(g)dt = \frac{4^{k+2}\pi^{k+1}}{(k+2)^2} r^{2k+4} \\
\Phi = e^{\mathrm{Vol}}: \qquad & E = \int_0^1 e^{\mathrm{Vol}} \int_{S^2} r_t^2 \,\mathrm{vol}(g)dt = \frac{1}{\pi}(e^{2\pi r^2} - 1)^2.
\end{aligned}$$

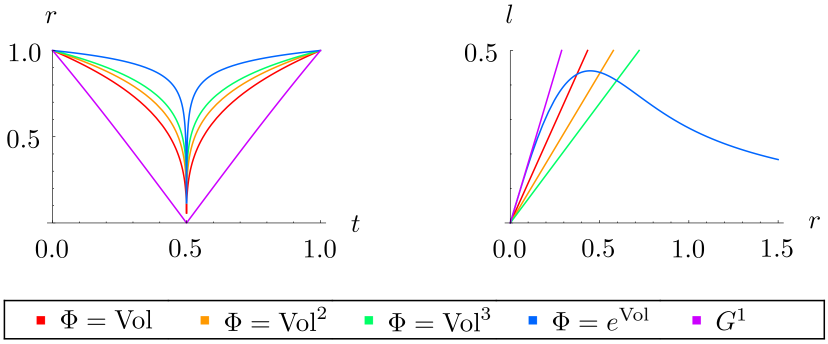

Figure 4.10: Left: Shrinking a sphere to zero along a geodesic path and blowing it up again. Right: Pairs of ℓ and r such that translating a sphere of radius r by distance ℓ needs as much energy as shrinking it to zero and blowing it up again. G^1 stands for the G^A metric with $A = 1$.

The energy of the two different paths are the same when

$$\Phi = \mathrm{Vol}^k : \qquad \ell = \frac{2\sqrt{3}r}{k+2}$$

$$\Phi = e^{\mathrm{Vol}} : \qquad \ell = \frac{\sqrt{3}(1 - e^{-2\pi r^2})}{2r\pi}.$$

These curves are shown in figure 4.10 (right). We did not derive an analytic solution for the G^A metric, but for $A = 1$ one can see the solution curves in figure 4.10. In figure 4.9 one can see an example of this shrink and grow phenomenon. We could not determine numerically whether a collapse of the sphere to a point occurs or not. But the more time steps were used, the smaller the ellipsoid in the middle turned out. Also, the energy of the geodesic path comes very close to the energy needed to shrink the sphere to a point and blow it up again. It is remarkable that almost all of the translation is concentrated at a single time step, independently of the number of timesteps that were used. The reason for this behavior is that high volumes are penalized so much: In the case of figure 4.9, e^{Vol} is more than 1000 times smaller in the middle than at the boundary shapes.

4.3.2 Moving an optimal shape

In the following we want to determine whether pure translation of a sphere is a geodesic. Therefore let $f_t = f_0 + b(t) \cdot e_1$, where f_0 is a sphere of radius r and where $b(t)$ is constant on M. Plugging this into the geodesic equation from section 3.1 yields an ODE for $b(t)$ and a part which has to vanish identically. The latter is given by:

$$(\partial_1 \Phi)\frac{2}{r}4r^2\pi + (\partial_2 \Phi)\frac{2}{r^2} - (\partial_3 \Phi).\frac{2}{r}.\frac{1}{r^2} + \Phi\frac{2}{r} = 0 \tag{1}$$

For conformal metrics this equation is only satisfied if $\Phi = \mathrm{Vol}^{-1}$. Since this metric induces vanishing geodesic distance (see section 2.4.5) we are not interested in this case. For curvature weighted metrics the above equation reads as:

$$\Phi = 1 + A\,\mathrm{Tr}(L)^{2k}: \qquad \frac{4^k A(k-1)}{r^{4k}} = 1$$
$$\Phi = 1 + B\det(L)^{2l}: \qquad \frac{B(2l-1)}{r^{4l}} = 1$$

Solutions to this equations are given by:

$$\Phi = 1 + A\,\mathrm{Tr}(L)^{2k}: \qquad r = 2\sqrt[2k]{A(k-1)}, \quad k \geq 1.$$
$$\Phi = 1 + B\det(L)^{2l}: \qquad r = \sqrt[4l]{B(2l-1)}, \quad l \geq 1.$$

For the most prominent example the G^A metric this yields $r = 0$ and therefore translation can never be a geodesic for this type of metrics. The numerics have shown that the G^A metrics yields geodesics that resemble the geodesics of the G^A metric for planar curves from [36, section 5.2]. Namely, when the two spheres are sufficiently far apart, the geodesic passes through a cigar-like middle shape, see figure 4.11. As predicted by the theory (see section 3.6.2) geodesics for very high distances tend to have a similar behavior as Vol^k metrics, i.e. the geodesic first shrinks the sphere, then moves it, and then blows it up again (cf. section 4.3.1).

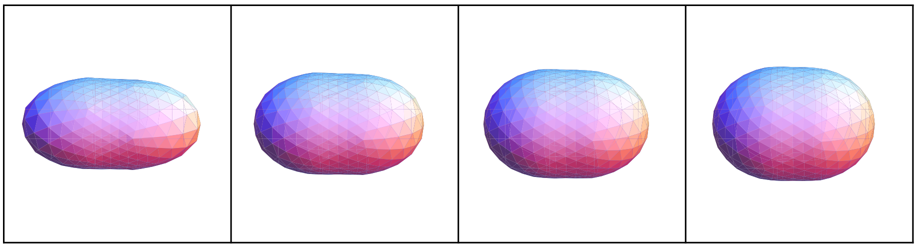

Figure 4.11: Middle figure of a geodesic between two unit spheres translated by distance 3 for $\Phi = 1 + A\,\mathrm{Tr}(L)^2$. From left to right: $A = 0.2$, $A = 0.4$, $A = 0.6$, $A = 0.8$. In each of the simulations 20 timesteps and a triangulation with 720 triangles were used.

For metrics weighted by higher factors of mean curvature and for Gauß curvature weighted metrics the above equation for the radius has a positive solution. For these metrics geodesics for translations tend to scale the sphere until it has reached the optimal radius and then translate it. If the radius is already optimal the resulting geodesic is a pure translation (see figure 4.12).

If the distance is not high enough there is still a scaling towards the optimal size, but the middle figure is not a perfect sphere anymore. Instead it is an ellipsoid as in figure 4.12.

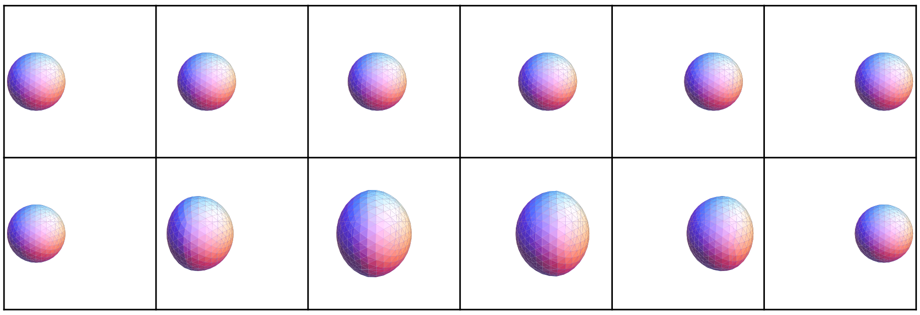

Figure 4.12: Geodesic between two unit spheres translated by distance 3 for $\Phi = 1 + \det(L)^2$ (first row) and $\Phi = 1 + \mathrm{Tr}(L)^6$ (second row). In each of the experiments 20 timesteps and a triangulation with 720 triangles were used. Time progresses from left to right. Boundary shapes $t = 0$ and $t = 1$ are not included.

4.4 Deformation of a shape

We will calculate numerically the geodesic between a shape and a deformation of the shape for various almost local metrics. Small deformations are handled well by all metrics, and they all yield similar results. An example of a geodesic resulting in a small deformation can be seen in figure 4.13, where a small bump is grown out of a sphere. The energy needed for this deformation is reasonable compared to the energy needed for a pure translation. Taking the $\Phi = \mathrm{Vol}$-metric as an example, growing a bump of size 0.4 as in figure 4.13 costs about a third of a translation of the sphere by 0.4.

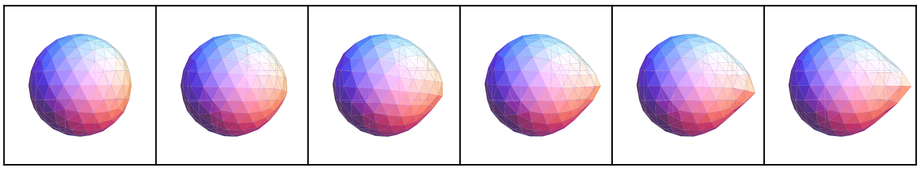

Figure 4.13: Geodesic between a sphere and a sphere with a small bump for $\Phi = \mathrm{Vol}$. 20 timesteps and a triangulation with 500 triangles were used. Time progresses from left to right.

Bigger deformations work well with Vol^k-metrics and curvature weighted metrics, but not with the e^{Vol}-metric, which tends to shrink the object and to concentrate almost all of the deformation at a single time step. In figure 4.14, a large deformation can be seen for the case of $\Phi = \mathrm{Vol}$ and $\Phi = e^{\mathrm{Vol}}$. Clearly one can see that the e^{Vol}-metric concentrates almost all of the deformation in a single time step. We have met this misbehavior of the e^{Vol}-metric already with translations. Again, the reason is that e^{Vol} is so sensitive to changes in volume.

In figure 4.15 one sees that using higher curvature weights smoothens the geodesics.

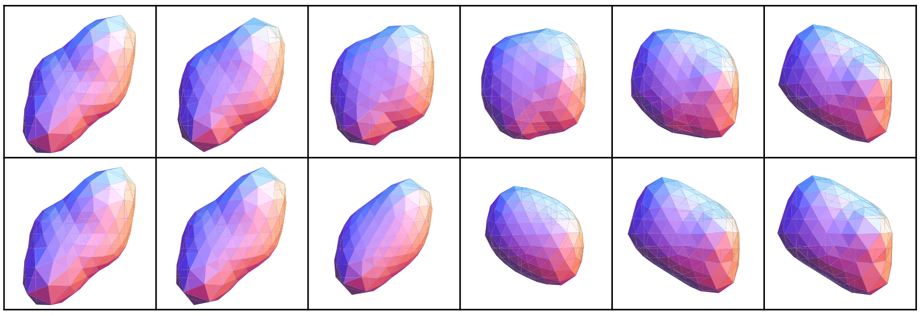

Figure 4.14: Large deformation of a shape for $\Phi = \mathrm{Vol}$ and $\Phi = e^{\mathrm{Vol}}$. 20 timesteps and a triangulation with 500 triangles were used. Time progresses from left to right.

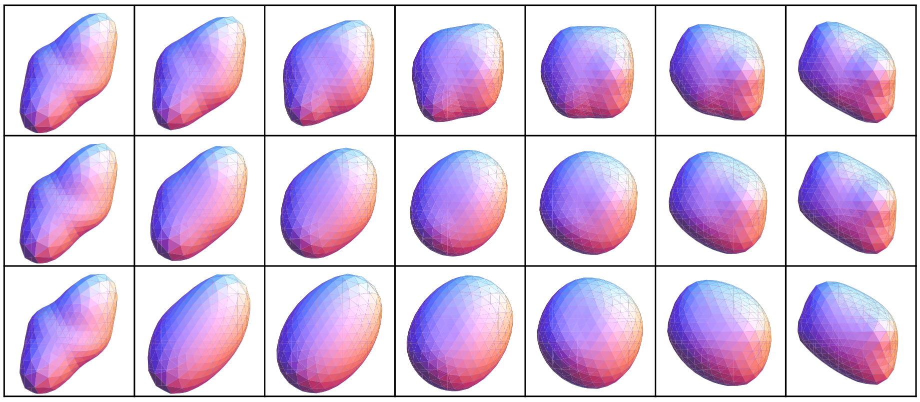

Figure 4.15: Large deformation of a shape for $\Phi = 1 + 0.1\,\mathrm{Tr}(L)^2$ (top), $\Phi = 1 + 10\,\mathrm{Tr}(L)^2$ (middle) and $\Phi = 1 + \mathrm{Tr}(L)^6$ (bottom) . 20 timesteps and a triangulation with 720 triangles were used. Time progresses from left to right.

The AMPL model file

Listing 1: AMPL model file

```
param A default 1;
param k default 1;
param B default 1;
param l default 1;
param C default 1;
param j default 1;
param D default 1;
param TimestepsN > 1 integer;
param VerticesN integer;
param PenaltyFactor default 1;
param PenaltyExponent default 2;
set VerticesI := 1..VerticesN;
set VerticesOfEdgesI within {VerticesI, VerticesI};
set VerticesOfFacesI within {VerticesI, VerticesI, VerticesI};
set FacesOfVerticesI {v in VerticesI} within VerticesOfFacesI;
set LinkOfVerticesI {VerticesI} within {VerticesOfFacesI,VerticesOfEdgesI,{-1,1}};
set AdjacentEdgesOfVerticesI {VerticesI} within {VerticesOfEdgesI,{1,-1},VerticesOfEdgesI,{1,-1}};
```

```
set EdgesOfFacesI {VerticesOfFacesI} within VerticesOfEdgesI;
set EdgesOfVerticesI {v in VerticesI} := setof {(f1,f2,f3,e1,e2,o) in LinkOfVerticesI[v]}(e1,e2);

param Pi default 3.141592653589793;
param PerfectAngle {v in VerticesI} default cos(2*Pi/card(FacesOfVerticesI[v]));
param InitialVertices {VerticesI ,1..3};
param FinalVertices {VerticesI,1..3};

var MiddleVertices {2..TimestepsN,VerticesI,1..3};

var Vertices {t in 1..TimestepsN+1,v in VerticesI,i in 1..3} =
  (if t=1 then InitialVertices[v,i]
   else if t=TimestepsN+1 then FinalVertices[v,i]
   else MiddleVertices[t,v,i]);

var VectorOfEdges {t in 1..TimestepsN+1, (v1,v2) in VerticesOfEdgesI,i in 1..3} =
         Vertices[t,v2,i] - Vertices[t,v1,i];

var LengthOfEdges {t in 1..TimestepsN+1, (v1,v2) in VerticesOfEdgesI} =
          sqrt(VectorOfEdges[t,v1,v2,1]^2+VectorOfEdges[t,v1,v2,2]^2+VectorOfEdges[t,v1,v2,3]^2);

var CrossOfFaces {t in 1..TimestepsN+1,(v1,v2,v3) in VerticesOfFacesI,i in 1..3} =
  if i=1 then (Vertices[t,v2,2]-Vertices[t,v1,2])*(Vertices[t,v3,3]-Vertices[t,v1,3]) -
              (Vertices[t,v2,3]-Vertices[t,v1,3])*(Vertices[t,v3,2]-Vertices[t,v1,2])
  else if i=2 then -(Vertices[t,v2,1]-Vertices[t,v1,1])*(Vertices[t,v3,3]-Vertices[t,v1,3]) +
                   (Vertices[t,v2,3]-Vertices[t,v1,3])*(Vertices[t,v3,1]-Vertices[t,v1,1])
  else (Vertices[t,v2,1]-Vertices[t,v1,1])*(Vertices[t,v3,2]-Vertices[t,v1,2]) -
       (Vertices[t,v2,2]-Vertices[t,v1,2])*(Vertices[t,v3,1]-Vertices[t,v1,1]) ;

var NormCrossOfFaces {t in 1..TimestepsN+1,(v1,v2,v3) in VerticesOfFacesI} =
  sqrt(CrossOfFaces[t,v1,v2,v3,1]^2 + CrossOfFaces[t,v1,v2,v3,2]^2 + CrossOfFaces[t,v1,v2,v3,3]^2);
```

```
var NuOfFaces {t in 1..TimestepsN+1,(v1,v2,v3) in VerticesOfFacesI,i in 1..3} =
  CrossOfFaces[t,v1,v2,v3,i]/NormCrossOfFaces[t,v1,v2,v3];

var AreaOfFaces {t in 1..TimestepsN+1,(v1,v2,v3) in VerticesOfFacesI} =
  NormCrossOfFaces[t,v1,v2,v3]/2;

var AreaOfVertices {t in 1..TimestepsN+1, v in VerticesI} =
  (sum {(f1,f2,f3) in FacesOfVerticesI[v]} AreaOfFaces[t,f1,f2,f3])/3;

var VectorAreaOfVertices {t in 1..TimestepsN+1, v in VerticesI, i in 1..3} =
  (sum {(v1,v2,v3) in FacesOfVerticesI[v]} CrossOfFaces[t,v1,v2,v3,i])/6;

var SquareOfNormOfVectorAreaOfVertices {t in 1..TimestepsN+1, v in VerticesI} =
  VectorAreaOfVertices[t,v,1]^2+VectorAreaOfVertices[t,v,2]^2+VectorAreaOfVertices[t,v,3]^2;

var NormOfVectorAreaOfVertices {t in 1..TimestepsN+1, v in VerticesI} =
  sqrt( SquareOfNormOfVectorAreaOfVertices[t,v]);

var Volume {t in 1..TimestepsN+1} =
  sum{(v1,v2,v3) in VerticesOfFacesI} AreaOfFaces[t,v1,v2,v3];

var GaussCurvature {t in 1..TimestepsN+1, v in VerticesI} =
  (2*Pi- sum{(v1,w1,o1,v2,w2,o2) in AdjacentEdgesOfVerticesI[v]}
          acos(( VectorOfEdges[t,v1,w1,1]*VectorOfEdges[t,v2,w2,1]
                +VectorOfEdges[t,v1,w1,2]*VectorOfEdges[t,v2,w2,2]
                +VectorOfEdges[t,v1,w1,3]*VectorOfEdges[t,v2,w2,3])
             * o1 * o2 / LengthOfEdges[t,v1,w1] / LengthOfEdges[t,v2,w2]))/AreaOfVertices[t,v];

var VectorMeanCurvatureOfVertices {t in 1..TimestepsN+1, v in VerticesI, i in 1..3} =
  if i=1 then
```

```
    sum {(f1,f2,f3,e1,e2,o) in LinkOfVerticesI[v]} o∗
      ( VectorOfEdges[t,e1,e2,2]∗NuOfFaces[t,f1,f2,f3,3] −
        VectorOfEdges[t,e1,e2,3]∗NuOfFaces[t,f1,f2,f3,2] )
  else if i=2 then
    sum {(f1,f2,f3,e1,e2,o) in LinkOfVerticesI[v]} o∗
      (−VectorOfEdges[t,e1,e2,1]∗NuOfFaces[t,f1,f2,f3,3] +
        VectorOfEdges[t,e1,e2,3]∗NuOfFaces[t,f1,f2,f3,1] )
  else
    sum {(f1,f2,f3,e1,e2,o) in LinkOfVerticesI[v]} o∗
      ( VectorOfEdges[t,e1,e2,1]∗NuOfFaces[t,f1,f2,f3,2] −
        VectorOfEdges[t,e1,e2,2]∗NuOfFaces[t,f1,f2,f3,1] ) ;

var SquareOfScalarMeanCurvatureOfVertices {t in 1..TimestepsN+1, v in VerticesI} =
  (VectorMeanCurvatureOfVertices[t,v,1]^2+VectorMeanCurvatureOfVertices[t,v,2]^2
    +VectorMeanCurvatureOfVertices[t,v,3]^2)/SquareOfNormOfVectorAreaOfVertices[t,v];

var PhiOfVertices {t in 1..TimestepsN+1,v in VerticesI} =
  1+ A∗(SquareOfScalarMeanCurvatureOfVertices[t,v])^k + B∗(GaussCurvature[t,v])^(2∗l)
   +C∗(Volume[t])^j+D∗ exp(Volume[t]);

var IncrementsOfVertices {t in 1..TimestepsN,v in VerticesI,i in 1..3} =
  TimestepsN∗(Vertices[t+1,v,i] − Vertices[t,v,i]);

var Energy = 1/ 12 / TimestepsN ∗(
  sum {t in 1..TimestepsN,v in VerticesI}
    PhiOfVertices[t,v] ∗ sum {(w1,w2,w3) in FacesOfVerticesI[v]}
        ( IncrementsOfVertices[t,v,1]∗CrossOfFaces[t,w1,w2,w3,1] +
          IncrementsOfVertices[t,v,2]∗CrossOfFaces[t,w1,w2,w3,2] +
          IncrementsOfVertices[t,v,3]∗CrossOfFaces[t,w1,w2,w3,3] )^2 /
        NormCrossOfFaces[t,w1,w2,w3] +
  sum {t in 1..TimestepsN,v in VerticesI}
```

```
  PhiOfVertices[t+1,v] * sum {(w1,w2,w3) in FacesOfVerticesI[v]}
      ( IncrementsOfVertices[t,v,1]*CrossOfFaces[t+1,w1,w2,w3,1] +
        IncrementsOfVertices[t,v,2]*CrossOfFaces[t+1,w1,w2,w3,2] +
        IncrementsOfVertices[t,v,3]*CrossOfFaces[t+1,w1,w2,w3,3] )^2 /
      NormCrossOfFaces[t+1,w1,w2,w3] )
;

var Penalty =
 sum {t in 1..TimestepsN+1, v in VerticesI,(v1,w1,o1,v2,w2,o2) in AdjacentEdgesOfVerticesI[v]}
  abs(
    ( VectorOfEdges[t,v1,w1,1]*VectorOfEdges[t,v2,w2,1] +
      VectorOfEdges[t,v1,w1,2]*VectorOfEdges[t,v2,w2,2] +
      VectorOfEdges[t,v1,w1,3]*VectorOfEdges[t,v2,w2,3] ) * o1 * o2
    / LengthOfEdges[t,v1,w1] / LengthOfEdges[t,v2,w2]
    - PerfectAngle[v]
  )^PenaltyExponent;

minimize f:
 Energy+Penalty*PenaltyFactor;
```

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Curriculum Vitae

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Zusammenfassung

In vielen Bereiche der Wissenschaft, Technik und Medizin ist es notwendig zwischen verschiedenen geometrischen Figuren zu unterscheiden. Daher ist es sehr wichtig signifikante Metriken für den Raum aller Figuren zu bestimmen. Wir modellieren Figuren als immersive unparametrisierte Untermannigfaltigkeiten. In dieser Arbeit betrachten wir Riemannsche Metriken auf dem Raum der Figuren die von Metriken auf der Mannigfaltigkeit der Immersionen induziert werden. Diese Metriken werden auch als innere Metriken bezeichnet. Unglüchierweise induziert die einfachste und natürlichste solche Metrik verschwinde geodätische Distanz am Raum aller Figuren. Diese Entdeckung von Michor und Mumford war der Startpunkt der Suche nach stärkeren aussagekräftigen Metriken. In dieser Arbeit betrachten wir eine bestimmten Klasse innerer Metriken auf Figurenräume beliebiger Dimension. Wir berechnen die Geodätengleichung und die Schnittkrümmung, zeigen dass sie positive geodaätische Distanz induzieren und vergleichen sie mit der Fréchet Distanz. Im letzten Teil studieren wir das Verhalten dieser Metriken anhand verschiedener numerischer Experimente.