

Extended Version of the proof of Prop. 10.48 in O'Neill, Semi-Riemannian Geometry.

M. Kunzinger, April 2015

Prop.: Let P be a spacelike semi-Riemannian sub-manifold of a Lorentzian manifold (or $P = \{p\}$) and let $\sigma \in S^1(P, q)$ a P -normal null geodesic.

If there is a focal point $\sigma(r)$ ($0 < r < b$, $\sigma(0) = p \in P$, $\sigma(b) = q$) of P along σ then there is a timelike curve from P to q arbitrarily close to σ .

Proof: We may suppose that $\sigma(r)$ is the first focal point of P along σ . It will suffice to show that for $\delta > 0$ sufficiently small \exists P -variation of $\sigma|_{[0, r+\delta]}$ which is timelike for small values of v .

Let J be a non-trivial P -Jacobi field ($=: JF$) along σ with $J(r) = 0$.

Then $J(0)$ is tangential to P and $\tan J'(0) = \tilde{\Pi}(J(0), \sigma'(0))$. Also, $J(u) \neq 0$ for all $u \in (0, r)$.

- 1.) $\exists \delta \in (0, b-r) \exists f \in C^\infty([0, b])$, $f > 0$ on $(0, r)$, $f < 0$ on $(r, r+\delta]$, $\exists U \in X(\sigma)$ a spacelike unit vector field with $J = fU$ on $[0, r+\delta]$.

We have $J \perp \sigma$ on $[0, b]$ and we show that J is nowhere tangent to σ on $(0, r)$: if $J(a) = c\sigma'(a)$ for some $a \in (0, r)$. Let $J_1 := u \mapsto J(u) - \frac{cu}{a}\sigma'(u)$.

Then $J_1'' = J''$ and $R_{J_1, \sigma} \sigma' = R_{J, \sigma'} \sigma'$, so J_1 is a JF along σ . Also, $J_1(0) = J(0)$ is tangential to P and $J_1'(0) = J'(0) - \frac{c}{a}\sigma'(0)$, so $\tan J_1'(0) = \tan J'(0) = \tilde{\Pi}(J(0), \sigma'(0))$. Thus J_1 is a P - JF . But since $J_1(a) = 0$, $\sigma(a)$ would be a focal point before $\sigma(r)$:

focal point before $\sigma(r)$: ↗

Since M is Lorentz, J is spacelike on $[0, r]$. Also, since $J(r) = 0$, \exists a vf Y_1 along σ with $J(u) = (r-u)Y_1(u)$. If, additionally, $J(0) = 0$ then also $Y_1(0) = 0$, so $Y_1(u) = u \cdot Y_1(u)$. Altogether, $\exists Y \in X(\sigma)$ s.t. $J(u) = y(u)Y(u)$, where

$$y(u) := \begin{cases} u(r-u) & \text{if } J(u) = 0 \\ r-u & \text{if } J(u) \neq 0 \end{cases}$$

Then Y is spacelike on $(0, r)$. If $J(0) \neq 0$ then also $Y(0) \neq 0$. If $J(0) = 0$ then $J'(0) \neq 0$, hence $0 \neq J'(0) = \underbrace{y'(0)}_{=r} Y(0)$, i.e. $Y(0) \neq 0$ also in this case. Analogously we get $Y(r) \neq 0$.

(2)

$\gamma(0)$ and $\gamma(r)$ are spacelike : we show this for $\gamma(0)$: If $J(0) \neq 0$ then

$\gamma(0) = \frac{1}{r} J(0) \in T_p P$: ✓ So let $J(0) = 0$. Since $\langle J, \sigma' \rangle = 0 \Rightarrow 0 = \frac{d}{dt} \langle J, \sigma' \rangle = \langle J', \sigma' \rangle$ $\Rightarrow J' \perp \sigma'$. If $\gamma(0) = c\sigma'(0) \Rightarrow J'(0) = \gamma'(0) \gamma(0) = r c \sigma'(0)$: but $\tilde{J}: u \mapsto r c u \sigma'(u)$ is a JF along σ with $\tilde{J}(0) = 0$ and $\tilde{J}'(0) = r c \sigma'(0) \Rightarrow J = \tilde{J}$: ↳ to $J(r) = 0$.

$\Rightarrow \gamma(0) = \frac{1}{r} J'(0)$ is spacelike. Summing up, γ is a spacelike vf on σ that vanishes nowhere on $[0, r]$. $\Rightarrow \exists \delta \in (0, b-r)$ s.t. the same is true on $[0, r+\delta]$. Now let $U := \frac{\gamma}{\|\gamma\|}$ and $f := \gamma \cdot \|U\|$. Since $\gamma \neq 0$ and spacelike, $\|\gamma\| = \sqrt{\langle \gamma, \gamma \rangle}$ positive and C^∞ . Hence f is C^∞ , $f > 0$ on $(0, r)$ and $f < 0$ on $(r, r+\delta)$.

- 2.) $\exists \delta \in (0, b-r)$ and $\exists V \in X(\sigma)$ s.t. $V(0) = J(0)$, $V(r+\delta) = 0$, $V \perp \sigma$ on $[0, r+\delta]$ and $\langle V'' - R_{V\sigma}, \sigma', V \rangle > 0$ on $(0, r+\delta)$.

We make the ansatz $V = J + gU = (f+g)U$, where g is to be determined.

Then

$$V' = J' + g'U + gU'$$

$$V'' = J'' + g''U + 2g'U' + gU''$$

$$R_{V\sigma}, \sigma' = \underbrace{R_{J\sigma}, \sigma'}_{= J''} + g R_{U\sigma}, \sigma' , \text{ so}$$

$$V'' - R_{V\sigma}, \sigma' = g''U + 2g'U' + g[U'' - R_{U\sigma}, \sigma']$$

Since $\langle U, U \rangle = 1$, $\langle U', U \rangle = 0$, hence

$$\begin{aligned} \langle V'' - R_{V\sigma}, \sigma', V \rangle &= \langle g''U + 2g'U' + g[U'' - R_{U\sigma}, \sigma'], (f+g)U \rangle = \\ &= (f+g)g'' + g(f+g) \underbrace{\langle U'' - R_{U\sigma}, \sigma', U \rangle}_{=: h} = (f+g)(g'' + gh). \end{aligned}$$

Let $a > 0$ s.t. $-a^2$ is a lower bound on h on $[0, r+\delta]$. Set $g(u) := b(e^{au} - 1)$, where $b > 0$ is s.t. $g(r+\delta) = -f(r+\delta)$ (> 0 by 1.1). Then $V(r+\delta) = 0$ and $V(0) = J(0)$. We have $f+g > 0$ on $[0, r]$ and $(f+g)(r+\delta) = 0$. By shrinking δ if necessary we can achieve that $r+\delta$ is the first positive zero of $f+g$, i.e., $f+g > 0$ on $(0, r+\delta)$ and $(f+g)(r+\delta) = 0$. Since $g'' = a^2(g+b)$, $g'' + gh = g(a^2 + h) + a^2 b > 0$ on $(0, r+\delta)$, so $\langle V'' - R_{V\sigma}, \sigma', V \rangle > 0$ on $(0, r+\delta)$, as claimed.

3.) $\exists A \in \mathbb{X}(\sigma)$ with $A(0) = \mathbb{II}(V(0), V(0))$, $A(r+\delta) = 0$ and

$$-\langle V'' - R_{V\sigma}, \sigma' \rangle, V \rangle + (\langle V, V' \rangle + \langle A, \sigma' \rangle)' < 0 \text{ on } [0, r+\delta]$$

$$\begin{aligned} \text{we have } \langle \mathbb{II}(J(0), J(0)), \sigma'(0) \rangle &= -\langle \tilde{\mathbb{II}}(J(0), \sigma'(0)), J(0) \rangle = -\langle \tan J'(0), J(0) \rangle \\ &= \underbrace{\langle J'(0), J(0) \rangle}_{J(0) \in T_{\sigma(0)} P}, \end{aligned}$$

and:

$$\begin{aligned} \langle V, V' \rangle &= \langle J + b(e^{at}-1)U, J' + ab e^{at}U + b(e^{at}-1)U' \rangle \\ &= \underbrace{\langle J, J' \rangle}_{\langle J, U' \rangle = f \langle U, U' \rangle = 0} + \langle V, ab e^{at}U \rangle + b(e^{at}-1) \langle U, J' \rangle \\ &\quad = \langle J(0), U(0) \rangle = f(0) = \|J(0)\| \end{aligned}$$

For $u=0$ this gives $\langle V, V' \rangle(0) = \langle J, J' \rangle(0) + ab \overbrace{\langle V(0), U(0) \rangle}^{\langle J(0), U(0) \rangle} = \langle J, J' \rangle(0) + ab \|J(0)\|$. (\star)

We distinguish 3 cases:

a) $\langle \mathbb{II}(J(0), J(0)), \sigma'(0) \rangle \neq 0$.

Write $\mathbb{II}(J(0), J(0)) = \alpha X_0$ with $\langle X_0, \sigma'(0) \rangle = -1$. Then

$$-\alpha = \langle \alpha X_0, \sigma'(0) \rangle = \langle \mathbb{II}(J(0), J(0)), \sigma'(0) \rangle = -\langle J'(0), J'(0) \rangle \Rightarrow \alpha = \langle J'(0), J'(0) \rangle.$$

Let X be the parallel vf along σ with $X(0) = X_0$. Then $\langle X(u), \sigma'(u) \rangle = -1 \forall u$.

We set

$$A(u) := (\langle V(u), V'(u) \rangle + \frac{ab \|J(0)\|}{r+\delta} (u-r-\delta)) \cdot X(u).$$

Then $\overset{(*)}{A}(0) = (\langle J(0), J'(0) \rangle + ab \|J(0)\| - ab \|J(0)\|) X(0) = \alpha X_0 = \mathbb{II}(J(0), J(0)) = V(0)$

and

$$A(r+\delta) = (\underbrace{\langle V(r+\delta), V'(r+\delta) \rangle}_{=0} + 0) X(r+\delta) = 0$$

Also, on $[0, r+\delta]$:

$$\begin{aligned} (\langle V, V' \rangle + \langle A, \sigma' \rangle)' &= (\langle V, V' \rangle - \cancel{\langle V, V' \rangle} - \frac{ab \|J(0)\|}{r+\delta} (u-r-\delta))' \\ &= -\frac{ab \|J(0)\|}{r+\delta} < 0 \end{aligned}$$

This gives the claim since, by 2.), $-\langle V'' - R_{V\sigma}, \sigma' \rangle, V \rangle \leq 0$ on $[0, r+\delta]$.

$$b) \langle \mathbb{I}(J(0), J(0)), \sigma'(0) \rangle = 0, \quad J(0) \neq 0.$$

Choose $X \in \mathcal{X}(\sigma)$ parallel along σ with $\langle X, \sigma' \rangle = -1$. Also, pick $Z \in \mathcal{X}(\sigma)$ parallel s.t. $Z(0) = \mathbb{I}(J(0), J(0))$ ($\Rightarrow \langle Z, \sigma' \rangle = 0$). Now let

$$A(u) := (\langle V(u), V'(u) \rangle + \frac{ab \|J(0)\|}{r+s} (u-r-s)) \cdot X(u) + \left(1 - \frac{u}{r+s}\right) Z(u)$$

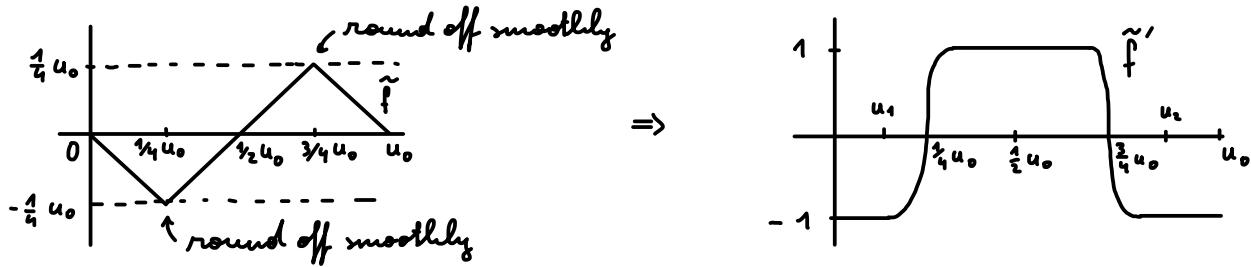
$$\text{Then } A(0) \stackrel{(*)}{=} \underbrace{\langle J(0), J'(0) \rangle}_{=0} X(0) + Z(0) = -\underbrace{\langle \mathbb{I}(J(0), J(0)), \sigma'(0) \rangle}_{=0} X(0) + \mathbb{I}(J(0), J(0)) \\ = \mathbb{I}(J(0), J(0))$$

and $A(r+s) = 0 \cdot X(r+s) + 0 \cdot Z(r+s)$, as well as

$$(\langle V, V' \rangle + \langle A, \sigma' \rangle)' = -\frac{ab \|J(0)\|}{r+s} < 0 \Rightarrow \text{claim as in a).}$$

$$c) \quad J(0) = 0$$

let $u_0 := r+s$ and let $\tilde{f} \in C^\infty([0, r+s])$ be the following function:



Pick $u_1, u_2 \in (0, r+s)$ s.t. $\tilde{f}' = -1$ on $[0, u_1]$ and $\tilde{f}' = 1$ on $[u_2, u_0]$.

Let $\varepsilon > 0$ s.t. $\varepsilon < \min_{u \in [u_1, u_2]} \langle V'' - R_{V\sigma}, \sigma', V \rangle$ and set $f := -\varepsilon \tilde{f}$.

Define X as in b) and set $A(u) := (\langle V, V' \rangle + f(u)) X(u)$. Then:

$$A(0) \stackrel{(*)}{=} (\underbrace{\langle J(0), J'(0) \rangle}_{=0} + ab \|J(0)\| + \underbrace{f(0)}_{=0}) = 0 \quad (= \mathbb{I}(V(0), V(0)))$$

$$A(r+s) = (0 + \underbrace{f(r+s)}_{=0}) X(r+s) = 0, \quad \text{and } (\langle V, V' \rangle + \langle A, \sigma' \rangle)' = \varepsilon \tilde{f}'. \quad \text{Thus}$$

$$-\langle V'' - R_{V\sigma}, \sigma', V \rangle + \underbrace{(\langle V, V' \rangle + \langle A, \sigma' \rangle)'}_{\varepsilon \tilde{f}'} \begin{cases} \leq \varepsilon \tilde{f}' = -\varepsilon & \text{on } [0, u_1] \\ < 0 & \text{on } [u_1, u_2] \quad (\text{by def. of } \varepsilon) \\ \leq \varepsilon \tilde{f}' = -\varepsilon & \text{on } [u_2, r+s] \end{cases}$$

hence < 0 on $[0, r+s]$.

4.) For $q := \sigma(r+\delta)$ $\exists (P, q)$ -variation x of $\sigma|_{[0, r+\delta]}$ whose longitudinal curves $\sigma_v := u \mapsto x(u, v)$ are timelike.

Let x be the (P, q) -variation of $\sigma|_{[0, r+\delta]}$ with variation-vf V from 2.) and transversal acceleration-vf A from 3.). Also, let $f(u, v) := \langle x_u(u, v), x_u(u, v) \rangle$. Then for $(u, v) \in [0, r+\delta] \times [-v_0, v_0]$ (with suitable $v_0 > 0$) we have

$$f(u, v) = f(u, 0) + v \frac{\partial f}{\partial v}(u, 0) + \frac{1}{2} v^2 \frac{\partial^2 f}{\partial v^2}(u, \theta v) \quad (\theta = \theta(u, v) \in (0, 1)).$$

Here, $f(u, 0) = \langle \sigma'(u), \sigma'(u) \rangle = 0$

$$\frac{\partial f}{\partial v}(u, 0) = 2 \left(-\underbrace{\langle V, \sigma'' \rangle}_{=0} + \underbrace{\langle V, \sigma' \rangle'}_{=0 \text{ (} V \perp \sigma \text{)}} \right) = 0, \text{ and}$$

$$\frac{\partial^2 f}{\partial v^2}(u, 0) = 2 \left(-\langle V'' - R_{V_0} \sigma', V \rangle + (\langle A, \sigma' \rangle + \langle V, V' \rangle)' \right) < 0 \text{ on } [0, r+\delta] \text{ by 3.).}$$

Since $[0, r+\delta]$ is compact $\exists v_1 > 0$ s.t. $f(u, v) < 0$ on $[0, r+\delta] \times [-v_1, v_1] \Rightarrow$ all σ_v for $v \in [-v_1, v_1]$ are timelike.

□