

# GEODESIC DISTANCE FOR RIGHT INVARIANT SOBOLEV METRICS OF FRACTIONAL ORDER ON THE DIFFEOMORPHISM GROUP

MARTIN BAUER, MARTINS BRUVERIS, PHILIPP HARMS, PETER W. MICHOR

ABSTRACT. We study Sobolev-type metrics of fractional order on the group of compactly supported diffeomorphisms  $\text{Diff}_c(M)$ , where  $M$  is a Riemannian manifold of bounded geometry. We prove that the geodesic distance, induced by the Riemannian metric, vanishes if the order  $s$  is between  $0 \leq s < \frac{1}{2}$ . For  $M \neq \mathbb{R}$  we show the vanishing of the geodesic distance also for  $s = \frac{1}{2}$  and for  $\dim(M) = 1$  we show that the distance doesn't vanish for  $\frac{1}{2} < s$ .

For  $M = \mathbb{R}^n$  we discuss the geodesic equations for these metrics. It is a known fact that for specific values of  $s$  one recovers well known PDEs of hydrodynamics: Burgers' equation for  $s = 0$ , the modified Constantin-Lax-Majda equation for  $s = \frac{1}{2}$  or the Camassa-Holm equation for  $s = 1$ .

## 1. INTRODUCTION

In the seminal paper [1] Arnold showed that the incompressible Euler equations can be seen as geodesic equations on the group of volume preserving diffeomorphisms with respect to the  $L^2$ -metric. This interpretation was extended to other PDEs used in hydrodynamics, some of which are: Burgers' equation as the geodesic equation for  $\text{Diff}_c(\mathbb{R})$  with the  $L^2$ -metric, Camassa-Holm equation for  $\text{Diff}_c(\mathbb{R})$  with the  $H^1$ -metric or KdV for the Virasoro-Bott group with the  $L^2$ -metric in [20]. Recently it was shown by Wunsch [27] that the modified Constantin-Lax-Majda equation (mCLM) is the geodesic equation on the homogeneous space  $\text{Diff}(S^1)/S^1$  with respect to the homogeneous  $\dot{H}^{1/2}$ -metric.

The geometric interpretation was used by Ebin and Marsden in [5] to show the well-posedness of the Euler equations. These techniques were expanded and applied to other equations, see e.g. [3, 4, 9, 10].

The interpretation of a PDE as the geodesic equation on an infinite dimensional manifold opens up a variety of geometrical questions, which may be asked about the manifold. What is the curvature of this manifold? Do geodesics have conjugate point? Do there exist totally geodesic submanifolds? The question we want to concentrate upon in this paper, that of geodesic distance, has a very simple answer in finite dimensions but shows a more nuanced behaviour in infinite dimensions. The geodesic distance between two points is defined as the infimum of the pathlength over all paths connecting the two points. In finite dimensions, because of the

---

*Date:* May 14, 2012 .

*2010 Mathematics Subject Classification.* Primary 35Q31, 58B20, 58D05.

*Key words and phrases.* diffeomorphism group, geodesic distance, Sobolev metrics of non-integral order.

All authors were supported by 'Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P 21030'. Martins Bruveris was partially supported by the European Research Council.

local invertibility of the exponential map, this distance is always positive and the topology of the resulting metric space is the same as the manifold topology.

However, in infinite dimensions, this does not always hold: the induced geodesic distance for weak Riemannian metrics on infinite dimensional manifolds may vanish. This surprising fact was first noticed for the  $L^2$ -metric on shape space  $\text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$  in [18, 3.10]. Here  $\text{Imm}(S^1, \mathbb{R}^2)/\text{Diff}(S^1)$  denotes the orbifold of all immersions  $\text{Imm}(S^1, \mathbb{R}^2)$  of  $S^1$  into  $\mathbb{R}^2$  modulo reparametrizations. In [17] it was shown that this result holds for the general shape space  $\text{Imm}(M, N)/\text{Diff}(M)$  for any compact manifold  $M$  and Riemannian manifold  $N$ , and also for the right invariant  $L^2$ -metric (or equivalently Sobolev-type metric of order zero) on each full diffeomorphism group with compact support  $\text{Diff}_c(N)$ . In particular, since Burgers' equation is related to the geodesic equation of the right invariant  $L^2$ -metric on  $\text{Diff}_c(\mathbb{R}^1)$ , it implies that solutions of Burgers' equation are critical points of the length functional, but they are not length-minimizing. A similar result was shown for the KdV equation in [2].

On the other hand it was shown in [17] that for Sobolev-type metrics on the diffeomorphism group of order one or higher the induced geodesic distance is positive. This naturally leads to the question whether one can determine the Sobolev order where this change of behavior occurs. The main result of this paper gives a complete answer in the case of  $M = S^1$  and a partial answer for other manifolds.

**Theorem** (Geodesic distance). *Let  $M$  be a Riemannian manifold of bounded geometry. We have:*

- (1) *The geodesic distance for the fractional order Sobolev type metric  $H^s$  on  $\text{Diff}_c(M)$  vanishes for*
  - $0 \leq s < \frac{1}{2}$  and  $M$  any Riemannian manifold of bounded geometry.
  - $s = \frac{1}{2}$  and  $M \neq \mathbb{R}$ .

*In these cases the induced Riemannian exponential mapping can not be a diffeomorphism.*

- (2) *For  $\dim(M) = 1$  the induced geodesic distance is positive for  $\frac{1}{2} < s$  and for general  $\dim(M) \geq 2$  the geodesic distance is positive for  $1 \leq s$ .*

Let us now briefly outline the structure of this work. In Section 2 we review the basic definition of Sobolev-type metrics on diffeomorphism groups. For a Riemannian manifold  $M$  of bounded geometry (see 2.4) we consider the Sobolev metric  $H^s$  of order  $s$  on the Lie algebra of vector fields and the induced right invariant metric on the diffeomorphism group  $\text{Diff}_c(M)$ . The main results regarding geodesic distance are contained in Section 3 and 4. In the final section 5 we derive the geodesic equations for different versions of the Sobolev metric of order  $s$  on  $\text{Diff}_c(\mathbb{R}^n)$  and discuss their relation to various well-known PDE's.

## 2. SOBOLEV METRICS $H^s$ WITH $s \in \mathbb{R}$

In this section we give definitions for Sobolev norms of fractional orders and state the properties, which we will need later to prove the vanishing geodesic distance results.

**2.1. Sobolev metrics  $H^s$  on  $\mathbb{R}^n$ .** For  $s > 0$  the Sobolev  $H^s$ -norm of an  $\mathbb{R}^n$ -valued function  $f$  on  $\mathbb{R}^n$  is defined as

$$(1) \quad \|f\|_{H^s(\mathbb{R}^n)}^2 = \|\mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}f\|_{L^2(\mathbb{R}^n)}^2,$$

where  $\mathcal{F}$  is the Fourier transform

$$\mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} f(x) dx$$

and  $\xi$  is the independent variable in the frequency domain. An equivalent norm is given by

$$(2) \quad \|f\|_{\dot{H}^s(\mathbb{R}^n)}^2 = \|f\|_{L^2(\mathbb{R}^n)}^2 + \| |\xi|^s \mathcal{F}f \|_{L^2(\mathbb{R}^n)}^2.$$

The fact that both norms are equivalent is based on the inequality

$$\frac{1}{C} \left(1 + \sum_j |\xi_j|^s\right) \leq \left(1 + \sum_j |\xi_j|^2\right)^{\frac{s}{2}} \leq C \left(1 + \sum_j |\xi_j|^s\right)$$

holding for some constant  $C$ . For  $s > 1$  this says that all  $\ell^s$ -norms on  $\mathbb{R}^{n+1}$  are equivalent. But the inequality is true also for  $0 < s < 1$ , even though the expression does not define a norm on  $\mathbb{R}^{n+1}$ . Using any of these norms we obtain the Sobolev spaces with non-integral  $s$

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \|f\|_{H^s(\mathbb{R}^n)} < \infty\}.$$

These spaces are also known under the name Liouville spaces or Bessel potential spaces. To make a connection with other families of function spaces, we note that the spaces  $H^s(\mathbb{R}^n)$  coincide with

$$H^s(\mathbb{R}^n) = B_{22}^s(\mathbb{R}^n) = F_{22}^s(\mathbb{R}^n)$$

the Besov spaces  $B_{22}^s(\mathbb{R}^n)$  and spaces of Triebel-Lizorkin type  $F_{22}^s(\mathbb{R}^n)$ . Definitions of these spaces and a nice introduction to the general theory of function spaces can be found in [25, section 1].

We will now collect some results about this family of spaces, which we will need in the coming sections. First a result, which states that point wise multiplication with a sufficiently smooth function is well defined.

**2.2. Theorem** (See theorem 4.2.2 in [25]). *Let  $s > 0$  and  $g \in C_c^\infty(\mathbb{R}^n)$ , a smooth function with compact support. Then multiplication  $f \mapsto gf$  is a bounded map of  $H^s(\mathbb{R}^n)$  into itself.*

We are also allowed to compose with diffeomorphisms.

**2.3. Theorem** (See theorem 4.3.2 in [25]). *Let  $s > 0$  and  $\varphi \in \text{Diff}_c(\mathbb{R}^n)$  be a diffeomorphism which equals the identity off some compact set. Then composition  $f \mapsto f \circ \varphi$  is an isomorphic mapping of  $H^s(\mathbb{R}^n)$  onto itself.*

**2.4. Sobolev metrics for Riemannian manifolds of bounded geometry.** Following [25, section 7.2.1] we will now introduce the spaces  $H^s(M)$  on a manifold  $M$ . If  $M$  is not compact we equip  $M$  with a Riemannian metric  $g$  of bounded geometry which exists by [11]. This means that

- (I) The injectivity radius of  $(M, g)$  is positive.
- ( $B_\infty$ ) Each iterated covariant derivative of the curvature is uniformly  $g$ -bounded:  $\|\nabla^i R\|_g < C_i$  for  $i = 0, 1, 2, \dots$ .

The following is a compilation of special cases of results collected in [7, chapter 1], who treats Sobolev spaces only for integral order.

**Proposition** ([12], [21], [6]). *If  $(M, g)$  satisfies (I) and ( $B_\infty$ ) then the following holds:*

- (1)  $(M, g)$  is complete.
- (2) There exists  $\varepsilon_0 > 0$  such that for each  $\varepsilon \in (0, \varepsilon_0)$  there is a countable cover of  $M$  by geodesic balls  $B_\varepsilon(x_\alpha)$  such that the cover of  $M$  by the balls  $B_{2\varepsilon}(x_\alpha)$  is still uniformly locally finite.
- (3) Moreover there exists a partition of unity  $1 = \sum_\alpha \rho_\alpha$  on  $M$  such that  $\rho_\alpha \geq 0$ ,  $\rho_\alpha \in C_c^\infty(M)$ ,  $\text{supp}(\rho_\alpha) \subset B_{2\varepsilon}(x_\alpha)$ , and  $|D_u^\beta \rho_\alpha| < C_\beta$  where  $u$  are normal (Riemann exponential) coordinates in  $B_{2\varepsilon}(x_\alpha)$ .
- (4) In each  $B_{2\varepsilon}(x_\alpha)$ , in normal coordinates, we have  $|D_u^\beta g_{ij}| < C'_\beta$ ,  $|D_u^\beta g^{ij}| < C''_\beta$ , and  $|D_u^\beta \Gamma_{ij}^m| < C'''_\beta$ , where all constants are independent of  $\alpha$ .

We can define the  $H^s$ -norm of a function  $f$  on  $M$ :

$$\begin{aligned} \|f\|_{H^s(M,g)}^2 &= \sum_{\alpha=0}^{\infty} \|(\rho_\alpha f) \circ \exp_{x_\alpha}\|_{H^s(\mathbb{R}^n)}^2 \\ &= \sum_{\alpha=0}^{\infty} \|\mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}((\rho_\alpha f) \circ \exp_{x_\alpha})\|_{L^2(\mathbb{R}^n)}^2. \end{aligned}$$

If  $M$  is compact the sum is finite. Changing the charts or the partition of unity leads to equivalent norms by the proposition above, see [25, theorem 7.2.3]. For integer  $s$  we get norms which are equivalent to the Sobolev norms treated in [7, chapter 2]. The norms depends on the choice of the Riemann metric  $g$ . This dependence is worked out in detail in [7].

For vector fields we use the trivialization of the tangent bundle that is induced by the coordinate charts and define the norm in each coordinate as above. This leads to a (up to equivalence) well-defined  $H^s$ -norm on the Lie algebra  $\mathfrak{X}_c(M)$ .

**2.5. Sobolev metrics on  $\text{Diff}_c(M)$ .** Given a norm on  $\mathfrak{X}_c(M)$  we can use the right-multiplication in the diffeomorphism group  $\text{Diff}_c(M)$  to extend this norm to a right-invariant Riemannian metric on  $\text{Diff}_c(M)$ . In detail, given  $\varphi \in \text{Diff}_c(M)$  and  $X, Y \in T_\varphi \text{Diff}_c(M)$  we define

$$G_\varphi^s(X, Y) = \langle X \circ \varphi^{-1}, Y \circ \varphi^{-1} \rangle_{H^s(M)}.$$

In chapter 3 and in chapter 4 we are interested in questions of vanishing and non-vanishing of geodesic distance. These properties are invariant under changes to equivalent inner products, since equivalent inner products on the Lie Algebra

$$\frac{1}{C} \langle X, Y \rangle_1 \leq \langle X, Y \rangle_2 \leq C \langle X, Y \rangle_1$$

imply that the geodesic distances will be equivalent metrics

$$\frac{1}{C} \text{dist}_1(\varphi, \psi) \leq \text{dist}_2(\varphi, \psi) \leq C \text{dist}_1(\varphi, \psi).$$

There the ambiguity in the definition of the  $H^s$ -norm is of no concern to us.

In chapter 5 we will study the geodesic equations of the Sobolev metrics on  $\text{Diff}_c(\mathbb{R}^n)$ . Equivalent norms may induce different geodesic equations. Therefore we will denote the metric that is induced by the  $H^s(\mathbb{R}^n)$ -norm (1) as  $G^s$  and the metric that is induced by the  $\overline{H}^s(\mathbb{R}^n)$ -norm (2) as  $\overline{G}^s$ .

## 3. VANISHING GEODESIC DISTANCE

**3.1. Theorem** (Vanishing geodesic distance). *The Sobolev metric of order  $s$  induces vanishing geodesic distance on  $\text{Diff}_c(M)$  if:*

- $0 \leq s < \frac{1}{2}$  and any  $M$ ,
- $s = \frac{1}{2}$  and  $M \neq \mathbb{R}$ .

*This means that any two diffeomorphisms in the same connected component of  $\text{Diff}_c(M)$  can be connected by a path of arbitrarily short  $G^s$ -length.*

**Remark.** *Note that this result also implies vanishing of geodesic distance on to the homogenous space  $\text{Diff}(S^1)/S^1$  equipped with a homogenous metric of order  $s \leq \frac{1}{2}$ . In particular the geodesic equation for the metric of order  $s = \frac{1}{2}$  on this space is the modified Constantin-Lax-Majda equation, see section 5.5.*

We will prove this theorem by first constructing paths from the identity to some diffeomorphisms with arbitrary short length and then using the simplicity of the diffeomorphism group to show that any diffeomorphism can be connected to the identity with paths of arbitrary short length.

The restriction to  $M \neq \mathbb{R}$  in the case  $s = \frac{1}{2}$  is due to technical reasons. We believe that the result holds also in this case, however it is more difficult to construct the required vector fields in the non-compact case.

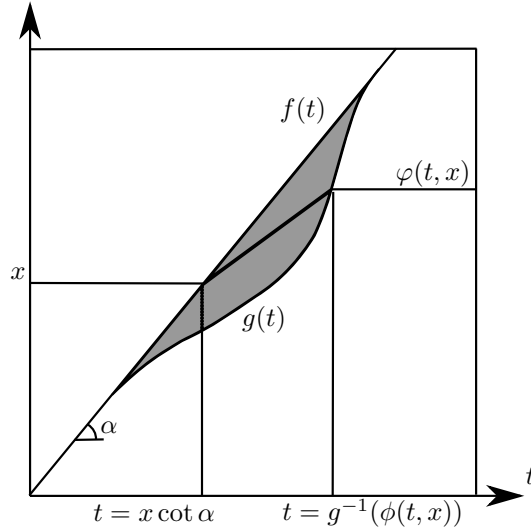


FIGURE 1. Sketch of the vector field  $u(t, x)$ . The gray area represents the support of  $u$  and one integral curve of  $u(\cdot, x)$  is shown.

**3.2. Lemma.** *Let  $\varphi \in \text{Diff}_c(\mathbb{R})$  be a diffeomorphism satisfying  $\varphi(x) \geq x$ . Then for  $0 \leq s < \frac{1}{2}$  the geodesic distance between  $\varphi$  and  $\text{id}$  with respect to the  $H^s$ -metric in  $\text{Diff}_c(\mathbb{R})$  is zero, i.e.,  $\varphi$  can be connected to the identity by a path of arbitrarily short  $G^s$ -length.*

*Proof.* The idea of the proof is as follows. Given the diffeomorphism  $\varphi$  with  $\varphi(x) > x$  we will construct a family of paths of the form

$$u(t, x) = \mathbb{1}_{[g(t), f(t)]} \star G_\varepsilon(x)$$

such that its flow  $\varphi(t, x)$  satisfies  $\varphi(0, x) = x$  and  $\varphi(T, x) = \varphi(x)$ . In a second step we will show, that when  $\|f - g\|_\infty$  is sufficiently small, so is also the  $H^s$ -length of the path  $\varphi(t, x)$ . Here  $G_\varepsilon(x) = \frac{1}{\varepsilon}G_1(\frac{x}{\varepsilon})$  is a smoothing kernel, where  $G_1$  is a smooth bump function.

So let us construct the vector field  $u(t, x)$ . If we could disregard continuity, we could choose an angle  $\alpha > \frac{\pi}{4}$  and set

$$\begin{aligned} f(t) &= t \tan \alpha \\ g^{-1}(x) &= x - (1 - \cot \alpha)\varphi^{-1}(x). \end{aligned}$$

The flow  $\varphi(t, x)$  of the unsmoothed vector field  $u(t, x) = \mathbb{1}_{[g(t), f(t)]}(x)$  satisfies

$$\varphi(t, x) = x + \int_0^t u(s, \varphi(s, x)) ds.$$

In this case we can write down the explicit solution, which is given by

$$\varphi(t, x) = \begin{cases} x, & t < \cot \alpha \\ t + (1 - \cot \alpha)x, & x \cot \alpha \leq t \leq \varphi(x) - (1 - \cot \alpha)x \\ \varphi(x), & t > \varphi(x) - (1 - \cot \alpha)x \end{cases}$$

and we see that it satisfies the boundary conditions. We also have the relation

$$f^{-1}(x) - g^{-1}(x) = -(1 - \cot \alpha)(x - \varphi^{-1}(x))$$

which implies that by choosing  $\alpha$  sufficiently close to  $\frac{\pi}{4}$  we can make  $\|f - g\|_\infty$  as small as necessary. By replacing  $u$  with the smoothed vector field  $\mathbb{1}_{[g(t), f(t)]} \star G_\varepsilon(x)$  we change the endpoint of the flow. However, by changing  $g$  suitably we can regain control of the endpoint. The necessary changes will be of order  $\varepsilon$  and hence we don't lose control over the difference  $\|f - g\|_\infty$ , which will be necessary later on.

Now we compute the norm of this vector field. Let  $u(t, x)$  have the form  $u(t, x) = \mathbb{1}_{[g(t), f(t)]} \star G_\varepsilon(x)$ , where  $f(t)$  and  $g(t)$  are smooth functions which coincide off a bounded interval. To compute the  $H^s$ -norm of  $u$ , we first need to compute its Fourier transform

$$\begin{aligned} \mathcal{F}\mathbb{1}_{[g(t), f(t)]}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{g(t)}^{f(t)} e^{i\xi x} dx = \frac{1}{\sqrt{2\pi}} \frac{e^{i\xi x}}{i\xi} \Big|_{x=g(t)}^{x=f(t)} = \frac{1}{\sqrt{2\pi}} \frac{e^{i\xi f(t)} - e^{i\xi g(t)}}{i\xi} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\xi} e^{i\xi \frac{f(t)+g(t)}{2}} \frac{e^{i\xi \frac{f(t)-g(t)}{2}} - e^{-i\xi \frac{f(t)-g(t)}{2}}}{i\xi} \\ &= \frac{2}{\sqrt{2\pi}} \frac{1}{\xi} e^{i\xi \frac{f(t)+g(t)}{2}} \sin(\xi \frac{f(t) - g(t)}{2}). \end{aligned}$$

Setting  $a = \frac{f(t)-g(t)}{2}$  we can now compute the norm

$$\begin{aligned} \|\xi^s \mathcal{F}u\|_{L^2}^2 &= \|\xi^s \mathcal{F}\mathbb{1}_{[g(t), f(t)]} \mathcal{F}G_\varepsilon\|_{L^2}^2 \\ &= \int_{\mathbb{R}} \frac{2}{\pi} \frac{1}{|\xi|^{2-2s}} \sin^2(a\xi) (\mathcal{F}G_1(\varepsilon\xi))^2 d\xi \\ &\leq \frac{2}{\pi} \|\mathcal{F}G_1\|_\infty^2 \int_{\mathbb{R}} \frac{\sin^2(a\xi)}{|\xi|^{2-2s}} d\xi \\ &\leq \frac{2}{\pi} \|\mathcal{F}G_1\|_\infty^2 \int_{\mathbb{R}} a^{1-2s} \frac{\sin^2(\xi)}{|\xi|^{2-2s}} d\xi. \end{aligned}$$

We get  $\|u\|_{L^2}^2$  by setting  $s = 0$  in the above calculation. We see that for  $s < \frac{1}{2}$  the  $H^s$ -norm of  $u(t, \cdot)$  is bounded by

$$\|u(t, \cdot)\|_{H^s}^2 \leq C_1|f(t) - g(t)| + C_2|f(t) - g(t)|^{1-2s} .$$

Now, putting everything together

$$\begin{aligned} \text{Len}(\varphi)^2 &= \left( \int_0^T \|u(t, \cdot)\|_{H^s} dt \right)^2 \leq T \int_0^T \|u(t, \cdot)\|_{H^s}^2 dt \\ &\leq T^2 (C_1\|f - g\|_\infty + C_2\|f - g\|_\infty^{1-2s}) . \end{aligned}$$

Since the geodesic length is defined as the infimum over all paths and since we have shown in the first part of the proof that by choosing the angle  $\alpha$  and the smoothing factor  $\varepsilon$ , we can control the norm  $\|f - g\|_\infty$ , the proof is complete.  $\square$

In the case  $s = \frac{1}{2}$  we will work on the circle  $S^1$ . Using a construction similar to that in Lemma 3.2 we will construct arbitrary short paths from the identity to the shift  $\varphi(x) = x + 1$ . The following lemma supplies us with functions that have small  $H^{1/2}$ -norm and large  $L^\infty$ -norm at the same time. We cannot use step functions as in the above proof, because they are not in  $H^{1/2}(S^1)$ .

**3.3. Lemma.** *Let  $\psi(x)$  be a non-negative, compactly supported  $C^\infty$ -function on  $\mathbb{R}$  and  $(b_j)_{j=0}^\infty$  a non-increasing sequence of non-negative numbers, with  $\sum b_j^2 < \infty$ . Then the  $H^{1/2}$ -norm of the function  $f(x) := \sum b_j \psi(2^j x)$  is bounded by*

$$\|f\|_{H^{1/2}}^2 \leq C \sum_{j=0}^\infty b_j^2 .$$

*Proof.* This result is shown in step 4 of the proof of theorem 13.2 in [26].  $\square$

The main difference between Lemma 3.2 and the following Lemma is that on  $S^1$  we don't have to worry about the diffeomorphisms having compact support. On  $\mathbb{R}$  the diffeomorphism  $\varphi(x) = x + 1$  is not element of  $\text{Diff}_c(\mathbb{R})$  and we would have to replace it by  $\varphi(x) = x + c(x)$ , where  $c(x)$  is some function with compact support. This makes working on the circle much easier.

**3.4. Lemma.** *Let  $\varphi \in \text{Diff}(S^1)$  be the shift by  $\sigma$ , i.e.  $\varphi(x) = x + \sigma$ . Then the geodesic distance between  $\varphi$  and  $\text{id}$  with respect to the  $H^{1/2}$ -metric in  $\text{Diff}(S^1)$  is zero.*

*Proof.* We will prove the lemma by constructing a sequence of vector fields with arbitrary small  $H^{1/2}$ -norms, whose flows at time  $t = T_{\text{end}}$  will be  $\varphi(T_{\text{end}}, x) = x + \sigma$ . First we apply Lemma 3.3 with  $b_j = \frac{1}{N}$  for  $j = 0, \dots, N - 1$  and 0 otherwise. By doing so we obtain  $\|f\|_{H^{1/2}}^2 \leq C \frac{1}{N}$ , while  $\|f\|_\infty = 1$ . For the basic function  $\psi(x)$  we choose  $\psi(x) = e^{\frac{1}{1-|x|^2}}$ . Note that  $\text{supp}(\psi) \subseteq [-1, 1]$  and that  $\psi$  is concave in a neighborhood around 0. Since each  $f$  is a finite sum of  $\psi(2^j x)$ , these properties hold also for  $f$ .

We define the vector field

$$u(t, x) = \lambda f(t - x) \quad \text{with} \quad 0 < \lambda < 1$$

for  $t \in [0, T_{\text{end}}]$ , where  $T_{\text{end}}$  will be specified later. The energy of this path is bounded by

$$E(u) = \int_0^{T_{\text{end}}} \|u(t, \cdot)\|_{H^{1/2}}^2 dt \leq CT_{\text{end}} \frac{1}{N}$$

and hence can be made as small as necessary. It remains to show that the flow of this vector field at time  $t = T_{\text{end}}$  is indeed  $\varphi(T_{\text{end}}, x) = x + \sigma$ .

We do this in several steps. First we consider this vector field defined on all of  $\mathbb{R}$  with time going from  $-\infty$  to  $\infty$ . The initial condition for the flow is  $\varphi(-\infty, x) = x$ . Since  $u(t, x)$  has compact support in  $x$ , this doesn't cause any analytical problems. As long as  $\lambda < 1$  each integral curve of  $u$  will leave the support of  $u$  after finite time. Therefore we can consider  $\varphi(\infty, x)$  to be the endpoint of the flow. Next we will establish that  $\varphi(\infty, x) = x + S$  is a uniform shift, that is independent of  $x$ . Then we show that by appropriately choosing  $\lambda$  we can control the amount of shifting, in particular we can always obtain  $S = \sigma$ . Then we find bounds for the time each integral curve spends in the support of  $u$ . By showing that this time is only dependent on  $S$ , but not on the specific form of  $f$  or  $\psi$ , we will know that  $T_{\text{end}}$  doesn't grow larger as we let  $N \rightarrow \infty$ . In the last step we go back to the circle, define  $T_{\text{end}}$ , start the flow at time  $t = 0$  and show that the resulting flow is a shift by 1 at time  $T_{\text{end}}$ . This will conclude the proof.

The flow  $\varphi(t, x)$  of  $u(t, x)$  is given by the equation

$$\partial_t \varphi(t, x) = u(t, \varphi(t, x))$$

with the initial condition  $\varphi(-\infty, x) = x$ . Define the function  $a_x(t) = t - \varphi(t, x)$ . Because  $\partial_t a_x(t) = 1 - \lambda f(a_x(t)) > 0$  the function  $a_x(\cdot)$  is a diffeomorphism in  $t$  for each fixed  $x$ . Since  $\text{supp}(f) \subseteq [-1, 1]$ , we have  $\partial_t \varphi(t, x) \neq 0$  only for  $t \in [a_x^{-1}(-1), a_x^{-1}(1)]$ . Let us define  $T_{\text{shift}} = a_x^{-1}(1) - a_x^{-1}(-1)$  to be the time necessary for the flow to pass through the vector field  $u$ .

**Claim A.**  $T_{\text{shift}}$  is independent of  $x$ .

This follows from the following symmetries of the flow  $\varphi(t, x)$  and the map  $a_x(t)$ . We have

$$\varphi(t, x) = \varphi(t - (x - y), y) + x - y$$

and

$$a_x(t) = a_y(t - (x - y)).$$

To prove the first identity assume  $x > y$  and note that at time  $t_0 = y - 1$  we have

$$\varphi(y - 1, x) = x = x + \varphi(y - 1 - (x - y), y) - y$$

since at time  $y - 1 - (x - y)$  the flow  $\varphi(t, y)$  still equals  $y$ . Now differentiate to see that both functions satisfy the same ODE. The second identity is an immediate consequence of the first one. To prove the claim that  $T_{\text{shift}}$  is independent of  $x$ , we will show that

$$\partial_x a_x^{-1}(t) = 1.$$

Start with  $a_x(a_x^{-1}(t)) = t$ , use the symmetry relation  $a_0(a_x^{-1}(t) - x) = t$  and differentiate with respect to  $x$  to obtain

$$\partial_t a_0(a_x^{-1}(t) - x)(\partial_x a_x^{-1}(t) - 1) = 0,$$

which concludes the proof of the claim.

For each  $x$ , the flow of the vector field performs a shift  $[a_x^{-1}(-1), a_x^{-1}(1)]$  given by

$$\varphi(\infty, x) = x + \int_{a_x^{-1}(-1)}^{a_x^{-1}(1)} \lambda f(t - \varphi(t, x)) dt = x + \int_{-1}^1 \frac{\lambda f(t)}{1 - \lambda f(t)} dt .$$

Define

$$I(\lambda) = \int_{-1}^1 \frac{\lambda f(t)}{1 - \lambda f(t)} dt$$

to be the amount of shifting that is taking place as a function of  $\lambda$ . We claim that we can always choose  $\lambda$  close enough to 1, to obtain any shift necessary.

**Claim B.**  $I : [0, 1) \rightarrow \mathbb{R}_{\geq 0}$  is a diffeomorphism

Obviously,  $\partial_\lambda I(\lambda) > 0$  and  $I(0) = 0$ . It remains to show that  $\lim_{\lambda \rightarrow 1} I(\lambda) = \infty$ . Each  $f$  that we choose in our construction is concave in some small neighborhood around 0. So choose  $a > 0$ , such that for  $t \in [0, a]$  we have  $f(t) \geq \frac{1}{2}$  and  $f(t) \geq 1 - ct$  for some constant  $c$ . Then we can estimate the integral by

$$\begin{aligned} I(\lambda) &\geq \int_0^a \frac{\lambda f(t)}{1 - \lambda f(t)} dt \geq \frac{\lambda}{2} \int_0^a \frac{1}{1 - \lambda + \lambda ct} dt \\ &\geq \frac{\lambda}{2} \log(1 - \lambda + \lambda ct) \Big|_{t=0}^{t=a} \\ &\geq \frac{\lambda}{2} \log \left( 1 + \frac{\lambda ca}{1 - \lambda} \right) . \end{aligned}$$

From this we can see that  $I(\lambda)$  grows towards infinity.

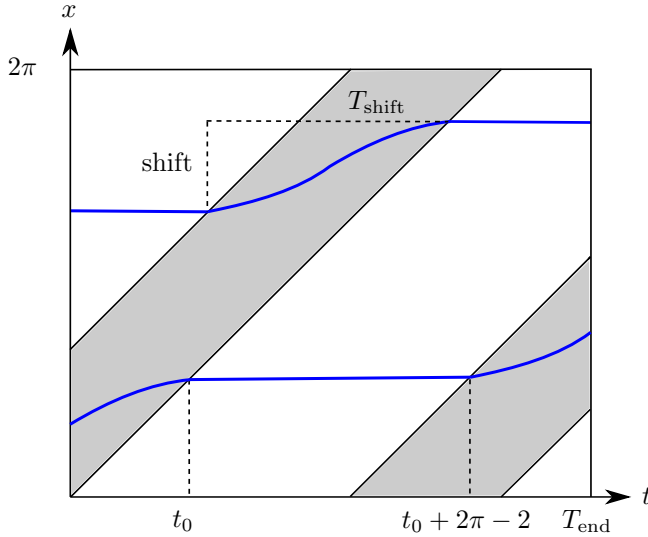


FIGURE 2. Sketch of the vector field  $u(t, x)$ . The gray area represents the support of  $u$  and the blue curves are integral curves of  $u(\cdot, x)$ .

**Claim C.** We have control over the time necessary to induce a shift  $I(\lambda)$ , via  $T_{\text{shift}} = 2 + I(\lambda)$ .

We had above  $T_{\text{shift}} = a_x^{-1}(1) - a_x^{-1}(-1)$  and also  $\partial_t a_x^{-1}(t) = \frac{1}{1-\lambda f(t)}$ . Hence

$$T_{\text{shift}} = \int_{-1}^1 \frac{1}{1-\lambda f(t)} dt = \int_{-1}^1 1 + \frac{\lambda f(t)}{1-\lambda f(t)} dt = 2 + I(\lambda).$$

This concludes the proof of the claim.

Now we choose  $\lambda$  such that  $I(\lambda) = \sigma$  and we define the flow  $\varphi(t, x)$  on the circle with period  $2\pi$  for time from 0 to  $T_{\text{end}} = T_{\text{shift}} + 2\pi - 2$ .

**Claim D.** *The endpoint of the resulting flow is a constant shift*

$$\varphi(T_{\text{end}}, x) = x + \sigma.$$

We have to consider two cases. First take a point  $x$ , such that  $x \notin \text{supp}(f)$ . W.l.o.g. we assume that  $\text{supp}(f)$  is the interval  $[0, 2]$  starting at 0, which implies that in this case  $x > 2$ . Then  $x$  meets the vector field at time  $t_0 = x - 2$  and leaves it again at time  $t_1 = x - 2 + T_{\text{shift}}$  after being shifted by one. It would meet the vector field again at time  $t_2 = x - 2 + T_{\text{shift}} + 2\pi - 2$ , but because  $x - 2 > 0$  we have  $t_2 > T_{\text{end}}$  and hence the point doesn't meet the vector field again.

Now take a point  $0 \leq x \leq 2$ . This point starts in the vector field, leaves it at some time  $t_0 < T_{\text{shift}}$  and meets the vector field again at time  $t_0 + 2\pi - 2$ . It then stays in the vector field until the  $T_{\text{end}}$  since  $T_{\text{end}} < t_0 + 2\pi - 2 + T_{\text{shift}}$ . Altogether the point has spent time  $t_0 + T_{\text{end}} - (t_0 + 2\pi - 2) = T_{\text{shift}}$  in the vector field, so it has also been shifted by  $\sigma$ . This concludes the proof.  $\square$

*Proof of theorem 3.1.* Let us denote by  $\text{Diff}_0(M)^{L=0}$  the set of all diffeomorphisms  $\varphi$  that can be reached from the identity by curves of arbitrarily short length, i.e., for each  $\varepsilon > 0$  there exists a curve from the identity to  $\varphi$  with length smaller than  $\varepsilon$ . In the following we will show that  $\text{Diff}_0(M)^{L=0}$  is a non-trivial normal subgroup of  $\text{Diff}_0(M)$ . By the simplicity of  $\text{Diff}_0(M)$  (see [8, 24, 13, 14]) it then coincides with  $\text{Diff}_0(M)$ , which concludes the proof.

**Claim A.**  *$\text{Diff}_0(M)^{L=0}$  is a normal subgroup of  $\text{Diff}_0(M)$ .*

Given a diffeomorphism  $\psi \in \text{Diff}_0(M)$ , we can choose a partition of unity  $\tau_j$  such that normal coordinates centered at  $x_j \in M$  are defined on  $\text{supp}(\tau_j)$  and such that normal coordinates centered at  $\psi(x_j)$  are defined on  $\psi(\text{supp}(\tau_j))$ . Then we can define  $\psi_j = \exp_{\psi(x_j)}^{-1} \circ \psi \circ \exp_{x_j}$ . For  $\varphi_1 \in \text{Diff}_0(M)^{L=0}$  we choose a curve  $t \mapsto \varphi(t, \cdot)$  from the identity to  $\varphi_1$  with length less than  $\varepsilon$ . Let  $u = \varphi_t \circ \varphi^{-1}$ . Then

$$\begin{aligned} \text{Len}(\psi^{-1} \circ \varphi \circ \psi) &\leq C_1(\tau) \int_0^1 \|(T\psi^{-1} \circ \varphi_t \circ \psi) \circ (\psi^{-1} \circ \varphi \circ \psi)^{-1}\|_{H^s(M, \tau)} dt \\ &= C_1(\tau) \int_0^1 \|(T\psi^{-1} \circ u \circ \psi)\|_{H^s(M, \tau)} dt \\ &= C_1(\tau) \int_0^1 \sqrt{\sum_{j=0}^{\infty} \|\exp_{x_j}^*(\tau_j \cdot T\psi^{-1} \circ u \circ \psi)\|_{H^s(\mathbb{R}^n)}^2} dt \\ &= C_1(\tau) \int_0^1 \sqrt{\sum_{j=0}^{\infty} \|T\psi_j^{-1} \cdot (\exp_{\psi(x_j)}^*(\tau_j \circ \psi^{-1} \cdot u)) \circ \psi_j\|_{H^s(\mathbb{R}^n)}^2} dt \\ &\leq C_2(\psi, \tau) \int_0^1 \sqrt{\sum_{j=0}^{\infty} \|(\exp_{\psi(x_j)}^*(\tau_j \circ \psi^{-1} \cdot u))\|_{H^s(\mathbb{R}^n)}^2} dt \end{aligned}$$

$$= C_2(\psi, \tau) \int_0^1 \|u\|_{H^s(M, \tau \circ \psi^{-1})} dt \leq C_3(\psi, \tau) \text{Len}(\varphi) .$$

Here we used that all partitions of unity  $\tau$  induce equivalent norms  $H^s(M, \tau)$  and that for  $h \in C^\infty(M)$  and  $\psi \in \text{Diff}_c(M)$  point wise multiplication  $f \mapsto h \cdot f$  and composition  $f \mapsto f \circ \psi$  are bounded linear operators on  $H^s(M)$ , as noted in theorems 2.2 and 2.3.

**Claim B.**  $\text{Diff}_0(M)^{L=0}$  is a nontrivial subgroup of  $\text{Diff}_0(M)$  .

For a one-dimensional manifold  $M$  the non-triviality of  $\text{Diff}_0(M)$  is shown in lemma 3.2 and lemma 3.4. It remains to prove the case  $\dim(M) > 1$ .

The idea is to use polar coordinates and a partition of unity to embed diffeomorphisms of  $S^1$  as diffeomorphisms of  $M$ . We choose a partition of unity  $(\tau_j)$  such that  $\tau_0 \equiv 1$  on some open subset  $U \subset M$ , where normal coordinates centered at  $x_0 \in U$  are defined. According to lemma 3.4, for  $0 \leq s \leq \frac{1}{2}$ , there exists a family of diffeotopies  $\varphi_{S^1}(\sigma, t, x)$  with arbitrarily small  $H^s$ -length which satisfies  $\varphi_{S^1}(0, t, x) = \varphi_{S^1}(\sigma, 0, x) = x$  and  $\varphi_{S^1}(\sigma, 1, x) = x + \sigma$ . Let  $u_{S^1}(\sigma, t, x) = (\varphi_{S^1})_t \circ \varphi_{S^1}^{-1}$  be the right-invariant velocity vector of the path. From the proof of lemma 3.4 we see that  $\varphi_{S^1}(\sigma, t, x)$  and  $u_{S^1}(\sigma, t, x)$  are also smooth in  $\sigma$ . Using a bump function  $b \in C^\infty(\mathbb{R})$  we define a vector field  $\tilde{u}$  on  $\mathbb{R}^+ \times S^1 \times \mathbb{R}^{n-2}$  via

$$\tilde{u}(t, r, \theta, x_1, \dots, x_{n-2}) := (0, u_{S^1}(b(r) \cdot b(x_1) \dots b(x_{n-2}), t, \theta), 0, \dots, 0) .$$

The flow at time  $t = 1$  of this vector field is the diffeomorphism

$$\tilde{\varphi}(r, \theta, x_1, \dots, x_{n-2}) = (r, \varphi_{S^1}(b(r) \cdot b(x_1) \dots b(x_{n-2}), \theta), x_1, \dots, x_{n-2})$$

and it depends only on the endpoint  $\varphi_{S^1}(\sigma, x)$  and not on the path in between. Using a transformation to polar coordinates

$$\Phi : \begin{cases} \mathbb{R}^+ \times S^1 & \rightarrow & \mathbb{R}^2 \setminus \{0\} \\ (r, \theta) & \mapsto & (r \cos(\theta), r \sin(\theta)) \end{cases}$$

we can define a vector field in  $\mathbb{R}^n$  coordinates

$$u_{\mathbb{R}^n}(t, x_1, \dots, x_n) := \tilde{u}(t, \Phi^{-1}(x_1, x_2), x_3, \dots, x_n) .$$

If  $b$  is chosen with sufficiently small support, then the vector field  $u_{\mathbb{R}^n}$  has support in  $\exp_{x_0}(U)$  and we can define the vector field  $u_M := (\exp_{x_0})_* u_{\mathbb{R}^n}$  on  $M$ . This vector field generates a path  $\varphi_M(t, \cdot) \in \text{Diff}_0(M)$  with arbitrarily small  $H^s$ -length:

$$\begin{aligned} \text{Len}(\varphi_M) &\leq C_1(\tau) \int_0^1 \|u_M\|_{H^s(M, \tau)} dt = C_1(\tau) \int_0^1 \|\exp_{x_0}^*(\tau_0 \cdot u_M)\|_{H^s(\mathbb{R}^n)} dt \\ &= C_1(\tau) \int_0^1 \|u_{\mathbb{R}^n}\|_{H^s(\mathbb{R}^n)} dt \\ &= C_1(\tau) \int_0^1 \|(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}u_{\mathbb{R}^n}\|_{L^2(\mathbb{R}^n)} dt \\ &\leq C_2(\tau, \Phi) \int_0^1 \|(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}\tilde{u}\|_{L^2(\mathbb{R}^+ \times S^1 \times \mathbb{R}^n)} dt \\ &= C_2(\tau, \Phi) \int_0^1 \|(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F}u_{S^1}(t, \xi_1) \mathcal{F}b(\xi_2) \dots \mathcal{F}b(\xi_n)\|_{L^2} dt \\ &\leq C_3(\tau, \Phi, b) \int_0^1 \|u_{S^1}\|_{H^s(S^1)} = C_3(\tau, \Phi, b) \cdot \text{Len}(\varphi_{S^1}) . \quad \square \end{aligned}$$

## 4. POSITIVE GEODESIC DISTANCE

**4.1. Theorem** (Positive geodesic distance). *For  $\dim(M) = 1$  the Sobolev-norm of order  $s$  induces positive geodesic distance on  $\text{Diff}_c(M)$  if  $s > \frac{1}{2}$ . For  $\dim(M) \geq 2$  it induces positive geodesic distance if  $s \geq 1$ .*

*Proof.* By the definition of the Sobolev metric it suffices to show the result for  $\text{Diff}_c(\mathbb{R}^n)$ .

For the case  $n = 1$  let  $\varphi_0, \varphi_1 \in \text{Diff}_c(\mathbb{R})$  with  $\varphi_0(x) \neq \varphi_1(x)$  for some  $x \in \mathbb{R}$ . For any path  $\varphi(t, \cdot)$ , with  $\varphi(0, \cdot) = \varphi_0$  and  $\varphi(1, \cdot) = \varphi_1$  we have

$$\begin{aligned} 0 \neq |\varphi_1(x) - \varphi_0(x)| &= \left| \int_0^1 \varphi_t(t, x) dt \right| = \left| \int_0^1 u(t, \varphi(t, x)) dt \right| \\ &\leq \int_0^1 |u(t, \varphi(t, x))| dt \leq \int_0^1 \|u(t, \cdot)\|_\infty dt \leq \int_0^1 \|u(t, \cdot)\|_{C^{0, s-1/2}} dt \\ &\leq \int_0^1 \|u(t, \cdot)\|_{H^s} dt. \end{aligned}$$

In the last step, we used the Sobolev embedding theorem, see [25] for example. The case  $\dim(M) \geq 2$  follows from [17, Theorem 5.7].  $\square$

5. THE GEODESIC EQUATION ON  $\text{Diff}_c(\mathbb{R}^n)$ 

In the upcoming parts we want to calculate the geodesic equation for the two equivalent Sobolev norms on  $\text{Diff}_c(\mathbb{R}^n)$ .

**5.1. The general setting.** According to [1], see [16, section 3] for a presentation directly applicable here, we have: For any right invariant metric  $G$  on a regular infinite dimensional Lie group, the geodesic equation reads as

$$u_t = -\text{ad}(u)^\top u.$$

Here  $\text{ad}(u)^\top$  denotes the adjoint of the adjoint representation  $\text{ad}$ , which is given by  $\langle \text{ad}(v)^\top u, w \rangle_G := \langle u, \text{ad}_v w \rangle_G$ . Note that for  $\text{Diff}_c(\mathbb{R}^n)$  we have  $\text{ad}_v w = -[v, w]$  for  $v, w \in \mathfrak{X}_c(\mathbb{R}^n)$ . The sectional curvature (for orthonormal  $u, v$ ) at the identity is then given by the formula

$$\langle R(u, v)v, u \rangle_G = \frac{1}{4} \|\beta(u)v - \beta(v)u - \text{ad}(u)v\|_G^2 + \langle [\beta(u), \beta(v)]u, v \rangle_G$$

where  $\beta(u)v := \text{ad}(u)^\top v + \text{ad}(u)v$ . This last expression is from [15, section 2.6]. The Jacobi equation for a right trivialized Jacobi field  $y$  along a geodesic with right trivialized velocity field  $u$  (which satisfies the geodesic equation) is derived in [19, 3.4 and 3.5] as:

$$y_{tt} = [\text{ad}(y)^\top + \text{ad}(y), \text{ad}(u)^\top]u - \text{ad}(u)^\top y_t - \text{ad}(y_t)^\top u + \text{ad}(u)y_t.$$

This will allow us to write down the curvature and the Jacobi equations for all metrics that we will treat below. Since this leads to complicated formulas we will not spell this out.

**5.2. Theorem.** *Let  $A : \mathfrak{X}_c(\mathbb{R}^n) \rightarrow \mathfrak{X}_c(\mathbb{R}^n)$  be an elliptic, scalar (pseudo)-differential operator that is positive and self-adjoint with respect to the  $L^2$ -metric. Then  $A$  induces a metric on  $\text{Diff}_c(\mathbb{R}^n)$  in the following way:*

$$G_\varphi^A(X, Y) := \langle A(X \circ \varphi^{-1}), Y \circ \varphi^{-1} \rangle_{L^2(\mathbb{R}^n)}.$$

The geodesic equation with respect to the  $G^A$ -metric is then given by

$$Au_t^k = - \sum_{i=1}^n (Au^i(\partial_k u^i) + (A(\partial_i u^k).u^i + Au^k.(\partial_i u^i))).$$

Equivalently it can be written in terms of the momentum  $m = Au$ :

$$m_t^k = - \sum_{i=1}^n (m^i(\partial_k u^i) + ((\partial_i m^k).u^i + m^k.(\partial_i u^i))), \quad u^k = A^{-1}m^k.$$

*Proof.* For  $u, v, w \in \mathfrak{X}_c(\mathbb{R}^n)$  we calculate

$$\begin{aligned} \langle u, -[v, w] \rangle_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \langle A_s u, -[v, w] \rangle_{\mathbb{R}^n} dx \\ &= \int_{\mathbb{R}^n} \sum_{k=1}^n A_s u^k \sum_{i=1}^n ((\partial_i v^k)w^i - v^i(\partial_i w^k)) dx \\ &= \int_{\mathbb{R}^n} \sum_{k=1}^n \sum_{i=1}^n A_s u^k (\partial_i v^k)w^i + \partial_i(A_s u^k.v^i)\partial_i w^k dx \\ &= \int_{\mathbb{R}^n} \sum_{i=1}^n \sum_{k=1}^n A_s u^i (\partial_k v^i)w^k + \sum_{k=1}^n \sum_{i=1}^n (A_s(\partial_i u^k).v^i + A_s u^k.(\partial_i v^i))w^k dx \\ &= \int_{\mathbb{R}^n} \sum_{k=1}^n w^k A_s (\text{ad}(v)^\top u)^k dx = \langle \text{ad}(v)^\top u, w \rangle_{H^s(\mathbb{R}^n)}. \\ \text{ad}(v)^\top u &= \sum_{i=1}^n A_s^{-1} (A_s u^i(\partial_k u^i) + (A_s(\partial_i u^k).u^i + A_s u^k.(\partial_i u^i))). \quad \square \end{aligned}$$

**5.3. Theorem** (Geodesic equation for the Sobolev metric  $G^s$ ). *The operator*

$$A_s : H^{k+2s}(\mathbb{R}^n) \rightarrow H^k(\mathbb{R}^n), \quad u(x) \mapsto (\mathcal{F}^{-1}(1 + |\xi|^2)^s \mathcal{F}u)(x)$$

*induces the Sobolev metric  $G^s$  of order  $s$  on  $\text{Diff}_c(\mathbb{R}^n)$ . The geodesic equation for this metric reads as:*

$$\begin{aligned} m_t^k &= - \sum_{i=1}^n (m^i(\partial_k u^i) + ((\partial_i m^k).u^i + m^k.(\partial_i u^i))), \\ u^k &= \begin{cases} (2\pi)^{\frac{n}{2}} \frac{2^{1-s} |\cdot|^{s-\frac{n}{2}}}{\Gamma(s)} K_{s-\frac{n}{2}}(|\cdot|) \star m^k, & s > \frac{n-1}{4} \\ (2\pi)^{\frac{n}{2}} |\cdot|^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(r|\cdot|) \frac{r^{\frac{n}{2}}}{(1+r^2)^s} dr \star m^k, & s \leq \frac{n-1}{4}. \end{cases} \end{aligned}$$

Here  $J_{n/2-1}$  denotes the Bessel function of the first kind, which is given by

$$J_\alpha(r) = \frac{1}{\pi} \int_0^\pi \cos(\alpha t - r \sin t) dt - \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty e^{-r \sinh(t) - \alpha t} dt,$$

and  $K_{s-\frac{n}{2}}$  denotes the modified Bessel function of second kind, which is given by

$$K_\nu(r) = \frac{\Gamma(\nu + \frac{1}{2})(2r)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\cos t}{(t^2 + r^2)^{\nu+\frac{1}{2}}} dt.$$

*Proof.* The operator  $A_s u = \mathcal{F}^{-1}(1 + |\xi|^2)^s \mathcal{F}u$  is an elliptic, scalar (pseudo)-differential operator that is positive and self-adjoint with respect to the  $L^2$ -metric. In

particular it is a linear isomorphism from  $H^{k+2s}(\mathbb{R}^n) \rightarrow H^k(\mathbb{R}^n)$ . By theorem 5.2 it remains to calculate the operator  $A_s^{-1}$ . This can be done as follows:

$$A_s^{-1}m(x) = \mathcal{F}^{-1}(1 + |\xi|^2)^{-s} \mathcal{F}m(x) = (2\pi)^{\frac{n}{2}} \mathcal{F}^{-1}((1 + |\xi|^2)^{-s}) \star m(x) .$$

Let  $f$  be a radial symmetric function on  $\mathbb{R}^n$ , i.e.,  $f(\xi) = f(|\xi|)$ . Then we have

$$\mathcal{F}^{-1}(f)(x) = |x|^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(r \cdot |x|) \cdot f(r) \cdot r^{\frac{n}{2}} dr .$$

For the function  $f(\xi) = (1 + |\xi|^2)^{-s}$  this yields:

$$\mathcal{F}^{-1}((1 + |\xi|^2)^{-s}) = |x|^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(r \cdot |x|) \frac{r^{\frac{n}{2}}}{(1 + r^2)^s} dr .$$

See for example the books [22, 23] for more details about Fourier transformation of radial symmetric functions. For  $s > \frac{n-1}{4}$  the last integral converges and we have

$$\mathcal{F}^{-1}((1 + |\xi|^2)^{-s}) = \frac{2^{1-s} |x|^{s-\frac{n}{2}}}{\Gamma(s)} K_{s-\frac{n}{2}}(|x|) . \quad \square$$

An immediate consequence of the above analysis is the geodesic equation for the equivalent Sobolev-metric  $\overline{G}^s$ .

**5.4. Theorem** (Geodesic equation for the Sobolev metric  $\overline{G}^s$ ). *The operator*

$$\overline{A}_s : H^{k+2s}(\mathbb{R}^n) \rightarrow H^k(\mathbb{R}^n), \quad u(x) \mapsto (\mathcal{F}^{-1}(1 + |\xi|^{2s}) \mathcal{F}u)(x)$$

*induces the Sobolev metric  $\overline{G}^s$  of order  $s$  on  $\text{Diff}_c(\mathbb{R}^n)$ . The geodesic equation for this metric reads as:*

$$m_t^k = - \sum_{i=1}^n (m^i (\partial_k u^i) + ((\partial_i m^k) \cdot u^i + m^k \cdot (\partial_i u^i))) ,$$

$$u^k = (2\pi)^{\frac{n}{2}} |\cdot|^{1-\frac{n}{2}} \int_0^\infty J_{\frac{n}{2}-1}(r \cdot |\cdot|) \frac{r^{\frac{n}{2}}}{(1 + r^{2s})} dr \star m^k .$$

**5.5. The Geodesic Equation in dimension one.** For the  $\overline{G}^s$ -metric on  $\text{Diff}_c(\mathbb{R})$  or  $\text{Diff}(S^1)$  the above expression for the geodesic equation simplifies to

$$m_t = -2u_x m - u m_x, \quad u = (2\pi)^{\frac{1}{2}} \int_0^\infty J_{-\frac{1}{2}}(r \cdot |\cdot|) \frac{r^{\frac{1}{2}}}{(1 + r^{2s})} dr \star m .$$

For  $s = k \in \mathbb{N}$  we can rewrite this equation as:

$$m_t = -2u_x m - u m_x, \quad m = u + \partial_x^{2k} u ,$$

where  $m(t, x)$  is the momentum corresponding to the velocity  $u(t, x)$ . In the case  $s = 0$  this becomes the inviscid Burger equation

$$u_t = -3u_x u$$

for  $s = 1$  it is the Camassa Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} .$$

and for  $s > 1$  they are related to the higher order Camassa Holm equations, see [4].

To study the Sobolev metric  $\overline{G}^s$ , for  $s = k + \frac{1}{2}$  we introduce the Hilbert transform, which is given by

$$\mathcal{F}(\mathcal{H}f)(\xi) = -i \operatorname{sgn}(\xi) \mathcal{F}f(\xi) .$$

Using this we can write the metric  $\overline{G}^{k+\frac{1}{2}}$  in the form

$$\overline{G}^{k+\frac{1}{2}}(u, v) = \int_{\mathbb{R}} (u + \mathcal{H}\partial_x^{2k+1}u)v \, dx = \int_{\mathbb{R}} (u + \partial_x^{2k+1}(\mathcal{H}u))v \, dx .$$

The geodesic equation is then given by

$$m_t = -2u_x m - um_x, \quad m = u + \mathcal{H}\partial_x^{2k+1}u .$$

If we pass to the homogenous space  $\text{Diff}(S^1)/S^1$  and consider the homogenous Sobolev metric of order one half the geodesic equation reduces to

$$m_t = -2u_x m - um_x, \quad m = \mathcal{H}u_x ,$$

which is the modified Constantin-Lax-Majda equation, see [27]. If we consider the homogenous Sobolev metric of order one the resulting geodesic equation

$$u_{xxt} = -2u_x u_{xx} - uu_{xxx}$$

is the Hunter-Saxton equation.

## 6. CONCLUSIONS.

In this paper we have provided a partial answer to the problem of vanishing geodesic distance for Sobolev-type metrics on diffeomorphism groups. Two cases remain open.

**Conjecture.** For  $\dim(M) \geq 2$  and  $\frac{1}{2} < s < 1$  the geodesic distance doesn't vanish.

For  $s \geq 1$  the non-vanishing of the geodesic distance has been shown in [17, Theorem 5.7]. In contrast to  $\dim(M) = 1$  the higher-dimensional behavior cannot be tied too much to the critical Sobolev-index  $s = \frac{\dim(M)}{2}$ , since for  $\dim(M) = 3$  the critical index equals  $s = \frac{3}{2}$  while [17, Theorem 5.7] proves that already for  $s = 1$  the geodesic distance doesn't vanish.

**Conjecture.** For  $M = \mathbb{R}$  and  $s = \frac{1}{2}$  the geodesic distance vanishes.

We believe that a similar construction as in Lemma 3.4 can be used to construct paths of arbitrary short length. However, since the diffeomorphisms need to have compact support, we need to adapt the construction to reach a diffeomorphism of the form  $\varphi(x) = x + c(x)$  as in Lemma 3.2. The difficulty lies in constructing a vector field, whose flow at time  $t = 1$  we can control.

Another interesting question is, wether this result carries on to the Virasoro-Bott group. In [2] it was shown that the right invariant  $L^2$ -metric on the Virasoro-Bott group has vanishing geodesic distance. We believe that the results of this paper can be extended to Sobolev metrics of fractional order on the Virasoro-Bott group similarly as in [2]. When working on the Virasoro-Bott group one has to control the central cocycle along the curve, which is expressed in terms of the diffeomorphism and its derivatives. All the constructions in this paper take place on the vector fields. In order to pass to the Virasoro-Bott group one would need to control the behavior of the flow via the vector field.

## REFERENCES

- [1] V. I. Arnold. Sur la géométrie différentielle des groupes de lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier*, 16:319–361, 1966.

- [2] M. Bauer, M. Bruveris, P. Harms, and P.W. Michor. Vanishing geodesic distance for the riemannian metric with geodesic equation the KdV-equation. *Ann. Global Analysis Geom.* 41, 4 (2012) 461–472.
- [3] A. Constantin, T. Kappeler, B. Kolev, and P. Topalov. On geodesic exponential maps of the Virasoro group. *Ann. Global Anal. Geom.*, 31(2):155–180, 2007.
- [4] Adrian Constantin and Boris Kolev. Geodesic flow on the diffeomorphism group of the circle. *Comment. Math. Helv.*, 78(4):787–804, 2003.
- [5] David G. Ebin and Jerrold Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. of Math. (2)*, 92:102–163, 1970.
- [6] Jürgen Eichhorn. The boundedness of connection coefficients and their derivatives. *Math. Nachr.*, 152:145–158, 1991.
- [7] Jürgen Eichhorn. *Global Analysis on Open Manifolds*. Nova Science Publishers Inc., New York, 2007.
- [8] D. B. A. Epstein. The simplicity of certain groups of homeomorphisms. *Compositio Math.*, 22:165–173, 1970.
- [9] Joachim Escher, Boris Kolev, and Marcus Wunsch. The geometry of a vorticity model equation. arXiv:1010.4844v1 [math.AP] 23 Oct 2010.
- [10] François Gay-Balmaz. Well-posedness of higher dimensional Camassa-Holm equations. *Bull. Transilv. Univ. Braşov Ser. III*, 2(51):55–58, 2009.
- [11] R. E. Greene. Complete metrics of bounded curvature on noncompact manifolds. *Arch. Math.*, 31(1):89–95, 1978.
- [12] Yu. A. Kordyukov.  $L^p$ -theory of elliptic differential operators on manifolds of bounded geometry. *Acta Appl. Math.*, 23(3):223–260, 1991.
- [13] John N. Mather. Commutators of diffeomorphisms. *Comment. Math. Helv.*, 49:512–528, 1974.
- [14] John N. Mather. Commutators of diffeomorphisms. II. *Comment. Math. Helv.*, 50:33–40, 1975.
- [15] Mario Micheli, Peter W. Michor, and David Mumford. Sobolev curvature of  $\text{Diff}(\mathbb{R}^n)$  and of some of its Chow quotients. *In preparation*.
- [16] Peter W. Michor. Some geometric evolution equations arising as geodesic equations on groups of diffeomorphisms including the Hamiltonian approach. In *Phase space analysis of partial differential equations*, volume 69 of *Progr. Nonlinear Differential Equations Appl.*, pages 133–215. Birkhäuser Boston, 2006.
- [17] Peter W. Michor and David Mumford. Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms. *Doc. Math.*, 10:217–245 (electronic), 2005.
- [18] Peter W. Michor and David Mumford. Riemannian geometries on spaces of plane curves. *J. Eur. Math. Soc. (JEMS)*, 8:1–48, 2006.
- [19] P.W. Michor and T. Ratiu. Geometry of the Virasoro-Bott group. *J. Lie Theory*, 8:293–309, 1998.
- [20] V. Y. Ovsienko and B. A. Khesin. Korteweg–de Vries superequations as an Euler equation. *Funct. Anal. Appl.*, 21:329–331, 1987.
- [21] M. A. Shubin. Spectral theory of elliptic operators on noncompact manifolds. *Astérisque*, 207(5):35–108, 1992. *Méthodes semi-classiques*, Vol. 1 (Nantes, 1991).
- [22] Elias M. Stein and Rami Shakarchi. *Fourier Analysis*, volume 1 of *Princeton Lectures in Analysis*. Princeton University Press, Princeton, NJ, 2003. An introduction.
- [23] Elias M. Stein and Guido Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32.
- [24] William Thurston. Foliations and groups of diffeomorphisms. *Bull. Amer. Math. Soc.*, 80:304–307, 1974.
- [25] Hans Triebel. *Theory of Function Spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.
- [26] Hans Triebel. *The Structure of Functions*, volume 97 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2001.
- [27] Marcus Wunsch. On the geodesic flow on the group of diffeomorphisms of the circle with a fractional Sobolev right-invariant metric. *J. Nonlinear Math. Phys.*, 17(1):7–11, 2010.

MARTIN BAUER, PHILIPP HARMS, PETER W. MICHOR: FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTRASSE 15, A-1090 WIEN, AUSTRIA.

MARTINS BRUVERIS: DEP. OF MATHEMATICS, IMPERIAL COLLEGE, LONDON SW7 2AZ, UK.

PHILIPP HARMS: EDLABS, HARVARD UNIVERSITY, 44 BRATTLE STREET, CAMBRIDGE, MA 02138

*E-mail address:* `bauer.martin@univie.ac.at`

*E-mail address:* `m.bruveris08@imperial.ac.uk`

*E-mail address:* `philipp.harms@univie.ac.at`

*E-mail address:* `peter.michor@esi.ac.at`