GEODESIC DISTANCE FOR RIGHT INVARIANT SOBOLEV METRICS OF FRACTIONAL ORDER ON THE DIFFEOMORPHISM GROUP. II

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ABSTRACT. The geodesic distance vanishes on the group $\operatorname{Diff}_c(M)$ of compactly supported diffeomorphisms of a Riemannian manifold M of bounded geometry, for the right invariant weak Riemannian metric which is induced by the Sobolev metric H^s of order $0 \le s < \frac{1}{2}$ on the Lie algebra $\mathfrak{X}_c(M)$ of vector fields with compact support.

1. Introduction

In the article [1] we studied right invariant metrics on the group $\operatorname{Diff}_c(M)$ of compactly supported diffeomorphisms of a manifold M, which are induced by the Sobolev metric H^s of order s on the Lie algebra $\mathfrak{X}_c(M)$ of vector fields with compact support. We showed that for $M = S^1$ the geodesic distance on $\operatorname{Diff}(S^1)$ vanishes if and only if $s \le \frac{1}{2}$. For other manifolds, we showed that the geodesic distance on $\operatorname{Diff}_c(M)$ vanishes for $M = \mathbb{R} \times N, s < \frac{1}{2}$ and for $M = S^1 \times N, s \le \frac{1}{2}$, with N being a compact Riemannian manifold.

Now we are able to complement this result by: *The geodesic distance vanishes on* $\operatorname{Diff}_c(M)$ *for any Riemannian manifold* M *of bounded geometry, if* $0 \le s < \frac{1}{2}$.

We believe that this result holds also for $s = \frac{1}{2}$, but we were able to overcome the technical difficulties only for the manifold $M = S^1$, in [1]. We also believe that it is true for the regular groups $\operatorname{Diff}_{\mathcal{H}^\infty}(\mathbb{R}^n)$ and $\operatorname{Diff}_{\mathcal{S}}(\mathbb{R}^n)$ as treated in [8], and for all Virasoro groups, where we could prove it only for $s = 0$ in [2].

In Section 2, we review the definitions for Sobolev norms of fractional orders on diffeomorphism groups as presented in [1] and extend them to diffeomorphism groups of manifolds of bounded geometry. Section 3 is devoted to the main result.

2. Sobolev metrics H^s with $s \in \mathbb{R}$

2.1. **Sobolev metrics H^s on $\mathbb{R}^n$.** For $s \ge 0$ the Sobolev H^s-norm of an $\mathbb{R}^n$-valued function f on $\mathbb{R}^n$ is defined as

$$\|f\|^2_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} f\|^2_{L^2(\mathbb{R}^n)} , \tag{1}$$

where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F} f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle x, \xi\rangle} f(x) \, \mathrm{d}x ,$$

2010 *Mathematics Subject Classification.* Primary 35Q31, 58B20, 58D05.

Martin Bauer was supported by ‘Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P 24625’.

and ξ is the independent variable in the frequency domain. An equivalent norm is given by

$$\|f\|^2_{\overline{H}^s(\mathbb{R}^n)} = \|f\|^2_{L^2(\mathbb{R}^n)} + \||\xi|^s \mathcal{F} f\|^2_{L^2(\mathbb{R}^n)}\ . \tag{2}$$

The fact that both norms are equivalent is based on the inequality

$$\frac{1}{C}\big(1+\sum_j |\xi_j|^s\big) \le \big(1+\sum_j |\xi_j|^2\big)^{\frac{s}{2}} \le C\big(1+\sum_j |\xi_j|^s\big)\ ,$$

holding for some constant C. For $s>1$ this says that all ℓ^s-norms on $\mathbb{R}^{n+1}$ are equivalent. But the inequality is true also for $0<s<1$, even though the expression does not define a norm on $\mathbb{R}^{n+1}$. Using any of these norms we obtain the Sobolev spaces with non-integral s

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \|f\|_{H^s(\mathbb{R}^n)} < \infty\}\ .$$

We will use the second version of the norm in the proof of the theorem, since it will make calculations easier.

2.2. **Sobolev metrics for Riemannian manifolds of bounded geometry.** Following [13, Section 7.2.1] we will now introduce the spaces $H^s(M)$ on a manifold M. If M is not compact we equip M with a Riemannian metric g of bounded geometry which exists by [5]. This means that

- (I) The injectivity radius of (M,g) is positive.
- (B_∞) Each iterated covariant derivative of the curvature is uniformly g-bounded: $\|\nabla^i R\|_g < C_i$ for $i=0,1,2,\dots$.

The following is a compilation of special cases of results collected in [3, Chapter 1], who treats Sobolev spaces only for integral order.

Proposition ([6], [10], [4])**.** *If (M,g) satisfies (I) and (B_∞) then the following holds:*

1. *(M,g) is complete.*
2. *There exists $\varepsilon_0>0$ such that for each $\varepsilon\in(0,\varepsilon_0)$ there is a countable cover of M by geodesic balls $B_\varepsilon(x_\alpha)$ such that the cover of M by the balls $B_{2\varepsilon}(x_\alpha)$ is still uniformly locally finite.*
3. *Moreover, there exists a partition of unity $1=\sum_\alpha \rho_\alpha$ on M such that $\rho_\alpha\ge 0$, $\rho_\alpha\in C^\infty_c(M)$, $\operatorname{supp}(\rho_\alpha)\subset B_{2\varepsilon}(x_\alpha)$, and $|D^\beta_u \rho_\alpha| < C_\beta$ where u are normal (Riemann exponential) coordinates in $B_{2\varepsilon}(x_\alpha)$.*
4. *In each $B_{2\varepsilon}(x_\alpha)$, in normal coordinates, we have $|D^\beta_u g_{ij}| < C'_\beta$, $|D^\beta_u g^{ij}| < C''_\beta$, and $|D^\beta_u \Gamma^m_{ij}| < C'''_\beta$, where all constants are independent of α.*

We can now define the H^s-norm of a function f on M:

$$\begin{aligned}\|f\|^2_{H^s(M,g)} &= \sum_{\alpha=0}^\infty \|(\rho_\alpha f)\circ \exp_{x_\alpha}\|^2_{H^s(\mathbb{R}^n)} = \\ &= \sum_{\alpha=0}^\infty \|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}((\rho_\alpha f)\circ\exp_{x_\alpha})\|^2_{L^2(\mathbb{R}^n)}\ .\end{aligned}$$

If M is compact the sum is finite. Changing the charts or the partition of unity leads to equivalent norms by the proposition above, see [13, Theorem 7.2.3]. For integer s we get norms which are equivalent to the Sobolev norms treated in [3, Chapter 2]. The norms depends on the choice of the Riemann metric g. This dependence is worked out in detail in [3].

For vector fields we use the trivialization of the tangent bundle that is induced by the coordinate charts and define the norm in each coordinate as above. This leads to a (up to equivalence) well-defined H^s-norm on the Lie algebra $\mathfrak{X}_c(M)$.

2.3. **Sobolev metrics on** $\operatorname{Diff}_c(M)$**.** A positive definite weak inner product on $\mathfrak{X}_c(M)$ can be extended to a right-invariant weak Riemannian metric on $\operatorname{Diff}_c(M)$. In detail, given $\varphi \in \operatorname{Diff}_c(M)$ and $X, Y \in T_\varphi \operatorname{Diff}_c(M)$ we define

$$G^s_\varphi(X,Y) = \langle X \circ \varphi^{-1}, Y \circ \varphi^{-1} \rangle_{H^s(M)} \,.$$

We are interested solely in questions of vanishing and non-vanishing of geodesic distance. These properties are invariant under changes to equivalent inner products, since equivalent inner products on the Lie algebra

$$\frac{1}{C}\langle X,Y\rangle_1 \le \langle X,Y\rangle_2 \le C\langle X,Y\rangle_1$$

imply that the geodesic distances will be equivalent metrics

$$\frac{1}{C}\operatorname{dist}_1(\varphi,\psi) \le \operatorname{dist}_2(\varphi,\psi) \le C\operatorname{dist}_1(\varphi,\psi)\,.$$

Therefore the ambiguity – dependence on the charts and the partition of unity – in the definition of the H^s-norm is of no concern to us.

3. Vanishing geodesic distance

3.1. **Theorem** (Vanishing geodesic distance)**.** *The Sobolev metric of order s induces vanishing geodesic distance on* $\operatorname{Diff}_c(M)$ *if:*

- $0 \le s < \frac{1}{2}$ *and M is any Riemannian manifold of bounded geometry.*

This means that any two diffeomorphisms in the same connected component of $\operatorname{Diff}_c(M)$ *can be connected by a path of arbitrarily short G^s-length.*

In the proof of the theorem we shall make use of the following lemma from [1].

3.2. **Lemma** ([1, Lemma 3.2])**.** *Let $\varphi \in \operatorname{Diff}_c(\mathbb{R})$ be a diffeomorphism satisfying $\varphi(x) \ge x$ and let $T > 0$ be fixed. Then for each $0 \le s < \frac{1}{2}$ and $\varepsilon > 0$ there exists a time dependent vector field $u^\varepsilon_{\mathbb{R}}$ of the form*

$$u^\varepsilon_{\mathbb{R}}(t,x) = \mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]} * G_\varepsilon(x),$$

with $f, g \in C^\infty([0,T])$, such that its flow $\varphi^\varepsilon(t,x)$ satisfies – independently of ε – the properties $\varphi^\varepsilon(0,x) = x$, $\varphi^\varepsilon(T,x) = \varphi(x)$ and whose H^s-length is smaller than ε, i.e.,

$$\operatorname{Len}(\varphi^\varepsilon) = \int_0^T \|u^\varepsilon_{\mathbb{R}}(t,\cdot)\|_{H^s} dt \le C\|f^\varepsilon - g^\varepsilon\|_\infty \le \varepsilon\,.$$

Furthermore $\{t : f^\varepsilon(t) < g^\varepsilon(t)\} \subseteq \operatorname{supp}(\varphi)$ and there exists a limit function $h \in C^\infty([0,T])$, such that $f^\varepsilon \to h$ and $g^\varepsilon \to h$ for $\varepsilon \to 0$ and the convergence is uniform in t.

Here, $G_\varepsilon(x) = \frac{1}{\varepsilon}G_1(\frac{x}{\varepsilon})$ is a smoothing kernel, defined via a smooth bump function G_1 with compact support.

Proof of Theorem 3.1. Consider the connected component $\operatorname{Diff}_0(M)$ of Id, i.e. those diffeomorphisms of $\operatorname{Diff}_c(M)$, for which there exists at least one path, joining them to the identity. Denote by $\operatorname{Diff}_c(M)^{L=0}$ the set of all diffeomorphisms φ that can be reached from the identity by curves of arbitrarily short length, i.e., for each $\varepsilon > 0$ there exists a curve from Id to φ with length smaller than ε.

Claim A. $\operatorname{Diff}_c(M)^{L=0}$ *is a normal subgroup of* $\operatorname{Diff}_0(M)$.

Claim B. $\operatorname{Diff}_c(M)^{L=0}$ *is a non-trivial subgroup of* $\operatorname{Diff}_0(M)$.

By [12] or [7], the group $\operatorname{Diff}_0(M)$ is simple. Thus claims A and B imply $\operatorname{Diff}_c(M)^{L=0} = \operatorname{Diff}_0(M)$, which proves the theorem.

The proof of claim A can be found in [1, Theorem 3.1] and works without change in the case of M being an arbitrary manifold and hence we will not repeat it here. It remains to show that $\operatorname{Diff}_c(M)^{L=0}$ contains a diffeomorphism $\varphi \neq \mathrm{Id}$.

We shall first prove claim B for $M = \mathbb{R}^n$ and then show how to extend the arguments to arbitrary manifolds. Choose a diffeomorphism $\varphi_{\mathbb{R}} \in \operatorname{Diff}_c(\mathbb{R})$ with $\varphi_{\mathbb{R}}(x) \geq x$ and $\operatorname{supp}(\varphi_{\mathbb{R}}) \subseteq [1, \infty)$. Then let

$$u^\varepsilon_{\mathbb{R}}(t,x) := \mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]} * G_\varepsilon(x)$$

be the family of vector fields constructed in Lemma 3.2, whose flows at time T equal $\varphi_{\mathbb{R}}$. We extend the vector field $u^\varepsilon_{\mathbb{R}}$ to a vector field $u^\varepsilon_{\mathbb{R}^n}$ on $\mathbb{R}^n$ via

$$u^\varepsilon_{\mathbb{R}^n}(t, x_1, \ldots, x_n) := \big(u^\varepsilon_{\mathbb{R}}(t, |x|), 0, \ldots, 0\big) \ .$$

The flow of this vector field is given by

$$\varphi^\varepsilon_{\mathbb{R}^n}(t, x_1, \ldots, x_n) = (\varphi^\varepsilon_{\mathbb{R}}(t, |x|), x_2, \ldots, x_n) \ ,$$

where $\varphi^\varepsilon_{\mathbb{R}}$ is the flow of $u^\varepsilon_{\mathbb{R}}$. In particular we see that at time $t = T$

$$\varphi^\varepsilon_{\mathbb{R}^n}(t, x_1, \ldots, x_n) = (\varphi_{\mathbb{R}}(|x|), x_2, \ldots, x_n) \ ,$$

the flow is independent of ε. So it remains to show that for the length of the path $\varphi^\varepsilon_{\mathbb{R}^n}(t, \cdot)$ we have

$$\operatorname{Len}(\varphi^\varepsilon_{\mathbb{R}^n}) \to 0 \quad \text{as} \quad \varepsilon \to 0 \ .$$

We can estimate the length of this path via

$$\begin{aligned}
\operatorname{Len}(\varphi^\varepsilon_{\mathbb{R}^n})^2 &= \left(\int_0^T \|u^\varepsilon_{\mathbb{R}^n}(t,.)\|_{H^s(\mathbb{R}^n)} \,\mathrm{d}t \right)^2 \leq T \int_0^T \|u^\varepsilon_{\mathbb{R}^n}(t,.)\|^2_{H^s(\mathbb{R}^n)} \,\mathrm{d}t \\
&= T \int_0^T \|u^\varepsilon_{\mathbb{R}}(t, |\cdot|)\|^2_{H^s(\mathbb{R}^n)} \,\mathrm{d}t = T \int_0^T \|\mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]} * G_\varepsilon(|x|)\|^2_{H^s(\mathbb{R}^n)} \,\mathrm{d}t \\
&\leq C(G_1, T) \int_0^T \|\mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]}(|\cdot|)\|^2_{H^s(\mathbb{R}^n)} \,\mathrm{d}t \ ,
\end{aligned}$$

where the last estimate follows from

$$\begin{aligned}
&\|\mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]} * G_\varepsilon(|x|)\|^2_{H^s(\mathbb{R}^n)} = \\
&\qquad = \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) \left[\mathcal{F}\left(\mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]}(|\cdot|)\right)(\xi)\right]^2 \left[\mathcal{F}\left(G_\varepsilon(|\cdot|)\right)(\xi)\right]^2 \mathrm{d}\xi \\
&\qquad = \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) \left[\mathcal{F}\left(\mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]}(|\cdot|)\right)(\xi)\right]^2 \left[\mathcal{F}\left(G_1(|\cdot|)\right)(\varepsilon\xi)\right]^2 \mathrm{d}\xi \\
&\qquad \leq \|\mathcal{F} G_1(|\cdot|)\|^2_{L^\infty} \cdot \|\mathbb{1}_{[f^\varepsilon(t), g^\varepsilon(t)]}(|\cdot|)\|^2_{H^s(\mathbb{R}^n)} \ .
\end{aligned}$$

Hence it is sufficient to show that

$$\left\|\mathbb{1}_{[f^\varepsilon(t),g^\varepsilon(t)]}(|\cdot|)\right\|_{H^s(\mathbb{R}^n)} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly in } t\,.$$

To compute the H^s-norm of $\mathbb{1}_{[f^\varepsilon(t),g^\varepsilon(t)]}(|\cdot|)$ we first Fourier-transform it. The Fourier-transform of a radially symmetric function $v(|\cdot|) \in L^1(\mathbb{R}^n)$ is again radially symmetric and given by the following formula, see [11, Theorem 3.3],

$$(\mathcal{F}v(|\cdot|))(\xi) = 2\pi|\xi|^{1-n/2} \int_0^\infty J_{n/2-1}(2\pi|\xi|s)v(s)s^{n/2}\,\mathrm{d}s\,,$$

with $J_{n/2-1}$ denoting the Bessel function of order $\frac{n}{2}-1$. To simplify notation we will omit the dependece of the vector field $\mathbb{1}_{[f^\varepsilon(t),g^\varepsilon(t)]}(|\cdot|)$ on t and ε. Changing coordinates, this becomes

$$(\mathcal{F}\mathbb{1}_{[f,g]}(|\cdot|))(\xi) = (2\pi)^{-n/2}|\xi|^{-n} \int_{2\pi f|\xi|}^{2\pi g|\xi|} J_{n/2-1}(s)s^{n/2}\,\mathrm{d}s\,.$$

This integral can be evaluated explicitly using the following integral identity for Bessel functions from [9, (10.22.1)]

$$\int z^{\nu+1}J_\nu(z)\,\mathrm{d}z = z^{\nu+1}J_{\nu+1}(z), \quad \nu \neq -\tfrac{1}{2}\,.$$

This gives us

$$(\mathcal{F}\mathbb{1}_{[f,g]}(|\cdot|))(\xi) = |\xi|^{-n/2}\left(J_{n/2}(2\pi g|\xi|)g^{n/2} - J_{n/2}(2\pi f|\xi|)f^{n/2}\right)\,.$$

The H^s-norm of $\mathbb{1}_{[f,g]}(|\cdot|)$ is given by

$$\left\|\mathbb{1}_{[f,g]}(|\cdot|)\right\|^2_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(1+|\xi|^{2s}\right)\mathcal{F}\mathbb{1}_{[f,g]}(|\cdot|)(\xi)^2\,\mathrm{d}\xi\,.$$

We will only consider the term involving $|\xi|^{2s}$, since the L^2-term can be estimated in the same way by setting $s=0$. Transforming to polar coordinates we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |\xi|^{2s}\left(\mathcal{F}\mathbb{1}_{[f,g]}(|\cdot|)(\xi)\right)^2\mathrm{d}\xi &= \\ &= \int_{\mathbb{R}^n} |\xi|^{2s-n}\left(J_{n/2}(2\pi g|\xi|)g^{n/2} - J_{n/2}(2\pi f|\xi|)f^{n/2}\right)^2\,\mathrm{d}\xi \\ &= \mathrm{Vol}(S^{n-1})\int_0^\infty r^{2s-1}\left(J_{n/2}(2\pi gr)g^{n/2} - J_{n/2}(2\pi fr)f^{n/2}\right)^2\,\mathrm{d}r\,.\end{aligned}$$

The above integral is non-zero only for those t, where $f^\varepsilon(t) \neq g^\varepsilon(t)$. From Lemma 3.2 and our assumptions on $\varphi_\mathbb{R}$ we know that

$$\{t\,:\,f^\varepsilon(t) < g^\varepsilon(t)\} \subseteq \mathrm{supp}(\varphi_\mathbb{R}) \subseteq [1,\infty)\,.$$

Thus both $f^\varepsilon(t)$ and $g^\varepsilon(t)$ are different and away from 0 and we can evaluate the above integral using the identity [9, (10.22.57)],

$$\int_0^\infty \frac{J_\mu(at)J_\nu(at)}{t^\lambda}\,\mathrm{d}t = \frac{\left(\frac{1}{2}a\right)^{\lambda-1}\Gamma\left(\frac{\mu}{2}+\frac{\nu}{2}-\frac{\lambda}{2}+\frac{1}{2}\right)\Gamma\left(\lambda\right)}{2\Gamma\left(\frac{\lambda}{2}+\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2}+\frac{\mu}{2}-\frac{\nu}{2}+\frac{1}{2}\right)\Gamma\left(\frac{\lambda}{2}+\frac{\mu}{2}+\frac{\nu}{2}+\frac{1}{2}\right)}\,,$$

which holds for $\operatorname{Re}(\mu+\nu+1) > \operatorname{Re}\lambda > 0$ and the identity [9, (10.22.56)],

$$\int_0^\infty \frac{J_\mu(at)J_\nu(bt)}{t^\lambda}\,\mathrm{d}t = \\ = \frac{a^\mu \Gamma\left(\frac{\nu}{2}+\frac{\mu}{2}-\frac{\lambda}{2}+\frac{1}{2}\right)}{2^\lambda b^{\mu-\lambda+1}\Gamma\left(\frac{\nu}{2}-\frac{\mu}{2}+\frac{\lambda}{2}+\frac{1}{2}\right)}\mathbf{F}\left(\tfrac{\nu}{2}+\tfrac{\mu}{2}-\tfrac{\lambda}{2}+\tfrac{1}{2}, \tfrac{\mu}{2}-\tfrac{\nu}{2}-\tfrac{\lambda}{2}+\tfrac{1}{2}; \mu+1; \tfrac{a^2}{b^2}\right),$$

which holds for $0 < a < b$ and $\operatorname{Re}(\mu+\nu+1) > \operatorname{Re}\lambda > -1$. Here $\mathbf{F}(a,b;c;d)$ is the regularized hypergeometric function. Using these identities with $\lambda = 1-2s$, $\mu = \nu = \frac{n}{2}$, $a = 2\pi f$ and $b = 2\pi g$ we obtain

$$\int_0^\infty r^{2s-1}J_{n/2}(2\pi f r)^2\,\mathrm{d}r = \frac{1}{2}(\pi f)^{-2s}\frac{\Gamma\left(\frac{n}{2}+s\right)\Gamma(1-2s)}{\Gamma(1-s)^2\Gamma\left(\frac{n}{2}+1-s\right)}$$

and

$$\int_0^\infty r^{2s-1}J_{n/2}(2\pi f r)J_{n/2}(2\pi g r)\,\mathrm{d}r = \\ = \frac{1}{2}(\pi g)^{-2s}\left(\frac{f}{g}\right)^{n/2}\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(1-s)}\mathbf{F}\left(\tfrac{n}{2}+s, s; \tfrac{n}{2}+1; \tfrac{f^2}{g^2}\right).$$

Putting it together results in

$$\int_{\mathbb{R}^n} |\xi|^{2s}(\mathcal{F}\mathbb{1}_{[f,g]}(|\cdot|))(\xi)^2\,\mathrm{d}\xi = \\ = \operatorname{Vol}(S^{n-1})\left(\frac{f^{-2s}+g^{-2s}}{2\pi^{2s}}\frac{\Gamma\left(\frac{n}{2}+s\right)\Gamma(1-2s)}{\Gamma(1-s)^2\Gamma\left(\frac{n}{2}+1-s\right)} - \right. \\ \left. - \frac{g^{-2s}}{\pi^{2s}}\frac{f^{n/2}}{g^{n/2}}\frac{\Gamma\left(\frac{n}{2}+s\right)}{\Gamma(1-s)}\mathbf{F}\left(\tfrac{n}{2}+s, s; \tfrac{n}{2}+1; \tfrac{f^2}{g^2}\right)\right).$$

In the limit $\varepsilon \to 0$ we know from Lemma 3.2 that $f^\varepsilon(t) \to h(t)$ and $g^\varepsilon(t) \to h(t)$ uniformly in t on $[0,T]$ and hence $\frac{f^\varepsilon(t)}{g^\varepsilon(t)} \to 1$. For the regularized hypergeometric function $\mathbf{F}(a,b;c;d)$ at $d=1$ we have the identity [9, (15.4.20)]

$$\mathbf{F}(a,b;c;1) = \frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

for $\operatorname{Re}(c-a-b) > 0$. Applying the identity with $a = \frac{n}{2}+s$, $b = s$ and $c = \frac{n}{2}+1$ we get

$$\mathbf{F}\left(\tfrac{n}{2}+s, s; \tfrac{n}{2}+1; 1\right) = \frac{\Gamma(1-2s)}{\Gamma(1-s)\Gamma\left(\frac{n}{2}+1-s\right)}.$$

Using the continuity of the hypergeometric function it follows that

$$\int_{\mathbb{R}^n} |\xi|^{2s}\left(\mathcal{F}\mathbb{1}_{[f,g]}(|\cdot|))(\xi)\right)^2\,\mathrm{d}\xi \to 0,$$

as $\varepsilon \to 0$ and the convergence is uniform in t. This concludes the proof that

$$\left\|\mathbb{1}_{[f^\varepsilon(t),g^\varepsilon(t)]}(|\cdot|)\right\|_{H^s(\mathbb{R}^n)} \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{uniformly in } t,$$

and hence we have established claim B for $\operatorname{Diff}_c(\mathbb{R}^n)$.

To prove this result for an arbitrary manifold M of bounded geometry we choose a partition of unity (τ_j) such that $\tau_0 \equiv 1$ on some open subset $U \subset M$, where normal coordinates centred at $x_0 \in M$ are defined. If $\varphi_{\mathbb{R}}$ is chosen with sufficiently small support, then the vector field $u^\varepsilon_{\mathbb{R}^n}$ has support in $\exp_{x_0}(U)$ and we can define the vector field $u^\varepsilon_M := (\exp_{x_0}^{-1})^* u^\varepsilon_{\mathbb{R}^n}$ on M. This vector field generates a path $\varphi^\varepsilon_M(t,\cdot) \in \operatorname{Diff}_0(M)$ with an endpoint $\varphi^\varepsilon_M(T,\cdot) = \varphi_M(\cdot)$ that doesn't depend on ε with arbitrarily small H^s-length since

$$\begin{aligned}\operatorname{Len}(\varphi^\varepsilon_M) &\le C_1(\tau)\int_0^T \|u^\varepsilon_M\|_{H^s(M,\tau)}\,\mathrm{d}t = C_1(\tau)\int_0^T \|\exp_{x_0}^*(\tau_0.u^\varepsilon_M)\|_{H^s(\mathbb{R}^n)}\,\mathrm{d}t \\ &= C_1(\tau)\int_0^T \|u^\varepsilon_{\mathbb{R}^n}\|_{H^s(\mathbb{R}^n)}\,\mathrm{d}t\,.\end{aligned}$$

Thus we can reduce the case of arbitrary manifolds to $\mathbb{R}^n$ and this concludes the proof. □

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