General Topology

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Preface

These are lecture notes for a four hour advanced course on general topology. They assume familiarity with the foundations of the subject, as taught in the two-hour introductory course offered at our faculty. In fact, a number of topics from the introductory course will be repeated here to keep prerequisites minimal. Based on this, detailed proofs are supplied for all results. Nevertheless, the approach taken is rather advanced and theory-oriented, and the overall style is in the Bourbaki spirit (which the subject matter lends itself to quite naturally). Throughout, we mainly follow the standard text [5], with occasional input from other sources (mainly [1] and [3]).

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Chapter 1

New spaces from old

1.1 Subspaces and products

1.1.1 Example. Let (X, d) be a metric space and let $U \subseteq X$. Then $d' := d|_{U \times U}$ is a metric on U. The ε-balls in (U, d') are given by

$$\{x \in U | d'(x_0, x) < \varepsilon\} = U \cap \{x \in X | d(x_0, x) < \varepsilon\},$$

hence are exactly the intersections of the ε-balls in X with U. We expect that d' defines the restriction of the topology of X to U.

More generally, we define:

1.1.2 Proposition. *Let $(X, \mathcal{O})$ be a topological space and $U \subseteq X$. Then $\mathcal{O}_U := \{O \cap U | O \in \mathcal{O}\}$ is a topology on U, the so-called* trace topology *(or induced topology, or subspace topology). $(U, \mathcal{O}_U)$ is called a (topological) subspace of X.*

Proof. We have to verify the axioms of a topology for $\mathcal{O}_U$: $U = X \cap U$, $\emptyset = \emptyset \cap U$, $\bigcup_{i \in I}(O_i \cap U) = \left(\bigcup_{i \in I} O_i\right) \cap U$, $\bigcap_{i \in I}(O_i \cap U) = \left(\bigcap_{i \in I} O_i\right) \cap U$. $\square$

The open (resp. closed) subsets of U thus are exactly the intersections of the open (resp. closed) subsets of X with U. If U_1 is open (closed) in U, it need *not* be open (closed) in X. For example, $[0, 1)$ is open in $[0, 2)$ and closed in $(-1, 1)$, but neither open nor closed in $\mathbb{R}$.

1.1.3 Examples.

(i) The natural topology on $\mathbb{C}$ is induced by the metric $d(z_1, z_2) = |z_1 - z_2| = ((x_2 - x_1)^2 + (y_2 - y_1)^2)^{1/2}$ for $z_k = x_k + iy_k$. The trace topology induced by this topology on $\mathbb{R}$ is the natural topology on $\mathbb{R}$.

(ii) Let $A \subseteq B \subseteq X$, each equipped with the trace topology of the respective superset. Then X induces on A the same topology as B.

The following result characterizes the trace topology by a universal property:

1.1.4 Theorem. *Let $(X, \mathcal{O})$ be a topological space, $U \subseteq X$ and $j : U \longrightarrow X$ the inclusion map. The trace topology $\mathcal{O}_U$ has the following properties:*

(i) For every topological space Y and any map $g : Y \to U$, g is continuous if and only if $j \circ g$ is continuous (i.e., $g : Y \to U$ is continuous $\Leftrightarrow$ $g : Y \to X$ is continuous).

(ii) $\mathcal{O}_U$ is the coarsest topology on U for which $j : U \to X$ is continuous.

Proof. (i) $j \circ g$ is continuous $\Leftrightarrow \forall\, O \in \mathcal{O}$: $(j \circ g)^{-1}(O) = g^{-1}(j^{-1}(O)) = g^{-1}(O \cap U)$ is open in $Y \Leftrightarrow g : Y \to (U, \mathcal{O}_U)$ is continuous.
(ii) Let $\tilde{\mathcal{O}}$ be any topology on U. Then $\mathcal{O}_U$ is coarser than $\tilde{\mathcal{O}} \Leftrightarrow g :=$ id: $(U, \tilde{\mathcal{O}}) \to (U, \mathcal{O}_U)$ is continuous $\overset{(i)}{\Leftrightarrow} j = g \circ j : (U, \tilde{\mathcal{O}}_U) \to X$ is continuous. . □

If X, Y are topological spaces and $f : X \to Y$ is continuous in $x \in A$, then also $f|_A : A \to Y$ is continuous in x (let V be a neighborhood of $f(x)$ in Y, then $f|_A^{-1}(V) = f^{-1}(V) \cap A$ is neighborhood of x in A). However, the converse is not true in general:

1.1.5 Example. Let $X = Y = \mathbb{R}$, $A := \mathbb{Q}$,

$$f(x) := \begin{cases} 0 & \text{for } x \in \mathbb{Q} \\ 1 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Then f is not continuous in any point although both $f|_{\mathbb{Q}}$ and $f|_{\mathbb{R}\setminus\mathbb{Q}}$ are continuous.

Nevertheless, under certain conditions the continuity of a map follows from the continuity of its restrictions to certain sets:

1.1.6 Proposition. *Let $X = \bigcup_{i=1}^{n} A$, where each A_i is closed in X. Let $f : X \to Y$, $f|_{A_i}$ continuous for each $1 \le i \le n$. Then f is continuous.*

Proof. Let $B \subseteq Y$ be closed. Then

$$f^{-1}(B) = f^{-1}(B) \cap \bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} (f^{-1}(B) \cap A_i) = \bigcup_{i=1}^{n} (f|_{A_i})^{-1}(B)$$

is closed in X. □

1.1.7 Definition. *A map $f : X \to Y$ is called an embedding of X into Y if f is a homeomorphism of X onto $f(X)$.*

1.1.8 Lemma. *$f : X \to Y$ is an embedding if and only if f is continuous and injective and $f(U)$ is open in $f(X)$ for every open set U in X.*

Proof.
($\Rightarrow$): f is clearly injective and $f : X \to Y$ is continuous by 1.1.4 (i). Also, if $U \subseteq X$ is open then so is $f(U)$ in $f(X)$.
($\Leftarrow$): $f : X \to f(X)$ is bijective, and continuous by 1.1.4 (i). Finally, $f^{-1} : f(X) \to X$ is continuous since for $U \in X$ open we have $(f^{-1})^{-1}(U) = f(U)$ is open in $f(X)$.
□

1.1.9 Examples.

(i) $f : \mathbb{R} \to \mathbb{R}^2$, $f(x) := (x, 0)$ is an embedding.

(ii) $f : [0, 2\pi) \to S^1 \in \mathbb{R}^2, x \mapsto (\cos x, \sin x)$ is continuous and injective, but is not an embedding. Indeed, for $0 < t < 2\pi$, $[0, t)$ is open in $[0, 2\pi)$, but $f([0, t))$ is not open in S^1

We now turn to the product of topological spaces.

1.1.10 Definition. *Let I be a set and for all $i \in I$ let $(X_i, \mathcal{O}_i)$ be a topological space. Let $X := \prod_{i\in I} X_i = \{(x_i)_{i\in I} | x_i \in X_i \ \forall i \in I\}$ and let $p_i : X \to X_i$, $p_i((x_j)_{j\in I}) := x_i$. The* product topology *$\mathcal{O}$ on X is defined by the basis*

$$\mathcal{B} := \left\{ \bigcap_{k\in K} p_k^{-1}(O_k) | O_k \in \mathcal{O}_k, K \subseteq I \text{ finite} \right\}.$$

$(X, \mathcal{O})$ is called the product (or topological product) of the spaces $(X_i, \mathcal{O}_i)$.

A sub-basis of $\mathcal{B}$ is given by $\mathcal{S} = \{p_i^{-1}(O_i) | O_i \in \mathcal{O}_i,\ i \in I\}$. A subset A of $\prod_{i\in I} X_i$ is in $\mathcal{B}$ if and only if $A = \prod_{i\in I} O_i$, where O_i is open in X_i for all i and $O_i = X_i$ for almost all $i \in I$.

1.1.11 Examples.

(i) The natural topology on $\mathbb{R}^n$ is precisely the product topology on $\mathbb{R}^n = \prod_{i=1}^n \mathbb{R}$.

(ii) Let $A_i \subseteq X_i$ for all $i \in I$. Then the trace topology of $\prod_{i\in I} X_i$ on $\prod_{i\in I} A_i$ is the product of the trace topologies of X_i on A_i. Indeed,

$$\prod_{i\in I} A_i \cap \prod_{i\in I} O_i = \prod_{i\in I} (A_i \cap O_i),$$

where $O_i = X_i$ and $(A_i \cap O_i) = A_i$ for almost all $i \in I$.

1.1.12 Theorem.

(i) For every $j \in I$, $p_j : \prod_{i\in I} X_i \to X_j$ is continuous and open.

(ii) The product topology on $\prod_{i\in I} X_i$ is the coarsest topology for which all projections p_j $(j \in I)$ are continuous.

Proof. (i) For every O_j open in X_j, $p_j^{-1}(O_j)$ is an element of $\mathcal{B}$, hence open, so p_j is continuous. Also, if $O = \prod_{i\in I} O_i \in \mathcal{B}$, then $p_i(O) = O_i$ is open, so p_i is open.
(ii) Let $\mathcal{O}'$ be a topology on $\prod_{i\in I} X_i$, for which all p_i are continuous. Then $\mathcal{O}'$ contains every $p_i^{-1}(O_i)$ and these sets form a subbasis of $\mathcal{O}$, $\mathcal{O} \subseteq \mathcal{O}'$. □

1.1.13 Proposition. *A map $g : Y \to \prod_{i\in I} X_i$ is continuous if and only if $g_i = p_i \circ g$ is continuous for every $i \in I$.*

Proof. ($\Rightarrow$): Clear since the p_i are continuous by 1.1.12.
($\Leftarrow$): Since the sets $p_i^{-1}(O_i)$, O_i open in X_i, form a subbasis of the product topology, it suffices to note that $g^{-1}(p_i^{-1}(O_i)) = g_i^{-1}(O_i)$ is open in Y. □

1.1.14 Theorem. *Let $f_i : X_i \to Y_i$, $X_i \neq \emptyset$ $(i \in I)$. The map*

$$f : \prod_{i\in I} X_i \to \prod_{i\in I} Y_i, \quad (x_i)_{i\in I} \mapsto (f_i(x_i))_{i\in I}$$

is continuous if and only if f_i is continuous for every $i \in I$.

Proof. Let $p_i : \prod_{j\in I} X_j \to X_i$ and $q_i : \prod_{j\in I} Y_j \to Y_i$ be the projections.
($\Leftarrow$): $f_i \circ p_i = q_i \circ f$ is continuous for every f, so f is continuous by 1.1.13.

(⇒): Fix $(a_i)_{i\in I}$ in $\prod_{i\in I} X_i$ and set, for $j \in I$:

$$s_j : X_j \to \prod_{i\in I} X_i, \quad s_j(x_j) = (z_i)_{i\in I} \text{ with } z_i := \begin{cases} a_i & \text{for } i \neq j \\ x_j & \text{for } i = j \end{cases}$$

Then s_j is continuous since $p_i \circ s_j$ is continuous for every i (it is even an embedding by 1.1.8) and the diagram

$$\begin{tikzcd} \prod_{i\in I} X_i \arrow[r, "f"] \arrow[d, shift right, "p_j"'] & \prod_{i\in I} Y_i \arrow[d, "q_j"] \\ X_j \arrow[r, "f_j"] \arrow[u, shift right, "s_j"'] & Y_j \end{tikzcd}$$

commutes. Hence $f_j = q_j \circ f \circ s_j$ is continuous for every $j \in I$. □

1.2 Initial topologies

Both trace and product topology are characterized as being the coarsest topology with respect to which certain maps (inclusions, projections) are continuous. In this section we will generalize this construction principle by means of so-called universal properties (as we have already encountered in 1.1.4 and 1.1.14).

1.2.1 Definition. *Let X be a set, $(X_i, O_i)_{i\in I}$ a family of topological spaces and, for each $i \in I$, $f_i : X \to X_i$ a map. A topology $\mathcal{I}$ on X is called* initial topology *with respect to $(f_i)_{i\in I}$ if it possesses the following universal property:*
If Y is a toplogical space and $g : Y \to X$ is a map then g is continuous if and only if $f_i \circ g$ is continuous for each $i \in I$.

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[dr, "f_i\circ g"'] & (X, \mathcal{I}) \arrow[d, "f_i"] \\ & X_i \end{tikzcd} \qquad g \text{ continuous} \Leftrightarrow \forall i: \; f_i \circ g \text{ continuous} \tag{1.2.1}$$

By 1.1.4 and 1.1.14, the trace and the product topology are initial topologies with respect to j and $(p_i)_{i\in I}$, respectively.

1.2.2 Theorem. *Under the assumptions of 1.2.1, there is a unique initial topology $\mathcal{I}$ on X with respect to $(f_i)_{i\in I}$. $\mathcal{I}$ is the coarsest topology on X such that $f_i : X \to Y_i$ is continuous for each $i \in I$. Setting $\mathcal{M}_i := \{f_i^{-1}(O) | O \in \mathcal{O}_i\}$, $\mathcal{S} := \bigcup_{i\in I} \mathcal{M}_i$ is a subbasis of $\mathcal{I}$.*

Proof. *Uniqueness:* Let $\mathcal{I}_1$, $\mathcal{I}_2$ be initial topologies, and consider the diagram

$$\begin{tikzcd} (X, \mathcal{I}_k) \arrow[r, "\mathrm{id}"] \arrow[dr, "f_i"'] & (X, \mathcal{I}_l) \arrow[d, "f_i"] \\ & X_i \end{tikzcd}$$

Setting $k = l = 1$, id is continuous, so (1.2.1) implies that $f_i : (X, \mathcal{I}_1) \to X_i$ is continuous for all $i \in I$.

Now setting $k = 1,\ l = 2$, since all f_i are continuous, (1.2.1) shows that id:$(X, \mathcal{I}_1) \to (X, \mathcal{I}_2)$ is continuous, hence $\mathcal{I}_1 \geq \mathcal{I}_2$. By symmetry, also $\mathcal{I}_2 \geq \mathcal{I}_1$, so $\mathcal{I}_1 = \mathcal{I}_2$.

Existence: If $\mathcal{I}$ is to be the initial topology on X with respect to $(f_i)_{i\in I}$ then from (1.2.1) and

$$\begin{tikzcd}
(X,\mathcal{I}) \arrow[r, "\mathrm{id}"] \arrow[dr, "f_i"'] & (X,\mathcal{I}) \arrow[d, "f_i"] \\
 & X_i
\end{tikzcd}$$

it follows that $f_i \circ \mathrm{id} = f_i$ is continuous for every $i \in I$. Consequently, $\mathcal{I}$ must contanin all elements of $\mathcal{S} = \bigcup_{i\in I}\{f_i^{-1}(O) | O \in O_i\}$. Now let $\mathcal{O}$ be the topology defined by the subbasis $\mathcal{S}$. Then $\mathcal{O}$ is the coarsest topology for which all f_i are continuous and by the above $\mathcal{I}$ is necessarily finer than $\mathcal{O}$. To finish the proof we show that $\mathcal{O}$ possesses the universal property (1.2.1).

To this end, first let $g : Y \to (X, \mathcal{O})$ be continuous. Then since the f_i are continuous, so is $f_i \circ g$ for every i. Conversely, let $f_i \circ g$ be continuous for every $i \in I$. To show that g is continuous, it suffices to show that $g^{-1}(S)$ is open for each $S \in \mathcal{S}$. Since $S = f_i^{-1}(O)$ for some $O \in \mathcal{O}_i$, we have $g^{-1}(S) = g^{-1}(f_i^{-1}(O)) = (f_i \circ g)^{-1}(O)$, which is open. $\square$

1.2.3 Theorem. *(Transitivity of initial topologies) Let X be a set, $(Z_i)_{i\in I}$ a family of toplogical spaces, $(J_\lambda)_{\lambda\in L}$ a partition of I and $(Y_\lambda)_{\lambda\in L}$ a family of sets. For every $\lambda \in L$ let $h_\lambda : X \to Y_\lambda$, and for each $\lambda \in L$ and each $i \in J_\lambda$ let $g_{i\lambda} : Y_\lambda \to Z_i$ and set $f_i := g_{i\lambda} \circ h_\lambda$. Suppose that every Y_λ is endowed with the initial topology with respect to $(g_{i\lambda})_{i\in J_\lambda}$. Then on X the initial topology with respect to $(f_i)_{i\in I}$ and the initial topology with respect to $(h_\lambda)_{\lambda\in L}$ coincide.*

$$\begin{tikzcd}
 & & & Z_i \\
 & & Y_\lambda \arrow[ur, "g_{i\lambda}"'] \arrow[r, "g_{j\lambda}"'] & Z_j \\
W \arrow[r, dashed, "k"] & X \arrow[uurr, bend left, "f_i"] \arrow[ur, "h_\lambda"] \arrow[r, "h_{\lambda'}"] \arrow[dr] & Y_{\lambda'} \arrow[r, "g_{k\lambda'}"] \arrow[dr] & Z_k \\
 & & &
\end{tikzcd}$$

Proof. Let $\mathcal{I}_1$ be the initial topology with respect to $(h_\lambda)_{\lambda\in L}$, and $\mathcal{I}_2$ the initial topology with respect to $(f_i)_{i\in I}$. By 1.2.2 it suffices to show that $\mathcal{I}_1$ possesses the universal property (1.2.1) with respect to $(f_i)_{i\in I}$. To see this, let W be a topological space and let $k : W \to X$. Then by (1.2.1) for $\mathcal{I}_1$, k is continuous with respect to $\mathcal{I}_1$ if and only if $h_\lambda \circ k : W \to Y_\lambda$ is continuous for all λ, which in turn is equivalent to the continuity of $g_{i\lambda} \circ h_\lambda \circ k = f_i \circ k$ for all $\lambda \in L$ and all $i \in J_\lambda$. Since the J_λ form a partition of I, this finally is equivalent to $f_i \circ k$ being continuous for every $i \in I$. $\square$

1.2.4 Remark. In particular, if $(J_\lambda)_{\lambda\in L}$ is a partition of I, then

$$\prod_{i\in I} X_i \cong \prod_{\lambda\in L}\prod_{i\in J_\lambda} X_i.$$

1.2.5 Example. Least upper bound of a family of topologies.

Let $(\mathcal{O}_i)_{i\in I}$ be a family of topologies on a set X. Then there exists a unique coarsest topology $\mathcal{O}$ on X that is finer than each $\mathcal{O}_i$. In fact, $\mathcal{O}$ is finer than $\mathcal{O}_i$ if and only if id: $(X,\mathcal{O}) \to (X,\mathcal{O}_i)$ is continuous. Therefore $\mathcal{O}$ is the initial topology on X with respect to all id: $X \to (X,\mathcal{O}_i)$.

1.3 Final topology, quotient topology

In this section we consider the dual problem of that treated in the previous one:

1.3.1 Definition. *Let X be a set, $(X_i,\mathcal{O}_i)_{i\in I}$ a family of topological spaces and for each $i\in I$ let $f_i : X_i \to X$ be a map. A topology $\mathcal{F}$ on X is called* final topology *with respect to $(f_i)_{i\in I}$ if it possesses the following universal property:*

If Y is any topological space, a map $g : X \to Y$ is continuous if and only if $g\circ f_i$ is continuous for each $i\in I$.

$$\begin{tikzcd} (X,\mathcal{F}) \arrow[r, "g"] & Y \\ X_i \arrow[u, "f_i"] \arrow[ur, "g\circ f_i"'] & \end{tikzcd} \qquad g \text{ continuous} \Leftrightarrow \forall i:\ g\circ f_i \text{ continuous} \tag{1.3.1}$$

1.3.2 Theorem. *Under the assumptions of 1.3.1 there is a unique final topology $\mathcal{F}$ on X. $\mathcal{F}$ is the finest topology on X for which all maps $f_i : X_i \to X$ are continuous. Moreover, $\mathcal{F} = \{O \subseteq X | f_i^{-1}(O) \in \mathcal{O}_i\ \forall i \in I\}$.*

Proof. *Uniqueness:* Let $\mathcal{F}_1, \mathcal{F}_2$ be topologies on X with the universal property (1.3.1) and consider

$$\begin{tikzcd} (X,\mathcal{F}_l) \arrow[r, "\mathrm{id}"] & (X,\mathcal{F}_k) \\ X_i \arrow[u, "f_i"] \arrow[ur, "f_i"'] & \end{tikzcd}$$

For $k = l = 1$, since id is continuous, (1.3.1) implies that $f_i : X_i \to (X,\mathcal{F}_1)$ is continuous for each i. Now set $k = 1$ and $l = 2$, then again by (1.3.1) it follows that id: $(X,\mathcal{F}_2) \to (X,\mathcal{F}_1)$ is continuous, i.e., $\mathcal{F}_1 \subseteq \mathcal{F}_2$. By symmetry, $\mathcal{F}_1 = \mathcal{F}_2$.

Existence: We first verify that $\mathcal{F}$ is indeed a topology. Since $f_i^{-1}(X) = X_i$ and $f_i^{-1}(\emptyset) = \emptyset$, both X and $\emptyset$ are in $\mathcal{F}$. If $O_\lambda \in \mathcal{F}$ for $\lambda \in L$ then $f_i^{-1}(\bigcup_{\lambda\in L} O_\lambda) = \bigcup_{\lambda\in L} f_i^{-1}(O_\lambda) \in \mathcal{O}_i$ for all i. Finally, if $O_1,\dots,O_n \in \mathcal{F}$ then $f_i^{-1}(\bigcap_{j=1}^n O_j) = \bigcap_{j=1}^n f_i^{-1}(O_j) \in \mathcal{O}_i$ for all i.

By construction, all f_i are continuous for $\mathcal{F}$. Conversely, if $\mathcal{O}$ is a topology on X for which all f_i are continuous then, for each $O \in \mathcal{O}$, $f_i^{-1}(O) \in \mathcal{O}_i$ for all $i \in I$, so $O \in \mathcal{F}$, which shows that $\mathcal{O} \le \mathcal{F}$. Thus $\mathcal{F}$ is the finest topology for which all f_i are continuous. It remains to show that $\mathcal{F}$ possesses the universal property (1.3.1). Thus let g: $(X,\mathcal{F}) \to Y$ be continuous. Then $g\circ f_i$ is continuous for each i. Conversely, let $g\circ f_i$ be continuous for each $i\in I$ and let $U \subseteq Y$ be open. Then $f_i^{-1}(g^{-1}(U)) = (g\circ f_i)^{-1}(U)$ is open in X_i for each $i\in I$, so $g^{-1}(U)$ is open in $(X,\mathcal{F})$, implying that g is continuous. □

Final topologies also satisfy a transitivity property analogous to 1.2.3

1.3.3 Theorem. *(Transitivity of final topologies) Let X be a set, $(Z_i)_{i\in I}$ a family of topological spaces, $(J_\lambda)_{\lambda\in L}$ a partition of I and $(Y_\lambda)_{\lambda\in L}$ a family of sets. Suppose* *that for every $\lambda \in L$, $h_\lambda : Y_\lambda \to X$ is a map and that for each $\lambda \in L$ and each $i \in J_\lambda$, $g_{\lambda i} : Z_i \to Y_\lambda$ is a map, and set $f_i := h_\lambda \circ g_{\lambda i}$. Let each Y_λ be equipped with the final topology with respect to $(g_{\lambda i})_{i \in J_\lambda}$. Then on X the final topologies with respect to $(f_i)_{i\in I}$ and with respect to $(h_\lambda)_{\lambda \in L}$ coincide.*

$$\begin{tikzcd}
Z_i \arrow[dr, "g_{\lambda i}"] \arrow[ddrr, bend left, "f_i"] & & & \\
Z_{i'} \arrow[r, "g_{\lambda i'}"] & Y_\lambda \arrow[dr, "h_\lambda"] & & \\
\vdots & & X \arrow[r, dashed, "k"] & W \\
Z_j \arrow[r, "g_{\lambda' j}"] & Y_{\lambda'} \arrow[ur, "h_{\lambda'}"'] & & \\
Z_{j'} \arrow[ur, "g_{\lambda' j'}"'] \arrow[uurr, bend right, "f_{j'}"'] & & &
\end{tikzcd}$$

Proof. Let $\mathcal{F}_1$ be the final topology with respect to $(h_\lambda)_{\lambda\in L}$, and $\mathcal{F}_2$ the one with respect to $(f_i)_{i\in I}$. By 1.3.2 it suffices to show that $\mathcal{F}_1$ possesses the universal property (1.3.1) with respect to $(f_i)_{i\in I}$. Let W be a topological space and let $k : X \to W$. Then k is continuous with respect to $\mathcal{F}_1$ if and only if $k \circ h_\lambda : Y_\lambda \to W$ is continuous for all $\lambda \in L$, which in turn is the case if and only if $k \circ h_\lambda \circ g_{\lambda i} = k \circ f_i : Z_i \to W$ is continuous for all $\lambda \in L$ and all $i \in J_\lambda$. Since the J_λ partition I, this holds if and only if $k \circ f_i : Z_i \to X$ is continuous for all $i \in I$. □

1.3.4 Remark. Analogously to 1.2.5, 1.3.2 can be used to determine the greatest lower bound of a family of topologies on X. In fact, if $(\mathcal{O}_i)_{i\in I}$ is a family of toplogies on X then 1.3.2, applied to id: $(X, \mathcal{O}_i) \to X$ shows that the finest topology coarser than all $\mathcal{O}_i$ is given by $\mathcal{O} = \bigcap_{i\in I} \mathcal{O}_i$.

An important special case of final topologies is the quotient topology:

1.3.5 Definition. *Let X be a topological space, $\sim$ an equivalence relation on X and $p : X \to X/\sim$ the canonical projection from X onto the set of equivalence classes, $p : x \mapsto [x]$. The final topology on $X/\sim$ with respect to ρ is called* quotient topology*. $X/\sim$ is called quotient space or factor space with respect to $\sim$.*

By 1.3.2 the quotient topology is the finest topology on $X/\sim$ for which p is continuous. $A \subseteq X/\sim$ is open if and only if $p^{-1}(A)$ is open in X. Note that

$$p^{-1}(A) = \{x \in X | p(x) \in A\} = \{x \in X | \exists a \in A \text{ with } x \sim a\}.$$

1.3.6 Example. On $\mathbb{R}$, consider the equivalence relation $x \sim y :\Leftrightarrow x - y \in \mathbb{Z}$. Let $S := \mathbb{R}/\sim$. Then S is homeomorphic to the unit circle $S^1 \subseteq \mathbb{R}^2$: Let $g : S \to S^1$, $g([x]) = (\cos 2\pi x, \sin 2\pi x)$. Then g is well-defined and bijective. g is continuous by (1.3.1) since $h := g \circ p : x \mapsto (\cos 2\pi x, \sin 2\pi x)$ is continuous. Moreover, g is open: We have $S = \{[x] | x \in [0, 1)\}$. For $x \in (0, 1)$, a neighborhood basis in S is given by $p((x - \varepsilon, x + \varepsilon))$ $(\varepsilon > 0)$. Then $g(p(x - \varepsilon, x + \varepsilon)) = h(x - \varepsilon, x + \varepsilon)$ is open in S^1. Now let $x = 0$. Then a neighborhood basis of $[x]$ is given by $p(U_\varepsilon)$ $(\varepsilon > 0)$ with $U_\varepsilon := [0, \varepsilon) \cup (\varepsilon, 1)$, so $g(U_\varepsilon)$ is open in S^1.

1.4 Identification topology, gluing of topological spaces

For any map $f : X \to Y$, $x \sim y :\Leftrightarrow f(x) = f(y)$ defines an equivalence relation. Let $p : X \to X/\sim$ be the corresponding projection, and set

$$\bar{f} : X/\sim \to f(X),\ \bar{f}([x]) := f(x)$$

Then $\bar{f}$ is well-defined and surjective, and $\bar{f}([x_1]) = \bar{f}([x_2]) \Rightarrow f(x_1) = f(x_2) \Rightarrow [x_1] = [x_2]$, so $\bar{f}$ is bijective. Also, let $j : f(X) \hookrightarrow Y$ be the inclusion map. Then the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & Y \\ X/\sim \arrow[r, "\bar{f}"'] & f(X) \arrow[u, "j"'] \end{tikzcd}$$

1.4.1 Definition. *Let $f : X \to Y$ be continuous and equip $X/\sim$ with the quotient topology, as well as $f(X)$ with the trace topology of Y. If $\bar{f} : X/\sim \to f(X)$ is a homeomorphism then f is called an identifying map. If f is, in addition, surjective, then the topology of Y is called identification topology.*

1.4.2 Theorem. *Let $f : X \to Y$ be continuous. Then:*

(i) p, $\bar{f}$ and j are continuous.

(ii) $\bar{f}$ is a homeomorphism if and only if for every open (closed) set of the form $f^{-1}(A)$ $(A \subseteq Y)$, the set $f(f^{-1}(A))$ is open (closed) in $f(X)$.

(iii) If $f : X \to Y$ is surjective and open or closed, then Y carries the identification topology with respect to f.

Proof.

(i) p and j are continuous by definition of the quotient and trace topology. Since f is continuous, $j \circ \bar{f} \circ p$ is continuous, and this by 1.1.4 implies that $\bar{f} \circ p$ is continuous. By (1.3.1) this gives that $\bar{f}$ is continuous.

(ii) Since $\bar{f}$ is continuous and bijective, $\bar{f}$ is a homeomorphism if and only if $\bar{f} : X/\sim \to f(X)$ is open (resp. closed). To see that the latter is equivalent to (ii) we need some preparations:

(1) $\forall V \subseteq X/\sim$ we have: $f^{-1}(f(p^{-1}(V))) = p^{-1}(V)$
Indeed, $x \in f^{-1}(f(p^{-1}(V))) \Leftrightarrow f(x) \in f(p^{-1}(V)) \Leftrightarrow \exists \tilde{x} \in p^{-1}(V)$ with $x \sim \tilde{x} \Leftrightarrow x \in p^{-1}(V)$

(2) $\forall A \subseteq Y$: $p^{-1}(p(f^{-1}(A))) = f^{-1}(A)$
In fact, $x \in p^{-1}(p(f^{-1}(A))) \Leftrightarrow p(x) \in p(f^{-1}(A)) \Leftrightarrow \exists \tilde{x} \in f^{-1}(A)$ with $x \sim \tilde{x} \Leftrightarrow f(x) \in A \Leftrightarrow x \in f^{-1}(A)$

Using this we can now show:

$\bar{f}$ open (closed) $\Leftrightarrow \forall A \subseteq Y$ with $f^{-1}(A)$ open (closed): $f(f^{-1}(A))$ is open (closed) in $f(X)$

($\Rightarrow$): Let $V := p(f^{-1}(A)) \Rightarrow p^{-1}(V) \stackrel{(2)}{=} f^{-1}(A)$ is open (closed) $\Rightarrow V$ is open (closed) in $X/\sim$ $\Rightarrow \bar{f}(V) = \bar{f} \circ p(f^{-1}(A)) = f(f^{-1}(A))$ is open (closed) in $f(X)$

($\Leftarrow$): Let $V \subseteq X/\sim$ be open (closed) and set $A := f(p^{-1}(V))$. Then $f^{-1}(A) \overset{(1)}{=} p^{-1}(V)$ is open (closed) $\Rightarrow V = p(p^{-1}(V)) = p(f^{-1}(A)) \Rightarrow \bar{f}(V) = \bar{f}(p(f^{-1}(A))) = f(f^{-1}(A))$ is open (closed).

(iii) Since $f(X) = Y$, by (ii) we have to show that $f(f^{-1}(A))$ is open (closed) for $f^{-1}(A)$ open (closed). This in turn is immediate from f open (closed). $\square$

1.4.3 Examples.

(i) Let $f : \mathbb{R} \to S^1 \subseteq \mathbb{R}^2$, $f(x) = (\cos 2\pi x, \sin 2\pi x)$. f is continuous, surjective and open. Let $x \sim y :\Leftrightarrow f(x) = f(y)$, i.e.: $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$. Then by 1.4.2 (iii), $\bar{f} : \mathbb{R}/\sim \to S^1$ is a homeomorphism, confirming 1.3.6.

(ii) Let $X := \mathbb{R}^3 \setminus \{0\}$, $x = (x_1, x_2, x_3)$, $\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $\bar{x} := \frac{x}{\|x\|} \in S^2 \subseteq \mathbb{R}^3 \setminus \{0\}$. We define an equivalence relation $\sim$ on X by

$$x \sim x' :\Leftrightarrow \exists \lambda \in \mathbb{R},\ \lambda > 0,\ \text{s.t. } x' = \lambda x$$

Let $[x]$ be the equivalence class of x and consider $f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$, $f(x) = \bar{x}$. Then $f(x) = f(x') \Leftrightarrow x \sim x'$. f is continuous and $f(X) = S^2$. That $f^{-1}(A)$ is open in $\mathbb{R}^3 \setminus \{0\}$ means that the cone which is determined by $A \cap S^2$ is open in $\mathbb{R}^3 \setminus \{0\}$. But then also $A \cap S^2$, so $f(f^{-1}(A))$ must be open. Hence by 1.4.2 $\bar{f} : X/\sim \to f(X) = S^2$, $\bar{f}([x]) = f(x) = \bar{x}$ is a homeomorphism. The trace topology on S^2 therefore is precisely the identification topology with respect to f.

(iii) For $x,\ y \in S^2$ let $x \sim y$ if $y = -x$. Then $S^2/\sim$ with the quotient topology is called the projective plane P^2. Alternatively, on $\mathbb{R}^3 \setminus \{0\}$ consider the equivalence relation $x \approx y \Leftrightarrow \exists \lambda \neq 0$ s.t. $x = \lambda y$. Let $[[x]]$ be the class of x with respect to $\approx$. Consider

$$\begin{aligned} g : S^2/\sim\ &\to \mathbb{R}^3 \setminus \{0\}/\approx \\ [x] &\mapsto [[x]]. \end{aligned}$$

g is well-defined and bijective. Let $p_1 : S^2 \to S^2/\sim$, $p_2 : \mathbb{R}^3 \setminus \{0\} \to (R^3 \setminus \{0\})/\approx$ be the projections. Then $g \circ p_1 = p_2|_{S^2}$ is continuous, so g is continuous and $g^{-1} \circ p_2 = x \mapsto \frac{x}{\|x\|} \mapsto [\frac{x}{\|x\|}]$ is continuous, hence g is a homeomorphism. It follows that $(\mathbb{R}^3 \setminus \{0\})/\approx$ is an equivalent representation of P^2.

1.4.4 Definition. *Let $(X_i, \mathcal{O}_i)_{i \in I}$ be a family of topological spaces that are pairwise disjoint. Let $j_i : X_i \hookrightarrow \bigcup_{k \in I} X_k$ be the canonical embedding. Then $\bigcup_{i \in I} X_i$, equipped with the final topology with respect to $(j_i)_{i \in I}$ is called the* topological sum *of the $(X_i)_{i \in I}$. If the X_i are not disjoint, we replace X_i by $(X_i \times \{i\})_{i \in I}$.*

By 1.3.2 a subset $O \subseteq \bigcup_{j \in I} X_j$ is open if and only if $O \cap X_i$ is open in X_i for all $i \in I$. Thus $\bigcup_{i \in I} X_i$ induces on every X_i its original topology $\mathcal{O}_i$.

We now turn to the task of gluing topological spaces.

1.4.5 Definition. *Let X and Y be disjoint topological spaces, $A \subseteq X$ closed, and let $f : A \to Y$. On the topological sum $X \cup Y$ we define the following equivalence relation:*

$$z_1 \sim z_2 :\Leftrightarrow \begin{cases} z_1, z_2 \in A \text{ and } f(z_1) = f(z_2) \text{ or} \\ z_1 \in A, z_2 \in f(A) \text{ and } f(z_1) = z_2 \text{ or} \\ z_2 \in A, z_1 \in f(A) \text{ and } f(z_2) = z_1 \text{ or} \\ z_2 = z_1 \end{cases}$$

Then $Y \cup_f X := (X \cup Y)/\sim$ is the topological space resulting from gluing X and Y along f.

Thus any point from $f(A)$ is identified with every element of its pre-image.

1.4.6 Examples.

(i) Let $X := [0,1],\ A = \{0\} \cup \{1\}, Y := [2,3], f(0) := 2, f(1) := 3$.
Then $Y \cup_f X$ is homeomorphic to S^1: Let $p : X \cup Y \to X \cup Y/\sim$ be the projection. For $0 < z < 1$ or $2 < z < 3$ there clearly exists a neighborhood basis in $X \cup Y/\sim$ that maps bijectively to a basis in X resp. Y. But also for $z = f(0) = 2$ there is a neighborhood basis whose inverse image under p consists of the sets $[0,\varepsilon) \cup [2,\varepsilon)$, which are open in $X \cup Y$, and similar for $z = f(1) = 3$.

(ii) Let $X = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\},\ A := \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$,
$Y = \{(0,2) \in \mathbb{R}^2\},\ f(x,y) := (0,2)\ \forall (x,y) \in A$. Then $Y \cup_f X \cong S^2$.

1.4.7 Definition. *Let D^n be the closed unit ball in $\mathbb{R}^n$, $e^n := (D^n)^\circ, S^{n-1} = \partial D^n = D^n \setminus e_n$, equipped with the trace topologies. D^n resp. e^n (and any spaces homeomorphic to them) are called n-dimensional ball resp. n-dimensiona cell. Let $f : S^{n-1} \to X$ a map into a topological space X. Then $X \cup_f D^n$ (as well as any space homeomorphic to it) is called a space resulting from X by gluing an n-cell along f to X.*

Note: Since $e^n \cap S^{n-1} = \emptyset$, for $z_1,\ z_2 \in e^n$ we have $z_1 \sim z_2 \Leftrightarrow z_1 = z_2$. Therefore (with $p : X \cup D^n \to X \cup_f D^n$), $p|_{e^n} : e^n \to p(e^n)$ is a homeomorphism.

1.4.8 Examples.

(i) Let $X := D^2$, $f := \mathrm{id}_{S^1}$. Then $X \cup_f D^2 \cong S^2$.

(ii) Let $X := \{(x,y) \in \mathbb{R}^2 | 0 \leq x \leq 1,\ 0 \leq y \leq 1\}, A := \{(x,y) \in X | x = 0 \text{ or } 1\},\ Y := [0,1]$. Let $f : A \to Y,\ f(0,y) := y,\ f(1,y) := 1 - y$. Then $M := Y \cup_f X$ is called the Moebius strip.

1.5 Manifolds and topological groups

Here we briefly introduce some notions that play an important role in several fields of Mathematics.

1.5.1 Definition. *Let M be a set such that there exists a cover $(U_i)_{i\in I}$ of M and a family of bijective maps $\varphi_i : U_i \to V_i$ with $V_i \subseteq \mathbb{R}^n$ open. Moreover, suppose that for all $i,\ j$ with $U_i \cap U_j \neq \emptyset$, $\varphi_i(U_i \cap U_j),\ \varphi_j(U_i \cap U_j)$ are open and that*

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is a homeomorphism. Equip M with the final topology with respect to the $\varphi_i^{-1} : V_i \to M$. If M is Hausdorff, it is called a topological manifold of dimension n.

The (φ_i, U_i) are called charts of M, the system $\{(\varphi_i, U_i) | i \in I\}$ is called atlas of M. Often one requires in addition that the topology of M should possess a countable basis (or is paracompact, see Ch. 7). The maps $\varphi_i \circ \varphi_j^{-1}$ are also called chart transition functions. If these maps are even diffeomorphisms of class C^k ($1 \leq k \leq \infty$), then M is called a C^k-manifold. If they are analytic, M is called a C^ω-manifold.

1.5.2 Lemma. *Every $\varphi_i : U_i \to V_i$ is a homeomorphism, and each U_i is open in M.*

Proof. By definition of the final topology, each $\varphi_i^{-1} : V_i \to M$ is continuous. Hence (by 1.1.4) also $\varphi_i^{-1} : V_i \to U_i$ is continuous. It remains to show that φ_i is continuous. Let $W \subseteq V_i$ be open. Then for all $j \in I$,

$$\begin{aligned}(\varphi_j^{-1})^{-1}(\varphi_i^{-1}(W)) &= \varphi_j(U_j \cap \underbrace{\varphi_i^{-1}(W)}_{\subseteq U_i}) = \varphi_j(U_j \cap U_i \cap \varphi_i^{-1}(W)) \\ &= \varphi_j(\varphi_i^{-1}(\varphi_i(U_i \cap U_j)) \cap \varphi_i^{-1}(W)) = \varphi_j \circ \varphi_i^{-1} \underbrace{(\varphi_i(U_i \cap U_j) \cap W)}_{\text{open in } \varphi_i(U_i \cap U_j)}\end{aligned}$$

is open in $\varphi_j(U_i \cap U_j) \subseteq \varphi_j(U_j) = V_j$. By 1.3.2 it follows that $\varphi_i^{-1}(W)$ is open in M, and since $\varphi_i^{-1}(W) \subseteq U_i$, it is also open in U_i. Consequently, $\varphi_i : U_i \to V_i$ is continuous.
Finally, $(\varphi_j^{-1})^{-1}(U_i) = \varphi_j(U_i \cap U_j)$ is open $\forall j$, so U_i is open in the final topology. $\square$

By suitably restricting the φ_i and composing them with translations and dilations we can achieve that $V_i = B_1(0)$ for all $i \in I$. Thus a manifold is a topological space that locally looks like a ball in $\mathbb{R}^n$. Moreover, $B_1(0)$ is homeomorphic to $\mathbb{R}^n$. Indeed, $f : B_1(0) \to \mathbb{R}^n$, $f(x) := \frac{x}{\|x\|} \tan(\frac{\pi}{2}\|x\|)$ is a homeomorphism: $B_1(0) \to \mathbb{R}^n$ with inverse $f^{-1}(y) = \frac{2}{\pi}$ arctan $(\|y\|) \cdot y \cdot \|y\|$. Thus manifolds can equivalently be viewed as topological spaces that locally look like $\mathbb{R}^n$.

1.5.3 Examples.

(i) Every $\mathbb{R}^n$ is a manifold with atlas $\{\text{id}\}$.

(ii) Let $M = S^1 = \{(x, y) | x^2 + y^2 = 1\}$. M is a manifold with the following charts:

$$\begin{aligned}&U_1 = S^1 \cap \{x > 0\},\ \varphi_1 = p_2 : (x, y) \mapsto y \\ &U_2 = S^1 \cap \{x < 0\},\ \varphi_2 = p_2 \\ &U_3 = S^1 \cap \{y > 0\},\ \varphi_3 = p_1 : (x, y) \mapsto x \\ &U_4 = S^1 \cap \{y < 0\},\ \varphi_4 = p_1\end{aligned}$$

E.g.: $\varphi(U_1 \cap U_3) = (0, 1)$, $\varphi(U_1 \cap U_3) = (0, 1)$ and: $\varphi_1 \circ \varphi_3^{-1}(x) = \sqrt{1 - x^2}$ is C^∞, and analogously for the other cases. Hence S^1 is a smooth manifold. Since all φ_i are homeomorphisms (when their domains are equipped with the trace topology from $\mathbb{R}^2$), S^1 carries the trace topology of $\mathbb{R}^2$ (in particular, S^1 is Hausdorff).

1.5.4 Definition. *Let G be a group that at the same time is a topological space such that:*

(i) $(x, y) \mapsto x \cdot y : G \times G \to G$ *is continuous.*

(ii) $x \mapsto x^{-1} : G \to G$ *is continuous.*

Then G is called a topological group.

1.5.5 Definition. *If G is a group that also is a C^∞-manifold, and such that (i), (ii) are smooth, then G is called a Lie group.*

1.5.6 Examples.

(i) $(\mathbb{R}^n, +)$, $(\mathbb{C}^n, +)$ are topological groups and Lie groups.

(ii) The matrix groups $GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $O(n)$, $U(n), \ldots$ are topological groups (and Lie groups) with respect to matrix multiplication. They all are equipped with the trace topology of $\mathbb{R}^{n^2}$ resp. $\mathbb{C}^{n^2}$, since $M(n)$ itself can be identified with $\mathbb{R}^{n^2}$ resp. $\mathbb{C}^{n^2}$. The group operations are continuous (even smooth) by the usual formulas for matrix multiplication/inversion.

Chapter 2

Filters and convergence

2.1 Nets

2.1.1 Definition. *A set I is called* directed*, if on I there is a relation $\leq$ satisfying*

(i) $i \leq i \ \forall i \in I$.

(ii) $i_1 \leq i_2 \wedge i_2 \leq i_3 \Rightarrow i_1 \leq i_3$.

(iii) $\forall i_1, i_2 \in I \ \exists i_3 \in I$ *with* $i_1 \leq i_3 \wedge i_2 \leq i_3$.

Also, $i_1 < i_2$ *means:* $i_1 \leq i_2$, *but not* $i_2 \leq i_1$.

2.1.2 Examples.

(i) $(\mathbb{N}, \leq), (\mathbb{R}, \leq)$ are directed sets.

(ii) Let X be a topological space, $x \in X$ and $\mathcal{U}(x)$ the set of all neighborhoods of x. $\mathcal{U}(x)$ is a directed set via $U_1 \leq U_2 \Leftrightarrow U_2 \subseteq U_1$.

(iii) The set $\mathcal{Z}$ of all partitions $Z = (x_0, x_1, \ldots, x_n)$, $a = x_0 < x_1 < \cdots < x_n = b$ of the interval $[a, b] \subseteq \mathbb{R}$ is a directed set via $Z_1 \leq Z_2 :\Leftrightarrow Z_1 \subseteq Z_2$

2.1.3 Definition.

(i) A net (or: Moore-Smith sequence) in a set X is a map $\Phi : I \to X$, $i \mapsto x_i$ of a directed set I into X. One also writes $(x_i)_{i \in I}$ for Φ.

(ii) A net $(x_i)_{i \in I}$ in a topological space X is called convergent to $x \in X$, $x_i \to x$, if: $\forall U \in \mathcal{U}(x) \ \exists i_0 \in I : \ x_i \in U \ \forall i \geq i_0$.

2.1.4 Examples.

(i) A sequence $(x_n)_{n \in \mathbb{N}}$ is a net on the directed set $\mathbb{N}$. The net $(x_n)_{n \in \mathbb{N}}$ converges if and only if the sequence $(x_n)_{n \in \mathbb{N}}$ converges.

(ii) Let X be a topological space, let $x \in X$ and let $\mathcal{U}(x)$ as in 2.1.2 (ii). If for any $U \in \mathcal{U}(x)$, x_U is a point in U then the net $(x_U)_{U \in \mathcal{U}(x)}$ converges to x. In fact, let $V \geq U$, i.e. $V \subseteq U$, then $x_V \in V \subseteq U$, so $x_V \in U \ \forall V \geq U$.

(iii) Let $\mathcal{Z}$ as in 2.1.2 (iii), $f : [a, b] \to \mathbb{R}$ and consider the nets

$$\Phi_1 : \mathcal{Z} \to \mathbb{R},\ (x_0, x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} (x_i - x_{i-1}) \sup \{f(x) | x \in [x_{i-1}, x_i]\}$$

$$\Phi_2 : \mathcal{Z} \to \mathbb{R},\ (x_0, \ldots, x_n) \mapsto \sum_{i=1}^{n} (x_i - x_{i-1}) \inf \{f(x) | x \in [x_{i-1}, x_i]\}$$

Then f is Riemann-integrable if Φ_1 and Φ_2 converge to the same number c. In this case, $c = \int\limits_a^b f(x)\, dx$.

2.1.5 Theorem. *Let X, Y be topological spaces.*

(i) Let $A \subseteq X$. Then $x \in \bar{A}$ if and only if there exists a net $(x_i)_{i \in I}$, $x_i \in A$ with $x_i \to x$.

(ii) Let $f : X \to Y$. Then f is continuous in x if and only if for each net $(x_i)_{i \in I}$ with $x_i \to x$, $f(x_i) \to f(x)$.

Proof.

(i) ($\Rightarrow$): $\forall U \in \mathcal{U}(x)\ \exists x_U \in U \cap A$. Thus $(x_U)_{U \in \mathcal{U}(x)}$ is a net in A that converges to x (by 2.1.4 (ii)).

($\Leftarrow$): Let $U \in \mathcal{U}(x)$ and $(x_i)_{i \in I}$ a net in A with $x_i \to x \Rightarrow \exists x_{i_0} \in U \Rightarrow A \cap U \neq \emptyset$.

(ii) ($\Rightarrow$): Let $V \in \mathcal{U}(f(x)) \Rightarrow \exists U \in \mathcal{U}(x)$ with $f(U) \subseteq V$. Since $x_i \to x$, there exists some i_0 such that $x_i \in U\ \forall i \geq i_0$. Then also $f(x_i) \in V\ \forall i \geq i_0$, so $f(x_i) \to f(x)$.

($\Leftarrow$): If f were not continuous in x there would exist some $V \in \mathcal{U}(f(x))$ such that $\forall U \in \mathcal{U}(x) : f(U) \nsubseteq V \Rightarrow \forall U \in \mathcal{U}(x)\ \exists x_U \in U$ with $f(x_U) \notin V$. But then $(x_U)_{U \in \mathcal{U}(x)}$ is a net with $x_U \to x$, but $f(x_U) \not\to f(x)$, a contradiction.

□

2.2 Filters

In 2.1 we have repeatedly considered nets over the directed set $\mathcal{U}(x)$. For a system of sets $\mathcal{F}$ that is directed by inclusion the above examples suggest to say that $\mathcal{F}$ converges to x if eventually (in the sense of this direction) the sets of $\mathcal{F}$ lie in any $U \in \mathcal{U}(x)$.

2.2.1 Definition. *Let X be a set and let $\mathcal{F} \subseteq \mathcal{P}(X)$ (the power set of X). $\mathcal{F}$ is called a* filter*, if:*

(i) $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$.

(ii) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$.

(iii) $F \in \mathcal{F}$, $F' \supseteq F \Rightarrow F' \in \mathcal{F}$.

2.2.2 Examples.

(i) Let X be a topological space and let $x \in X$. Then $\mathcal{U}(x)$ is a filter, the so-called *neighborhood filter* of x.

(ii) Let X be an infinite set and let $\mathcal{F} := \{F \subseteq X | X \setminus F \text{ finite }\}$. Then $\mathcal{F}$ is a filter: (i) and (iii) are clear, and (ii) follows from

$$X \setminus (F_1 \cap F_2) = (X \setminus F_1) \cup (X \setminus F_2) \text{ finite.}$$

$\mathcal{F}$ is called *Fréchet filter* on X.

(iii) Let $\emptyset \neq A \subseteq X$. Then $\mathcal{F} := \{B \subseteq X | A \subseteq B\}$ is a filter.

(iv) Any filter is a directed set by defining

$$F_1 \leq F_2 :\Leftrightarrow F_2 \subseteq F_1.$$

Then 2.1.1 (i) and (ii) are clear and 2.1.1 (iii) follows from: $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$ and $F_1 \leq F_1 \cap F_2,\ F_2 \leq F_1 \cap F_2$. For any $F \in \mathcal{F}$ choose an $x_F \in F$, then $(x_F)_{F \in \mathcal{F}}$ is a net.

(v) Conversely, let $(x_i)_{i \in I}$ be a net and set

$$\mathcal{F} := \{F \subseteq X | \exists i_0 \in I \text{ with } \{x_i | i \geq i_0\} \subseteq F\}.$$

Then $\mathcal{F}$ is a filter: (i), (iii) from 2.2.1 are clear, and (ii) follows from: $F_1, F_2 \in \mathcal{F} \Rightarrow \{x_i | i \geq i_1\} \subseteq F_1,\ \{x_i | i \geq i_2\} \subseteq F_2$. Pick $i_0 \in I$ such that $i_1 \leq i_0$ and $i_2 \leq i_0$, then $\{x_i | i \geq i_0\} \subseteq F_1 \cap F_2 \Rightarrow F_1 \cap F_2 \in \mathcal{F}$.

We will return to the relation between filters and nets below when we look at convergence.

2.2.3 Definition. *Let X be a set and let $\mathcal{F}, \mathcal{F}'$ be filters on X. $\mathcal{F}$ is called* finer *than $\mathcal{F}'$, and $\mathcal{F}'$* coarser *than $\mathcal{F}$, if $\mathcal{F} \supseteq \mathcal{F}'$ (i.e., $F \in \mathcal{F}' \Rightarrow F \in \mathcal{F}$).*

2.2.4 Example. Let I be a set and for each $i \in I$ let $\mathcal{F}_i$ be a filter on X. Then $\bigcap_{i \in I} \mathcal{F}_i$ is a filter on X, namely the finest filter that is coarser than every $\mathcal{F}_i$.

Next we want to answer the question under what conditions for a given system $\mathcal{S}$ of subsets of X there exists a filter containing $\mathcal{S}$.

2.2.5 Proposition. *Let X be a set, $\mathcal{S} \subseteq \mathcal{P}(X)$. The following are equivalent:*

(i) There exists a filter $\mathcal{F}$ containing $\mathcal{S}$.

(ii) If $\mathcal{S}' \subseteq \mathcal{S}$ is finite, then $\bigcap_{S \in \mathcal{S}'} S \neq \emptyset$.

Then $\mathcal{F} := \{F \subseteq X | \exists \mathcal{S}' \subseteq \mathcal{S}$ finite with $\bigcap_{S \in \mathcal{S}'} S \subseteq F\}$ is the coarsest filter on X containing $\mathcal{S}$.

Proof. (i)$\Rightarrow$(ii): Clear by 2.2.1 (i) and (ii).
(ii)$\Rightarrow$(i): We show that $\mathcal{F}$ is a filter: 2.2.1 (i), (iii) are clear. Ad (ii): Let $F_1, F_2 \in \mathcal{F}$, $\bigcap_{S \in \mathcal{S}_1'} S \subseteq F_1$, $\bigcap_{S \in \mathcal{S}_2'} S \subseteq F_2 \Rightarrow \bigcap_{S \in (\mathcal{S}_1' \cup \mathcal{S}_2')} S \subseteq F_1 \cap F_2 \Rightarrow F_1 \cap F_2 \in \mathcal{F}$. Let $\mathcal{F}'$ be another filter containing $\mathcal{S} \Rightarrow \mathcal{F}'$ contains all $\bigcap_{S \in \mathcal{S}} S$ and thereby all $F \in \mathcal{F} \Rightarrow \mathcal{F} \subseteq \mathcal{F}'$. $\square$

2.2.6 Definition. *Let X be a set and suppose that $\mathcal{S} \subseteq \mathcal{P}(X)$ satisfies 2.2.5 (ii). Let $\mathcal{F}$ be the coarsest filter containing $\mathcal{S}$. Then $\mathcal{S}$ is called* a subbasis *of $\mathcal{F}$.*

The following result clarifies the dual question to 2.2.4:

2.2.7 Corollary. *Let I, X be sets, and for $i \in I$ let $\mathcal{F}_i$ be a filter on X. TFAE:*

(i) There exists a coarsest filter that is finer than every $\mathcal{F}_i$.

(ii) For every finite set $J \subseteq I$ and every family $(F_j)_{j \in J}$ with $F_j \in \mathcal{F}_j$ for every j, $\bigcap_{j \in J} F_j \neq \emptyset$.

Proof. By 2.2.5, (i) $\Leftrightarrow$ $\exists$ coarsest filter $\mathcal{F}$ with $\mathcal{S} := \bigcup_{i \in I} \mathcal{F}_i \subseteq \mathcal{F} \Leftrightarrow$ (ii). □

2.2.8 Corollary. *Let X be a set and let $(\mathcal{F}_i)_{i \in I}$ be a family of filters on X that is totally ordered (with respect to inclusion). Then there exists a coarsest filter $\mathcal{F}$ on X that is finer than every $\mathcal{F}_i$.*

Proof. Let $J \subseteq I$ be finite, $J = \{i_1, \dots, i_n\}$ such that $\mathcal{F}_{i_1} \subseteq \mathcal{F}_{i_2} \subseteq \dots \mathcal{F}_{i_n}$. Let $F_{i_j} \in \mathcal{F}_{i_j}$ $(1 \le j \le n)$. Then $F_{i_j} \in \mathcal{F}_{i_n} \forall j \Rightarrow \bigcap_{j=1}^n F_{i_j} \neq \emptyset$, yielding the claim. □

2.2.9 Proposition. *Let X be a set, $\mathcal{B} \subseteq \mathcal{P}(X)$ and $\mathcal{F} := \{F \subseteq X | \exists B \in \mathcal{B}$ with $B \subseteq F\}$. TFAE:*

(i) $\mathcal{F}$ is a filter.

(ii) (a) $\forall B_1, B_2 \in \mathcal{B}\ \exists B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$.
(b) $\emptyset \notin \mathcal{B} \neq \emptyset$.

Then $\mathcal{F}$ is the coarsest filter on X that contains $\mathcal{B}$.

Proof. (i) $\Rightarrow$ (ii): (a): Let $B_1, B_2 \in \mathcal{B} \Rightarrow B_1, B_2 \in \mathcal{F} \Rightarrow B_1 \cap B_2 \in \mathcal{F} \Rightarrow$ claim. (b): Clear since $\mathcal{F}$ is a filter.
(ii) $\Rightarrow$ (i): 2.2.1 (i), (iii) are clear. Ad (ii): $F_1 \supseteq B_1, F_2 \supseteq B_2 \Rightarrow F_1 \cap F_2 \supseteq B_1 \cap B_2 \supseteq B_3$.
Finally, if $\mathcal{F}'$ is a filter that contains $\mathcal{B}$, then $\mathcal{F}' \supseteq \mathcal{F}$. □

2.2.10 Definition. *Let X be a set and let $\mathcal{B} \subseteq \mathcal{P}(X)$ as in 2.2.9 (ii). Let $\mathcal{F}$ be the coarsest filter that contains $\mathcal{B}$. Then $\mathcal{B}$ is called a filter basis of $\mathcal{F}$.*

Any filter basis of $\mathcal{F}$ is also a subbasis of $\mathcal{F}$. Conversely, if $\mathcal{S}$ is a subbasis of a filter $\mathcal{F}$ then by 2.2.5 $\{\bigcap_{S \in \mathcal{S}'} S | \mathcal{S}' \subseteq \mathcal{S}$ finite $\}$ is a basis of $\mathcal{F}$.

2.2.11 Examples.

(i) The tail ends $\{x_i | i \ge i_0\}$ of a net form a basis of the filter defined in 2.2.2 (v).

(ii) The sets $\{(n, \infty) | n \in \mathbb{N}\}$ form a basis of the Fréchet filter on $\mathbb{N}$.

2.2.12 Proposition. *Let X be a set, $\mathcal{F}$ a filter on X and $\mathcal{B} \subseteq \mathcal{F}$. TFAE:*

(i) $\mathcal{B}$ is a filter basis of $\mathcal{F}$.

(ii) $\forall F \in \mathcal{F}\ \exists B \in \mathcal{B}$ with $B \subseteq F$.

Proof. (i) $\Rightarrow$ (ii): Clear by 2.2.9.
(ii) $\Rightarrow$ (i): $\mathcal{B}$ is a filter basis: (a): $B_1, B_2 \in \mathcal{B} \Rightarrow B_1, B_2 \in \mathcal{F} \Rightarrow \exists B_3 \subseteq B_1 \cap B_2$.
(b): $\emptyset \notin \mathcal{B} \neq \emptyset$ is clear.
By 2.2.9, $\mathcal{B}$ is a basis of $\mathcal{G} := \{F \subseteq X | \exists B \in \mathcal{B} : B \subseteq F\} \Rightarrow \mathcal{G} \subseteq \mathcal{F}$. Conversely, $\mathcal{B} \subseteq \mathcal{G} \overset{(ii)}{\Rightarrow} \mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{F} = \mathcal{G}$. □

In particular, 2.2.12 implies that the filter bases of the neighborhood filter $\mathcal{U}(x)$ are precisely the neighborhood bases of x.

2.2.13 Proposition. *Let X be a set, $\mathcal{F}, \mathcal{F}'$ filters on X with bases $\mathcal{B}, \mathcal{B}'$. TFAE:*

(i) $\mathcal{F}$ is finer than $\mathcal{F}'$, i.e. : $\mathcal{F} \supseteq \mathcal{F}'$.

(ii) $\forall B' \in \mathcal{B}'$ $\exists B \in \mathcal{B}$ with $B \subseteq B'$.

Proof. (i) $\Rightarrow$ (ii): $\mathcal{B}' \subseteq \mathcal{F}' \subseteq \mathcal{F} = \{F \subseteq X | \exists B \in \mathcal{B} \text{ with } B \subseteq F\}$.
(ii) $\Rightarrow$ (i): Let $F' \in \mathcal{F}' \Rightarrow \exists B' \in \mathcal{B}'$ with $B' \subseteq F' \Rightarrow \exists B \in \mathcal{B}$ with $B \subseteq B' \subseteq F' \Rightarrow F' \in \mathcal{F}$. □

Before we turn to the definition of ultrafilters, let us first recall some notions from set theory.

Let $(X, \leq)$ be an ordered set, $A \subseteq X$, $x \in X$. x is called an *upper bound* of A if $a \leq x$ for all $a \in A$. $m \in X$ is called *maximal element* of X, if $m \leq x$, $x \in X$ implies $x = m$. If $(X, \leq)$ is an ordered set in which any totally ordered subset possesses an upper bound, then there exists a maximal element in X (Zorn's Lemma).

2.2.14 Definition. *Let X be a set and let $\mathcal{F}$ be a maximal element in the set of all filters on X. Then $\mathcal{F}$ is called an* ultrafilter *on X.*

Thus $\mathcal{F}$ is an ultrafilter if and only if for every filter $\mathcal{F}'$ with $\mathcal{F}' \supseteq \mathcal{F}$ it follows that $\mathcal{F}' = \mathcal{F}$.

2.2.15 Theorem. *Let X be a set and let $\mathcal{F}$ be a filter on X. Then there exists an ultrafilter $\mathcal{F}'$ on X with $\mathcal{F}' \supseteq \mathcal{F}$.*

Proof. Let $\Psi := \{\mathcal{F}' | \mathcal{F}' \text{ is a filter on } X \text{ with } \mathcal{F}' \supseteq \mathcal{F}\}$. By 2.2.8, any totally ordered subset of Ψ has an upper bound. Thus by Zorn's Lemma, Ψ has a maximal element $\mathcal{F}'$. This is the ultrafilter we wanted to find. □

2.2.16 Theorem. *Let X be a set and $\mathcal{F}$ a filter on X. TFAE:*

(i) $\mathcal{F}$ is an ultrafilter.

(ii) $\forall A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$.

Proof. (i) $\Rightarrow$ (ii): Since $A \cap (X \setminus A) = \emptyset$ there can be no two sets F_1, F_2 in $\mathcal{F}$ with $F_1 \subseteq A$ and $F_2 \subseteq X \setminus A$. Thus either all $F \in \mathcal{F}$ intersect A or all of them meet $X \setminus A$. In the first case, $\{F \cap A | F \in \mathcal{F}\}$ is the basis of a filter $\mathcal{G}$ which by 2.2.13 is finer than $\mathcal{F} \Rightarrow \mathcal{F} = \mathcal{G} \Rightarrow A \in \mathcal{F}$. If $F \cap (X \setminus A) \neq \emptyset$ $\forall F \in \mathcal{F}$ it follows analogously that $X \setminus A \in \mathcal{F}$.
(ii)$\Rightarrow$ (i): Suppose that $\mathcal{F}$ is not an ultrafilter. Then there exists a filter $\mathcal{G} \supset \mathcal{F} \Rightarrow \exists G \in \mathcal{G}, G \notin \mathcal{F} \Rightarrow X \setminus G \in \mathcal{F} \subseteq \mathcal{G} \Rightarrow G, X \setminus G \in \mathcal{G}$. But this is impossible since $\mathcal{G}$ is a filter. □

2.2.17 Definition. *Let X be a set and $\mathcal{F}$ a filter on X. $\mathcal{F}$ is called* free *if $\bigcap_{F \in \mathcal{F}} F = \emptyset$. Otherwise $\mathcal{F}$ is called* fixed.

2.2.18 Corollary. *Let $\mathcal{F}$ be a filter on a set X. TFAE:*

(i) $\mathcal{F}$ is a fixed ultrafilter.

(ii) $\exists x \in X$ with $\mathcal{F} = \{F \subseteq X | x \in F\}$.

Proof. (i) $\Rightarrow$ (ii): Let $x \in \bigcap_{F\in\mathcal{F}} F \Rightarrow$ all $F \in \mathcal{F}$ meet $A := \{x\}$. Hence by 2.2.16, (i) $\Rightarrow$ (ii), $\{\{x\} \cap F | F \in \mathcal{F}\} = \{\{x\}\}$ is a basis of $\mathcal{F}$.

(ii) $\Rightarrow$ (i): $\{x\}$ is a basis of $\mathcal{F}$. Hence $A \subseteq X$ is in $\mathcal{F} \Leftrightarrow x \in A$. Since either $x \in A$ or $x \in X \setminus A$, we get: for any $A \subseteq X$, either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F} \Rightarrow \mathcal{F}$ is an ultrafilter. □

Fixed ultrafilters are in fact the only ones that can be described explicitly.

2.2.19 Corollary. *Let $\mathcal{F}$ be an ultrafilter on X and let $A, B \subseteq X$ and $A \cup B \in \mathcal{F}$. Then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.*

Proof. Suppose indirectly that $A \notin \mathcal{F}$ and $B \notin \mathcal{F}$. Then by 2.2.16 $X \setminus A \in \mathcal{F}$, $X \setminus B \in \mathcal{F} \Rightarrow (X \setminus A) \cap (X \setminus B) = X \setminus (A \cup B) \in \mathcal{F} \Rightarrow A \cup B \notin \mathcal{F}$. □

By induction it immediately follows that if $\bigcup_{i=1}^n A_i \in \mathcal{F} \Rightarrow \exists j \in \{1, \dots, n\}$ with $A_j \in \mathcal{F}$.

2.2.20 Proposition. *Let $\mathcal{F}$ be a filter on X. Then*

$$\mathcal{F} = \bigcap \{\mathcal{G} \supseteq \mathcal{F} | \mathcal{G} \text{ ultrafilter}\}$$

Proof. $\subseteq$: follows from 2.2.15.

$\supseteq$: Let $A \notin \mathcal{F}$. Then $F \nsubseteq A\ \forall F \in \mathcal{F} \Rightarrow F \cap (X \setminus A) \neq \emptyset\ \forall F \in \mathcal{F}$. Then by 2.2.5, $\{X \setminus A\} \cup \{F | F \in \mathcal{F}\}$ is a subbasis of a filter $\mathcal{F}'$ that is finer than $\mathcal{F}$. Let $\mathcal{G}$ be an ultrafilter with $\mathcal{G} \supseteq \mathcal{F}'$. Then $X \setminus A \in \mathcal{G} \Rightarrow A \notin \mathcal{G} \Rightarrow A \notin \bigcap\{\mathcal{G} \supseteq \mathcal{F} | \mathcal{G} \text{ ultrafilter}\}$. □

Next we look at restrictions of filters to subsets.

2.2.21 Proposition. *Let X be a set, $A \subseteq X$ and $\mathcal{F}$ a filter on X. TFAE:*

(i) $\mathcal{F}_A := \{F \cap A | F \in \mathcal{F}\}$ *is a filter on* A.

(ii) $F \cap A \neq \emptyset\ \forall F \in \mathcal{F}$.

Proof. (i)$\Rightarrow$(ii): clear.

(ii)$\Rightarrow$(i): We check 2.6 (i) – (iii) : (i): $\emptyset \notin \mathcal{F}_A$ by (ii), $A = X \cap A \in \mathcal{F}_A$. (ii): $(F_1 \cap A) \cap (F_2 \cap A) = (F_1 \cap F_2) \cap A$. (iii): Let $A \supseteq B \supseteq F \cap A \Rightarrow F \cup B \in \mathcal{F}$ and therefore $B = (F \cap A) \cup (B \cap A) = (F \cup B) \cap A \in \mathcal{F}_A$. □

2.2.22 Proposition. *Let X be a set, $A \subseteq X$ and $\mathcal{F}$ an ultrafilter on X. TFAE:*

(i) $\mathcal{F}_A$ *is a filter on* A.

(ii) $A \in \mathcal{F}$.

In this case $\mathcal{F}_A$ is an ultrafilter on A.

Proof. (i) $\Rightarrow$ (ii): If $A \notin \mathcal{F}$, then by 2.2.16 $X \setminus A \in \mathcal{F} \Rightarrow \emptyset = (X \setminus A) \cap A \in \mathcal{F}_A$, a contradiction.

(ii) $\Rightarrow$ (i): $A \in \mathcal{F} \Rightarrow A \cap F \neq \emptyset\ \forall F \in \mathcal{F} \Rightarrow \mathcal{F}_A$ is a filter by 2.2.21.

To see that $\mathcal{F}_A$ is an ultrafilter on A we use 2.2.16: Let $B \subseteq A$, $B \notin \mathcal{F}_A$. We need to show that $A \setminus B \in \mathcal{F}_A$. Since $B \notin \mathcal{F}_A$ it follows that $B \notin \mathcal{F}$ (otherwise $B = B \cap A \in \mathcal{F}_A$). Therefore, since $\mathcal{F}$ is an ultrafilter, $X \setminus B \in \mathcal{F} \Rightarrow A \setminus B = (X \setminus B) \cap A \in \mathcal{F}_A$. □

2.2.23 Definition. *Let X be a set, $A \subseteq X$, $\mathcal{F}$ a filter on X and $F \cap A \neq \emptyset\ \forall F \in \mathcal{F}$. Then $\mathcal{F}_A$ is called the* trace *of $\mathcal{F}$ on A.*

2.2.24 Example. Let X be a topological space, $A \subseteq X$, $x \in X$. Consider $\mathcal{U}(x)_A$. $\mathcal{U}(x)_A$ is a filter on $A \Leftrightarrow \forall V \in \mathcal{U}(x)$, $V \cap A \neq \emptyset \Leftrightarrow x \in \bar{A}$.

2.2.25 Proposition. *Let X, Y be sets and $\mathcal{B}$ a basis of a filter $\mathcal{F}$ on X. Let $f : X \to Y$ be a map. Then $f(\mathcal{B})$ is a basis of a filter $f(\mathcal{F})$ on Y. If $\mathcal{F}$ is an ultrafilter then also $f(\mathcal{B})$ is a basis of an ultrafilter (hence $f(\mathcal{F})$ is an ultrafilter).*

Proof. Clearly, $\emptyset \notin f(\mathcal{B}) \neq \emptyset$. Let $B_1, B_2 \in \mathcal{B} \Rightarrow \exists B_3 \subseteq B_1 \cap B_2 \Rightarrow f(B_3) \subseteq f(B_1) \cap f(B_1)$. If $\mathcal{F}$ is an ultrafilter, we employ 2.2.16: Let $A \subseteq Y$, then there are two possibilities:

1) $f^{-1}(A) \in \mathcal{F} \Rightarrow \exists B \in \mathcal{B}$ with $B \subseteq f^{-1}(A) \Rightarrow f(B) \subseteq A \Rightarrow A \in f(\mathcal{F})$.
2) $X \setminus f^{-1}(A) = f^{-1}(Y \setminus A) \in \mathcal{F} \Rightarrow \exists B \subseteq f^{-1}(Y \setminus A) \Rightarrow f(B) \subseteq Y \setminus A \Rightarrow Y \setminus A \in f(\mathcal{F})$. □

2.2.26 Theorem and Definition. *Let $(X_i)_{i\in I}$ be a family of sets and for each $i \in I$ let $\mathcal{B}_i$ be a basis of a filter $\mathcal{F}_i$ on X_i. Then $\mathcal{B} := \{\prod_{i\in I} B_i | B_i = X_i$ for almost all $i, B_i \in \mathcal{B}_i$ otherwise$\}$ is a basis of a filter $\mathcal{F}$ on $\prod_{i\in I} X_i$. $\mathcal{F}$ is called the product of the filters $(\mathcal{F}_i)_{i\in I}$.*

Proof. Obviously, $\emptyset \notin \mathcal{B} \neq \emptyset$.
Now let $B_i = X_i$ for $i \notin H$ (finite), $B_i \in \mathcal{B}_i$ otherwise, $B_i' = X_i$ for $i \notin H'$ (finite), $B_i' \in \mathcal{B}_i$ otherwise. For $i \in H \cap H'$ let $B_i'' \in \mathcal{B}$ such that $B_i'' \subseteq B_i \cap B_i'$

$$\text{Let } C_i := \begin{cases} B_i & i \in H \setminus H' \\ B_i' & i \in H' \setminus H \\ B_i'' & i \in H \cap H' \\ X_i & i \notin H \cup H' \end{cases} \Rightarrow \prod_{i\in I} C_i \subseteq \prod_{i\in I} B_i \cap \prod_{i\in I} B_i'$$

□

2.2.27 Example. Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Then the product of the $(\mathcal{U}(x_i))_{i\in I}$ is precisely $\mathcal{U}((x_i)_{i\in I})$.

2.3 Convergence

2.3.1 Definition. *Let X be a topological space, $\mathcal{F}$ a filter on X with basis $\mathcal{B}$ and $x \in X$. x is called* limit *or* limit point *of $\mathcal{F}$ (or also of $\mathcal{B}$), if $\mathcal{F}$ is finer than the neighborhood filter $\mathcal{U}(x)$ of x. We write $\mathcal{F} \to x$.*

Thus $\mathcal{F}$ converges to x if any neighborhood $\mathcal{U}$ of x contains some $B \in \mathcal{B}$. Together with $\mathcal{F}$, also any finer filter $\mathcal{F}'$ converges to x.

2.3.2 Proposition. *Let X be a topological space, $\mathcal{F}$ a filter on X and $x \in X$. TFAE:*

(i) $\mathcal{F}$ converges to x.

(ii) Every ultrafilter $\mathcal{G} \supseteq \mathcal{F}$ converges to x.

Proof. (i)$\Rightarrow$(ii): clear.
(ii)$\Rightarrow$(i): $\mathcal{U}(x) \subseteq \bigcap_{\mathcal{G} \supseteq \mathcal{F}} \mathcal{G} = \mathcal{F}$ (using 2.2.20). $\square$

Here, for one last time we look at nets, to clarify the relation between convergence of nets and of filters. For the following, cf. 2.2.2 (iv) and (v).

2.3.3 Theorem. *Let X be a topological space, $x \in X$.*

(i) Let $(x_i)_{i \in I}$ be a net in X and $\mathcal{F}$ the filter whose basis are the tail ends $\{x_i | i \geq i_0\}$ of $(x_i)_{i \in I}$ (cf. 2.2.2 (v)). Then:

$$x_i \to x \Leftrightarrow \mathcal{F} \to x.$$

(ii) Let $\mathcal{F}$ be a filter on X and for any map $s : (\mathcal{P}(X) \supseteq)\mathcal{F} \to X$ consider the net $(s(F))_{F \in \mathcal{F}} \equiv (s_F)_{F \in \mathcal{F}}$. Then:

$$\mathcal{F} \to x \Leftrightarrow s_F \to x \quad \forall s : \mathcal{F} \to X \text{ with } s_F \in F \ \forall F \in \mathcal{F}.$$

Proof. (i) $x_i \to x \Leftrightarrow$ every $V \in \mathcal{U}(x)$ contains a tail end of $(x_i)_{i \in I} \Leftrightarrow \mathcal{F} \to x$.
(ii) ($\Rightarrow$): Let $V \in \mathcal{U}(x) \Rightarrow \exists F_0 \in \mathcal{F}$ with $F_0 \subseteq V$. Let $s : \mathcal{F} \to X$ with $s_F \in F$ for all $F \in \mathcal{F}$. Then $\{s_F | F \geq F_0\} \subseteq V$, because $s_F \in F \subseteq F_0 \subseteq V \ \forall F \geq F_0$.
($\Leftarrow$): If $\mathcal{F} \not\to x$, then there exists some $V \in \mathcal{U}(x)$ such that $F \not\subseteq V \ \forall F \in \mathcal{F} \Rightarrow \forall F \in \mathcal{F} \ \exists s_F \in F \setminus V \Rightarrow (s_F)_{F \in \mathcal{F}} \not\to x$. $\square$

From 2.3.3 it can be seen that a satisfactory theory of convergence can be based both on nets or on filters. Generally, filters are preferred in the literature since systems of sets possess a number of technical advantages as compared to nets. In particular, the notion of ultrafilter is more versatile than the corresponding notion of universal net. Henceforth we will therefore exclusively work with filters.

2.3.4 Definition. *Let X be a topological space, $\mathcal{B}$ a basis of a filter $\mathcal{F}$ on X and $x \in \bigcap_{B \in \mathcal{B}} \bar{B}$. Then x is called a cluster point of $\mathcal{F}$.*

Thus x is a cluster point of $\mathcal{F} \Leftrightarrow x \in \bar{B} \ \forall B \in \mathcal{B} \Leftrightarrow \forall B \in \mathcal{B} \ \forall \mathcal{U} \in \mathcal{U}(x) : B \cap U \neq \emptyset$. If $\mathcal{B}_1, \mathcal{B}_2$ are bases of the same filter $\mathcal{F}$, then $\bigcap_{B \in \mathcal{B}_1} \bar{B} = \bigcap_{B \in \mathcal{B}_2} \bar{B}$: Let $B_1 \in \mathcal{B}_1 \Rightarrow \exists B_2 \in \mathcal{B}_2$ with $B_2 \subseteq B_1 \Rightarrow \bigcap_{B_2 \in \mathcal{B}_2} \bar{B}_2 \subseteq \bigcap_{B_1 \in \mathcal{B}_1} \bar{B}_1$, and analogously for the converse direction. In particular: $\bigcap_{B \in \mathcal{B}} \bar{B} = \bigcap_{F \in \mathcal{F}} \bar{F}$. The set of cluster points of $\mathcal{F}$ is closed, being the intersection of closed sets.

2.3.5 Proposition. *Let X be a topological space, $\mathcal{F}$ a filter on X, $x \in X$. TFAE:*

(i) x ist a cluster point of $\mathcal{F}$.

(ii) There exists a filter $\mathcal{G}$ on X, $\mathcal{G} \supseteq \mathcal{F}$ with $\mathcal{G} \to x$.

In particular, any limit point of $\mathcal{F}$ is also a cluster point of $\mathcal{F}$.

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{B} := \{F \cap U | U \in \mathcal{U}(x), F \in \mathcal{F}\}$. Then $\mathcal{B}$ is a basis of a filter $\mathcal{G}$ with $\mathcal{G} \supseteq \mathcal{F}$ and $\mathcal{G} \supseteq \mathcal{U}(x)$, i.e. $\mathcal{G} \to x$.
(ii) $\Rightarrow$ (i): Since $\mathcal{G} \supseteq \mathcal{F} \cup \mathcal{U}(x)$, $F \cap U \neq \emptyset \ \forall F \in \mathcal{F} \ \forall U \in \mathcal{U}(x)$. $\square$

2.3.6 Corollary. *Let X be a topological space, $\mathcal{F}$ an ultrafilter on X, $x \in X$. TFAE:*

(i) $\mathcal{F} \to x$

(ii) *x is a cluster point of $\mathcal{F}$.*

Proof. (i) $\Rightarrow$ (ii): Clear from 2.3.5.
(ii) $\Rightarrow$ (i): 2.3.5 $\Rightarrow \exists \mathcal{G} \supseteq \mathcal{F}$ with $\mathcal{G} \to x$. Since $\mathcal{F}$ is an ultrafilter, $\mathcal{G} = \mathcal{F}$. $\square$

2.3.7 Theorem. *Let X be a topological space, $A \subseteq X$, $x \in X$. TFAE:*

(i) $x \in \bar{A}$.

(ii) *There exists a filter $\mathcal{F}$ on X with $A \in \mathcal{F}$ and $\mathcal{F} \to x$.*

(iii) *x is a cluster point in X of some filter on A.*

Proof. (i) $\Rightarrow$ (ii): $\{A \cap U | U \in \mathcal{U}(x)\}$ is a basis of a filter $\mathcal{F}$ that contains A and is finer than $\mathcal{U}(x)$.
(ii) $\Rightarrow$ (iii): Since $A \in \mathcal{F}$, by 2.2.21 $\mathcal{F}_A := \{F \cap A | F \in \mathcal{F}\}$ is a filter on A (hence also a filter basis on X). To see that x is a cluster point of $\mathcal{F}_A$ on X, let $U \in \mathcal{U}(x)$. Then $U \in \mathcal{F}$ because $\mathcal{F} \to x$. Thus for any $F \in \mathcal{F}$, $(F \cap A) \cap U \in \mathcal{F}$, hence is nonempty.
(iii) $\Rightarrow$ (i): Let $\mathcal{F}$ be a filter on A with x as a cluster point of $\mathcal{F}$ on X. This means that

$$x \in \bigcap_{F \in \mathcal{F}} \bar{F}.$$

It follows that for any $U \in \mathcal{U}(x), F \in \mathcal{F}$ we have $A \cap U \supseteq F \cap U \neq \emptyset$. Therefore, $x \in \bar{A}$. $\square$

2.3.8 Definition. *Let X be a set , Y a topological space, $\mathcal{F}$ a filter on $X, f : X \to Y$ and $y \in Y$. y is called limit (resp. cluster point) of f with respect to $\mathcal{F}$ if y is a limit (resp. cluster point) of the filter $f(\mathcal{F})$ (cf. 2.2.25). If y is a limit of f with respect to $\mathcal{F}$ then we write $y = \lim_{\mathcal{F}} f$.*

Note that, in general a map f can have more than one limit with respect to a filter. Hence from $y = \lim_{\mathcal{F}} f$, $y' = \lim_{\mathcal{F}} f$ it does *not* follow that $y = y'$ (unless Y is Hausdorff, cf. 3.1.5 below).

2.3.9 Proposition. *Under the assumptions of 2.3.8, TFAE:*

(i) $y = \lim_{\mathcal{F}} f$.

(ii) $\forall V \in \mathcal{U}(y)\ \exists F \in \mathcal{F}$ *with* $f(F) \subseteq V$.

as well as

(i') *y is a cluster point of f with respect to $\mathcal{F}$.*

(ii') $\forall V \in \mathcal{U}(y)\ \forall F \in \mathcal{F}\ \exists x \in F$ *with* $f(x) \in V$.

Proof. Immediate from the definitions. $\square$

2.3.10 Example. Let $X = \mathbb{N}$, $\mathcal{F}$ the Fréchet filter on $\mathbb{N}$ (cf. 2.2.2 (ii)). Let $\omega : \mathbb{N} \to Y$, $\omega = (x_n)_{n\in\mathbb{N}}$ be a sequence in Y. If x is a limit of ω with respect to $\mathcal{F}$, we write $x = \lim_{n\to\infty} x_n$. A cluster point of ω with respect to $\mathcal{F}$ is called an accumulation point of ω.

The filter that is associated to $(x_n)_{n\in\mathbb{N}}$ by 2.3.3 (i) is called the *elementary filter* of ω. By 2.3.9, $x = \lim_{\mathcal{F}} \omega \Leftrightarrow \forall V \in \mathcal{U}(x)\ \exists F \in \mathcal{F}$ with $\omega(F) \subseteq V \Leftrightarrow$ every $V \in \mathcal{U}(x)$ contains a tail end of $\omega \Leftrightarrow x = \lim_{n\to\infty} x_n$. Also, x is a cluster point of ω with respect to $\mathcal{F} \Leftrightarrow \forall V \in \mathcal{U}(x)\ \forall k \in \mathbb{N}\ \exists n_k \geq k$ with $x_{n_k} \in V \Leftrightarrow x$ is an accumulation point (in the sense of analysis) of the sequence ω. However, in general (unless Y is first countable) this does not imply that ω possesses a subsequence that converges to x, as would be the case for sequences in $\mathbb{R}$ (there only needs to be a finer *filter* that converges to x, see 2.3.11 below).

For a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of ω, the family $(\{x_{n_k} | k \geq l\})_{l\in\mathbb{N}}$ is a basis of a filter that is finer than the elementary filter of ω. More generally we have:

2.3.11 Proposition. *Let X be a set, Y a topological space, $\mathcal{F}$ a filter on X, $f : X \to Y$, and $y \in Y$. TFAE:*

(i) y is a cluster point of f with respect to $\mathcal{F}$.

(ii) There exists a filter $\mathcal{G} \supseteq \mathcal{F}$ on X such that $y = \lim_{\mathcal{G}} f$.

Proof. (i) $\Rightarrow$ (ii): $\mathcal{B} := \{f^{-1}(V) \cap F | V \in \mathcal{U}(y), F \in \mathcal{F}\}$ is a filter basis on X: In fact, $f^{-1}(V_1) \cap F_1 \cap f^{-1}(V_1) \cap F_2 = f^{-1}(V_1 \cap V_2) \cap F_1 \cap F_2$, so it suffices to show that $f^{-1}(V) \cap F \neq \emptyset\ \forall V, F$. By 2.3.9 (ii'), there exists some $x \in F$ with $f(x) \in V \Rightarrow x \in F \cap f^{-1}(V)$. Let $\mathcal{G}$ be the filter with basis $\mathcal{B}$. Then $\mathcal{G} \supseteq \mathcal{F}$ and $\forall V \in \mathcal{U}(y)$ we have: $f(f^{-1}(V) \cap F) \subseteq f(f^{-1}(V)) \subseteq V \Rightarrow \lim_{\mathcal{G}} f = y$.
(ii) $\Rightarrow$ (i): By 2.3.5, y is a cluster point of $f(\mathcal{G})$, hence also of $f(\mathcal{F}) \subseteq f(\mathcal{G})$. □

2.3.12 Definition. *Let X, Y be topological spaces, $a \in X$, $f : X \to Y$, $y \in Y$. y is called limit (resp. cluster point) of f in a, if y is a limit (resp. cluster point) of f with respect to $\mathcal{U}(a)$. We write $y = \lim_{x\to a} f(x)$.*

2.3.13 Theorem. *Let X, Y be topological spaces, $f : X \to Y$, $a \in X$. TFAE:*

(i) f is continuous in a.

(ii) $f(a) = \lim_{x\to a} f(x)$.

(iii) If $\mathcal{F}$ is a filter on X with $\mathcal{F} \to a$, then $f(\mathcal{F}) \to f(a)$.

Proof. (i) $\Rightarrow$ (ii): Let $V \in \mathcal{U}(f(a)) \Rightarrow \exists U \in \mathcal{U}(a)$ with $f(U) \subseteq V \Rightarrow f(\mathcal{U}(a)) \supseteq \mathcal{U}(f(a))$.
(ii) $\Rightarrow$ (iii): $\mathcal{F} \supseteq \mathcal{U}(a) \Rightarrow f(\mathcal{F}) \supseteq f(\mathcal{U}(a)) \supseteq \mathcal{U}(f(a)) \Rightarrow f(\mathcal{F}) \to f(a)$.
(iii) $\Rightarrow$ (i): Set $\mathcal{F} := \mathcal{U}(a)$. Then $\mathcal{F} \to a \Rightarrow f(\mathcal{F}) = f(\mathcal{U}(a)) \supseteq \mathcal{U}(f(a)) \Rightarrow \forall V \in \mathcal{U}(f(a))\ \exists U \in \mathcal{U}(a)$ with $f(U) \subseteq V \Rightarrow f$ continuous in a. □

2.3.14 Corollary. *Let X, Y be topological spaces, Z a set, $f : X \to Y$, $g : Z \to X$, $\mathcal{F}$ a filter on Z. Let $a = \lim_{\mathcal{F}} g$ and f continuous in a. Then $f(a) = \lim_{\mathcal{F}} f \circ g$.*

Proof. By 2.3.13, $g(\mathcal{F}) \to a$ implies $f(g(\mathcal{F})) \to f(a)$, i.e., $f(a) = \lim_{\mathcal{F}} f \circ g$. □

2.3.15 Definition. *Let X, Y be topological spaces, $A \subseteq X$, $a \in \bar{A}$, $f : A \to Y$, $y \in Y$. y is called limit of f in a* with respect to A *if y is a limit of f with respect to $\mathcal{U}(a)_A$ (cf. 2.2.21). We then write $y = \lim\limits_{\substack{x\to a\\ x\in A}} f(x)$.*

Note that, by 2.2.21 we need $a \in \bar{A}$ to secure that $\mathcal{U}(a)_A$ is a filter. Also, if $y = \lim\limits_{\substack{x\to a\\ x\in A}} f(x)$, then $y \in \overline{f(A)}$: In fact, if $V \in \mathcal{U}(y)$, then there exists some $U \in \mathcal{U}(a)$ with $f(U \cap A) \subseteq V$, and $U \cap A \neq \emptyset$ since $a \in \bar{A}$, hence $V \cap f(A) \neq \emptyset$. (Alternatively, apply 2.3.7 (ii) to $f(\mathcal{U}(a)_A)$.)

If $A = X \setminus \{a\}$ and $a \in \bar{A}$ (i.e., a is not an isolated point of X), then instead of $y = \lim\limits_{\substack{x\to a\\ x\in X\setminus\{a\}}} f(x)$ we write $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x)$. If $g : X \to Y$, for $y = \lim\limits_{\substack{x\to a\\ x\in A}} g|_A(x)$ we simply write $y = \lim\limits_{\substack{x\to a\\ x\in A}} g(x)$.

2.3.16 Lemma. *Let X, Y be topological spaces, $f : X \to Y$.*

(i) If $B \subseteq A \subseteq X$, $a \in \bar{B}$ and $y = \lim\limits_{\substack{x\to a\\ x\in A}} f(x)$, then $y = \lim\limits_{\substack{x\to a\\ x\in B}} f(x)$.

(ii) Let $a \in \bar{A} \subseteq X$, $y \in Y$, V a neighborhood of a, $y = \lim\limits_{\substack{x\to a\\ x\in V\cap A}} f(x)$. Then $y = \lim\limits_{\substack{x\to a\\ x\in A}} f(x)$.

Proof. (i): Let $V \in \mathcal{U}(y) \Rightarrow \exists W \in \mathcal{U}(a)$ with $f(W \cap A) \subseteq V \Rightarrow f(W \cap B) \subseteq V$.

(ii): Let $U \in \mathcal{U}(y) \Rightarrow \exists W \in \mathcal{U}(a)$ with $f(V \cap A \cap W) \subseteq U$. Since $V \cap W \in \mathcal{U}(a)$, this gives the claim. □

2.3.17 Proposition. *Let X, Y be topological spaces, a a non-isolated point of X, $f : X \to Y$. TFAE:*

(i) f is continuous in a.

(ii) $\lim\limits_{\substack{x\to a\\ x\neq a}} f(x) = f(a)$.

Proof. (i) $\Rightarrow$ (ii): By 2.3.13, $f(a) = \lim_{x\to a} f(x)$. Thus, 2.3.16 (i) with $B = X\setminus\{a\}$ and $A = X$ gives $f(a) = \lim\limits_{\substack{x\to a\\ x\neq a}} f(x)$.

(ii) $\Rightarrow$ (i): Let $V \in \mathcal{U}(f(a)) \Rightarrow \exists U \in \mathcal{U}(a)$ with $f(U \setminus \{a\}) \subseteq V \Rightarrow f(U) \subseteq V \Rightarrow f$ continuous in a. □

2.3.18 Theorem. *Let $(X_i)_{i\in I}$ be a family of topological spaces, X a set and $f_i : X \to X_i$ a map ($i \in I$). Let X be endowed with the initial topology with respect to $(f_i)_{i\in I}$. Then for any filter $\mathcal{F}$ on X, TFAE:*

(i) $\mathcal{F} \to x$.

(ii) $f_i(\mathcal{F}) \to f_i(x)$ $\forall i \in I$.

Proof. (i) $\Rightarrow$ (ii): This follows from 2.3.13 since every f_i is continuous.

(ii) $\Rightarrow$ (i): By 1.2.2, $\{\bigcap_{k\in K} f_k^{-1}(U_k) | K \subseteq I$ finite, $U_k \in \mathcal{U}(f_k(x))\}$ is a basis of neighborhoods of x. By assumption, $\forall k \in K$ $\exists F_k \in \mathcal{F}$ with $f_k(F_k) \subseteq U_k$. Hence $F := \bigcap_{k\in K} F_k \in \mathcal{F}$ and $F \subseteq \bigcap_{k\in K} f_k^{-1}(U_k)$. □

2.3.19 Corollary. *Let $\mathcal{F}$ be a filter on $\prod_{i\in I} X_i$ and let $x \in \prod_{i\in I} X_i$. TFAE:*

(i) $\mathcal{F} \to x$.

(ii) $p_i(\mathcal{F}) \to p_i(x)$ $\forall i \in I$.

Proof. $\prod_{i\in I} X_i$ carries the initial topology with respect to $(p_i)_{i\in I}$. $\square$

2.3.20 Corollary. *Let $A_i \subseteq X_i$ $\forall i \in I$. Then $(\overline{\prod_{i\in I} A_i}) = \prod_{i\in I} \overline{A}_i$. Thus $\prod_{i\in I} A_i$ is closed if and only if each A_i is closed.*

Proof. Let $x = (x_i)_{i\in I} \in (\overline{\prod_{i\in I} A_i})$. Then by 2.3.7 there exists a filter $\mathcal{F}$ on $\prod_{i\in I} X_i$ with $\mathcal{F} \to x$ and $A \in \mathcal{F}$. By 2.3.19, $p_i(\mathcal{F}) \to x_i$, and $A_i = p_i(A) \in p_i(\mathcal{F})$. Since any limit point is also a cluster point, it follows that $x_i \in \overline{A}_i$ $\forall i$.

Conversely, let $x \in \prod_{i\in I} \bar{A}_i$ and let $\prod_{i\in I} U_i$ be a neighborhood of x ($U_i \in \mathcal{U}(x_i)$, $U_i = X_i$ for almost every i). Then $\prod_{i\in I} U_i \cap \prod_{i\in I} A_i = \prod_{i\in I}(U_i \cap A_i) \neq \emptyset$, so $x \in \overline{\prod_{i\in I} A_i}$. $\square$

2.3.21 Corollary. *Let X, I be sets and for $i \in I$, let Y_i be a topological space. Let $f : X \to \prod_{i\in I} Y_i$, $\mathcal{F}$ a filter on X and $y \in \prod_{i\in I} Y_i$. TFAE:*

(i) $\lim_{\mathcal{F}} f = y$.

(ii) $\forall i \in I$, $\lim_{\mathcal{F}} p_i \circ f = p_i(y)$.

Proof. By 2.3.19, $f(\mathcal{F}) \to y \Leftrightarrow \forall i \in I : p_i(f(\mathcal{F})) \to p_i(y)$. $\square$

Chapter 3

Separation properties

In $\mathbb{R}^n$ or, more generally, in metric spaces, any two disjoint closed sets can be separated by disjoint open sets. However, in general topological spaces, this property may be lost. As an example, consider a topological space $(X, \mathcal{O})$ with $\mathcal{O} = \{\emptyset, X\}$ the indiscrete topology. Then in X no two closed sets can be separated openly. It has proved to be useful to classify topological spaces on account of their separation properties.

3.1 Separation axioms

3.1.1 Definition. *A topological space is called*

T_0, *if whenever two points are distinct, one of them possesses a neighborhood not containing the other.*

T_1, *if whenever two points are distinct, each of them possesses a neighborhood not containing the other.*

T_2, *if any two distinct points possess disjoint neighborhoods. T_2-spaces are also called Hausdorff.*

T_3, *if any closed set $A \subseteq X$ and any $x \in X \setminus A$ possess disjoint neighborhoods.*

T_{3a}, *if for any closed set $A \subseteq X$ and any $x \in X \setminus A$ there exists a continuous function $f : X \to [0,1]$ with $f(x) = 1$ and $f|_A \equiv 0$.*

T_4, *if any two disjoint closed sets possess disjoint neighborhoods.*

These properties are also called separation axioms.

3.1.2 Remark. Relations between the separation axioms.

(i) Any T_1-space is a T_0-space, but not conversely: Consider on $\mathbb{R}$ the topology $\mathcal{O}_< := \{\emptyset, \mathbb{R}\} \cup \{(-\infty, a) | a \in \mathbb{R}\}$. Then $(\mathbb{R}, \mathcal{O}_<)$ is T_0, but not T_1.

(ii) Any T_2-space is a T_1-space, but not conversely: Let X be an infinite set and $\mathcal{O} := \{X, \emptyset\} \cup \{X \setminus A | A \subseteq X, A \text{ finite}\}$. Then $(X, \mathcal{O})$ is T_1: Let $x \neq y \Rightarrow X \setminus \{y\} \in \mathcal{U}(x),\ X \setminus \{x\} \in \mathcal{U}(y)$. However, $(X, \mathcal{O})$ is not T_2: Suppose that U, V are open, $x \in U,\ y \in V,\ U \cap V = \emptyset$. Then $X = X \setminus (U \cap V) = X \setminus U \cup X \setminus V$ would be finite, a contradiction.

(iii) A T_3-space need not be either T_1 or T_2: Let X be a set , $|X| > 1$ with the indiscrete topology $\mathcal{O} = \{\emptyset, X\}$. Then the only closed sets are $\emptyset$ and X, so T_3 is satisfied trivially. However, $(X, \mathcal{O})$ is neither T_1 nor T_2.

(iv) Any T_{3a}-space also is a T_3-space: Let $A \neq \emptyset$ closed and $x \in X \setminus A$. Let $f : X \to [0,1]$ with $f|_A \equiv 0$ and $f(x) = 1$. Then $f^{-1}([0, \frac{1}{2}))$ and $f^{-1}((\frac{1}{2}, 1])$ are open, disjoint neighborhoods of A and x, respectively.

(v) $T_4 \not\Rightarrow T_3$: Let $X = \{1, 2, 3, 4\}$ and $\mathcal{O} := \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, X\}$. The closed sets of X then are: $\emptyset, X, \{4\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}$. Thus two closed sets of X are disjoint only if one of them is empty. It follows that X is T_4. However, the point 1 and the closed set $\{4\}$ cannot be separated by open sets, so X is not T_3.

3.1.3 Proposition. *Let (X, d) be a metric space. Then X is T_2, T_4 and T_{3a} (and thereby also T_0, T_1 and T_3).*

Proof. Let $B(x, r)$ be the open ball of radius r.

T_2: $x \neq y$, $r := d(x, y) \Rightarrow B(x, \frac{r}{2}) \cap B(y, \frac{r}{2}) = \emptyset$.

T_4: Let A_1, A_2 be closed and disjoint. Then for any $x \in A_i$ there exists some $r_x > 0$ such that $B(x, 2r_x) \cap A_j = \emptyset \ \forall j \neq i$. Let $O_i := \bigcup_{x \in A_i} B(x, r_x)$. Then O_i is open, $A_i \subseteq O_i$ and $O_1 \cap O_2 = \emptyset$: In fact, suppose there were some $z \in O_1 \cap O_2$. Then there exist $x \in A_1$ and $y \in A_2$ with $z \in B(x, r_x) \cap B(y, r_y)$. Thus $d(x, y) \leq d(x, z) + d(z, y) < r_x + r_y \leq 2 \max(r_x, r_y)$, but $d(x, y) \geq \max(2r_x, 2r_y)$ by the definition of r_x and r_y, a contradiction.

T_{3a}: Let A be closed, and $x \notin A$. Then $d(x, A) := \inf_{a \in A} d(x, a) > 0$ (otherwise $\exists a_n \to x \Rightarrow x \in A$ since A is closed). Let $f(y) := \min\left(1, \frac{1}{d(x,A)} \cdot d(y, A)\right)$. Then $f : X \to [0, 1]$ is continuous, $f|_A \equiv 0$, and $f(x) = 1$. □

3.1.4 Theorem. *Let X be a topological space. TFAE:*

(i) X is T_1.

(ii) Any singleton in X is closed.

(iii) Any $A \subseteq X$ is the intersection of all its neighborhoods.

Proof. (i) $\Rightarrow$ (ii): Let $x \in X$, and $y \in X \setminus \{x\}$. Then there exists some $U_y \in \mathcal{U}(y)$ with $U_y \subseteq X \setminus \{x\} \Rightarrow X \setminus \{x\}$ open $\Rightarrow \{x\}$ closed.

(ii) $\Rightarrow$ (iii): Clearly, $A \subseteq \bigcap_{U \in \mathcal{U}(A)} U$. Conversely, let $x \notin A \Rightarrow X \setminus \{x\}$ is an open neighborhood of $A \Rightarrow x \notin X \setminus \{x\} \supseteq \bigcap_{U \in \mathcal{U}(A)} U$.

(iii) $\Rightarrow$ (i): Let $y \neq x \Rightarrow y \notin \bigcap_{U \in \mathcal{U}(x)} U = \{x\} \Rightarrow \exists U \in \mathcal{U}(x)$ with $y \notin U$. □

3.1.5 Theorem. *Let X be a topological space. TFAE:*

(i) X is Hausdorff.

(ii) Any convergent filter on X has a unique limit.

(iii) $\forall x \in X$, $\{x\} = \bigcap_{U \in \mathcal{U}(x)} \bar{U}$.

(iv) $\Delta := \{(x, x) | x \in X\} \subseteq X \times X$ is closed.

Proof. (i) $\Rightarrow$ (ii): Suppose that $\mathcal{F} \to x$ and $\mathcal{F} \to y$, but $x \neq y$. Since X is T_2, there exist $U \in \mathcal{U}(x)$, $V \in \mathcal{U}(y)$ with $U \cap V = \emptyset$. Since $\mathcal{F} \supseteq \mathcal{U}(x) \cup \mathcal{U}(y) \Rightarrow \exists F_1 \subseteq U$, $F_2 \subseteq V \Rightarrow F_1 \cap F_2 = \emptyset$, a contradiction.

(ii) $\Rightarrow$ (iii): Clearly, $x \in \bigcap_{U \in \mathcal{U}(x)} \bar{U}$. Conversely, let $y \in \bigcap_{U \in \mathcal{U}(x)} \bar{U}$. Then y is a cluster point of $\mathcal{U}(x)$, so by 2.3.5 there exists a filter $\mathcal{F}$ with $\mathcal{F} \supseteq \mathcal{U}(x)$ and $\mathcal{F} \to y$. Thus both y and x are limits of $\mathcal{F}$, so $y = x$.

(iii) $\Rightarrow$ (i): Let $x \neq y \Rightarrow \exists U \in \mathcal{U}(x)$ with $y \notin \bar{U} \Rightarrow \bar{U}$ and $X \setminus \bar{U}$ are neighborhoods of x resp. y that are disjoint.

(i) $\Rightarrow$ (iv): Let $(x, y) \notin \Delta \Rightarrow x \neq y \Rightarrow \exists U \in \mathcal{U}(x)$, $V \in \mathcal{U}(y)$ with $U \cap V = \emptyset$. Therefore, $U \times V$ is a neighborhood of (x, y) in $X \times X$ and $(U \times V) \cap \Delta = \emptyset$. Consequently, $(x, y) \notin \bar{\Delta} \Rightarrow \bar{\Delta} \subseteq \Delta \Rightarrow \bar{\Delta} = \Delta \Rightarrow \Delta$ is closed.

(iv) $\Rightarrow$ (i): If $x \neq y$ then $(x, y) \notin \Delta \Rightarrow \exists U \in \mathcal{U}(x)$, $V \in \mathcal{U}(y)$ with $(U \times V) \cap \Delta = \emptyset \Rightarrow U \cap V = \emptyset$. $\square$

3.1.6 Theorem. *Let X be a topological space. TFAE:*

(i) X is T_3.

(ii) $\forall x \in X$ $\forall O$ open with $x \in O$ $\exists U \in \mathcal{U}(x)$ with $x \in U \subseteq \bar{U} \subseteq O$ (i.e.: the closed neighborhoods of x form a neighborhood basis of x).

Proof. (i) $\Rightarrow$ (ii): Let O be open, $x \in O$. Then $X \setminus O$ is closed and $x \notin X \setminus O$, so there exist U, W open with $x \in U$, $X \setminus O \subseteq W$, and $U \cap W = \emptyset$. Thus $U \subseteq X \setminus W \subseteq O \Rightarrow \bar{U} \subseteq O$.

(ii) $\Rightarrow$ (i): Let A be closed, $x \notin A$. Then $O := X \setminus A$ is open and $x \in O$. Hence there exists some $U \in \mathcal{U}(x)$ with $x \in U \subseteq \bar{U} \subseteq X \setminus A$. Therefore, $A \subseteq X \setminus \bar{U}$, $x \in U$, and $(X \setminus \bar{U}) \cap U = \emptyset$. $\square$

3.1.7 Theorem. *Let X be a topological space. TFAE:*

(i) X is a T_{3a}-space.

(ii) $\mathcal{B} := \{f^{-1}(U) | U \subseteq \mathbb{R}$ open, $f : X \to \mathbb{R}$ continuous and bounded$\}$ is a basis of the topology of X.

(iii) $\mathcal{B}' := \{f^{-1}(U) | U \subseteq \mathbb{R}$ open, $f : X \to \mathbb{R}$ continuous$\}$ is a basis of the topology of X.

(iv) $\tilde{\mathcal{B}} := \{f^{-1}(0) | f : X \to \mathbb{R}$ continuous$\}$ is a basis for the closed sets of X (i.e., any closed set is an intersection of sets from $\tilde{\mathcal{B}}$).

(v) $\hat{\mathcal{B}} := \{f^{-1}(0) | f : X \to \mathbb{R}$ continuous and bounded$\}$ is a basis for the closed sets of X.

Proof. (i) $\Rightarrow$ (ii): It suffices to show that, for each $x \in X$, $\{B \in \mathcal{B} \mid x \in B\}$ forms a neighborhood basis of x. To see this, let $V \in \mathcal{U}(x)$ be open. Then $X \setminus V$ is closed and $x \notin X \setminus V$, so (i) implies that there exists some $f : X \to [0, 1] \subseteq \mathbb{R}$ continuous (and bounded) with $f(X \setminus V) \subseteq \{0\}$, and $f(x) = 1$. Hence $f^{-1}(\mathbb{R} \setminus \{0\}) = f^{-1}((0, 1])$ is an open neighborhood of x with $f^{-1}(\mathbb{R} \setminus \{0\}) \subseteq V$.

(ii) $\Rightarrow$ (iii): clear.

(iii) $\Rightarrow$ (iv): Let $A \subseteq X$ be closed. Then since $X \in \tilde{\mathcal{B}}$ (take $f \equiv 0$), we have $A \subseteq \bigcap_{\tilde{B} \in \tilde{\mathcal{B}}, A \subseteq \tilde{B}} \tilde{B}$. Conversely, let $x \notin A$. We have to show that there exists some $\tilde{B} \in \tilde{\mathcal{B}}$ with $x \notin \tilde{B}$ and $A \subseteq \tilde{B}$. By (iii), $X \setminus A = \bigcup_{i \in I} f_i^{-1}(U_i)$, where each $f_i : X \to \mathbb{R}$ is continuous, and $U_i \subseteq \mathbb{R}$ is open. Thus

$$A = \bigcap_{i \in I} (X \setminus f_i^{-1}(U_i)). \tag{3.1.1}$$

As $x \notin A$, there exists some $k \in I$ with $x \in f_k^{-1}(U_k)$. Also $\mathbb{R}$, being a metric space, is T_{3a}, so there exists a continuous $g : \mathbb{R} \to [0,1]$ with $g(f_k(x)) = 1$ and $g(\mathbb{R} \setminus U_k) \subseteq \{0\}$. Consequently, by (3.1.1) we obtain

$$g \circ f_k(A) \subseteq g \circ f_k(X \setminus f_k^{-1}(U_k)) \subseteq g(\mathbb{R} \setminus U_k) \subseteq \{0\} \text{ and } g \circ f_k(x) = 1.$$

$\Rightarrow x \notin \tilde{B} := (g \circ f_k)^{-1}(\{0\}),\ A \subseteq \tilde{B}$.

(iv) $\Rightarrow$ (i): Let $A \subseteq X$ be closed and $x \notin A$. Then by (iv) there exists some $f : X \to \mathbb{R}$ continuous with $A \subseteq f^{-1}(0)$ and $f(x) \neq 0$. Set $f_1 := |f|$ and

$$g(y) := \min\left(\frac{f_1(y)}{f_1(x)}, 1\right).$$

Then $g : X \to [0,1]$ is continuous, $g(A) \subseteq \{0\}$, and $g(x) = 1$.

(v) $\Rightarrow$ (iv): is clear since $\hat{\mathcal{B}} \subseteq \tilde{\mathcal{B}}$.

(iv) $\Rightarrow$ (v): It suffices to show that for every $f : X \to \mathbb{R}$ continuous there exists some $\tilde{f} : X \to \mathbb{R}$ continuous and bounded with $f^{-1}(0) = \tilde{f}^{-1}(0)$. To this end, since $\arctan : \mathbb{R} \to (\frac{\pi}{2}, \frac{\pi}{2})$ is bijective, it suffices to set $\tilde{f} := \arctan \circ f$. □

3.1.8 Theorem. *Let X be a topological space. TFAE:*

(i) X is T_4.

(ii) If $A \subseteq X$ is closed then for any neighborhood U of A there exists some open O with $A \subseteq O \subseteq \bar{O} \subseteq U$ (i.e.: the closed neighborhoods form a neighborhood basis of A).

Proof. (i) $\Rightarrow$ (ii): Let $A \subseteq X$ be closed, and let U be an open neighborhood of A. Then $X \setminus U$ is closed and disjoint from A. Hence there exist W_1, W_2 open, $W_1 \cap W_2 = \emptyset$, $A \subseteq W_1$, $X \setminus U \subseteq W_2$. Thus $A \subseteq W_1 \subseteq \overline{W_1} \subseteq \overline{X \setminus W_2} = X \setminus W_2 \subseteq U$.

(ii) $\Rightarrow$ (i): Let A_1, A_2 be closed with $A_1 \cap A_2 = \emptyset$. Then A_1 is contained in the open set $X \setminus A_2$, so there exists soem open set O with $A_1 \subseteq O \subseteq \bar{O} \subseteq X \setminus A_2$. Consequently, O, $X \setminus \bar{O}$ are open and disjoint, $A_1 \subseteq O$, and $A_2 \subseteq X \setminus \bar{O}$. □

3.1.9 Definition. *Let X be a T_1-space. X is called* regular *if X is T_3,* completely regular *if X is T_{3a}, and* normal *if X is T_4.*

Then we have the following interrelations between the various separation properties:

$$\begin{tikzcd}
\text{normal} \arrow[r, Rightarrow, "4.1.4"] \arrow[d, Rightarrow, "\text{Def}"'] & \text{compl. reg.} \arrow[r, Rightarrow, "3.1.2\ (iv)"] \arrow[d, Rightarrow, "\text{Def}"'] & \text{reg.} \arrow[r, Rightarrow, "3.1.4"] \arrow[d, Rightarrow, "\text{Def}"'] & T_2 \arrow[r, Rightarrow] & T_1 \arrow[r, Rightarrow] & T_0 \\
T_4 & T_{3a} \arrow[r, Rightarrow, "3.1.2\ (iv)"] & T_3 & & &
\end{tikzcd}$$

3.1.10 Theorem. *Any completely regular space X can be embedded into a product space of the form $\prod_{f \in L} I_f$. Here L is a suitable index set and any I_f is a closed, bounded interval in $\mathbb{R}$.*

Proof. We will in fact show the following stronger statement: Let L be a set of continuous and bounded functions $X \to \mathbb{R}$ with:

(i) L separates points, i.e.: $\forall x, y \in X$ with $x \neq y$ $\exists f \in L$ with $f(x) \neq f(y)$.

(ii) If $\mathcal{B}$ is a basis of the topology of $\mathbb{R}$ then $\{f^{-1}(B)|B \in \mathcal{B},\ f \in L\}$ is a basis of the topology of X.

Such sets L do exist, e.g. $L := \{f : X \to \mathbb{R}| f$ continuous and bounded$\}$: In fact, since X is T_1 and T_{3a}, for $x \neq y$ there exists some $f : X \to [0,1]$ continuous with $f(x) = 0,\ f(y) = 1$, and (ii) follows from 3.1.7 (ii).

We will show that if we pick for every $f \in L$ a closed and bounded interval $I_f \subseteq \mathbb{R}$ with $f(X) \subseteq I_f$ then

$$e : X \longrightarrow \prod_{f \in L} I_f, \quad x \longmapsto (f(x))_{f \in L}$$

is an embedding (cf. 1.1.7).

First, e is injective by (i): $e(x) = e(y) \Rightarrow f(x) = f(y)\ \forall f \in L \Rightarrow x = y$.

e is continuous by 1.1.13: Let $p_g : \prod_{f \in L} I_f \to I_g$ be the projection. Then $p_g \circ e = g$ is continuous for every $g \in L$, so e is continuous.

By 1.1.8 it remains to show that for every U from a basis of the topology of X the set $e(U)$ is open in $e(X)$. By (ii) we have that $\{g^{-1}(V)|g \in L,\ V \subseteq \mathbb{R}$ open$\}$ is a basis for X. Thus let $U := g^{-1}(V)$. Then (viewing p_g as a map into $\mathbb{R}$):

$$e^{-1}(p_g^{-1}(V)) = (p_g \circ e)^{-1}(V) = g^{-1}(V) = U \Rightarrow e(U) = p_g^{-1}(V) \cap e(X).$$

Hence $e(U)$ is open in $e(X)$. $\square$

3.1.11 Remark. In fact, every I_f can be chosen to be $[0,1]$: Indeed, for every $f \in L$ there exists a homeomorphism $h_f : I_f \to [0,1]$ (composing a translation with a dilation). Then $(x_f)_{f \in L} \longmapsto (h_f(x_f))_{f \in L}$ is a homeomorphism $\prod_{f \in L} I_f \to [0,1]^L$ (cf. 1.1.14).

3.2 Inheritability of separation properties

3.2.1 Theorem. *Any subspace of a T_i-space ($i \in \{0,1,2,3,3a\}$) is itself T_i. In particular, any subspace of a (completely) regular space is (completely) regular.*

Proof. To begin with, let X be T_3, $Y \subseteq X$ and A closed in Y. Then there exists some $B \subseteq X$ closed with $A = Y \cap B$. If $y \in Y \setminus A \subseteq X \setminus B$, then there exist $U,\ V$ open in X such that $U \cap V = \emptyset,\ y \in U,\ B \subseteq V$. Hence $U \cap Y,\ V \cap Y$ are disjoint and open (in Y) neighborhoods of y resp. A.

The cases $i = 0, 1, 2$ follow in the same way.

Finally, let X be T_{3a}. Then by the above it follows that there exists some continuous $f : X \to [0,1]$ such that $f(y) = 1,\ f|_B \equiv 0$. Hence $g := f|_Y : Y \to [0,1]$ is continuous, $g(y) = 1$, and $g|_A \equiv 0$. $\square$

On the other hand, general subspaces of normal spaces do not have to be normal. We have, however:

3.2.2 Theorem. *Any closed subset of a normal (resp. T_4-) space is normal (resp. T_4).*

Proof. Let $Y \subseteq X$ be closed, $A,\ B \subseteq Y$ closed, $A \cap B = \emptyset$. Since Y is closed, $A,\ B$ are closed in X. Hence there are $U,\ V$ open in X, $A \subseteq U,\ B \subseteq V,\ U \cap V = \emptyset$, and so $U \cap Y,\ V \cap Y$ are disjoint neighborhoods of $A,\ B$ in Y. $\square$

3.2.3 Theorem. *Let $(X_k)_{k\in I}$ be a family of topological spaces and let $i \in \{0,1,2,3,3a\}$. TFAE:*

(i) $\prod_{k\in I} X_k$ *is* T_i.

(ii) All X_k *are* T_i.

Analogous equivalences therefore hold for (completely) regular spaces.

Proof. (i) $\Rightarrow$ (ii): Fix $k_0 \in I$ and for $j \neq k_0$ pick any $x_j \in X_j$. Then

$$f : x \mapsto (x_k)_{k\in I}, \quad x_k := \begin{cases} x_j & k = j \neq k_0 \\ x & k = k_0 \end{cases}$$

is a homeomorphism from X_{k_0} onto the subspace $\prod_{k\in I} Y_k$ of $\prod_{k\in I} X_k$ (cf. 1.1.14), where

$$Y_k = \begin{cases} \{x_k\} & k \neq k_0 \\ X_{k_0} & k = k_0 \end{cases} .$$

As this space is T_i by 3.2.1, the claim follows.

(ii)$\Rightarrow$ (i): We suppose that all X_k are T_3 (the cases T_0, T_1, T_2 follow analogously). We use 3.1.6: Let $x = (x_k)_{k\in I} \in X = \prod_{k\in I} X_k$ and let $U \in \mathcal{U}(x)$. Then by 1.1.10 there exists a finite subset K of I and $U_k \in \mathcal{U}(x_k)$ such that $\bigcap_{k\in K} p_k^{-1}(U_k) \subseteq U$. Now 3.1.6 $\Rightarrow \forall k \in K\ \exists A_k$ closed $\in \mathcal{U}(x_k)$ with $A_k \subseteq U_k$, so $\bigcap_{k\in K} p_k^{-1}(A_k)$ is a closed neighborhood of x in U.

Suppose now that all X_k are T_{3a}, $A \subseteq X$ closed, and $x = (x_k)_{k\in I} \notin A$. Then there exists some $U = \bigcap_{k\in K} p_k^{-1}(U_k) \in \mathcal{U}(x)$, $K \subseteq I$ finite, with $U \cap A = \emptyset$. For each $k \in K$ there exists some $f_k : X_k \to [0,1]$ continuous with $f_k(x_k) = 1$, $f_k|_{X_k\setminus U_k} \equiv 0$. Let $g : \prod_{k\in I} X_k \to [0,1]$, $(y_k)_{k\in I} \mapsto \min\{f_k(y_k)|k \in K\}$. Thus $g = \min\{f_k \circ p_k | k \in K\}$, so g is continuous: $X \to [0,1]$. Moreover, $g(x) = 1$ and $g|_{X\setminus U} \equiv 0$, so $g|_A \equiv 0$. □

Note, however, that products of normal spaces need not be normal.

Finally, we turn to inheritability of separation properties by quotients. These are, generally speaking, rather poor.

3.2.4 Example. Let $X := [0,1] \subseteq \mathbb{R}$ and denote by $\sim$ the equivalence relation

$$x \sim y :\Leftrightarrow x = y \text{ or } x, y \in \mathbb{Q} \cap [0,1].$$

X is metric, so it satisfies every T_i ($i \in \{0,1,2,3,3a,4\}$) by 3.1.3. We set $Y := X/\sim$.

Y is T_0: Let $p(x) \neq p(y) \in X/\sim$. Then $x \neq y$ and (w.l.o.g.) $x \notin \mathbb{Q}$. Let $U := Y \setminus \{p(x)\}$. Then $p(y) \in U$ and $p^{-1}(U) = [0,1]\setminus\{x\}$ is open $\Rightarrow U \in \mathcal{U}(p(y))$, $p(x) \notin U$.

However, Y is *not* T_i for $i \in \{1,2,3,3a,4\}$:

Y is not $\mathbf{T_1}$: Let $x \in [0,1]\cap\mathbb{Q} \Rightarrow p^{-1}(p(x)) = [0,1]\cap\mathbb{Q}$, which is not closed in $[0,1]$ $\Rightarrow \{p(x)\}$ not closed in y. Consequently also:

Y is not $\mathbf{T_2}$

Y is not $\mathbf{T_3}$: Let $y \in [0,1] \setminus \mathbb{Q} \Rightarrow \{p(y)\}$ is closed in Y since $p^{-1}(p(y)) = y$ is closed in $[0,1]$. Let $x \in [0,1] \cap \mathbb{Q} \Rightarrow p(x) \notin \{p(y)\}$, but every neighborhood of $p(y)$ contains $p(x)$: let $U \in \mathcal{U}(p(y))$ be open, then $p^{-1}(U)$ is open in $[0,1]$ and contains y, so it must also contain some $z \in \mathbb{Q}$. Therefore, U contains $p(z) = p(x)$. In particular:

Y is not $\mathbf{T_{3a}}$

Y is not $\mathbf{T_4}$: Let $x \neq y \in [0,1] \setminus \mathbb{Q} \Rightarrow \{p(x)\}, \{p(y)\}$ are closed and disjoint, but if U, V are neighborhoods of $\{p(x)\}, \{p(y)\}$ then both contain $p(\mathbb{Q})$, hence are not disjoint.

The following theorem states conditions under which separation properties are inherited by quotient spaces. To formulate it we need the following auxiliary result on closed maps:

3.2.5 Lemma. *Let X, Y be topological spaces and $f : X \to Y$ a closed map. Let $B \subseteq Y$ and let $U \subseteq X$ be an open neighborhood of $f^{-1}(B)$. Then there exists an open W in Y with $B \subseteq W$ and $f^{-1}(W) \subseteq U$.*

Proof. We have $X \setminus U \subseteq X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$, so $f(X \setminus U)$ is closed and $\subseteq Y \setminus B$. $\Rightarrow W := Y \setminus f(X \setminus U)$ is open and $\supseteq B$. It remains to show that $f^{-1}(W) \subseteq U$: $f^{-1}(W) = f^{-1}(Y \setminus f(X \setminus U)) = X \setminus f^{-1}(f(X \setminus U)) \subseteq X \setminus (X \setminus U) = U$. □

3.2.6 Theorem. *Let $\sim$ be an equivalence relation on a topological space X, let $p : X \to X/\sim =: Y$ be the canonical projection and set $R := \{(x, y) \in X \times X | x \sim y\}$. Then:*

(i) $X/\sim$ is T_1 if and only if every equivalence class in X is closed.

(ii) If $X/\sim$ is T_2 then R is closed in $X \times X$.

(iii) If p is open then $X/\sim$ is T_2 if and only if R is closed $X \times X$.

(iv) If X is regular and p is open and closed, then $X/\sim$ is T_2.

(v) Let X be regular and $A \subseteq X$ closed. If $\sim$ is the equivalence relation $x \sim y :\Leftrightarrow x = y$ or $x, y \in A$, then $X/\sim$ is T_2.

(vi) If X is normal (resp. T_4) and p is closed, then also $X/\sim$ is normal (resp. T_4).

Proof.

(i) $X/\sim$ is $T_1 \Leftrightarrow p(x)$ closed in $X/\sim$ $\forall x \in X \Leftrightarrow p^{-1}(p(x))$ closed in X $\forall x$.

(ii) We have $R = (p \times p)^{-1}(\Delta_Y)$, where $p \times p : X \times X \to Y \times Y$ and $\Delta_Y = \{(y, y) | y \in Y\}$. Since Y is T_2, by 3.1.5 (iv) Δ_Y is closed, hence so is R.

(iii) ($\Rightarrow$): follows from (ii)

($\Leftarrow$): Let $p(x) \neq p(y) \in Y$. Then $(x, y) \in (X \times X) \setminus R \Rightarrow \exists$ neighborhood $U \times V$ of (x, y) with $(U \times V) \cap R = \emptyset$. Thus, since p is open, $p(U)$, $p(V)$ are neighborhoods of $p(x)$, $p(y)$ in Y, and $p(U) \cap p(V) = \emptyset$, since $p(u) = p(v)$ would entail $(u, v) \in R$.

(iv) By (iii) it suffices to show that R is closed in $X \times X$. Thus let $(x, y) \in (X \times X) \setminus R$. Then $x \notin p^{-1}(p(y))$, which is closed since X is T_1 and p is closed and continuous. As X is T_3, there exist U, V open and disjoint with $x \in U$ and $p^{-1}(p(y)) \subseteq V$. Again since p is closed, by 3.2.5 there exists an open neighborhood W of $p(y)$ with $p^{-1}(p(y)) \subseteq p^{-1}(W) \subseteq V$. Therefore, $U \times p^{-1}(W)$ is a neighborhood of (x, y) and $(U \times p^{-1}(W)) \cap R = \emptyset$: in fact, if $(u, z) \in R$, where $u \in U$, $z \in p^{-1}(W) \Rightarrow p(u) = p(z) \in W \Rightarrow u \in p^{-1}(W) \subseteq V$, a contradiction to $U \cap V = \emptyset$. Hence R is closed.

(v) Let $p(x) \neq p(y) \in Y$, $x \in A$, $y \notin A$ (the case $x, y \in A$, $x \neq y$ is treated similarly using that X is T_1 and T_3, hence T_2). Since X is T_3, there exist U, V open in X, $U \cap V = \emptyset$, $A \subseteq U$, $y \in V$. Then $p(U) \cap p(V) = \emptyset$: if $p(u) = p(v)$, then $u = v$ or $u, v \in A$, a contradiction. Also, $p(x) \in p(U)$, $p(y) \in p(V)$. It remains to show that $p(U)$, $p(V)$ are open. To see this, note that since $A \subseteq U$, $p^{-1}(p(U)) = U$ and since $V \subseteq X \setminus U$, $p^{-1}(p(V)) = V$.

(vi) Let X T_4, A, $B \subseteq Y$ closed, $A \cap B = \emptyset$. Then $p^{-1}(A)$, $p^{-1}(B)$ are closed, $p^{-1}(A) \cap p^{-1}(B) = \emptyset$, and so there exist U, V open and disjoint, $p^{-1}(A) \subseteq U$, $p^{-1}(B) \subseteq V$. By 3.2.5 it follows that there exist W_1, W_2 open in Y, $A \subseteq W_1$, $B \subseteq W_2$, $p^{-1}(W_1) \subseteq U$, $p^{-1}(W_2) \subseteq V$. Hence $\Rightarrow p^{-1}(W_1 \cap W_2) = p^{-1}(W_1) \cap p^{-1}(W_2) \subseteq U \cap V = \emptyset$, and thereby $W_1 \cap W_2 = p(p^{-1}(W_1 \cap W_2)) = \emptyset$, so Y is indeed T_4.

If, finally, X is normal (i.e., T_4 and T_1), then it remains to show that Y is T_1. Let $p(x) \in Y$. Since X is T_1, $\{x\}$ is closed, hence $\{p(x)\}$ is closed, giving the claim by 3.1.4.

□

3.3 Extension by continuity

3.3.1 Theorem. *Let X, Y be topological spaces, Y T_2, f, $g : X \to Y$ continuous. Then:*

(i) $\{x \in X | f(x) = g(x)\}$ *is closed in* X.

(ii) If $D \subseteq X$ *is dense in* X *and* $f|_D = g|_D$, *then* $f = g$.

(iii) The graph of f, $\Gamma_f = \{(x, f(x)) | x \in X\}$ *is closed in* $X \times Y$.

(iv) If f is injective, then X is Hausdorff.

Proof. (i) Let $h : X \longrightarrow Y \times Y$, $h(x) := (f(x), g(x))$. Then h is continuous and therefore $\{x \in X | f(x) = g(x)\} = h^{-1}(\Delta_Y)$ is closed by 3.1.5 (iv).
(ii) $A := \{x \in X | f(x) = g(x)\}$ is closed and $A \supseteq D \Rightarrow A \supseteq \bar{D} = X$.
(iii) $f \times \mathrm{id} : X \times Y \to Y \times Y$ is continuous and $\Gamma_f = (f \times \mathrm{id})^{-1}(\Delta_Y)$.
(iv) Let $x_1 \neq x_2 \in X$. Then $f(x_1) \neq f(x_2) \Rightarrow \exists V_1 \in \mathcal{U}(f(x_1))$, $V_2 \in \mathcal{U}(f(x_2))$, $V_1 \cap V_2 = \emptyset \Rightarrow f^{-1}(V_1) \in \mathcal{U}(x_1)$, $f^{-1}(V_2) \in \mathcal{U}(x_2)$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) = \emptyset$. □

3.3.2 Corollary. *Let X, Y be topological spaces, Y T_2 and $f : X \to Y$ continuous. Let $\sim$ be the equivalence relation $x \sim x' :\Leftrightarrow f(x) = f(x')$. Then $X/\!\sim$ is T_2.*

Proof. In Section 1.4 we derived the decomposition

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"] & Y \\ X/\!\sim \arrow[r, "\bar{f}"'] & f(X) \arrow[u, "j"'] \end{tikzcd}$$

Here, $j \circ \bar{f} : X/\!\sim \to Y$ is continuous and injective, and Y is T_2, so by 3.3.1 (iv), $X/\!\sim$ is T_2. □

If $D \subseteq X$ is dense, Y is T_2 and $f : D \to X$ is continuous, then by 3.3.1 f can be extended in at most one way to a continuous function $F : X \to Y$. However, such an extension in general need not exist! As an example, consider $f(x) = \frac{1}{x}$ on $D = (0,1) \subseteq [0,1]$. The following result provides a necessary and sufficient condition for the existence of a continuous extension.

3.3.3 Theorem. *Let X be a topological space, $D \subseteq X$ dense in X, Y regular, and $f : D \to Y$ a map (a priori not assumed to be continuous). TFAE:*

(i) $\exists F : X \to Y$ *continuous with* $F|_D = f$.

(ii) $\forall x \in X$ *there exists* $\lim\limits_{\substack{y \to x \\ y \in D}} f(y)$.

F then is uniquely determined.

Proof. (i)$\Rightarrow$ (ii): Since F is continuous, by 2.3.13 and 2.3.16 (i) we have for every $x \in X$:

$$F(x) = \lim_{y \to x} F(y) = \lim_{\substack{y \to x \\ y \in D}} F(y) = \lim_{\substack{y \to x \\ y \in D}} f(y).$$

(ii)$\Rightarrow$ (i): Let $F(x) := \lim\limits_{\substack{y \to x \\ y \in D}} f(y)$ (this is well-defined since Y is T_2).

F is continuous: Let V be a closed neighborhood of $F(x)$ (by 3.1.6 these form a neighborhood basis). Then there exists some open $U \in \mathcal{U}(x)$ with $f(U \cap D) \subseteq V$. Let $z \in U$. Then $U \in \mathcal{U}(z)$ and by 2.3.16 (ii) and the remark following 2.3.15 we obtain:

$$F(z) = \lim_{\substack{y \to z \\ y \in D}} f(y) = \lim_{\substack{y \to z \\ y \in D \cap U}} f(y) \in \overline{f(D \cap U)} \subseteq \bar{V} = V.$$

Consequently, $F(U) \subseteq V$, so F is continuous.

$F|_D = f$: to see this, we show that for any $x \in D$ we have $f(x) = \lim\limits_{\substack{y \to x \\ y \in D}} f(y) =: a$. In fact, suppose that $f(x) \neq a$. Then since X is T_1, there exists some $V \in \mathcal{U}(a)$ with $f(x) \notin V$. However, by definition of a there exists some $U \in \mathcal{U}(x)$ with $f(U \cap D) \subseteq V$. But then also $f(x) \in V$, a contradiction.

F is unique: Since Y is regular, it is also T_2, so the claim follows from 3.3.1 (ii). $\square$

Chapter 4

Normal spaces

In general, a topological space need not possess many continuous functions. E.g., on a space with the indiscrete topology only the constant functions are continuous. On normal spaces, however, there are in fact many continuous functions, as we shall see in this section.

4.1 Urysohn's lemma

4.1.1 Lemma. *Let (X,d) be a metric space and let $A,\ B \subseteq X$ be closed, $A \cap B = \emptyset$. Then there exists a continuous function $f : X \to [0,1]$ with $f|_A \equiv 0,\ f|_B \equiv 1$.*

Proof. Let $d(x,A) := \inf_{a \in A} d(x,a)$. Then it suffices to set $f(x) := \frac{d(x,A)}{d(x,A)+d(x,B)}$. □

This property is characteristic of T_4-spaces:

4.1.2 Theorem. *(Urysohn): Let X be a topological space. TFAE:*

(i) X is a T_4-space.

(ii) $\forall A,\ B \subseteq X$ closed with $A \cap B = \emptyset$ there exists a continuous map $f : X \to [0,1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.

Proof. (ii)⇒ (i): The sets $U := f^{-1}([0, \frac{1}{2}))$ and $V := f^{-1}((\frac{1}{2}, 1])$ are disjoint open neighborhoods of A and B.
(i)⇒ (ii): Let A, B be closed, $A \cap B = \emptyset$. Then $G_1 := X \setminus B$ is an open neighborhood of A. By 3.1.8 there exists an open neighborhood G_0 of A with $\overline{G_0} \subseteq G_1$. Again by 3.1.8 there exists an open neighborhood $G_{\frac{1}{2}}$ of $\overline{G_0}$ with $\overline{G_{\frac{1}{2}}} \subseteq G_1$, hence:

$$\overline{G_0} \subseteq G_{\frac{1}{2}}, \quad \overline{G_{\frac{1}{2}}} \subseteq G_1.$$

By 3.1.8 there is an open neighborhood $G_{\frac{1}{4}}$ of $\overline{G_0}$ with $\overline{G_{\frac{1}{4}}} \subseteq G_{\frac{1}{2}}$ and $\exists G_{\frac{3}{4}}$ open neighborhood of $\overline{G_{\frac{1}{2}}}$ with $\overline{G_{\frac{3}{4}}} \subseteq G_1$, i.e:

$$\overline{G_0} \subseteq G_{\frac{1}{2^2}}, \quad \overline{G_{\frac{1}{2^2}}} \subseteq G_{\frac{2}{2^2}}, \quad \overline{G_{\frac{2}{2^2}}} \subseteq G_{\frac{3}{2^2}}, \quad \overline{G_{\frac{3}{2^2}}} \subseteq G_1.$$

Iterating this procedure, we obtain:

$$\overline{G_{\frac{k}{2^n}}} \subseteq G_{\frac{k+1}{2^n}}, \quad k = 0,1,2,\dots,2^n-1,\ G_{\frac{j}{2^n}} \text{ open for } j = 0,1,\dots,2^n. \tag{4.1.1}$$

Next we show that for all dyadic numbers $r < r' \in [0,1] \cap \{\frac{k}{2^n} | k, n \in \mathbb{N}\} =: D$ we have

$$\overline{G_r} \subseteq G_{r'}$$

To see this, note that without loss of generality (expanding fractions) we can assume that $r = \frac{k}{2^n}$ and $r' = \frac{k'}{2^n}$. Since $r < r'$ it follows that $k < k'$, so (4.1.1) yields the claim.

For any $t \in [0,1]$ we set

$$G_t := \bigcup_{\substack{r \in D \\ r \leq t}} G_r.$$

Then G_t is an open set and for $t,\ t' \in [0,1]$ we have:

$$\overline{G_t} \subseteq G_{t'} \text{ if } t < t'. \tag{4.1.2}$$

Indeed, pick $n \in \mathbb{N}$ with $\frac{1}{2^n} < \frac{1}{2}(t'-t) \Rightarrow 2\frac{1}{2^n} < t'-t \Rightarrow \exists k \in \mathbb{N}$ with:

$$\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \subseteq (t,t'), \text{ i.e.: } t < \frac{k}{2^n} < \frac{k+1}{2^n} < t'.$$

Thus $\overline{G_t} \subseteq \overline{G_{\frac{k}{2^n}}} \subseteq G_{\frac{k+1}{2^n}} \subseteq G_{t'}$, which gives (4.1.2).

In addition, we set $G_t := X$ for $t > 1$ and $G_t := \emptyset$ for $t < 0$. Then (4.1.2) holds for $t, t' \in \mathbb{R},\ t < t'$.

For $x \in X$ let

$$f(x) := \inf\{t \in \mathbb{R} \mid x \in G_t\}.$$

Since $G_t = X$ for $t > 1$, $f(x) \leq 1$ for all x and since $G_t = \emptyset$ for $t < 0$, $f(x) \geq 0$. Also, $A \subseteq G_0$ implies $f|_A \equiv 0$, and $B = X \setminus G_1$, together with $f \leq 1$ gives $f|_B \equiv 1$.

It remains to show that $f : X \to \mathbb{R}$ is continuous.

Let $x_0 \in X$ and $\varepsilon > 0$. Then it suffices to show that there exists some $U \in \mathcal{U}(x_0)$ with $|f(x) - f(x_0)| \leq \varepsilon\ \forall x \in U$. For $x \in G_{f(x_0)+\varepsilon}$ we have $f(x) \leq f(x_0) + \varepsilon$ and for $x \in X \setminus \overline{G_{f(x_0)-\varepsilon}}$ we get $f(x) \geq f(x_0) - \varepsilon$ (in fact, $f(x) < f(x_0) - \varepsilon$ would imply $x \in G_{f(x_0)-\varepsilon}$). Hence for $x \in U := G_{f(x_0)+\varepsilon} \setminus \overline{G_{f(x_0)-\varepsilon}}$ we get $f(x_0) - \varepsilon \leq f(x) \leq f(x_0) + \varepsilon$, i.e.: $|f(x) - f(x_0)| \leq \varepsilon$.

Finally, $U \in \mathcal{U}(x_0)$: U is open and $x_0 \in U$, since we have

$$f(x_0) < f(x_0) + \varepsilon \Rightarrow x_0 \in G_{f(x_0)+\varepsilon}, \quad \text{and}$$
$$f(x_0) - \frac{\varepsilon}{2} < f(x_0) \Rightarrow x_0 \notin G_{f(x_0)-\frac{\varepsilon}{2}} \supseteq \overline{G_{f(x_0)-\varepsilon}}.$$

□

4.1.3 Corollary. *Any metric space X is normal.*

Proof. X is clearly T_1. By 4.1.1 and 4.1.2 it is also T_4. See also 3.1.3. □

4.1.4 Corollary. *Any normal space is completely regular.*

Proof. Immediate from 4.1.2 and the definition of completely regular ($T_1 + T_{3a}$). □

In the following we want to characterize zero sets $f^{-1}(0)$ of continuous functions $f : X \to \mathbb{R}$. □

4.1.5 Definition. *Let X be a topological space. $A \subseteq X$ is called a G_δ-set if A is the intersection of countably many open sets: $A = \bigcap_{i=1}^\infty G_i,\ G_i$ open. $B \subseteq X$ is called an F_σ-set if B is the union of countably many closed sets: $B = \bigcup_{i=1}^\infty F_i,\ F_i$ closed.*

Here G stands for 'Gebiet', F for 'fermé' (closed), σ for 'sum' (union), δ for 'intersection'.

4.1.6 Proposition. *Let X be a T_4-space and $\emptyset \neq A \subseteq X$ closed. TFAE:*

(i) $\exists f : X \to [0,1]$ *continuous with* $f^{-1}(\{0\}) = A$.

(ii) *A is a G_δ-set.*

Proof. (i)$\Rightarrow$ (ii):

$$A = f^{-1}(\{0\}) = f^{-1}(\bigcap_{n=1}^{\infty}(-\frac{1}{n},\frac{1}{n})) = \bigcap_{n=1}^{\infty} f^{-1}((-\frac{1}{n},\frac{1}{n})).$$

(ii)$\Rightarrow$ (i): Let $A = \bigcap_{i=0}^{\infty} G_i,\ G_i$ open. 4.1.2 $\Rightarrow \forall i \in \mathbb{N}\ \exists h_i : X \to [0,1]$ continuous with $h_i(A) = \{0\}$ and $h_i(X \setminus G_i) = \{1\}$. Let $f_n := \sum_{i=0}^{n} 2^{-i} h_i$. Then for every $x \in X$, $(f_n(x))$ is a Cauchy sequence, hence converges to some $f(x)$. Furthermore, $f_n \to f$ uniformly on X and therefore f is continuous: Let $\varepsilon > 0$, $x_0 \in X$ and n_0 such that $|f_{n_0}(x) - f(x)| < \frac{\varepsilon}{3}\ \forall x \in X$. Let $U \in \mathcal{U}(x_0)$ be such that $|f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{3}\ \forall x \in U$. Then for all $x \in U$:

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| < \varepsilon.$$

Moreover, $f(x) = 0 \Leftrightarrow h_i(x) = 0\ \forall i \Rightarrow x \notin X \setminus G_i\ \forall i \Rightarrow x \in \bigcap_{i=0}^{\infty} G_i = A$. Conversely, $x \in A \Rightarrow h_i(x) = 0\ \forall i \Rightarrow f(x) = 0$. Thus $A = f^{-1}(\{0\})$. $\square$

Based on this result we can strengthen Urysohn's Lemma as follows:

4.1.7 Proposition. *Let X be T_4, A, B closed and nonempty, $A \cap B = \emptyset$. If A resp. B is a G_δ-set, then there exists a continuous map $f : X \to [0,1]$ with $f(A) = \{0\}$ and $f(B) = \{1\}$ such that $f(x) \neq 0\ \forall x \in X \setminus A$ resp. $f(x) \neq 1\ \forall x \in X \setminus B$. If both A and B are G_δ-sets, then there exists some continuous $f : X \to [0,1]$ with $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$.*

Proof. Let A be G_δ. Then by 4.1.6 there exists some $g : X \to [0,1]$ continuous with $g^{-1}(\{0\}) = A$. Also, by 4.1.2 there exists some $f : X \to [0,1]$ continuous with $f(A) = \{0\}$, $f(B) = \{1\}$. Let $h := \max\{f, g\}$. Then $h : X \to [0,1]$ is continuous with $h(A) = \{0\}$, $h(B) = \{1\}$ and $h(x) \neq 0\ \forall x \notin A$.

Let B be G_δ. Then by 4.1.6 there exists some $k : X \to [0,1]$ continuous with $k^{-1}(\{1\}) = B$. Then $l := \min(f,k)$ is continuous $X \to [0,1]$, $l(A) = \{0\}$, $l(B) = \{1\}$ and $l \neq 1$ on $X \setminus B$.

Finally, suppose that both A and B are G_δ, and set $e := \frac{1}{2}(h + \min(f,k))$. Then $e : X \to [0,1]$ is continuous, and

$$e(x) = 0 \Leftrightarrow \underbrace{h(x) = 0}_{\Leftrightarrow f(x)=0 \wedge \underbrace{g(x)=0}_{\Leftrightarrow x \in A}} \wedge (f(x) = 0 \vee k(x) = 0) \Leftrightarrow x \in A,$$

$$e(x) = 1 \Leftrightarrow h(x) = 1 \wedge f(x) = 1 \wedge \underbrace{k(x) = 1}_{\Leftrightarrow x \in B} \Leftrightarrow x \in B.$$

$\square$

4.2 Extension of continuous maps

Urysohn's Lemma 4.1.2 can also be read as follows: X is T_4 if and only if for all A, B closed with $A \cap B = \emptyset$ the function $f : A \cup B \to [0,1]$, $f|_A = 0, f|_B = 1$ can be extended continuously to all of X. (As A and B are open in $A \cup B$, f is continuous on $A \cup B$ in the trace topology) . Hence 4.1.2 is a special case of the following result:

4.2.1 Theorem. *(Tietze) Let X be a topological space. TFAE:*

(i) X *is* T_4.

(ii) Let $A \subseteq X$ be closed and $f : A \to \mathbb{R}$ continuous. Then there exists a continuous $F : X \to \mathbb{R}$ with $F|_A = f$.

For the proof of 4.2.1 we require two auxilliary results:

4.2.2 Lemma. *Let X be T_4 and $A \subseteq X$ closed. If $f : A \to [-1,1]$ is continuous, then there exists a sequence (g_n) of continuous functions $g_n : X \to \mathbb{R}$ with:*

(i) $-1 + (\frac{2}{3})^n \le g_n(x) \le 1 - (\frac{2}{3})^n \quad \forall x \in X,$

(ii) $|f(x) - g_n(x)| \le (\frac{2}{3})^n \quad \forall x \in A,$

(iii) $|g_{n+1}(x) - g_n(x)| \le \frac{1}{3}(\frac{2}{3})^n \quad \forall x \in X,$

(iv) $|g_n(x) - g_m(x)| \le (\frac{2}{3})^p \quad \forall x \in X\ \forall m, n \ge p.$

Proof. Proceeding by induction, we first set $g_0(x) = 0\ \forall x \in X$, so g_0 satisfies (i), (ii). Suppose that $g_0, \dots, g_n$ have already been defined in such a way that (i),(ii),(iii) are satisfied. Let

$$B_{n+1} := \left\{ x \in A \mid f(x) - g_n(x) \ge \frac{1}{3}\left(\frac{2}{3}\right)^n \right\}$$
$$C_{n+1} := \left\{ x \in A \mid f(x) - g_n(x) \le -\frac{1}{3}\left(\frac{2}{3}\right)^n \right\}.$$

Then $B_{n+1},\ C_{n+1}$ are closed and disjoint. By 4.1.2 there exists a continuous function $v_n : X \to [-\frac{1}{3}(\frac{2}{3})^n, \frac{1}{3}(\frac{2}{3})^n]$ with $v_n(B_{n+1}) = \{-\frac{1}{3}(\frac{2}{3})^n\},\ v_n(C_{n+1}) = \{\frac{1}{3}(\frac{2}{3})^n\}$. Let $g_{n+1} := g_n - v_n$. Then by (i) for g_n, we get

$$-1 + \underbrace{\left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^n}_{=(\frac{2}{3})^{n+1}} \le g_n - v_n = g_{n+1} \le 1 \underbrace{- \left(\frac{2}{3}\right)^n + \frac{1}{3}\left(\frac{2}{3}\right)^n}_{=-(\frac{2}{3})^{n+1}} \Rightarrow \text{ (i) for } g_{n+1}$$

and:

$$|f(x) - g_{n+1}(x)| \le |f(x) + g_n(x)| + |v_n(x)| =: (*)$$

There are now two possibilites:

1.) $|f(x) - g_n(x)| < \frac{1}{3}(\frac{2}{3})^n \Rightarrow (*) \le \frac{2}{3}(\frac{2}{3})^n = (\frac{2}{3})^{n+1}$.

2.) $|f(x) - g_n(x)| \ge \frac{1}{3}(\frac{2}{3})^n \Rightarrow x \in B_{n+1}$ or $x \in C_{n+1}$.

  a) Let $x \in B_{n+1} \Rightarrow f(x) - g_n(x) \ge \frac{1}{3}(\frac{2}{3})^n$. By induction hypothesis (ii) we have $f(x) - g_n(x) \le (\frac{2}{3})^n$. Due to $v_n|_{B_{n+1}} = -\frac{1}{3}(\frac{2}{3})^n$ we get:

$$f(x) - g_{n+1}(x) = f(x) - g_n(x) - \frac{1}{3}\left(\frac{2}{3}\right)^n \in \left[0, \frac{2}{3} \cdot \left(\frac{2}{3}\right)^n\right]$$

  $\Rightarrow |f(x) - g_{n+1}(x)| \le (\frac{2}{3})^{n+1}$.

b) Let $x \in C_{n+1} \Rightarrow f(x) - g_n(x) \le -\frac{1}{3}(\frac{2}{3})^n$. Also, by induction hypothesis (ii), $f(x) - g_n(x) \ge -(\frac{2}{3})^n$. Now $v_n|_{C_{n+1}} = \frac{1}{3}(\frac{2}{3})^n$, so:

$$f(x) - g_{n+1}(x) = f(x) - g_n(x) + \frac{1}{3}\left(\frac{2}{3}\right)^n \in \left[-\frac{2}{3}\left(\frac{2}{3}\right)^n, 0\right].$$

$\Rightarrow |f(x) - g_{n+1}(x)| \le (\frac{2}{3})^{n+1} \Rightarrow$ (ii) for g_{n+1}.

(iii) follows from $|g_{n+1} - g_n| \le |v_n| \le \frac{1}{3}(\frac{2}{3})^n$.

(iv) By (iii),

$$\begin{aligned} |g_{n+k}(x) - g_n(x)| &\le \sum_{i=1}^{k} |g_{n+i} - g_{n+i-1}(x)| \\ &\overset{(iii)}{\le} \frac{1}{3}\left(\frac{2}{3}\right)^n \underbrace{\sum_{i=1}^{k}\left(\frac{2}{3}\right)^{i-1}}_{<(1-\frac{2}{3})^{-1}=3} < \left(\frac{2}{3}\right)^n \end{aligned}$$

Let $n \ge m \ge p \Rightarrow |g_n(x_0) - g_m(x)| \le (\frac{2}{3})^m \le \left(\frac{2}{3}\right)^p \Rightarrow$ (iv).

□

4.2.3 Lemma. *Let X be T_4 and $A \subseteq X$ closed. Then any continuous function $f : A \to (-1, 1)$ can be extended to a continuous function $F : X \to (-1, 1)$, and analogously for $f : A \to [-1, 1]$.*

Proof. Let $(g_n)_{n\in\mathbb{N}}$ be as in 4.2.2. By 4.2.2 (iv) g_n converges uniformly. Thus $\tilde{F}(x) := \lim_{n\to\infty} g_n(x)$ is continuous on X (cf. the proof of 4.1.6). Let $x \in A$. Then

$$|f(x) - \tilde{F}(x)| = |f(x) - \lim_{n\to\infty} g_n(x)| = \lim_{n\to\infty} |f(x) - g_n(x)| = 0$$

by 4.2.2 (ii). Hence $\tilde{F}|_A = f$. Also, 4.2.2 (i) implies $-1 \le \tilde{F}(x) \le 1 \ \forall x \in X$. $\Rightarrow \tilde{F} : X \to [1, 1]$ (which gives the claim for $f : A \to [-1, 1]$). Let $B := \{x \in X | |\tilde{F}(x)| = 1\}$. Then B is closed and $A \cap B = \emptyset$, so by 4.1.2 there exists some continuous $g : X \to [0, 1]$ with $g(A) = \{1\}$, $g(B) = \{0\}$. Then $F := \tilde{F} \cdot g$ is continuous, $F|_A = f$ and $|F(x)| < 1 \ \forall x \in X$. □

After these preparations we are now ready to give the

Proof of 4.2.1: (i) $\Rightarrow$ (ii): Let $A \subseteq X$ be closed, $f : A \to \mathbb{R}$ continuous. Let $h : \mathbb{R} \to (-1, 1)$ be a homeomorphism. (e.g.: $h(x) := \frac{x}{\sqrt{1+x^2}}$). By 4.2.3 there exists a continuous extension $\tilde{F} : X \to (-1, 1)$ of $\tilde{f} := h \circ f$. Then $F := h^{-1} \circ \tilde{F}$ is continuous: $X \to \mathbb{R}$ and $F|_A = h^{-1} \circ \tilde{f} = f$.

(ii) $\Rightarrow$ (i): Let A, $B \subseteq X$ be closed and disjoint. Extend $f : A \cup B \to [0, 1]$, $f|_A = 0$, $f|_B = 1$ continuously to $F : X \to [0, 1]$. Then $F^{-1}((-\infty, \frac{1}{2}))$ and $F^{-1}((\frac{1}{2}, \infty))$ are disjoint open neighborhoods of A and B. □

4.2.4 Lemma. *Let A be a closed G_δ-set in a T_4-space X. Then any continuous function $f : A \to [-1, 1]$ can be extended to a continuous function $F : X \to [-1, 1]$ such that $|F(x)| < 1 \ \forall x \notin A$.*

Proof. By 4.2.3 there exists some $\tilde{F} : X \to [-1, 1]$ continuous with $\tilde{F}|_A = f$. By 4.1.6 there exists some $g : X \to \mathbb{R}$ continuous such that $A = g^{-1}(\{0\})$. Let $F(x) := \frac{\tilde{F}(x)}{1+|g(x)|} \Rightarrow |F(x)| \le |\tilde{F}(x)| \le 1$, $F|_A = \tilde{F}|_A = f$ and for $x \notin A$ we have $|g(x)| > 0$, so $|F(x)| < |\tilde{F}(x)| \le 1$. □

4.3 Locally finite systems and partitions of unity

In many areas of mathematics (analysis, differential geometry, measure theory, ...) one requires decompositions of functions into functions with small supports in order to derive global properties from local ones.

4.3.1 Definition. *Let X be a topological space and $(U_i)_{i\in I}$ a family of subsets of X. $(U_i)_{i\in I}$ is called a cover of $A \subseteq X$ if $A \subseteq \bigcup_{i\in I} U_i$. The cover is called open resp. closed, if all U_i are open resp. closed. It is called finite resp. countable if I is finite resp. countable. A family $\mathcal{A} = (A_i)_{i\in I}$ of subsets of X is called locally finite if any $x \in X$ has a neighborhood $U \in \mathcal{U}(x)$ that intersects only finitely many A_i. $\mathcal{A}$ is called point-finite if any $x \in X$ is contained in only finitely many A_i.*

4.3.2 Example. Clearly, locally finite implies point-finite, but not conversely: Let $X := \{\frac{1}{n}|n \in \mathbb{N}\} \cup \{0\}$ with the trace topology of $\mathbb{R}$ and set $\mathcal{A} := \{\{\frac{1}{n}\}|n \in \mathbb{N}\} \cup \{X\}$. Then $\mathcal{A}$ is point-finite, but not locally finite: any neighborhood of $\{0\}$ contains infinitely many $\frac{1}{n}$.

4.3.3 Definition. *An open cover $\mathcal{U} = (U_i)_{i\in I}$ of some subset $F \subseteq X$ is called shrinkable if there exists an open cover $\mathcal{V} = (V_i)_{i\in I}$ of F such that $\overline{V_i} \subseteq U_i$ for all $i \in I$. Then $\mathcal{V}$ is called a shrinking of $\mathcal{U}$.*

4.3.4 Theorem. *Let X be a toplogical space. TFAE:*

(i) X is T_4.

(ii) Any point-finite open cover of any closed subset F of X is shrinkable.

(iii) Any point-finite open cover of X is shrinkable.

Proof. (i)⇒(ii): Let $F \subseteq X$ be closed and $\mathcal{A} = (A_i)_{i\in I}$ a point-finite system of open sets covering F. Let $\mathcal{M}$ be the family of all open covers of F of the form $\{B_k \mid k \in K\} \cup \{A_l \mid l \in L\}$ with $K \cup L = I,\ K \cap L = \emptyset$ and $\overline{B_k} \subseteq A_k\ \forall k \in K$. Then $\mathcal{M} \neq \emptyset$, since one can take $K = \emptyset,\ L = I$. We introduce an ordering on $\mathcal{M}$ as follows: For $\mathcal{C} = \{B_k \mid k \in K\} \cup \{A_l \mid l \in L\}$, $\mathcal{C}' = \{B'_k \mid k \in K'\} \cup \{A_l \mid l \in L'\} \in \mathcal{M}$ let $\mathcal{C} \leq \mathcal{C}'$ if $K \subseteq K'$ and $B_k = B'_k\ \forall k \in K$. We want to apply Zorn's lemma in order to find a maximal element in $\mathcal{M}$. Thus let $(\mathcal{C}^s)_{s\in S}$ be a totally ordered subset of $\mathcal{M}$, $\mathcal{C}^s = \{B^s_k \mid k \in K_s\} \cup \{A_l \mid l \in L_s\}$, $K_s \cup L_s = I$, $K_s \cap L_s = \emptyset$. Let $K := \bigcup_{s\in S} K_s$, $L := \bigcap_{s\in S} L_s$. Then

$$K \cap L = \bigcup_{s\in S} (K_s \cap \bigcap_{s'\in S} L_{s'}) \subseteq \bigcup_{s\in S} (K_s \cap L_s) = \emptyset$$

and

$$I = \bigcap_{s\in S} (\underbrace{K_s \cup L_s}_{=I}) \subseteq \bigcap_{s\in S} (\bigcup_{s'\in S} K_{s'} \cup L_s) = \bigcup_{s'\in S} K_{s'} \cup \bigcap_{s\in S} L_s = K \cup L,$$

so $K \cup L = I$.

Set $\mathcal{C} := \{B_k \mid k \in K\} \cup \{A_l \mid l \in L\}$ with $B_k = B^s_k$ for $k \in K_s$. The B_k are well-defined: if $k \in K_s$ and $k \in K_{s'}$ and w.l.o.g. $\mathcal{C}_s \leq \mathcal{C}_{s'}$ then by definition of the order relation, $K_s \subseteq K_{s'}$ and $B^s_k = B^{s'}_k$.

We now claim that $\mathcal{S}$ is an upper bound, i.e., $\mathcal{C} \in \mathcal{M}$, and $\mathcal{C} \geq \mathcal{C}^s\ \forall s \in S$.

To establish this we only need to prove that $\mathcal{C}$ is an open cover of F. Thus let $x \in F$ and set $P(x) := \{i \in I \mid x \in A_i\}$. Since $\mathcal{A}$ is point-finite, $P(x)$ is finite. This leaves two possibilities:

1.) $P(x) \cap L \neq \emptyset$. Then let $i \in P(x) \cap L \Rightarrow A_i \in \mathcal{C}$ and $x \in A_i$.

2.) $P(x) \cap L = \emptyset$. Then due to $K \cup L = I$, $P(x) \subseteq K$. Since $P(x)$ is finite and $(K_s)_{s \in S}$ is totally ordered it follows that $P(x) \subseteq K_s$ for some $s \in S \Rightarrow \exists k \in K_s \subseteq K$ with $x \in B_k = B_k^s \in \mathcal{C}^s \subseteq \mathcal{C}$ (otherwise, since $\mathcal{C}^s$ is a cover of F, we would have $x \in A_l$ for some $l \in L_s = I \setminus K_s$, contradicting $P(x) \cap I \setminus K_s = \emptyset$).

By Zorn's Lemma, therefore, there exists a maximal element

$$\mathcal{C}^* = \{B_k \mid k \in K^*\} \cup \{A_l \mid l \in L^*\}$$

in $\mathcal{M}$.

We claim that $L^* = \emptyset$ (which will finish the proof). Suppose to the contrary that $L^* \neq \emptyset \Rightarrow \exists i \in L^*$. Let

$$D := F \setminus \left(\bigcup_{k \in K^*} B_k \cup \bigcup \{A_l \mid l \in L^*,\ l \neq i\} \right).$$

Then D is closed. Since $\mathcal{C}^*$ is a cover of F we must have $D \subseteq A_i$. Next, since X is T_4, it follows from 3.1.8 that there exists some open set B_i with $D \subseteq B_i \subseteq \bar{B}_i \subseteq A_i$. But then

$$\mathcal{C}' := \{B_k \mid k \in K^*\} \cup \{B_i\} \cup \{A_l \mid l \in L^*, l \neq i\} \in \mathcal{M},\ \mathcal{C}' \geq \mathcal{C}^*,\ \mathcal{C}' \neq \mathcal{C}^*,$$

contradicting the maximality of $\mathcal{C}^*$.

(ii)$\Rightarrow$(iii) is clear.

(iii)$\Rightarrow$(i): Let A and B be closed and disjoint. Then $\mathcal{U} := \{X \setminus A,\ X \setminus B\}$ is a point-finite open cover of X. Let $\{U, V\}$ be a shrinking of $\mathcal{U}$. Then $X \setminus \bar{U}$ and $X \setminus \bar{V}$ openly separate A and B. □

4.3.5 Lemma. *Let X be a topological space and $\mathcal{A} = (A_i)_{i \in I}$ a locally finite family of subsets of X. Then also $\overline{\mathcal{A}} := (\overline{A_i})_{i \in I}$ is locally finite and*

$$\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}. \tag{4.3.1}$$

Proof. Let $x \in X$ and pick U open in $\mathcal{U}(x)$ such that $U \cap A_i \neq \emptyset$ only for $i \in H$, where $H \subseteq I$ is finite. If $U \cap \overline{A_i} \neq \emptyset$, there exists some $y \in U \cap \overline{A_i}$. Hence $U \in \mathcal{U}(y)$ and since $y \in \overline{A_i}$ we get that $U \cap A_i \neq \emptyset \Rightarrow i \in H$. Thus $\overline{\mathcal{A}}$ is locally finite. It remains to show (4.3.1):

$\supseteq$: is clear.

$\subseteq$: Let $x \notin \bigcup_{i \in I} \overline{A_i}$ and $U \in \mathcal{U}(x)$ such that $U \cap A_i \neq \emptyset$ only for $i \in H \subseteq I$ finite. Then $V := U \setminus \bigcup_{i \in H} \overline{A_i} \in \mathcal{U}(x)$ and

$$V \cap \bigcup_{i \in I} A_i = (U \cap \bigcup_{i \in I} A_i) \setminus \bigcup_{i \in H} \overline{A_i} = \emptyset \Rightarrow x \notin \overline{\bigcup_{i \in I} A_i}.$$

□

4.3.6 Definition. *Let X be a topological space, $f : X \to \mathbb{R}$. Then the set*

$$\operatorname{supp} f := \overline{\{x \in X \mid f(x) \neq 0\}}$$

is called the support of f.

If $(f_i)_{i\in I}$ is a family of continuous functions: $X \to \mathbb{R}$, such that $(\operatorname{supp} f_i)_{i\in I}$ is locally finite, then

$$f(x) := \sum_{i\in I} f_i(x)$$

is well-defined and continuous: In fact, for each $x \in X$ the sum is finite. Moreover, for each $x \in X$ there exists some $U \in \mathcal{U}(x)$ and some finite set H such that $f|_U = \sum_{i\in H} f_i|_U$, which is continuous.

4.3.7 Definition. *Let X be a topological space, and $\mathcal{U} = (U_i)_{i\in I}$ an open cover of X. A family $(f_i)_{i\in I}$ of continuous functions is called a partition of unity subordinate to $\mathcal{U}$, if:*

(i) $f_i(x) \geq 0 \ \forall x \in X \ \forall i \in I$.

(ii) $(\operatorname{supp} f_i)_{i\in I}$ *is locally finite.*

(iii) $\operatorname{supp} f_i \subseteq U_i \ \forall i \in I$.

(iv) $\sum_{i\in I} f_i(x) = 1 \ \forall x \in X$.

4.3.8 Theorem. *Let X be T_4 and let $\mathcal{U} = (U_i)_{i\in I}$ be a locally finite open cover of X. Then there exists a partition of unity subordinate to $\mathcal{U}$.*

Proof. $\mathcal{U}$ is locally finite, hence point-finite. Therefore 4.3.4 guarantees the existence of an open cover $\mathcal{B} = (B_i)_{i\in I}$ of X with $\overline{B_i} \subseteq U_i \ \forall i \in I$. Also, since X is T_4, by 3.1.8 $\forall i \in I \ \exists C_i$ open with $\overline{B_i} \subseteq C_i \subseteq \bar{C}_i \subseteq U_i$. Thus Urysohn's Lemma 4.1.2 gives: $\forall i \in I \ \exists g_i : X \to [0, 1]$ with $g_i(x) = 1 \ \forall x \in \overline{B_i}$ and $g_i(x) = 0$ for $x \in X \setminus C_i$. Hence $\operatorname{supp} g_i \subseteq \bar{C}_i \subseteq U_i$, implying that $g(x) = \sum_{i\in I} g_i(x)$ is well-defined and continuous. For any $x \in X$ there exists some $i \in I$ with $x \in B_i$, so $g_i(x) = 1$, and consequently $g(x) \geq 1$. Finally, setting $f_i := \frac{g_i}{g}$, it follows that $(f_i)_{i\in I}$ satisfies (i) – (iv). □

4.3.9 Corollary. *Let X be T_4, $F \subseteq X$ closed and $(U_i)_{i\in I}$ a locally finite open cover of F. Then there exists a family $(f_i)_{i\in I}$ of continuous functions $f_i : X \to [0, 1]$ such that $f_i(x) = 0$ for $x \notin U_i$ and $\sum_{i\in I} f_i(x) = 1 \ \forall x \in F$.*

Proof. $(U_i)_{i\in I} \cup \{X \setminus F\}$ is a locally finite open cover of X, so by 4.3.8 there exists a partition of unity $(f_i)_{i\in I} \cup \{f\}_{X\setminus F}$ subordinate to this cover. These f_i have the required properties. □

Chapter 5

Compactness

Compactness is of central importance in all applications of topology, in particular in analysis (PDEs, functional analysis, global analysis,...) as it is one of the most common sources of existence results.

5.1 Compact spaces

5.1.1 Definition. *A topological space X is called compact if any open cover of X contains a finite sub-cover, i.e.:*

$$\bigcup_{i\in I} U_i = X,\ U_i \text{ open in } X \Rightarrow \exists I' \subseteq I,\ |I'| < \infty : \bigcup_{i\in I'} U_i = X.$$

A subset $A \subseteq X$ is called compact if A is compact in the trace topology.

5.1.2 Remark. (i) Some authors, (e.g., Bourbaki) call the above property quasi-compact and for compactness require, in addition, that X be T_2.

(ii) An elementary, yet fundamental observation is that compactness is independent of the subspace, in the following sense: Let $K \subseteq A \subseteq X$. Then K is compact in X if and only if K is compact in A. This is immediate from the form of open sets in the trace topology.

5.1.3 Theorem. *Let X be a topological space. TFAE:*

(i) X is compact.

(ii) Any family $(A_i)_{i\in I}$ of closed sets in X with $\bigcap_{i\in I} A_i = \emptyset$ contains a finite family $(A_i)_{i\in I'}$ with $\bigcap_{i\in I'} A_i = \emptyset$ (finite intersection property).

(iii) Any filter on X possesses a cluster point.

(iv) Any ultrafilter on X is convergent.

Proof. (i) $\Rightarrow$ (ii): Let $U_i := X \setminus A_i$. Then U_i is open, and

$$\bigcup_{i\in I} U_i = \bigcup_{i\in I}(X \setminus A_i) = X \setminus \bigcap_{i\in I} A_i = X,$$

so there exists some finite set $I' \subseteq I$ with

$$X = \bigcup_{i\in I'} U_i = X \setminus \bigcap_{i\in I'} A_i \Rightarrow \bigcap_{i\in I'} A_i = \emptyset.$$

(ii) $\Rightarrow$ (iii): Suppose there existed a filter $\mathcal{F}$ on X without an accumulation point. Then $\bigcap_{F\in\mathcal{F}} \bar{F} = \emptyset$, and so by (ii) there exist $F_1, \dots, F_k \in \mathcal{F}$ with $\emptyset = \bigcap_{i=1}^k \bar{F}_i \supseteq \bigcap_{i=1}^k F_i$, but this is impossible since $\mathcal{F}$ is a filter.

(iii) $\Rightarrow$ (iv): Let $\mathcal{F}$ be an ultrafilter on X. Then $\mathcal{F}$ possesses an accumulation point, hence is convergent by 2.3.6.

(iv) $\Rightarrow$ (i): Suppose to the contrary that there exists an open cover $(U_i)_{i\in I}$ of X that does not contain a finite sub-cover. For $L \subseteq I$ finite let $B_L := X \setminus \bigcup_{i\in L} U_i$. Then B_L is nonempty and $B_L \cap B_{L'} = X \setminus \bigcup_{i\in L\cup L'} U_i = B_{L\cup L'} \neq \emptyset$. Hence by 2.2.9 $\mathcal{B} := \{B_L \mid L \subseteq I \text{ finite }\}$ is a basis of a filter $\mathcal{F}'$ on X, and so by 2.2.15 there exists an ultrafilter $\mathcal{F} \supseteq \mathcal{F}'$. (iv) now implies the existence of some $x \in X$ with $\mathcal{F} \to x$, i.e. $\mathcal{F} \supseteq \mathcal{U}(x)$. As $(U_i)_{i\in I}$ is an open cover of X there exists some $i \in I$ with $U_i \in \mathcal{U}(x)$. Consequently, $U_i \in \mathcal{F}$. However, for $L = \{i\}$ also $B_L = X \setminus U_i \in \mathcal{F}$, contradicting the fact that $\mathcal{F}$ is a filter. □

5.1.4 Corollary. *Let X be a compact topological space. Then any sequence $(x_n)_{n\in\mathbb{N}}$ possesses an accumulation point in X.*

Proof. Let $\mathcal{F}$ be the filter with basis $(\{x_m \mid m \geq n\})_{n\in\mathbb{N}}$. By 5.1.3 (ii), $\mathcal{F}$ possesses an accumulation point, which by 2.3.10 is an accumulation point of the sequence $(x_n)_{n\in\mathbb{N}}$. □

Note, however, that the converse of 5.1.4 is false in general.

The following result shows that it suffices to check the defining property 5.1.1 of compact spaces for elements of a subbasis.

5.1.5 Theorem. *(Alexander) Let $\mathcal{S}$ be a subbasis of the topological space X. TFAE:*

(i) X is compact.

(ii) Any cover of X by sets from $\mathcal{S}$ contains a finite sub-cover.

Proof. (i) $\Rightarrow$ (ii): is clear.

(ii) $\Rightarrow$ (i): Suppose to the contrary that X is not compact. Then by 5.1.3 there exists an ultrafilter $\mathcal{F}$ on X that does not converge. We claim that $\forall x \in X\ \exists U_x \in \mathcal{S}$ with $x \in U_x$ and $U_x \notin \mathcal{F}$.

In fact, otherwise there would exist some $x \in X$ such that all $S \in \mathcal{S}$ with $x \in S$ lie in $\mathcal{F}$. But then also all finite intersections of such S lie in $\mathcal{F}$. As these form a neighborhood basis of x we conclude that $\mathcal{F} \supseteq \mathcal{U}(x) \Rightarrow \mathcal{F} \to x$, a contradiction.

These sets U_x form an open cover of X by sets from $\mathcal{S}$, so by (ii) there exists some $Y \subseteq X$ finite with $X = \bigcup_{y\in Y} U_y$. As $U_y \notin \mathcal{F}$, 2.2.16 implies that $X \setminus U_y \in \mathcal{F}$. But then

$$\emptyset = X \setminus \bigcup_{y\in Y} U_y = \bigcap_{y\in Y} X \setminus U_y \in \mathcal{F},$$

a contradiction. □

5.1.6 Example. Let $I = [a, b] \subseteq \mathbb{R}$. The intervals $[a, c)$ and $(d, b]$ with $a < c, d < b$ form a subbasis of the trace topology on I. Let $\mathcal{U}$ be a cover by sets of this subbasis. Let $\tilde{c} := \sup\{c \mid [a, c) \in \mathcal{U}\}$. Then there exists a $d_1 < \tilde{c}$ with $(d_1, b] \in \mathcal{U}$ (otherwise $\tilde{c}$ would not be contained in any element of $\mathcal{U}$).

Since $d_1 < \tilde{c}$, there exists some $c_1 > d_1$ with $[a, c_1) \in \mathcal{U}$, and so $[a, c_1)$ and $(d_1, b]$ cover $[a, b]$. By 5.1.5 this implies that $[a, b]$ is compact.

5.1.7 Lemma. *Let X be T_2 and K compact in X. Then for any $x \in X \setminus K$ there exists a neighborhood U of K and a neighborhood V of x with $U \cap V = \emptyset$.*

Proof. Let $x \in X \setminus K$. Then since X is T_2, for any $y \in K$ there exists some U_y open $\in \mathcal{U}(y)$ and some $V_y \in \mathcal{U}(x)$ with $U_y \cap V_y = \emptyset$. Then $(U_y)_{y \in K}$ is an open cover of K, so there exist $y_1, \ldots, y_n \in K$ with $K \subseteq \bigcup_{i=1}^n U_{y_i} =: U$. Let $V := \bigcap_{i=1}^n V_{y_i}$. Then $V \in \mathcal{U}(x)$, $U \in \mathcal{U}(K)$, and $U \cap V = \emptyset$. $\square$

5.1.8 Theorem.

(i) Any closed subset of a compact space is compact.

(ii) Any compact subset of a Hausdorff space is closed.

Proof. (i) Let $A \subseteq X$, A closed, X compact. Let $A_i \subseteq A$ be closed, $\bigcap_{i \in I} A_i = \emptyset$. Then since A_i is closed in X, 5.1.3 (ii) shows the existence of some finite set $I' \subseteq I$ with $\bigcap_{i \in I'} A_i = \emptyset$. Thus, again by 5.1.3 (ii), A is compact.

(ii) Let $K \subseteq X$ be compact and $x \notin K$. Then by 5.1.7 there exists some $U \in U(x)$ with $U \cap K = \emptyset$. Thus $x \notin \bar{K}$, and so $\bar{K} \subseteq K \subseteq \bar{K}$, i.e., $K = \bar{K}$. $\square$

5.1.9 Theorem. *Any compact Hausdorff space is normal.*

Proof. Let $A,\ B \subseteq X$ be closed and disjoint. By 5.1.8 (i), A and B are compact. Also, by 5.1.7, $\forall x \in A\ \exists U_x$ open, $U_x \in \mathcal{U}(x)$, V_x open neighborhood of B with $U_x \cap V_x = \emptyset$. Since A is compact, $\exists x_1, \ldots, x_n$ with $A \subseteq \bigcup_{i=1}^n U_{x_i} =: U$. Let $V := \bigcap_{i=1}^n V_{x_i}$. Then $U,\ V$ are disjoint neighborhoods of A, B. $\square$

5.1.10 Theorem. *Let X be compact and $f : X \to Y$ continuous. Then $f(X)$ is compact.*

Proof. Let $(U_i)_{i \in I}$ be an open cover of $f(X)$, i.e., $f(X) \subseteq \bigcup_{i \in I} U_i$. Then $X = f^{-1}(f(X)) \subseteq \bigcup_{i \in I} f^{-1}(U_i)$, so there exist $i_1, \ldots, i_n$ with $X \subseteq \bigcup_{j=1}^n f^{-1}(U_{i_j})$. It follows that $f(X) \subseteq \bigcup_{j=1}^n f(f^{-1}(U_{i_j})) \subseteq \bigcup_{j=1}^n U_{i_j}$. $\square$

5.1.11 Corollary. *Let X be compact and $f : X \to \mathbb{R}$ continuous. Then there exist $x_1,\ x_2 \in X$ with $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in K$, i.e., f attains a minimum and a maximum on X.*

Proof. By 5.1.10, $f(X)$ is compact, hence bounded: the sets $U_m := \{y \mid |y| < m\}$ cover $f(X)$, so there exists a finite sub-cover, and thereby there exists some m_0 with $f(X) \subseteq U_{m_0}$.

It follows that there exist real numbers $m = \inf f(X)$, $M = \sup f(X)$. By definition of the supremum there is a sequence $x_n \in X$ with $f(x_n) \to M$. As $\mathbb{R}$ is T_2, by 5.1.8 (ii) $f(X)$ is closed, and so by 2.3.7 and 2.3.10 $M = \lim f(x_n) \in f(X)$. This means that the maximum is attained, and analogously for the minimum. $\square$

5.1.12 Proposition. *Let X be compact, Y T_2, and $f : X \to Y$ continuous. Then f is closed. If f is injective (or bijective), then f is an embedding (a homeomorphism).*

Proof. Let $A \subseteq X$ be closed. Then by 5.1.10 $f(A)$ is compact, hence closed by 5.1.8 (ii). It follows that f is closed.

Let f be injective. Then $f : X \to f(X)$ is open: if $O \subseteq X$ is open, then $f(X \setminus O) = f(X) \setminus f(O)$ is closed in $f(X)$, so $f(O)$ is open in $f(X)$. By 1.1.8, therefore, f is an embedding. $\square$

5.1.13 Definition. *Let X be a topological space and $A \subseteq X$. A is called relatively compact if there exists a compact set $B \supseteq A$.*

5.1.14 Proposition. *Let X be Hausdorff, $A \subseteq X$. TFAE:*

(i) A is relatively compact.

(ii) $\bar{A}$ is compact.

Proof. (ii) $\Rightarrow$ (i): is clear.
(i) $\Rightarrow$ (ii): Let $B \supseteq A$, B compact. Then by 5.1.8 (ii), $B = \bar{B} \Rightarrow \bar{A} \subseteq B$. From this, 5.1.8 (i) implies that $\bar{A}$ is compact. □

5.1.15 Theorem. *(Tychonoff) Let $(X_i)_{i\in I}$ be a family of nonempty topological spaces. TFAE:*

(i) $\prod_{i\in I} X_i$ is compact.

(ii) X_i is compact $\forall i \in I$.

Proof. (i) $\Rightarrow$ (ii): Since $p_j : \prod_{i\in I} X_i \to X_j$ is continuous for each $j \in I$, $X_j = p_j(\prod_{i\in I} X_i)$ is compact by 5.1.10.
(ii) $\Rightarrow$ (i): Let $\mathcal{F}$ be an ultrafilter on $\prod_{i\in I} X_i$. Then by 2.2.25 each $p_i(\mathcal{F})$ is an ultrafilter on X_i, and therefore converges by 5.1.3 (iv). Now 2.3.19 implies that $\mathcal{F}$ converges, and this by 5.1.3 (iv) establishes the compactness of $\prod_{i\in I} X_i$. □

5.1.16 Theorem. *(Heine-Borel) Let $A \subseteq \mathbb{R}^n$. TFAE:*

(i) A is compact.

(ii) A is closed and bounded.

Proof. (i) $\Rightarrow$ (ii): A bounded follows as in 5.1.11. A is closed by 5.1.8 (ii).
(ii) $\Rightarrow$ (i): Since A is bounded there exists some $a > 0$ such that $A \subseteq [-a, a]^n$. $[-a, a]^n$ is compact by 5.1.6 and 5.1.15. Thus since A is closed, it is compact by 5.1.8 (i). □

5.2 Locally compact spaces

5.2.1 Definition. *A topological space is called locally compact if it is Hausdorff and any point has a compact neighborhood.*

5.2.2 Example.

(i) Any compact Hausdorff space is locally compact.

(ii) $\mathbb{R}^n$ is locally compact.

(iii) In functional analysis it is shown that a normed space (or, more generally, a topological vector space) is locally compact if and only if it is of finite dimension.

5.2.3 Theorem. *Any locally compact space is regular.*

Proof. By 3.1.6 it suffices to show that for any $x \in X$ the closed neighborhoods form a neighborhood basis. Let $K \in \mathcal{U}(x)$ be compact. Then K is closed by 5.1.8 (ii) and by 5.1.9 K is normal, hence also regular. Let $U \in \mathcal{U}(x)$, then $U \cap K$ is a neighborhood of x in K. As K is regular, by 3.1.6 there exists a neighborhood V of x in K with $V \subseteq \overline{V}^K \subseteq U \cap K$. V is of the form $W \cap K$ for some $W \in \mathcal{U}(x)$, so $V \in \mathcal{U}(x)$. Furthermore, $\overline{V}^K = \bar{V} \cap K$ is closed in X, so $\overline{V}^K = \bar{V} \Rightarrow x \in V \subseteq \bar{V} \subseteq U$, so the closed neighborhoods form a neighborhood basis. □

In 5.2.8 we will even show that X is completely regular.

5.2.4 Proposition. *Let X be locally compact and let $x \in X$. Then the compact neighborhoods of x form a neighborhood basis of x.*

Proof. By 5.2.3, X is regular, so by 3.1.6 the closed neighborhoods of x form a neighborhood basis. Let K be a compact neighborhood of x, then $\{K \cap V \mid V \in \mathcal{U}(x),\ V \text{ closed}\}$ is a neighborhood basis, which by 5.1.8 (i) consists of compact sets. □

5.2.5 Proposition. *Let X be locally compact, $A \subseteq X$ closed, $U \subseteq X$ open. Then $A \cap U$ is locally compact.*

Proof. We first note that A is locally compact (let $K \in \mathcal{U}(x)$ be compact, then $A \cap K$ is a compact neighborhood of x in A). Also, U is locally compact since, by 5.2.4, $\forall x \in U\ \exists K \in \mathcal{U}(x)$ compact with $x \in K \subseteq U$. Finally, $A \cap U$ is a closed subset of the locally compact space U, hence is itself locally compact. □

5.2.6 Example. Stereographic projection. As is well-known, this is the projection of the north pole N of S^2 (with S^2 attached to the origin of $\mathbb{R}^2$) onto $\mathbb{R}^2$. Then $S^2 \setminus N$ is homeomorphic to $\mathbb{R}^2$, S^2 is compact, and $S^2 \setminus N$ is dense in S^2.

An analogous construction can be carried out for *any* locally compact space:

5.2.7 Theorem. *(Alexandroff-compactification). Let X be locally compact. Then there exists a compact Hausdorff space Y, unique up to homeomorphism, that contains a subspace X_1 homeomorphic to X and such that $Y \setminus X_1$ consists of a single point, denoted by ∞. If X is not compact, then X_1 is dense in Y. The space Y is called one-point or Alexandroff-compactification of X. ∞ is called the point at infinity.*

Proof. Assume first that such a Y has already been constructed. Then $\{\infty\}$ is closed in Y, so X_1 is open in Y. Whence $U \subseteq X_1$ is open in Y if and only if it is open in X_1. All the other open sets in Y contain the point ∞. Let U be such a set, then $Y \setminus U$ is closed in Y, hence $Y \setminus U$ is compact and $\subseteq X_1$. Conversely, for $K \subseteq X_1$ compact, $Y \setminus K$ is open. Thus we have no other choice than defining the topology $\mathcal{O}_Y$ on Y as follows (where $\mathcal{O}_{X_1}$ is the one on X_1):

$$\mathcal{O}_Y := \{O \subseteq Y \mid O \in \mathcal{O}_{X_1} \vee \exists K \text{ compact in } X_1 \text{ with } O = Y \setminus K\}.$$

Also, let $Y := X_1 \cup \{\infty\}$ ($X_1 := X$). Then:

$\mathcal{O}_Y$ is a topology: $K = \emptyset \Rightarrow Y = Y \setminus K \in \mathcal{O}_Y$, $\emptyset \in \mathcal{O}_{X_1} \subseteq \mathcal{O}_Y$. Let $(O_i)_{i \in I} \in \mathcal{O}_Y$. If all $O_i \in \mathcal{O}_{X_1} \Rightarrow O := \bigcup_{i \in I} O_i \in \mathcal{O}_{X_1} \subseteq \mathcal{O}_Y$. Otherwise there exists some i_0 such that $\infty \in O_{i_0}$. Then $Y \setminus O = \bigcap_{i \in I} Y \setminus O_i \subseteq Y \setminus O_{i_0}$, so $Y \setminus O = \bigcap_{i \in I} X_1 \setminus O_i$ is closed and contained in the compact set $X_1 \setminus O_{i_0}$, hence $Y \setminus O$ is compact, so $O \in \mathcal{O}_Y$.

Let $O_1, \dots, O_n \in \mathcal{O}_Y$. If there exists some k with $O_k \in \mathcal{O}_{X_1}$, then $\bigcap_{i=1}^n O_i \subseteq X_1$, so

$$\bigcap_{i=1}^n O_i = \bigcap_{i=1}^n (O_i \cap X_1) \in \mathcal{O}_{X_1} \subseteq \mathcal{O}_Y.$$

Otherwise, $X_1 \setminus O_i$ is compact for every $1 \le i \le n$, so

$$Y \setminus \bigcap_{i=1}^n O_i = X_1 \setminus \bigcap_{i=1}^n O_i = \bigcup_{i=1}^n X_1 \setminus O_i$$

is compact in X_1, implying that $\bigcap_{i=1}^n O_i \in \mathcal{O}_Y$.

$\mathcal{O}_Y$ induces $\mathcal{O}_{X_1}$ on X_1: Clearly, $\mathcal{O}_{X_1} \subseteq \mathcal{O}_Y|_{X_1}$. Conversely, let $O \in \mathcal{O}_Y$. If $O \in \mathcal{O}_{X_1}$, then $O \cap X_1 \in \mathcal{O}_{X1}$. Otherwise $O = Y \setminus K \Rightarrow X_1 \cap O = X_1 \setminus K \in \mathcal{O}_{X_1}$.

Uniqueness: Let $Y' = X_1' \cup \{\infty'\}$ be another Alexandroff-compactification. Then since $X_1 \cong X \cong X_1'$, there exists a homeomorphism $f : X_1 \to X_1'$. Let $F : Y \to Y'$, $F|_{X_1} = f$, $F(\infty) := \infty'$. Then F is bijective and $F|_{X_1}$ is continuous. Let $Y' \setminus K'$ be a neighborhood of $F(\infty) = \infty'$. Then $F^{-1}(Y' \setminus K') = Y \setminus f^{-1}(K')$ is a neighborhood of ∞, so F is continuous. By symmetry, also F^{-1} is continuous, implying that in fact F is a homeomorphism.

Y is T_2 : any $x,\ y \in X_1$ can already be separated in X_1 since X_1 is T_2. If $x \ne \infty$, choose K compact $\in \mathcal{U}(x)$. Then $K,\ Y \setminus K$ separate x and ∞.

Y is compact: Let $(O_i)_{i \in I}$ be an open cover of $Y \Rightarrow \exists i_0$, such that $\infty \in O_{i_0}$. Then $O_{i_0} = Y \setminus K$ for some compact K. Here, $K \subseteq \bigcup_{i \in I} O_i \cap X_1 \Rightarrow \exists i_1, \dots, i_n$ with $K \subseteq \bigcup_{k=1}^n O_{i_k} \Rightarrow Y \subseteq \bigcup_{k=0}^n O_{i_k}$.

If X is not compact, then X_1 is not compact, hence not closed in $Y \Rightarrow \overline{X}_1 \supset X_1 \Rightarrow \overline{X}_1 = Y$. □

Note that if X is compact, then X_1 is open and closed in Y, so Y is the topological sum of X_1 and $\{\infty\}$ (cf. 1.4.4).

5.2.8 Corollary. *Any locally compact space is completely regular.*

Proof. Y is a compact T_2-space, hence normal by 5.1.9. From this, the result follows by 4.1.4 and 3.2.1. □

5.2.9 Proposition. *Let $(X_i)_{i \in I}$ be a family of topological spaces, $X_i \ne \emptyset$ $\forall i$. TFAE:*

(i) $\prod_{i \in I} X_i$ is locally compact.

(ii) $\forall i \in I$, X_i is locally compact and for almost all $i \in I$, X_i is compact and T_2.

Proof. (i) $\Rightarrow$ (ii): Let $a_i \in X_i$ $\forall i \in I$. Then by 1.1.14, every map

$$s_j : X_j \to \prod_{i \in I} X_i =: X,\ s_j(x_j) = (z_i)_{i \in I},\ z_i = \begin{cases} a_i & i \ne j \\ x_j & i = j \end{cases}$$

is an embedding and $s_j(X_j) = \prod_{i \in I} A_i$, with

$$A_i = \begin{cases} \{a_i\} & i \ne j \\ X_j & i = j \end{cases},$$

so by 2.3.20 (and 3.2.3), $s_j(X_j)$ is closed. It follows that each X_j is homeomorphic to a closed subspace of X, hence is itself locally compact. Let K be a compact

neighborhood of $(a_i)_{i\in I}$. Then $p_i(K)$ is compact $\forall i$. By definition of the product topology, $p_i(K) = X_i$ for almost all $i \Rightarrow X_i$ is compact for almost all i.
(ii) $\Rightarrow$ (i): Let $(a_i)_{i\in I} \in \prod_{i\in I} X_i$. Let $H \subseteq I$ finite such that X_i is compact $\forall i \notin H$. For $i \in H$ let K_i be a compact neighborhood of a_i. Then by 5.1.15,

$$\prod_{i\in I} O_i \text{ with } O_i = \begin{cases} K_i & i \in H \\ X_i & i \notin H \end{cases}$$

is a compact neighborhood of $(a_i)_{i\in I}$ in $\prod_{i\in I} X_i$. Finally, X is T_2 by 3.2.3. □

5.2.10 Definition. *A locally compact space X is called σ-compact (or countable at infinity) if X is a countable union of compact sets.*

5.2.11 Examples.

(i) $\mathbb{R}^n$ is σ-compact: $\mathbb{R}^n = \bigcup_{m=1}^{\infty} \overline{B_m(0)}$.

(ii) Let $U \subseteq \mathbb{R}^n$ be open. Then U is σ-compact since we can write it as

$$U = \bigcup_{m=1}^{\infty} \{x \in U \mid |x| \leq m \wedge d(x, \partial U) \geq \frac{1}{m}\}.$$

5.2.12 Theorem. *Let X be locally compact. TFAE:*

(i) X is σ-compact.

(ii) There exists a sequence $(U_n)_{n\in\mathbb{N}}$ of open sets in X such that:

(1) $\overline{U_n}$ is compact $\forall n \in \mathbb{N}$.

(2) $\overline{U_n} \subseteq U_{n+1}$ $\forall n \in \mathbb{N}$.

(3) $X = \bigcup_{n\in\mathbb{N}} U_n$.

Proof. (ii) $\Rightarrow$ (i): $X = \bigcup_{n\in\mathbb{N}} \overline{U_n}$.
(i) $\Rightarrow$ (ii): We first show that if X is locally compact then for any compact set K in X there exists some open set O and some compact set K' with $K \subseteq O \subseteq K'$. Indeed, $\forall x \in K$ $\exists U_x$ open in $\mathcal{U}(x)$ with $\overline{U}_x$ compact. K compact $\Rightarrow \exists x_1, \dots, x_n \in K$ with $K \subseteq U_{x_1} \cup \dots \cup U_{x_n} =: O$. Then $\overline{O} = \bigcup_{i=1}^{n} \overline{U_{x_i}} =: K'$ is compact.
By (i), $X = \bigcup_{n=1}^{\infty} K_n$, K_n compact. By what we have just shown, there exists some U_1 open with $K_1 \subseteq U_1$ and $\overline{U}_1$ compact. Suppose that $U_1, \dots, U_n$ have already been constructed. Then $\overline{U_n} \cup K_{n+1}$ is compact, so there exists some open set $U_{n+1} \supseteq \overline{U_n} \cup K_{n+1}$ with $\overline{U_{n+1}}$ compact. Thus $(U_n)_{n\in\mathbb{N}}$ satisfies (1)–(3). □

5.2.13 Corollary. *Let X be locally compact. TFAE:*

(i) X is σ-compact.

(ii) X possesses a compact exhaustion: $\exists$ K_n compact in X, $X = \bigcup_{n\in\mathbb{N}} K_n$, $K_n \subseteq K_{n+1}^{\circ}$ for all $n \in \mathbb{N}$.

Proof. (i)$\Rightarrow$(ii): Set $K_n := \overline{U_n}$ in 5.2.12.
(ii) $\Rightarrow$ (i): Clear. □

5.2.14 Corollary. *Let X be locally compact . TFAE:*

(i) X is σ-compact.

(ii) If Y is the Alexandroff-compactification of X, then ∞ possesses a countable neighborhood basis.

Proof. (i)⇒(ii): Let $(U_n)_n$ be as in 5.2.12 (ii). Then each $Y \setminus \overline{U_n}$ is a neighborhood of ∞. Let $Y \setminus K$ be any open neighborhood of ∞. Then $K \subseteq \bigcup_{n\in\mathbb{N}} U_n$, so there exists some n_0 such that $K \subseteq U_{n_0} \Rightarrow Y \setminus \overline{U_{n_0}} \subseteq Y \setminus K$. Thus $(Y \setminus \overline{U_n})_{n\in\mathbb{N}}$ is a neighborhood basis of ∞.

(ii)⇒(i): Let $(Y \setminus K_n)_{n\in\mathbb{N}}$ be a countable neighborhood basis of ∞, and let $x \in X$. Then $\{x\}$ is compact, so $Y \setminus \{x\}$ is a neighborhood of ∞, and therefore there exists some n such that $Y \setminus \{x\} \supseteq Y \setminus K_n$. This implies that $X = \bigcup_{n\in\mathbb{N}} K_n$. □

The previous result explains the expression 'countable at infinity'.

5.2.15 Proposition. *Let X, Y be locally compact with Alexandroff-compactifications X', Y'. Let $f : X \to Y$ be continuous and $F : X' \to Y'$, $F|_X = f$, $F(\infty) := \infty'$. TFAE:*

(i) F is continuous.

(ii) $\forall K'$ compact in Y, $f^{-1}(K')$ is compact in X.

Proof. F is continuous on X. Thus F is continuous ⇔ F continuous in ∞ ⇔ for every open neighborhood $Y' \setminus K'$ of ∞', $F^{-1}(Y' \setminus K') = X' \setminus f^{-1}(K')$ is an open neighborhood of ∞ ⇔ (ii). □

5.2.16 Definition. *A map $f : X \to Y$ between topological spaces X, Y is called proper if it satisfies 5.2.15 (ii).*

5.2.17 Proposition. *Let X, Y be locally compact and $f : X \to Y$ a continuous proper map. Then f is closed and $f(X)$ is locally compact.*

Proof. By 5.2.15, $F : X' \to Y'$ is continuous and by 5.1.12, F is also closed. Let $A \subseteq X$ be closed. Then $A \cup \{\infty\}$ is closed in X' (indeed, $X' \setminus (A \cup \{\infty\}) = X \setminus A \in \mathcal{O}_X \subseteq \mathcal{O}_{X'}$). Thus $F(A \cup \{\infty\}) = f(A) \cup \{\infty'\}$ is closed in Y'. Since $\mathcal{O}_{Y'}|_Y = \mathcal{O}_Y$, we conclude that $f(A)$ is closed in Y. Finally, $f(X) = F(X') \cap Y$ is locally compact by 5.2.5. □

Chapter 6

Algebras of continuous functions, the Stone-Weierstrass theorem

The classical theorem of Weierstrass states that any continuous function on a compact interval can be approximated uniformly by polynomials. In this short chapter we ask a more general question: Let X be compact and let $D \subseteq C(X)$. Under which conditions can we approximate any $f \in C(X)$ uniformly by elements of D?

6.1 The Stone-Weierstrass theorem

6.1.1 Definition. *Let X be a compact topological space and $C(X)$ the set of continuous functions on X. Then $d(f,g) := \sup_{x\in X} |f(x) - g(x)|$ defines a metric on $C(X)$. The corresponding topology is called the topology of uniform convergence.*

In fact, $d(f_n, f) \to 0 \Leftrightarrow f_n \to f$ uniformly on X. Also, $A \subseteq C(X)$ is dense in $C(X) \Leftrightarrow \forall f \in C(X)\ \exists$ sequence (f_n) in A with $d(f_n, f) \to 0$. Our aim is to find algebraic criteria for a subset A of $C(X)$ to be dense in $C(X)$.

6.1.2 Lemma. *There exists a sequence of polynomials $p_n : \mathbb{R} \to \mathbb{R}$ with $p_n(0) = 0$ that converges to $t \mapsto \sqrt{t}$, uniformly on $[0,1]$.*

Proof. Let

$$p_0(t) := 0,\ p_{n+1}(t) := p_n(t) + \frac{1}{2}(t - p_n^2(t)) \tag{6.1.1}$$

Then

$$\sqrt{t} - p_{n+1}(t) = (\sqrt{t} - p_n(t))\ (1 - \frac{1}{2}(\sqrt{t} + p_n(t))) \tag{6.1.2}$$

We show by induction that for all $n \in \mathbb{N}$ we have

$$p_n(t) \geq 0,\ p_n(0) = 0 \quad (t \in [0,1], n \in \mathbb{N}) \tag{6.1.3}$$

$$0 \leq \sqrt{t} - p_n(t) \leq \frac{2\sqrt{t}}{2 + n\sqrt{t}} \quad (t \in [0,1], n \in \mathbb{N}) \tag{6.1.4}$$

$n = 0$: clear.

$n \to n+1$: By (6.1.1), (6.1.3), and (6.1.4), $0 \le p_{n+1}(t)$, as well as $p_{n+1}(0) = 0$. Hence

$$0 \le \sqrt{t} - p_{n+1}(t) \overset{(6.1.2)}{=} (\underbrace{\sqrt{t} - p_n(t)}_{\overset{(6.1.4)}{\ge} 0})\underbrace{(1 - \frac{1}{2}(\sqrt{t} + p_n(t)))}_{\ge 1-\sqrt{t} \ge 0} \overset{(6.1.4)}{\le}$$

$$\frac{2\sqrt{t}}{2+n\sqrt{t}} \cdot \frac{1}{2}(2 - \sqrt{t} - p_n(t)) \le \frac{2\sqrt{t}}{2+n\sqrt{t}} \cdot \frac{1}{2} \cdot (2 - \sqrt{t}) \le \frac{2\sqrt{t}}{2+(n+1)\sqrt{t}}$$

Using $\frac{2\sqrt{t}}{2+n\sqrt{t}} \le \frac{2}{n}$, (6.1.4) implies $\sup_{t\in[0,1]} |\sqrt{t} - p_n(t)| \le \frac{2}{n}$, so $p_n \to \sqrt{t}$ uniformly on $[0,1]$. □

6.1.3 Lemma. *There exists a sequence of polynomials (q_n) with $q_n(0) = 0$ that converges uniformly on $[-a,a]$ $(a > 0)$ to $t \mapsto |t|$.*

Proof. Let p_n as in 6.1.2 and set $q_n(t) := a \cdot p_n(\frac{t^2}{a^2})$. Then for $t \in [-a,a]$ we get

$$||t| - q_n(t)| = \left|a\sqrt{\frac{t^2}{a^2}} - q_n(t)\right| = a\left|\sqrt{\frac{t^2}{a^2}} - p_n\left(\frac{t^2}{a^2}\right)\right| \overset{(6.1.4)}{\le} \frac{2a\sqrt{\frac{t^2}{a^2}}}{2+n\sqrt{\frac{t^2}{a^2}}} \le \frac{2a}{n}.$$

□

6.1.4 Remark. Using the pointwise operations: $f+\lambda g := t \mapsto f(t)+\lambda g(t)$, $f\cdot g := t \mapsto f(t)g(t)$, $C(X)$ is an algebra. Let $D \subseteq C(X)$. The algebra $A(D) \subseteq C(X)$ generated by D is the smallest algebra containing D. Thus

$$A(D) = \{ \sum_{0\le\nu_1,\dots,\nu_r\le n} a_{\nu_1\dots v_r} d_1^{\nu_1}\dots d_r^{\nu_r} \mid r,n \in \mathbb{N},\ d_i \in D,$$
$$1 \le i \le r,\ a_{\nu_1\dots\nu_r} \in \mathbb{R},\ a_{0\dots 0} = 0\}.$$

6.1.5 Lemma. *Let X be a compact space and A a closed subalgebra of $C(X)$. Then together with any f, g, A also contains $|f|$, $\max(f,g)$, and $\min(f,g)$.*

Proof. Since $\min(f,g) = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$ and $\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ it suffices to show that together with f also $|f|$ lies in A.
Let $a := \sup\{|f(x)| \mid x \in X\}$. Then $a < \infty$ by 5.1.11, so by 6.1.3 for any $\varepsilon > 0$ there exists a polynomial p_ε with $p_\varepsilon(0) = 0$ and $||f(x)| - p_\varepsilon(f(x))| < \varepsilon$ $\forall x \in X$. By 6.1.4, $p_\varepsilon \circ f \in A$. Also, $p_\varepsilon \circ f \to |f|$ in $C(X)$, and since A is closed we conclude that $|f| \in A$. □

6.1.6 Lemma. *Let X be a compact space and A a subalgebra of $C(X)$. If f, $g \in \bar{A}$, then also $f+g$, $f\cdot g$ and $c\cdot f$ $(c \in \mathbb{R})$ are in $\bar{A}$. Hence $\bar{A}$ is a subalgebra as well.*

Proof. f, $g \in \bar{A} \Rightarrow \exists f_n \in A$, $g_n \in A$ with $f_n \to f$, $g_n \to g$ uniformly on X. $\Rightarrow f_n + g_n \to f+g$, $cf_n \to cf$, $f_n \cdot g_n \to f\cdot g$. □

After these preparations we are now ready to prove the main result of this chapter:

6.1.7 Theorem. *(Stone-Weierstrass) Let X be a compact space and let $D \subseteq C(X)$ be such that*

(i) $\forall x \in X\ \exists f_x \in D$ *with* $f_x(x) \ne 0$.

(ii) $\forall x,y \in X,\ x \ne y\ \exists f \in D$ *with* $f(x) \ne f(y)$.

Then the subalgebra $A(D)$ generated by D is dense in $C(X)$, i.e., $\overline{A(D)} = C(X)$.

Proof. Let $f \in C(X)$ and let $\varepsilon > 0$. We have to show that there exists some $g_\varepsilon \in \overline{A(D)}$ with $|f(x) - g_\varepsilon(x)| < \varepsilon \ \forall x \in X$, i.e.: $f(x) - \varepsilon < g_\varepsilon(x) < f(x) + \varepsilon \ \forall x \in X$ (because then $\overline{A(D)}$ and thereby $A(D)$ itself is dense in $C(X)$).

Claim 1: $\forall y, z \in X \ \exists h \in A(D)$ with $h(y) = f(y)$ and $h(z) = f(z)$.

In fact, (i) $\Rightarrow \exists f_1, f_2 \in D$ with $f_1(y) \neq 0, f_2(z) \neq 0$. Let

$$f_y := \frac{1}{f_1(y)} f_1, \quad f_z := \frac{1}{f_2(z)} f_2.$$

Then $f_y,\ f_z \in A(D)$ and $f_y(y) = f_z(z) = 1$. Let $h_1 := f_y + f_z - f_y \cdot f_z \Rightarrow h_1 \in A(D)$ and $h_1(y) = h_1(z) = 1$.

If $y = z$, set $h := f(y) \cdot f_y$. If $y \neq z$ then by (ii) there exists some $h_2 \in D$ with $h_2(y) \neq h_2(z)$. Then let

$$h := \frac{f(y) - f(z)}{h_2(y) - h_2(z)}\, h_2 \ - \ \frac{f(y)h_2(z) - f(z)h_2(y)}{h_2(y) - h_2(z)}\, h_1$$

$\Rightarrow h \in A(D),\ h(y) = f(y)$ and $h(z) = f(z)$.

Claim 2: $\forall \varepsilon > 0 \ \forall z \in X \ \exists h_z \in \overline{A(D)}$ with $h_z(z) = f(z)$ and $h_z(x) < f(x) + \varepsilon$ for all $x \in X$.

To see this, note that by claim 1 $\forall y \in X \ \exists g_y \in A(D)$ with $g_y(z) = f(z)$ and $g_y(y) = f(y)$. $f,\ g_y$ continuous $\Rightarrow \exists U_y \in \mathcal{U}(y)$ such that $g_y(x) < f(x) + \varepsilon \ \forall x \in U_y$. X compact $\Rightarrow \exists L \subseteq X$ finite with $X \subseteq \bigcup_{y \in L} U_y$. Let $h_z := \min\{g_y | y \in L\}$. Then by 6.1.5, $h_z \in \overline{A(D)}$. Also, $h_z(z) = \min_{y \in L} g_y(z) = f(z)$ and for any $x \in X$ there exists some $y \in L$ with $x \in U_y$. Therefore, $h_z(x) \leq g_y(x) < f(x) + \varepsilon$.

Now for every $z \in X$ pick some h_z as in claim 2. Since $h_z(z) = f(z)$ and since both functions are continuous, there exists some $W_z \in \mathcal{U}(z)$ with $h_z(x) > f(x) - \varepsilon \ \forall x \in W_z$. X compact $\Rightarrow \exists K \subseteq X$ finite with $X = \bigcup_{z \in K} W_z$. Let $g_\varepsilon := \max\{h_z | z \in K\}$. Then $h_z \in \overline{A(D)}$ by 6.1.5. Now let $x \in X \Rightarrow \exists z \in K$ with $x \in W_z$, and so

$$f(x) - \varepsilon < h_z(x) \leq g_\varepsilon(x) = \max_{z \in K} h_z(x) < f(x) + \varepsilon$$

by claim 2. □

6.1.8 Corollary. *(Weierstrass) Let $[a, b]$ be a compact interval in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ continuous. Then:*

$$\forall \varepsilon > 0 \ \exists \text{ polynomial } p_\varepsilon : \mathbb{R} \to \mathbb{R} \text{ with } |p_\varepsilon(x) - f(x)| < \varepsilon \ \forall x \in [a, b].$$

Proof. Set $X = [a, b]$ and $D = \{f_1, f_2\}$ with $f_1 \equiv 1$ and $f_2(x) = x$. □

6.1.9 Corollary. *Let $X \subseteq \mathbb{R}^n$ be compact. Then any continuous function on X can be uniformly approximated by polynomials.*

Proof. Let $f_0 \equiv 1$ and $f_i = p_i : \mathbb{R}^n \to \mathbb{R}$ the projections $p_i(x_1, \ldots, x_n) = x_i$ for $1 \leq i \leq n$. Let $D := \{f_0, f_1, \ldots, f_n\}$. Then $A(D)$ is the set of all polynomials on $\mathbb{R}^n$. □

Let us analyze the proof of 6.1.7 more closely: setting $D' := A(D)$, we have shown that $\overline{D'} = C(X)$. Apart from 6.1.7 (ii) we only used the following properties:

(1) D' is a linear subspace of $C(X)$.

(2) (cf. claim 1 in 6.1.7): $\forall y, z \in X\ \exists h \in \overline{D'}$ with $h(y) = h(z) = 1$.

(3) $g_1, g_2 \in \overline{D'} \Rightarrow \min(g_1, g_2) \in \overline{D'}$ and $\max(g_1, g_2) \in \overline{D'}$.

Moreover, consider the following property:

(3') $f \in D' \Rightarrow |g| \in \overline{D'}$

Then: $(1) \wedge (3) \Rightarrow (3')$ since $|g| = 2\max(g, 0) - g$, and: $(1) \wedge (3') \Rightarrow (3)$: in fact, $\max(g_1, g_2) = \frac{1}{2}(g_1 + g_2 + |g_1 - g_2|)$, $\min(g_1, g_2) = \frac{1}{2}(g_1 + g_2 - |g_1 - g_2|)$. Let $g_1, g_2 \in \overline{D'} \Rightarrow \exists h_n,\ k_n \in D',\ h_n \to g_1,\ k_n \to g_2$ uniformly. Thus $\max(\text{resp.}\min)(h_n, k_n) \to \max(\text{resp.}\min)(g_1, g_2)$ uniformly $\Rightarrow$ (3). Note also that (2) is certainly satisfied if $f : X \to \mathbb{R},\ f \equiv 1$ is contained in D'. Thus we obtain:

6.1.10 Theorem. *(M. H. Stone) Let X be a compact space and D a linear subspace of $C(X)$ with:*

(i) D contains the constant function $f : X \to \mathbb{R},\ f(x) = 1 \forall x$.

(ii) $\forall x \neq y \in X\ \exists h \in D$ with $h(x) \neq h(y)$.

(iii) $h \in D \Rightarrow |h| \in \overline{D}$.

Then D is dense in $C(X)$. □

Chapter 7

Paracompactness and metrizability

7.1 Paracompactness

7.1.1 Definition.

(i) Let $\mathcal{A} = (A_i)_{i \in I}$ and $\mathcal{B} = (B_j)_{j \in J}$ be systems of subsets of a set X. $\mathcal{B}$ is called finer than $\mathcal{A}$ or a refinement *of $\mathcal{A}$, if:*

$$\forall j \in J\ \exists i \in I \text{ with } B_j \subseteq A_i.$$

(ii) A Hausdorff-space X is called paracompact*, if for any open cover $\mathcal{U}$ of X there exists a finer locally finite open cover $\mathcal{V}$.*

Obviously, any compact Hausdorff-space is paracompact.

7.1.2 Lemma. *Let X be paracompact, A, $B \subseteq X$ closed and disjoint. Suppose that for any $x \in A$ there exists some open $U_x \in \mathcal{U}(x)$ and an open $V_x \supseteq B$ with $U_x \cap V_x = \emptyset$. Then A and B possess disjoint open neighborhoods.*

Proof. $\mathcal{U} := \{U_x | x \in A\} \cup \{X \setminus A\}$ is an open cover of X, so there exists an open locally finite finite cover $(T_i)_{i \in I}$ finer than $\mathcal{U}$. Let $T := \bigcup_{A \cap T_i \neq \emptyset} T_i$. Then T is an open neighborhood of A. $(T_i)_{i \in I}$ is locally finite, so for every $y \in B$ there exists some open $W_y \in \mathcal{U}(y)$ such that W_y intersects at most finitely many T_i. Let j be such that $T_j \cap W_y \neq \emptyset$ and $A \cap T_j \neq \emptyset$. Since $(T_i)_{i \in I}$ is a refinement of $\mathcal{U}$, there exists some $x_j \in A$ with $T_j \subseteq U_{x_j}$.

Then $J(y) := \{j \in I \mid T_j \cap W_y \neq \emptyset \text{ and } T_j \cap A \neq \emptyset\}$ is finite and we set $\widetilde{W}_y := W_y \cap \bigcap_{j \in J(y)} V_{x_j}$. Then $\widetilde{W}_y$ is an open neighborhood of y and we claim that $\widetilde{W}_y \cap T = \emptyset$.

To see this, suppose that there existed some $z \in \widetilde{W}_y \cap T$. Then by definition of T there exists some $j \in I$ with $z \in T_j$ and $A \cap T_j \neq \emptyset$. $z \in \widetilde{W}_y \Rightarrow z \in W_y \Rightarrow z \in T_j \cap W_y \Rightarrow j \in J(y)$. But then $z \in T_j \cap \widetilde{W}_y \subseteq U_{x_j} \cap V_{x_j} = \emptyset$, a contradiction.

Thus $W := \bigcup_{y \in B} \widetilde{W}_y$ is an open neighborhood of B that is disjoint from T. $\square$

7.1.3 Theorem. *Any paracompact space X is normal.*

Proof. Let $C,\ D \subseteq X$ be closed and disjoint. Since X is T_2, any $c \in C$ and any $d \in D$ possess disjoint open neighborhoods. Applying 7.1.2 to $A := D$ and $B := \{c\}$ it follows that there exist an open neighborhood of c and an open neighborhood of D that do not intersect. Again from 7.1.2 we conclude that C and D can be separated by open sets. □

In applications, the most important property of paracompact spaces is the existence of partitions of unity for arbitrary open covers (not only for locally finite ones as in the case of normal spaces, cf. 4.3.8). In fact, this property is characteristic:

7.1.4 Theorem. *Let X be T_1. TFAE:*

(i) X is paracompact

(ii) For every open cover $\mathcal{U}$ of X there exists a subordinate partition of unity.

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{B} = (B_j)_{j\in J}$ be a locally finite open cover finer than $\mathcal{U} = (U_i)_{i\in I}$. Then $\forall j \in J\ \exists \Phi(j) \in I$ with $B_j \subseteq U_{\Phi(j)}$. This defines a map $\Phi : J \to I$. By 7.1.3, X is normal, so 4.3.8 shows the existence of a partition of unity $(g_j)_{j\in J}$ subordinate to $\mathcal{B}$. Let

$$f_i(x) := \begin{cases} \sum_{\Phi(j)=i} g_j(x) & \text{if } \Phi^{-1}(i) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

Since $\mathcal{B}$ is locally finite, any point possesses a neighborhood in which only finitely many g_j do not vanish identically, so f_i is continuous. Moreover,

$$\operatorname{supp} f_i \subseteq \bigcup_{j\in\Phi^{-1}(i)} \operatorname{supp}\ g_j \subseteq U_i.$$

To see that $(\operatorname{supp} f_i)_{i\in I}$ is locally finite, let $x \in X$. Then since $(g_j)_{j\in J}$ is locally finite, there exists some open neighborhood U of x that intersects only finitely many $\operatorname{supp} g_j$, say those with $j \in \{j_1, \dots, j_k\}$. Hence if $i \notin \{\Phi(j_1), \dots, \Phi(j_k)\}$ it follows that $f_i|_U \equiv 0$, so $U \cap \operatorname{supp} f_i = \emptyset$. Finally,

$$\sum_{i\in I} f_i(x) = \sum_{i\in I} \sum_{\Phi(j)=i} g_j(x) = \sum_{j\in J} g_j(x) = 1.$$

(ii) $\Rightarrow$ (i): X is T_2: Let $x \neq y$. Then $\mathcal{U} := \{X \setminus \{x\}, X \setminus \{y\}\}$ is an open cover of X, so there exists a subordinate partition of unity $(f_i)_{i\in I}$ for $\mathcal{U}$. Then there exists some i_0 such that $f_{i_0}(x) = a > 0$. This implies that $\operatorname{supp} f_{i_0} \subseteq X \setminus \{y\}$, so $f_{i_0}(y) = 0$. Consequently, the open sets $f_{i_0}^{-1}((\frac{a}{2}, 1])$ and $f_{i_0}^{-1}([0, \frac{a}{2}))$ separate x and y.

Next, let $\mathcal{U} = (U_i)_{i\in I}$ be an open cover of X. We need to show that there exists a locally finite open cover that refines $\mathcal{U}$. Let $(f_i)_{i\in I}$ be a partition of unity subordinate to $\mathcal{U}$. We first claim:

If $g : X \to \mathbb{R}$ is continuous and $g(x_0) > 0$ then there exists some $U_0 \in \mathcal{U}(x_0)$ and some finite set $I_0 \subseteq I$ such that

$$f_i(x) < g(x) \quad \forall x \in U_0\ \forall i \in I \setminus I_0. \tag{7.1.1}$$

To see this, note that since $\sum_{i\in I} f_i(x_0) = 1$, there exists some $I_0 \subseteq I$ finite with $1 - \sum_{i\in I_0} f_i(x_0) < g(x_0)$. Let $U_0 := \{x \in X \mid 1 - \sum_{i\in I_0} f_i(x) < g(x)\}$. Then U_0 is open and $f_i(x) < g(x)$ for all $i \in I \setminus I_0$ and all $x \in U_0$.

Let $f(x) := \sup_{i\in I} f_i(x)$. For any $x_0 \in X$ there exists some $i_0 \in I$ with $f_{i_0}(x_0) > 0$. Setting $g := f_{i_0}$ in (7.1.1), it follows that there exists some $U_0 \in \mathcal{U}(x_0)$ and

$I_0 \subseteq I$ finite with $f_i(x) < f_{i_0}(x)$ $\forall i \in I \setminus I_0$ $\forall x \in U_0$. Thus on U_0 we have $f(x) = \max(f_{i_0}(x), \max_{i \in I_0} f_i(x))$, so $f|_{U_0}$ is continuous. Since x_0 was arbitrary, $f : X \to \mathbb{R}$ is continuous.

Let $V_i := \{x \in X | f_i(x) > \frac{1}{2} f(x)\}$. Then V_i is open and since $\operatorname{supp} f_i \subseteq U_i$, $V_i \subseteq U_i \Rightarrow (V_i)_{i \in I}$ is a refinement of $(U_i)_{i \in I}$. Also, $(V_i)_i$ is a cover of X: Let $x \in X$. Due to $f(x) = \sup_{i \in I} f_i(x) > 0$, there exists some $i \in I$ with $f_i(x) > \frac{1}{2} f(x) \Rightarrow x \in V_i$.

Finally, $(V_i)_i$ is locally finite: Let $x_0 \in X$. Setting $g(x) := \frac{1}{2} f(x)$ in (7.1.1), it follows that there exists some $U_0 \in \mathcal{U}(x_0)$ and some finite $I_0 \subseteq I$ with $f_i(x) < \frac{1}{2} f(x)$ $\forall x \in U_0$ $\forall i \in I \setminus I_0$. Therefore, $x \notin V_i$ $\forall x \in U_0$ $\forall i \in I \setminus I_0 \Rightarrow U_0 \cap V_i = \emptyset$ $\forall i \in I \setminus I_0$. □

7.1.5 Definition. *A system $\mathcal{A}$ of subsets of a topological space X is called σ-locally finite, if $\mathcal{A}$ is a countable union of locally finite sub-systems: $\mathcal{A} = \bigcup_{i=1}^{\infty} \mathcal{A}_i$.*

7.1.6 Example.

(i) Any locally finite system is σ-locally finite.

(ii) If $\mathcal{B} = \{B_1, B_2, \dots\}$ is a countable basis of X, then $\mathcal{B}$ is σ-locally finite: $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{A}_i$, with $\mathcal{A}_i = \{B_i\}$.

7.1.7 Proposition. *Let X be regular. TFAE:*

(i) X is paracompact.

(ii) Any open cover of X possesses a finer σ-locally finite open cover.

(iii) Any open cover of X possesses a finer (not necessarily open) locally finite cover.

(iv) Any open cover possesses a finer locally finite closed cover.

Proof. (i) $\Rightarrow$ (ii): clear, since locally finite implies σ-locally finite.

(ii) $\Rightarrow$ (iii): Let $\mathcal{U}$ be an open cover of X and $\mathcal{S} = \bigcup_{n \in \mathbb{N}} \mathcal{S}_n$ a finer open cover such that any $\mathcal{S}_n$ is locally finite. Let $X_n := \bigcup_{S \in \mathcal{S}_n} S$, $Y_m := \bigcup_{n=0}^{m} X_n$ and $A_0 := Y_0$, $A_n := Y_n \setminus Y_{n-1}$ $(n \geq 1)$. Let $\mathcal{A} := (A_n)_{n \in \mathbb{N}}$. Then $X = \bigcup_{n \in \mathbb{N}} X_n = \bigcup_{n \in \mathbb{N}} Y_n = \bigcup_{n \in \mathbb{N}} A_n$. We claim that $\mathcal{Z} := \{A_n \cap S | S \in \mathcal{S}_n,\ n \in \mathbb{N}\}$ ist a locally finite cover finer than $\mathcal{U}$.

To see this, note first that $\mathcal{Z}$ is a refinement of $\mathcal{U}$: In fact, $A_n \cap S \subseteq S$ and $S \subseteq U$ for some $U \in \mathcal{U}$ since $\mathcal{S}$ is a refinement of $\mathcal{U}$. Also, $\mathcal{Z}$ is a cover: Let $x \in X$. Then since $X = \bigcup_{n \in \mathbb{N}} A_n$, there exists some $n \in \mathbb{N}$ with $x \in A_n = Y_n \setminus Y_{n-1} \subseteq X_n$, and since $X_n = \bigcup_{S \in \mathcal{S}_n} S$, there exists some $S \in \mathcal{S}_n$ with $x \in S$, hence $x \in A_n \cap S \in \mathcal{Z}$.

$\mathcal{Z}$ is locally finite: Let $x \in X$. Then due to $X = \bigcup_{n \in \mathbb{N}} Y_n$, there exists some n with $x \in Y_n$. Y_n is open and $Y_n \cap A_k = \emptyset$ $\forall k > n$, so Y_n can only intersect a set of the form $A_m \cap S \in \mathcal{Z}$ if $m \leq n$. Thus let $m \leq n$. Since $\mathcal{S}_m$ is locally finite, there exists some $V_m \in \mathcal{U}(x)$, $V_m \subseteq Y_n$, that intersects only finitely many $S \in \mathcal{S}_m$. Hence $V := \bigcap_{m=0}^{n} V_m$ meets only finitely many $S \in \bigcup_{m=0}^{n} \mathcal{S}_m$. Due to $V \subseteq Y_n$, V therefore only meets finitely many sets from $\mathcal{Z}$.

(iii) $\Rightarrow$ (iv): Let $\mathcal{U}$ be an open cover of X. Let $x \in X$ and $U_x \in \mathcal{U}$ such that $x \in U_x$. As X is regular, 3.1.6 implies that there exists some open W_x with $x \in W_x \subseteq \overline{W}_x \subseteq U_x$. Let $\mathcal{W} := (W_x)_{x \in X}$. Then by (iii), there exists a locally finite cover $\mathcal{A}$ finer than $\mathcal{W}$. By 4.3.5 also $\overline{\mathcal{A}} := (\bar{A})_{A \in \mathcal{A}}$ is a locally finite cover of X. $\bar{\mathcal{A}}$ is a refinement of $\mathcal{U}$ since any $A \in \mathcal{A}$ lies in some W_x and $W_x \subseteq \overline{W}_x \subseteq U_x$.

(iv) $\Rightarrow$ (i): Let $\mathcal{U}$ be an open cover of X and $\mathcal{V}$ a locally finite cover finer than $\mathcal{U}$. For $x \in X$ let $W_x \in \mathcal{U}(x)$ be open such that W_x meets only finitely many $V \in \mathcal{V}$.

(iv) $\Rightarrow \exists \mathcal{A}$ locally finite closed cover finer than $\mathcal{W} := (W_x)_{x\in X}$. Since any W_x meets only finitely many $V \in \mathcal{V}$, we also have:

$$\text{Any } A \in \mathcal{A} \text{ meets only finitely many } V \in \mathcal{V}. \tag{7.1.2}$$

Fixing any $V \in \mathcal{V}$, the set $\{A \in \mathcal{A} | A \cap V = \emptyset\} \subseteq \mathcal{A}$, is locally finite as well. By 4.3.5,

$$\overline{\bigcup_{\substack{A\in\mathcal{A}\\ A\cap V=\emptyset}} A} = \bigcup_{\substack{A\in\mathcal{A}\\ A\cap V=\emptyset}} \overline{A} = \bigcup_{\substack{A\in\mathcal{A}\\ A\cap V=\emptyset}} A,$$

and so $V' := X \setminus (\bigcup_{\substack{A\in\mathcal{A}\\ A\cap V=\emptyset}} A)$ is open. Due to $\bigcup_{\substack{A\in\mathcal{A}\\ A\cap V=\emptyset}} A \subseteq X \setminus V \Rightarrow V \subseteq V' \Rightarrow \mathcal{V}' := (V')_{V\in\mathcal{V}}$ is an open cover of X.

Next we claim that $\mathcal{V}'$ is locally finite. To see this, note first that since $\mathcal{A}$ is locally finite, for any $x \in X$ there exists some $T_x \in \mathcal{U}(x)$ such that T_x intersects only finitely many $A_1, \dots, A_n \in \mathcal{A}$. $\mathcal{A}$ is a cover, so $T_x \subseteq \bigcup_{k=1}^n A_k$. Now suppose that $T_x \cap V' \neq \emptyset \Rightarrow \exists k \in \{1, \dots, n\}$ with $A_k \cap V' \neq \emptyset$ (otherwise we would have $V' \cap \bigcup_{k=1}^n A_k = \emptyset$). By definition of V', therefore, $A_k \cap V \neq \emptyset$ (otherwise $A_k \subseteq \bigcup_{\substack{A\in\mathcal{A}\\ A\cap V=\emptyset}} A \subseteq X \setminus V'$). By (7.1.2), A_k meets only finitely many $V \in \mathcal{V} \Rightarrow T_x$ meets only finitely many $V' \in \mathcal{V}'$, which gives the claim.

For every $V \in \mathcal{V}$ there exists some $U_V \in \mathcal{U}$ with $V \subseteq U_V$. To conclude the proof, we show that $\mathcal{V}'' := \{U_V \cap V' | V \in \mathcal{V}\}$ is a locally finite open cover finer than $\mathcal{U}$. First, $\mathcal{V}''$ is locally finite since $\mathcal{V}'$ is. Also, clearly $\mathcal{V}''$ is open. Moreover, $\mathcal{V}''$ is finer than $\mathcal{U}$ since $U_V \cap V' \subseteq U_V \in \mathcal{U}$. Finally, $\mathcal{V}''$ is a cover: in fact, $V \subseteq U_V \cap V'$ and $\mathcal{V}$ is a cover, so also $\mathcal{V}''$ is one. □

7.1.8 Theorem. *Let X be a topological space. TFAE:*

(i) X is paracompact.

(ii) X is regular and for any open cover of X there exists a finer σ-locally finite open cover.

Proof. (i)⇒(ii): X is normal, hence regular by 7.1.3, so the claim follows from 7.1.6 (i).

(ii)⇒(i): follows from 7.1.7. □

7.1.9 Theorem. *Let X be locally compact, connected and paracompact. Then X is σ-compact.*

Proof. We will in fact show that X possesses a compact exhaustion (see 5.2.13). Since X is locally compact and paracompact we may choose a locally finite open cover $(U_\alpha)_{\alpha\in A}$ of X such that each $\overline{U_\alpha}$ is compact and non-empty. Fix $\alpha_0 \in A$ and put $A_0 := \{\alpha_0\}$. Then for $k \in \mathbb{N}$ we recursively define $A_k \subseteq A$ by

$$A_{k+1} := \{\alpha \in A \mid \exists \beta \in A_k : \ U_\alpha \cap U_\beta \neq \emptyset\}.$$

Since $\overline{U_\beta}$ is compact and $(U_\alpha)_{\alpha\in A}$ is locally finite, only finitely many U_α can have non-empty intersection with U_β. Thus each A_k is finite. Also, $A_k \subseteq A_{k+1}$ for all k. Now set $U := \bigcup_{k\in\mathbb{N}} \bigcup_{\alpha\in A_k} U_\alpha$. Then U is open. If $x \notin U$ then we pick some $\alpha \in A$ with $x \in U_\alpha$. It then follows that $U_\alpha \cap U = \emptyset$: in fact, otherwise there would exist some $k \in \mathbb{N}$ and some $\beta \in A_k$ with $U_\alpha \cap U_\beta \neq \emptyset$. But then $\alpha \in A_{k+1}$ and $x \in U$, a contradiction. Therefore U is also closed, and since X is connected, $U = X$.

We show that the sets $K_n := \bigcup_{\alpha\in A_n} \overline{U_\alpha}$ form a compact exhaustion of X. Every K_n is compact and $X = U = \bigcup_{n\in\mathbb{N}} K_n$. Finally, if $x \in K_n$ then there exists some $\alpha \in A$ with $x \in U_\alpha$ and there exists some $\beta \in A_n$ with $x \in \overline{U_\beta}$. Hence $U_\alpha \cap U_\beta \neq \emptyset$. It follows that $\alpha \in A_{n+1}$ and so $x \in \bigcup_{\alpha\in A_{n+1}} U_\alpha \subseteq K^\circ_{n+1}$. □

7.1.10 Definition. *A topological space is called* Lindelöf *if any open cover possesses a countable sub-cover.*

7.1.11 Proposition. *Any σ-compact space is Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of X and $(K_n)_{n\in\mathbb{N}}$ a cover by compact sets. Then any K_n is covered by finitely many elements of $\mathcal{U}$ and the union of these countably many finite collections of sets is a countable cover of X. □

7.1.12 Corollary. *(Classes of paracompact spaces)*

(i) Any regular space with countable basis is paracompact.

(ii) Any regular Lindelöf-space is paracompact.

(iii) The union of countably many closed subsets of a paracompact space is paracompact.

Proof. (i) Let $\mathcal{B}$ be a countable basis of X and let $\mathcal{U}$ be an open cover of X. Then $\mathcal{V} := \{B \in \mathcal{B} \mid \exists U \in \mathcal{U} :\ B \subseteq U\}$ is an open cover of X (since $\mathcal{B}$ is a basis) and refines $\mathcal{U}$. Also, since $\mathcal{V} \subseteq \mathcal{B}$, it is countable, hence σ-locally finite by 7.1.6 (ii). Thus 7.1.7 gives the claim.

(ii) By definition, any open cover $\mathcal{U}$ contains a countable, hence in particular σ-locally finite sub-cover $\mathcal{V}$, so 7.1.8 gives the claim.

(iii) Let $F = \bigcup_{n=1}^\infty F_n$, F_n closed $\forall n \in \mathbb{N}$. Let $\mathcal{U} = (U_i)_{i\in I}$ be an (in F) open cover of F. Then $\forall i \in I\ \exists V_i$ open in X such that $U_i = F \cap V_i$. For any $n \in \mathbb{N}$,

$$\mathcal{V}_n := \{V_i \mid i \in I\} \cup \{X \setminus F_n\}$$

is an open cover of X and therefore possesses a locally finite open refinement $\mathcal{A}_n$ that covers X. Let $\mathcal{B}_n := \{F \cap A \mid A \in \mathcal{A}_n \text{ and } A \cap F_n \neq \emptyset\}$. Then $\mathcal{B}_n$ is locally finite (since $\mathcal{A}_n$ is), so $\mathcal{B} := \bigcup_{n=1}^\infty \mathcal{B}_n$ is σ-locally finite. $\mathcal{B}$ covers F because $\mathcal{B}_n$ covers F_n. In addition, $\mathcal{B}$ is finer than $\mathcal{U}$: Let $F \cap A \in \mathcal{B}_n$. Then since $\mathcal{A}_n$ refines $\mathcal{V}_n$ and $A \cap F_n \neq \emptyset$ implies that $A \not\subseteq X \setminus F_n$, there has to exist some i such that $A \subseteq V_i \Rightarrow F \cap A \subseteq F \cap V_i = U_i$. The claim therefore follows from 7.1.8 since F is regular by 3.2.1. □

7.2 Metrizability

7.2.1 Definition. *A topological space X is called metrizable if there exists a metric on X that generates the topology of X.*

Our first aim in this section is to show that any metrizable space is paracompact. For this, we need some preparations from set theory.

An ordered set is called well-ordered if any nonempty subset possesses a smallest element. On any set there exists a well-ordering (this is a theorem of Zermelo, and is equivalent to the axiom of choice). Any well-ordered set is totally ordered (consider subsets of two elements).

If (X, d) is a metric space and $\emptyset \neq A, B \subseteq X$, let

$$d(A, B) := \inf_{\substack{x \in A \\ y \in B}} d(x, y).$$

Then $|d(A, x) - d(A, y)| \leq d(x, y)$, so $x \mapsto d(A, x)$ is uniformly continuous. Moreover, $x \in \overline{A} \Leftrightarrow \forall \varepsilon > 0\ \exists a \in A$ with $d(x, a) < \varepsilon \Leftrightarrow d(x, A) = 0$. After these preparations we can now prove our first main result:

7.2.2 Theorem. *(M. H. Stone) Any metrizable space X is paracompact.*

Proof. Since X is regular by 3.1.3, by 7.1.8 it suffices to show that for any open cover $\mathcal{U} = (U_i)_{i \in I}$ there exists a finer σ-locally finite open cover. For $(n, i) \in \mathbb{N} \times I$ set $F_{ni} := \{x \in X \mid d(x, X \setminus U_i) \geq 2^{-n}\}$. Then $F_{ni} \subseteq U_i$ and $U_i = \bigcup_{n=1}^{\infty} F_{ni}$ ($x \in U_i \Rightarrow x \notin X \setminus U_i \Rightarrow d(x, X \setminus U_i) > 0 \Rightarrow \exists n$ such that $d(x, X \setminus U_i) \geq 2^{-n} \Rightarrow x \in F_{ni}$).

Let $\leq$ be a well-ordering on I and set $G_{ni} := F_{ni} \setminus \bigcup_{j<i} F_{n+1,j}$. Let $V_{ni} := \{x \in X \mid d(x, G_{ni}) < 2^{-n-3}\}$ and $\mathcal{S}_n := \{V_{ni} \mid i \in I\}$. Then since $x \mapsto d(x, G_{ni})$ is continuous, V_{ni} is open.

We first show that $\bigcup_{n=1}^{\infty} \mathcal{S}_n$ is a refinement of $\mathcal{U}$.

Let $x \in V_{ni} \Rightarrow d(x, G_{ni}) < 2^{-n-3} \Rightarrow \exists y \in G_{ni}$ with $d(x, y) < 2^{-n-3}$. Since $G_{ni} \subseteq F_{ni} \Rightarrow d(y, X \setminus U_i) \geq 2^{-n}$. Consequently, $d(x, X \setminus U_i) \geq d(y, X \setminus U_i) - d(x, y) \geq 2^{-n} - 2^{-n-3} > 0 \Rightarrow x \notin X \setminus U_i \Rightarrow x \in U_i \Rightarrow V_{ni} \subseteq U_i$.

Next, $\bigcup_{n=1}^{\infty} \mathcal{S}_n$ covers X:

Let $x \in X = \bigcup_{i \in I} U_i$. Since I is well-ordered, there exists a minimal $i \in I$ with $x \in U_i$, and since $U_i = \bigcup_{n \in \mathbb{N}} F_{ni}$, there exists some $n \in \mathbb{N}$ with $x \in F_{ni}$. As $x \notin U_j\ \forall j < i$, $x \notin F_{n+1,j} (\subseteq U_j)\ \forall j < i \Rightarrow x \in G_{ni} \Rightarrow d(x, G_{ni}) = 0 \Rightarrow x \in V_{ni} \in \mathcal{S}_n$.

Finally, any $\mathcal{S}_n$ is locally finite.

To show this we first prove two auxilliary statements:

a) For $i \neq j$, $d(G_{ni}, G_{nj}) \geq 2^{-n-1}$: To see this, since I is totally ordered, we may suppose $j < i$, $x \in G_{ni}$, $y \in G_{nj}$. Then $x \notin F_{n+1,j} \Rightarrow d(x, X \setminus U_j) < 2^{-n-1}$. Also, $y \in F_{nj} \Rightarrow d(y, X \setminus U_j) \geq 2^{-n} \Rightarrow d(x, y) \geq d(y, X \setminus U_j) - d(x, X \setminus U_j) \geq 2^{-n} - 2^{-n-1} = 2^{-n-1}$.

b) For $i \neq j$, $d(V_{ni}, V_{nj}) \geq 2^{-n-2}$: Again, let $j < i$, $x \in V_{ni}$, $y \in V_{nj} \Rightarrow d(x, G_{ni}) < 2^{-n-3}$, $d(y, G_{nj}) < 2^{-n-3} \Rightarrow \exists x' \in G_{ni}, \exists y' \in G_{nj}$ with $d(x, x') < 2^{-n-3}$, $d(y, y') < 2^{-n-3} \Rightarrow d(x, y) \geq d(x', y') - d(x, x') - d(y, y') \geq d(G_{ni}, G_{nj}) - 2^{-n-3} - 2^{-n-3} \overset{a)}{\geq} 2^{-n-1} - 2^{-n-2} = 2^{-n-2}$.

Now let $x \in X$ and set $B_{2^{-n-3}}(x) := \{y \in X \mid d(x, y) < 2^{-n-3}\} \Rightarrow B_{2^{-n-3}}(x) \in \mathcal{U}(x)$. We show that $B_{2^{-n-3}}(x)$ intersects only finitely many V_{ni} (in fact, at most one): Let $V_{ni} \cap B_{2^{-n-3}}(x) \neq \emptyset$, $V_{nj} \cap B_{2^{-n-3}}(x) \neq \emptyset$, $i \neq j$. Then there exist $y \in V_{ni}$, $z \in V_{nj}$ with $d(y, x) < 2^{-n-3}$, $d(z, x) < 2^{-n-3} \Rightarrow d(V_{ni}, V_{nj}) \leq d(y, z) \leq d(y, x) + d(x, z) < 2^{-n-2}$, a contradiction. Thus

$$|\{i \in I \mid V_{ni} \cap B_{2^{-n-3}}(x) \neq \emptyset\}| \leq 1.$$

□

Conversely we now want to analyze the question which topological spaces are metrizable. As a preparation we establish a necessary condition for metrizability:

7.2.3 Lemma. *Let (X, d) be a metric space. Then:*

(i) If $A \subseteq X$ is closed $\Rightarrow \forall n \in \mathbb{N}\ \exists\ G_n$ open with $A = \bigcap_{n=1}^{\infty} G_n$ (i.e.: A is a G_δ-set).

(ii) If $A \subseteq X$ is open $\Rightarrow \forall n \in \mathbb{N}\ \exists\ F_n$ closed with $A = \bigcup_{n=1}^{\infty} F_n$ (i.e.: A is an F_σ-set).

Proof. (i) $A = \bar{A} = \{y \mid d(y, A) = 0\} = \bigcap_{n=1}^{\infty}\{y \mid d(y, A) < \frac{1}{n}\}$
(ii) By (i), $X \setminus A = \bigcap_{n=1}^{\infty} G_n \Rightarrow A = X \setminus \bigcap_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty}(X \setminus G_n)$. □

7.2.4 Lemma. *Let X be regular with a σ-locally finite basis. Then any open subset of X is an F_σ-set.*

Proof. Let $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$ be a σ-locally finite basis, where $\mathcal{S}_n = \{S_{ni} \mid i \in I_n\}$ is locally finite for each n. Let $\emptyset \neq O$ be open in X. As X is regular, $\forall x \in O\ \exists V_x \in \mathcal{U}(x)$ with $x \in V_x \subseteq \overline{V_x} \subseteq O$. Since $\mathcal{S}$ is a basis $\Rightarrow \exists S_{n(x)i(x)} \in \mathcal{S}$ with $x \in S_{n(x)i(x)} \subseteq V_x \Rightarrow \overline{S_{n(x)i(x)}} \subseteq \overline{V_x} \subseteq O$. Let

$$S_k := \bigcup_{\substack{n(x)=k \\ x \in O}} S_{n(x)i(x)}.$$

Because $\mathcal{S}_k$ is locally finite, 4.3.5 implies that $\overline{S_k} = \bigcup_{\substack{n(x)=k \\ x \in O}} \overline{S_{n(x)i(x)}} \Rightarrow \bigcup_{k\in\mathbb{N}} \overline{S_k} = O$. □

7.2.5 Theorem. *(Metrization theorem of Bing-Nagata-Smirnow) Let X be a topological space. TFAE:*

(i) X is metrizable.

(ii) X is regular and possesses a σ-locally finite basis.

Proof. (i)$\Rightarrow$(ii): Let d be a metric on X that generates the topology. Let $\mathcal{V}_n$ be the cover of X by all open balls of radius 2^{-n}. Then by 7.2.2 there exists a locally finite open cover $\mathcal{E}_n$ refining $\mathcal{V}_n$. Then $\mathcal{E} := \bigcup_{n=1}^{\infty} \mathcal{E}_n$ is σ-locally finite and it is a basis: Let $x \in X$, $U \in \mathcal{U}(x) \Rightarrow \exists n$ such that $B_{2^{-n}}(x) \subseteq U$. Since $\mathcal{E}_{n+2}$ is a cover of $X \Rightarrow \exists E \in \mathcal{E}_{n+2}$ with $x \in E$. The diameter of E is $\leq 2 \cdot 2^{-(n+2)} = 2^{-(n+1)}$, so $E \subseteq B_{2^{-n}}(x) \subseteq U$. Finally, X is regular by 3.1.3.
(ii)$\Rightarrow$(i): We first show that X is paracompact. To see this, let $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$ be a basis of X such that each $\mathcal{S}_n = \{S_{ni} \mid i \in I_n\}$ is locally finite. Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover of X and set $\mathcal{V}_n := \{V \in \mathcal{S}_n \mid \exists i \text{ with } V \subseteq U_i\}$. Since $\mathcal{V}_n \subseteq \mathcal{S}_n$, $\mathcal{V}_n$ is locally finite. Let $\mathcal{V} := \bigcup_{n=1}^{\infty} \mathcal{V}_n$. Then $\mathcal{V}$ is σ-locally finite, refines $\mathcal{U}$ and is a cover because any U_i is a union of elements of $\mathcal{S}$ (which is a basis!). Hence 7.1.8 gives the claim.
By 7.2.4, any S_{ni} is an F_σ-set $\Rightarrow X \setminus S_{ni}$ is a G_δ-set. As X is normal by 7.1.3, by 4.1.6 there exists a continuous function $\varphi_{ni} : X \to [0, 1]$ with $S_{ni} = \{x \mid \varphi_{ni}(x) > 0\}$. $\mathcal{S}_n$ is locally finite $\Rightarrow \sum_{j\in I_n} \varphi_{nj}(x)$ is well-defined and continuous and thereby the same is true for

$$\psi_{ni}(x) := 2^{-n} \frac{\varphi_{ni}(x)}{1 + \sum_{j\in I_n} \varphi_{nj}(x)}.$$

Then $0 \leq \psi_{ni}(x) < 2^{-n}$, $S_{ni} = \{x \mid \psi_{ni}(x) > 0\}$, $0 \leq \sum_{i\in I_n} \psi_{ni}(x) \leq 2^{-n}$, and so

$$d(x, y) := \sum_{n=1}^{\infty} \sum_{i\in I_n} |\psi_{ni}(x) - \psi_{ni}(y)|$$

is well-defined.

d is a metric: Clearly, $d(x,y) = d(y,x)$, as is the triangle inequality for d, and that $d(x,x) = 0$. Let $x \neq y$. Then since $\mathcal{S}$ is a basis and X is T_2, there exists some S_{ni} with $x \in S_{ni},\ y \notin S_{ni} \Rightarrow \psi_{n,i}(x) > 0,\ \psi_{ni}(y) = 0 \Rightarrow d(x,y) > 0$.

It remains to show that the topology $\mathcal{O}_d$ induced by d equals the topology $\mathcal{O}$ of X.

$\mathcal{O}_d \subseteq \mathcal{O}$: $\forall x_0 \in X$, $B_\varepsilon(x_0) = \{y \mid d(x_0,y) < \varepsilon\} \in \mathcal{O}$ because $y \mapsto d(x_0,y)$ is continuous as the uniform limit of continuous functions (cf. the proof of 4.1.6).

$\mathcal{O} \subseteq \mathcal{O}_d$: Let $x \in X$ and $U \in \mathcal{U}(x)$. Then since $\mathcal{S}$ is a basis, there exists some $x \in S_{ni} \subseteq U$. Let $\varepsilon := \psi_{ni}(x) > 0$. If $d(x,y) < \varepsilon \Rightarrow |\psi_{ni}(x) - \psi_{ni}(y)| < \varepsilon = \psi_{ni}(x)$ $\Rightarrow \psi_{ni}(y) > 0 \Rightarrow y \in S_{ni} \subseteq U$, i.e., $B_\varepsilon(x) \subseteq U$. $\square$

7.2.6 Corollary. *(Urysohn's second metrization theorem) Let X be compact and T_2. TFAE:*

(i) X is metrizable.

(ii) X possesses a countable basis.

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{S} = \bigcup_{n=1}^\infty \mathcal{S}_n$ be a σ- locally finite basis of X. Any $\mathcal{S}_n$ is locally finite and therefore finite: $\forall x \in X\ \exists U_x \in \mathcal{U}(x)$ open that intersects only finitely many $S \in \mathcal{S}_n$. Since finitely many U_x cover X, the claim follows. Consequently, $\mathcal{S}$ is also countable.

(ii) $\Rightarrow$ (i): X is compact and $T_2 \Rightarrow$ normal $\Rightarrow$ regular. Thus the claim follows from 7.1.6 (ii). $\square$

7.2.7 Theorem. *Let X be locally compact. TFAE:*

(i) X possesses a countable basis.

(ii) The Alexandroff compactification $X' = X \cup \{\infty\}$ of X is metrizable.

(iii) X is metrizable and σ-compact.

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{B}$ be a countable basis of X. Let $x \in X,\ U \in \mathcal{U}(x) \Rightarrow \exists K \in \mathcal{U}(x)$ compact with $x \in K \subseteq U \Rightarrow \exists B \in \mathcal{B}$ with $x \in B \subseteq \bar{B} \subseteq K \subseteq U$. Therefore $\tilde{\mathcal{B}} := \{B \in \mathcal{B} \mid \bar{B} \text{ is compact}\}$ is also a countable basis of X. It follows that X is a countable union of compact sets, hence is σ-compact. By 5.2.14 we conclude that $\infty \in X'$ possesses a countable neighborhood basis $\mathcal{U}$ of open sets, and so $\mathcal{U} \cup \mathcal{B}$ is a countable basis of the compact space X'. By 7.2.6, therefore, X' is metrizable.

(ii) $\Rightarrow$ (iii): Being a subspace of X', X is metrizable. Moreover, ∞ possesses a countable neighborhood basis, so X is σ-compact by 5.2.14.

(iii) $\Rightarrow$ (i): 5.2.12 $\Rightarrow \exists U_n$ open in $X, \overline{U_n}$ compact, $\subseteq U_{n+1},\ X = \bigcup_{n\in\mathbb{N}} U_n$. Any $\overline{U_n}$ is compact and metrizable, so by 7.2.6 there exists a countable basis $(V_{nm})_{m\in\mathbb{N}}$ of $\overline{U_n} \Rightarrow \{U_n \cap V_{nm} \mid m \in \mathbb{N}\}$ is a basis of U_n. $\Rightarrow \{U_n \cap V_{nm} \mid n, m \in \mathbb{N}\}$ is a countable basis of X. $\square$

7.2.8 Definition. *A topological space X is called* separable *if X possesses a countable dense subset.*

Some authors (e.g. Bourbaki) define separability only for metrizable spaces, namely as follows: a metrizable space X is called separable if it possesses a countable basis. The following result clarifies the interrelation between these notions:

7.2.9 Theorem. *(Urysohn's first metrization theorem). Let X be a topological space. TFAE:*

(i) X is regular and possesses a countable basis.

(ii) X is metrizable and separable.

(iii) X can be embedded in $[0,1]^{\mathbb{N}}$.

Proof. (i)⇒(ii): X is metrizable by 7.2.5 (and 7.1.6 (ii)). Let $\mathcal{B}$ be a countable basis of X and $\emptyset \notin \mathcal{B}$. For each $B \in \mathcal{B}$ pick some $x_B \in B$. Then $\{x_B \mid B \in \mathcal{B}\}$ is countable and dense.

(ii)⇒(iii): Let d' be a metric that generates the topology of X. Then $d(x,y) := \min(1, d'(x,y))$ is a metric (in fact, $d = h \circ d'$, where $h : \mathbb{R}^+ \to \mathbb{R}^+$, $h(u) = \min(1,u)$, $h(u+v) \le h(u) + h(v)$, and h is monotonically increasing). Also, d generates the same topology as d' since $B_\varepsilon^d(x_0) = B_\varepsilon^{d'}(x_0)$ for all $\varepsilon \le 1$.

Now let $(a_n)_{n\in\mathbb{N}}$ be dense in X and $\varphi : X \to [0,1]^{\mathbb{N}}$, $\varphi(x) := (d(x,a_n))_{n\in\mathbb{N}}$. Then φ is continuous since any $x \mapsto d(x,a_n)$ is continuous.

φ is injective: Let $\varphi(x) = \varphi(y)$, i.e., $d(x,a_n) = d(y,a_n)$ $\forall n \in \mathbb{N}$. Since $A := \{a_n \mid n \in \mathbb{N}\}$ is dense $\Rightarrow \exists a_{n_j} \to x$ $(j \to \infty)$ $\Rightarrow 0 = d(x,x) = \lim_{j\to\infty} d(x,a_{n_j}) = \lim_{j\to\infty} d(y,a_{n_j}) = d(y,x) \Rightarrow x = y$.

By 1.1.8 it remains to show that $\varphi : X \to \varphi(X)$ is open.

Claim: The topology $\mathcal{O}$ of X is the initial topology with respect to $(p_n \circ \varphi)_{n\in\mathbb{N}}$.

To prove this, let $\mathcal{O}'$ be a topology on X such that all $p_n \circ \varphi = x \mapsto d(x,a_n)$ are continuous with respect to $\mathcal{O}'$. By 1.2.2 we have to show that $\mathcal{O} \subseteq \mathcal{O}'$.

To begin with, $d : (X,\mathcal{O}') \times (X,\mathcal{O}') \to [0,1]$ is continuous: Let $(x_0,y_0) \in X \times X$ and let $\varepsilon > 0$. Since A is dense $\Rightarrow \exists n,\ m \in \mathbb{N}$ with $d(x_0,a_n) < \frac{\varepsilon}{6}$ and $d(y_0,a_m) < \frac{\varepsilon}{6}$. Since $x \mapsto d(x,a_n)$, $x \mapsto d(x,a_m)$ are continuous $\Rightarrow \exists V \in \mathcal{U}(x_0)$, $W \in \mathcal{U}(y_0)$ in $\mathcal{O}'$ such that for $x \in V$, $y \in W$: $|d(x,a_n) - d(x_0,a_n)| < \frac{\varepsilon}{6}$ and $|d(y,a_m) - d(y_0,a_m)| < \frac{\varepsilon}{6}$. For $x \in V$ it follows that $d(x,a_n) < \frac{\varepsilon}{6} + d(x_0,a_n) < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}$ and for $y \in W$ we analogously obtain: $d(y,a_m) < \frac{\varepsilon}{3}$. Consequently, $d(x,x_0) \le d(x,a_n) + d(a_n,x_0) < \frac{\varepsilon}{3} + \frac{\varepsilon}{6} = \frac{\varepsilon}{2}$. In the same way we get that $d(y,y_0) < \frac{\varepsilon}{2}$. Summing up,

$$\begin{aligned}|d(x,y) - d(x_0,y_0)| &\le |d(x,y) - d(x_0,y)| + |d(x_0,y) - d(x_0,y_0)| \\ &\le\ d(x,x_0) + d(y,y_0) < \varepsilon\end{aligned}$$

for $(x,y) \in V \times W$. It follows that $d : (X,\mathcal{O}') \times (X,\mathcal{O}') \to [0,1]$ is continuous. Hence for every $\varepsilon > 0$, $B_\varepsilon(x_0) = \{y \in X \mid d(x_0,y) < \varepsilon\} \in \mathcal{O}'$, implying that $\mathcal{O} \subseteq \mathcal{O}'$, and thereby the claim.

Since $[0,1]^{\mathbb{N}}$ carries the initial topology with respect to $(p_n)_{n\in\mathbb{N}}$ and $(X,\mathcal{O})$ the initial topology with respect to $(p_n \circ \varphi)_{n\in\mathbb{N}}$, 1.2.3 implies that $\mathcal{O}$ is the initial topology with respect to φ. Consequently, $\varphi : X \to \varphi(X)$ is open: In fact, by 1.2.2, $\mathcal{O} = \{\varphi^{-1}(U) \mid U \text{ open in } [0,1]^{\mathbb{N}}\}$. For $\varphi^{-1}(U) \in \mathcal{O}$ we have that $\varphi(\varphi^{-1}(U)) = \varphi(X) \cap U$ is open in $\varphi(X)$, so indeed $\varphi : X \to \varphi(X)$ is open. Thus $\varphi : X \to \varphi(X)$ is a homeomorphism, so $\varphi : X \to [0,1]^{\mathbb{N}}$ is an embedding.

(iii)⇒(i): $[0,1]$ possesses a countable basis $\mathcal{B}$, so a basis of $[0,1]^{\mathbb{N}}$ is given by $\{\prod_{n\in\mathbb{N}} B_n \mid B_n \in \mathcal{B},\ B_n = X \text{ for almost all } n\}$, which is itself countable. Furthermore, $[0,1]^{\mathbb{N}}$ is regular by 3.2.3, and so is any of its subspaces by 3.2.1. □

Chapter 8

Topological manifolds

In this chapter, following [3], we want to derive some fundamental properties of topological (and thereby also of smooth) manifolds.

8.1 Locally Euclidean spaces

8.1.1 Definition. *A topological space X is called locally Euclidean if any point possesses a neighborhood that is homeomorphic to some $\mathbb{R}^n$, for some $n \in \mathbb{N}$.*

Equivalently, any point is supposed to possess a neighborhood that is homeomorphic to some open subset of some $\mathbb{R}^n$. 8.1.1 allows the number n to vary between points, consider, e.g., the topological sum of $\mathbb{R}^n$ and $\mathbb{R}^m$. The following non-trivial result from algebraic topology shows that n must be locally constant:

8.1.2 Theorem. *(Brouwer) Let U, V be subsets of $\mathbb{R}^n$ that are homeomorphic. Then if U is open, so is V.*

For a proof, we refer to [4].

8.1.2 indeed implies that n is locally constant: suppose that a point $x \in X$ has a neighborhood that is homeomorphic to both $U \subseteq \mathbb{R}^n$ and to $V \subseteq \mathbb{R}^m$, with $m > n$. Then since $\mathbb{R}^n$ can be viewed as a subspace of $\mathbb{R}^m$ it would follow from 8.1.2 that U has to be open in $\mathbb{R}^m$, which is impossible. Thus we may assign a natural number $n := \dim_x(X)$ to any point $x \in X$. Then the map $\dim(X) : X \to \mathbb{N}$, $x \mapsto \dim_x(X)$ is locally constant, hence constant on the connected components of X.

The following remark collects some basic properties of locally Euclidean spaces.

8.1.3 Remark. (i) Any locally Euclidean space is T_1. This is immediate from 3.1.4 since singletons are closed in $\mathbb{R}^n$.

(ii) However, locally Euclidean spaces need not be Hausdorff in general. As an example, let $Z := \{0, 1\}$ with the discrete topology and set $\tilde{X} := \mathbb{R} \times Z$. On $\tilde{X}$ we define an equivalence relation by $(x_1, z_1) \sim (x_2, z_2)$ if either $(x_1, z_1) = (x_2, z_2)$ or $x_1 = x_2 \neq 0$ and $z_1 \neq z_2$. Then $X := X/\sim$, equipped with the quotient topology, can be viewed as a real line with two different origins. It follows that X is locally Euclidean but is not Hausdorff because any neighborhood of U of $(0, 0)$ intersects any neighborhood V of $(0, 1)$.

(iii) Any locally Euclidean T_2-space X is completely regular. To see this, by (i) it remains to show that X is T_{3a}. Thus let $A \subseteq X$ be closed and let $x \notin A$. We need to construct a function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f|_A \equiv 0$. To do this, pick a homeomorphism $\varphi : U \to \mathbb{R}^n$, where U is an open neighborhood of x with $U \cap A = \emptyset$ and $\varphi(x) = 0$. The set $K := \{x \in U \mid \|\varphi(x)\| \le 1\}$ is compact in U, hence also compact in X. Since X is T_2, 5.1.8 implies that it is closed in X. Therefore $V := X \setminus K$ is open in X, and so the function

$$f : X \to [0,1], \qquad f(y) := \begin{cases} \max\{0, 1 - \|\varphi(y)\|\} & \text{if } y \in U \\ 0 & \text{if } y \in V \end{cases}$$

is continuous. By definition, $f(x) = 1$ and $f|_A \equiv 0$.

Since X is T_{3a}, 3.1.7 implies that $\mathcal{B} := \{f^{-1}(U) | U \subseteq \mathbb{R} \text{ open}, f : X \to \mathbb{R} \text{ continuous and bounded}\}$ is a basis of the topology of X, so by 1.2.2 the topology of X is the initial topology with respect to these maps.

(iv) By (i) and 7.1.4, a locally Euclidean space X is paracompact if and only if for any open cover of X there exists a subordinate partition of unity.

(v) Any locally Euclidean T_2-space is locally compact. Indeed, for any $x \in X$, K as in (iii) is a compact neighborhood of x.

(vi) Any locally Euclidean space is first countable, i.e., any point possesses a countable neighborhood basis.

(vii) A locally Euclidean space need not be second countable, however: as an example, consider the topological sum of uncountably many copies of $\mathbb{R}$. In fact, there are even connected locally Euclidean spaces that are not second countable, e.g., the so-called long line, cf. [7].

8.2 Topological manifolds

Recall from 1.5.1 that a topological manifold is a set M such that there exists a cover $(U_i)_{i\in I}$ of M and a family of bijective maps $\varphi_i : U_i \to V_i$ with $V_i \subseteq \mathbb{R}^n$ open. Moreover, for all $i,\ j$ with $U_i \cap U_j \neq \emptyset$, $\varphi_i(U_i \cap U_j),\ \varphi_j(U_i \cap U_j)$ is open and

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is a homeomorphism. M is equipped with the final topology with respect to the $\varphi_i^{-1} : V_i \to M$. If M is Hausdorff, it is called a topological manifold of dimension n. Moreover, by 1.5.2, every $\varphi_i : U_i \to V_i$ is a homeomorphism, each U_i is open in M, and w.l.o.g., any V_i can be chosen to be $\mathbb{R}^n$. It follows that any topological manifold is a locally Euclidean space.

Conversely, if M is a locally Euclidean Hausdorff space of constant dimension, then M is a topological manifold: in fact, pick homeomorphisms $\varphi_i : U_i \to \mathbb{R}^n$ such that $(U_i)_{i\in I}$ is a cover of M. Then whenever $U_i \cap U_j \neq \emptyset$, $\varphi_i(U_i \cap U_j),\ \varphi_j(U_i \cap U_j)$ are open and $\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is a homeomorphism. For any $x \in U_i$, the sets $\varphi_i^{-1}(B_\varepsilon(\varphi_i(x)))$ $(\varepsilon > 0)$ form a neighborhood basis of x in the given toplogy of the locally Euclidean space M. Since, by 1.5.2, they also form a neighborhood basis in the manifold topology induced by the atlas $(\varphi_i, U_i)_{i\in I}$, these two topologies coincide.

Our next aim is to provide various characterizations of topological properties of locally Euclidean spaces (and thereby of topological manifolds). For this, we will require the following observation.

8.2.1 Remark. Suppose that $(X_i, d_i)_{i\in I}$ is a family of (disjoint) metric spaces. Then also the topological direct sum (cf. 1.4.5) $\bigcup_{i\in I} X_i$ can be equipped with a metric that induces d_i on each X_i. In fact, it suffices to define

$$d(x,y) := \begin{cases} d_i(x,y) & \text{if } x, y \in X_i \\ \infty & \text{otherwise.} \end{cases}$$

8.2.2 Theorem. *Let X be a locally Euclidean Hausdorff space. TFAE:*

(i) Any connected component of X possesses a compact exhaustion (cf. 5.2.13).

(ii) Any connected component of X is σ-compact.

(iii) Any connected component of X is Lindelöf.

(iv) Any connected component of X is second countable.

(v) X is completely metrizable, i.e., there exists a metric d on X such that (X, d) is complete and d induces the topology of M.

(vi) X is metrizable.

(vii) X is paracompact.

Proof. (i)⇔(ii): This follows from 5.2.13.

(ii)⇒(iii): By 7.1.11, any σ-compact space is Lindelöf.

(iii)⇒(iv): Let Y be a connected component of X. Since Y is Lindelöf, we may extract a countable cover $(U_m)_{m\in\mathbb{N}}$ from a cover of Y by sets that are homeomorphic to some $\mathbb{R}^n$. Since $\mathbb{R}^n$ is second countable, so is each U_m. Let $(V_{mk})_{k\in\mathbb{N}}$ be a basis of the topology of U_m. Then $\{V_{mk} \mid k, m \in \mathbb{N}\}$ is a countable basis for the topology of Y.

(iv)⇒(vi): This follows from 7.2.7 and 8.2.1.

(vi)⇒(vii): See 7.2.2.

(vii)⇒(ii): See 7.1.9.

(v)⇒(vi): is clear.

(vi)⇒(v): By 8.2.1 we may without loss of generality assume that X is connected. Let d be a metric on X that induces the topology of X. Now consider the space $X\times\mathbb{R}$ with the metric $d'((x, s), (y, t)) := \max(d(x, y), |t-s|)$. Our aim is to construct a continuous and proper (cf. 5.2.16) function $f : X \to \mathbb{R}$. Suppose for the moment that we already have such an f. Then consider the map $\iota : X \to X \times \mathbb{R}$, $\iota(x) := (x, f(x))$. It is an embedding since $\mathrm{pr}_1|_{\iota(X)}$ is an inverse to ι on $\iota(X)$. Therefore the metric $d''(x, y) := d'(\iota(x), \iota(y))$ induces the topology of X. This metric is complete since f is proper: in fact, if (x_n) is a Cauchy sequence with respect to d'', then $(f(x_n))$ is bounded, so (x_n) lies in a compact set. It therefore possesses a convergent subsequence, hence is itself convergent since it is Cauchy.

It remains to construct such a function f. We first note that since we already proved that (vi)⇒(i), we can conclude that there exists a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of X. Then for each $n \in \mathbb{N}$, K_n and $X \setminus K^\circ_{n+1}$ are closed and disjoint. Thus by 4.1.1 there exist continuous functions $f_n : X \to [0, 1]$ with $f_n|_{K_n} \equiv 0$ and $f_n|_{X\setminus K^\circ_{n+1}} \equiv 1$. Then $f := \sum_{n\in\mathbb{N}} f_n$ is continuous, being a locally finite sum. Moreover, since $f^{-1}([-n, n]) \subseteq K_{n+1}$ for every n, it is also proper. □

8.2.3 Corollary. *Let M be a paracompact topological manifold. TFAE:*

(i) M is second countable.

(ii) M has at most countably many connected components.

(iii) M is separable.

Proof. (i)$\Leftrightarrow$(ii): is immediate from 8.2.2 (iv).

(ii)$\Rightarrow$(iii): Any connected component is separable by 8.2.2 (iv) and 7.2.9, hence so is X itself.

(iii)$\Rightarrow$(ii): Since any dense subset must intersect each connected component of M, there can be at most countably many such components. $\square$

There are, however, locally Euclidean spaces that are connected and separable, but are not second countable ([6]).

Finally, we show that any compact manifold can be embedded in some $\mathbb{R}^N$:

8.2.4 Theorem. *Let M be a compact topological manifold. Then there exists some $N \in \mathbb{N}$ and an embedding of M into $\mathbb{R}^N$.*

Proof. Let $\dim(M) = n$. For any $x \in M$ there exists a chart φ_x that homeomorphically maps some open neighborhood U_x of x onto $\mathbb{R}^n$. Since M is compact, there are finitely many points $x_1, \dots, x_m$ such that the sets $V_i := \varphi_{x_i}^{-1}(B_1(0))$ $(i = 1, \dots, m)$ cover M. Let $h : \mathbb{R}^n \to [0,1]$, $h(z) := \max(0, 1 - \|z\|)$ and set, for $i = 1, \dots, m$:

$$f_i : M \to \mathbb{R}^{n+1}, \qquad f_i(x) := \begin{cases} (\varphi_{x_i}(x) h(\varphi_{x_i}(x)), h(\varphi_{x_i}(x))) & \text{if } x \in U_{x_i} \\ 0 & \text{otherwise.} \end{cases}$$

Then f_i is continuous and $f_i|_{V_i} : V_i \to \mathbb{R}^{n+1}$ is injective. Also, $f_i(x) \neq 0$ if and only if $x \in V_i$. Therefore $F := (f_1, \dots, f_m) : M \to \mathbb{R}^{m(n+1)}$ is continuous and injective. Since M is compact, 5.1.12 shows that F is an embedding of M into $\mathbb{R}^N$ for $N = m(n+1)$. $\square$

One can in fact show that any n-dimensional second countable topological manifold can be embedded into $\mathbb{R}^{2n+1}$.

Chapter 9

Uniform spaces

There are a number of fundamental notions from analysis, like uniform continuity, uniform convergence, Cauchy sequence, or completeness, that cannot be described purely in terms of topological notions. Instead, a concept of 'nearness' of two points that is symmetric in the points (as opposed to one point lying in a neighborhood of the other) is required. This leads to the introduction of uniform structures, which we will study in this chapter.

9.1 Uniform structures

9.1.1 Example. Let (X, d) be a metric space and $U_\varepsilon := \{(x, y) \in X \times X \mid d(x, y) < \varepsilon\}$. Then $\Delta = \{(x, x) \mid x \in X\} \subseteq U_\varepsilon$ and U_ε is open since $d : X \times X \to \mathbb{R}^+$ is continuous. A map $f : (X, d) \to (Y, d')$ is uniformly continuous if and only if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $d(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < \varepsilon \Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0 :$ $(x_1, x_2) \in U_\delta \Rightarrow (f(x_1), f(x_2)) \in U'_\varepsilon$ (where the latter is defined with respect to d'). A sequence in X is a Cauchy sequence if $\forall \varepsilon > 0\ \exists n_0\ \forall n, m \geq n_0 : (x_n, x_m) \in U_\varepsilon$. Thus the sets U_ε allow to characterize uniform continuity and Cauchy sequences in metric spaces. We therefore want to study some general properties of the system $\mathcal{B} := \{U_\varepsilon \mid \varepsilon > 0\}$. If $x \in X \Rightarrow \{y \in X \mid (x, y) \in U_\varepsilon\} = U_\varepsilon(x)$ is an ε-ball around x. Furthermore,

- $U_{\varepsilon_1}, \dots, U_{\varepsilon_n} \in \mathcal{B} \Rightarrow \bigcap_{i=1}^k U_{\varepsilon_i} = U_{\min\{\varepsilon_i \mid 1 \leq i \leq k\}} \in \mathcal{B}$.
- $U_\varepsilon \in \mathcal{B} \Rightarrow \Delta \subseteq U_\varepsilon$.
- $U_\varepsilon \in \mathcal{B} \Rightarrow \{(x, y) \mid (y, x) \in U_\varepsilon\} = U_\varepsilon \in \mathcal{B}$ (since $d(x, y) = d(y, x)$).
- $U_\varepsilon \in \mathcal{B} \Rightarrow \exists V \in \mathcal{B}$ with $(x, y) \in U_\varepsilon$ if $\exists z$ with $(x, z),\ (z, y) \in V$. In fact, by the triangle inequality we may set $V := U_{\varepsilon/2}$.

Generalizing these properties we are going to define *uniform structures* on general sets. For this we need some preparations.

9.1.2 Definition. *(Operations on relations) For X a set and $A, B \subseteq X \times X$ let*

$$A^{-1} := \{(x, y) \in X \times X \mid (y, x) \in A\} \qquad \textit{(reflection on } \Delta)$$

$$AB := \{(x, y) \in X \times X \mid \exists z \in X \textit{ with } (x, z) \in B \wedge (z, y) \in A\} \ \textit{(composition)}$$

$$A^2 = AA,\ A^n = AA^{n-1}.$$

$A \subseteq X \times X$ is called symmetric if $A = A^{-1}$. Subsets of $X \times X$ are called relations.

As an important special case, functions can be viewed as relations by identifying them with their graphs. Composition of relations then reduces to composition of functions: $A = \{(x, f(x)) \mid x \in X\}$, $B = \{(x, g(x)) \mid x \in X\} \Rightarrow AB = \{(x, f \circ g(x)) \mid x \in X\}$.

9.1.3 Lemma. *(Calculating with relations) Let $A, A', B, B', C \subseteq X \times X$. Then:*

(i) $A \subseteq B \Rightarrow A^{-1} \subseteq B^{-1}$

(ii) $A \subseteq A', B \subseteq B' \Rightarrow AB \subseteq A'B'$

(iii) $(A^{-1})^{-1} = A$

(iv) $A_i \subseteq X \times X \ \forall i \in I \Rightarrow (\bigcup_{i\in I} A_i)^{-1} = \bigcup_{i\in I} A_i^{-1}, \ (\bigcap_{i\in I} A_i)^{-1} = \bigcap_{i\in I} A_i^{-1}$

(v) $A(BC) = (AB)C$

(vi) $(AB)^{-1} = B^{-1}A^{-1}$

(vii) $A = A^{-1} \Rightarrow A^n = (A^n)^{-1} \quad \forall n \in \mathbb{N}$

(viii) $\Delta \circ A = A \circ \Delta = A$

(ix) $\Delta \subseteq A \Rightarrow A \subseteq A^n \quad \forall n$

Proof. (i) – (iv) are clear.
(v) $(x, y) \in A(BC) \Leftrightarrow \exists z : (x, z) \in BC \wedge (z, y) \in A \Leftrightarrow \exists z, z' : (x, z') \in C \wedge (z', z) \in B \wedge (z, y) \in A \Leftrightarrow \exists z' : (z', y) \in AB \wedge (x, z') \in C \Leftrightarrow (x, y) \in (AB)C$.
(vi) $(x, y) \in (AB)^{-1} \Leftrightarrow (y, x) \in AB \Leftrightarrow \exists z : (y, z) \in B \wedge (z, x) \in A \Leftrightarrow \exists z : (x, z) \in A^{-1} \wedge (z, y) \in B^{-1} \Leftrightarrow (x, y) \in B^{-1}A^{-1}$.
(vii) By (vi), $(A^n)^{-1} = (A^{-1})^n = A^n$.
(viii) $(x, y) \in \Delta \circ A \Leftrightarrow \exists z : (x, z) \in A \wedge \underbrace{(z, y) \in \Delta}_{\Leftrightarrow y = z} \Leftrightarrow (x, y) \in A$.
(ix) By (ii), $\Delta = \Delta^{n-1} \subseteq A^{n-1}$, so (viii) gives $A \subseteq A^n$. □

9.1.4 Definition. *Let X be a set and $\emptyset \neq \mathcal{U}$ a system of subsets of $X \times X$. $\mathcal{U}$ is called uniform structure or uniformity on X, if:*

(i) $U \in \mathcal{U}, \ A \subseteq X \times X$ *and* $U \subseteq A \Rightarrow A \in \mathcal{U}$.

(ii) $U_1, \dots, U_k \in \mathcal{U} \Rightarrow \bigcap_{i=1}^k U_i \in \mathcal{U}$.

(iii) $U \in \mathcal{U} \Rightarrow U \supseteq \Delta$.

(iv) $U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$.

(v) $U \in \mathcal{U} \Rightarrow \exists V \in \mathcal{U} : V^2 \subseteq U$.

The elements of $\mathcal{U}$ are called entourages. $(X, \mathcal{U})$ is called a uniform space. If $(x, y) \in U$, then x and y are called U-close.

9.1.5 Remark. Properties (iii), (iv), (v) correspond to reflexivity, symmetry, and triangle inequality of the metric in 9.1.1. Also, by (i), (ii) and (iii), $\mathcal{U}$ is a filter on $X \times X$.

9.1.6 Lemma. *Supposing (i)–(iii), (iv) and (v) from 9.1.4 are equivalent to:*

(iv') $\forall U \in \mathcal{U}\ \exists V \in \mathcal{U}$ with $V = V^{-1}$ and $V^2 \subseteq U$.

Proof. (iv)$\wedge$(v) $\Rightarrow$ (iv'): Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Then (ii) and (iv) imply $V \cap V^{-1} \in \mathcal{U}$. Also, $(V \cap V^{-1})^{-1} = (V \cap V^{-1})$ and $(V \cap V^{-1})^2 \subseteq V^2 \subseteq U$.
(iv') $\Rightarrow$ (iv) $\wedge$ (v): (v) is clear. Let $U \in \mathcal{U}$ and $V = V^{-1} \in \mathcal{U}$ with $V^2 \subseteq U$. Then by 9.1.3 (ix) and (vii), $V \subseteq V^2 = (V^2)^{-1} \subseteq U^{-1}$. Hence (i) gives $U^{-1} \in \mathcal{U}$, implying (iv). □

9.1.7 Definition. *Let $(X,\mathcal{U})$ be a uniform space and $\mathcal{B} \subseteq \mathcal{U}$. Then $\mathcal{B}$ is called a fundamental system of entourages, if $\forall U \in \mathcal{U}\ \exists B \in \mathcal{B}$ with $B \subseteq U$.*

Thus $\mathcal{B}$ is a fundamental system of entourages if and only if $\mathcal{B}$ is a basis of the filter $\mathcal{U}$.

9.1.8 Lemma. *Let $(X,\mathcal{U})$ be a uniform space and $\mathcal{B}$ a fundamental system of entourages of $\mathcal{U}$. Then also*

$$\mathcal{B}' := \{A \cap A^{-1} \mid A \in \mathcal{B}\} \text{ and } \mathcal{B}_n := \{A^n \mid A \in \mathcal{B}\} \quad (n \geq 1)$$

are fundamental systems of entourages of $\mathcal{U}$. In particular, for any uniform structure $\mathcal{U}$ and any $n \geq 1$ the system $\{A^n \mid A \in \mathcal{U},\ A = A^{-1}\}$ is a fundamental system of entourages.

Proof. Clearly, $\mathcal{B}' \subseteq \mathcal{U}$, and $\mathcal{B}'$ is a fundamental system of entourages.
Turning now to $\mathcal{B}_n$, let $n \geq 1$. Pick $k \in \mathbb{N}$ such that $2^k > n$. Let $U \in \mathcal{U}$. Then by 9.1.4 (v) and since $\mathcal{B}$ is a basis, there exists some $A_1 \in \mathcal{B}$ with $A_1^2 \subseteq U$. Also, $\exists A_2 \in \mathcal{B}$ with $A_2^2 \subseteq A_1, \ldots, \exists A_k \in \mathcal{B}$ with $A_k^2 \subseteq A_{k-1}$. Hence: $A_k^{2^k} \subseteq U$. Now for $m \geq 1$, $A_k^m = A_k A_k^{m-1} \subseteq A_k^2 A_k^{m-1} = A_k^{m+1}$, so $A_k^n \subseteq A_k^{2^k} \subseteq U$. It follows that $\mathcal{B}_n$ is a fundamental system of entourages. □

9.1.9 Proposition. *Let X be a set and $\emptyset \neq \mathcal{B} \subseteq \mathcal{P}(X \times X)$. TFAE:*

(i) $\mathcal{B}$ is a fundamental system of entourages of a uniform structure $\mathcal{U}$ on X.

(ii) (a) $B_1, B_2 \in \mathcal{B} \Rightarrow \exists B_3 \in \mathcal{B}$ with $B_3 \subseteq B_1 \cap B_2$.

(b) $\Delta \subseteq B\ \forall B \in \mathcal{B}$.

(c) $\forall B \in \mathcal{B}\ \exists B' \in \mathcal{B}$ with $B'^{-1} \subseteq B$.

(d) $\forall B \in \mathcal{B}\ \exists B' \in \mathcal{B}$ with $B'^2 \subseteq B$.

Proof. (i) $\Rightarrow$ (ii) is clear.
(ii) $\Rightarrow$ (i): Let $\mathcal{U} := \{U \subseteq X \times X \mid \exists B \in \mathcal{B} \text{ with } B \subseteq U\}$. Then 9.1.4 (i) – (iii) for $\mathcal{U}$ are obvious. (iv): Let $U \in \mathcal{U}$, $B \subseteq U$. Then by (c), there exists some B' with $B'^{-1} \subseteq B \subseteq U$, so $B' \subseteq U^{-1} \Rightarrow U^{-1} \in \mathcal{U}$. Finally, let $U \in \mathcal{U}$, $B \subseteq U$, B' such that $B'^2 \subseteq B \subseteq U$. Then since $B' \in \mathcal{U}$, (v) follows. □

9.1.10 Example. Let (X,d) be a metric space. Then $\mathcal{B} := \{U_\varepsilon \mid \varepsilon > 0\}$ and $\mathcal{B}' = \{U_{\frac{1}{n}} \mid n \geq 1\}$ are fundamental systems of entourages of the same uniform structure $\mathcal{U}$, the so-called uniform structure of the metric space (X,d). A uniform space whose uniform structure is generated in this way by a metric is called metrizable.

As we have seen in 9.1.1, the sets $U_\varepsilon(x) := \{y \in X \mid (x,y) \in U_\varepsilon\}$ generate the topology of (X,d). We now want to generalize this to general uniform spaces:

9.1.11 Definition. *Let X be a set and let $U \in \mathcal{P}(X \times X)$. For $x \in X$ let*

$$U(x) := \{y \in X \mid (x, y) \in U\}.$$

Then $U(x)$ is called the image of x under U. Analogously, for $A \subseteq X$

$$U(A) := \{y \in X \mid \exists x \in A \text{ with } (x, y) \in U\} = \bigcup_{x \in A} U(x)$$

is called the image of A under U. $U(A)$ is also called a uniform neighborhood of A.

9.1.12 Example. Let $f : X \to X$, $U := \{(x, f(x)) \mid x \in X\} \Rightarrow U(x) = \{y \mid (x, y) \in U\} = \{f(x)\}$, and $U(A) = f(A)$.

9.1.13 Theorem. *Let $(X,\mathcal{U})$ be a uniform space and for $x \in X$ let $\mathcal{U}(x) := \{U(x) \mid U \in \mathcal{U}\}$. Then there is a unique topology $\mathcal{O}_{\mathcal{U}}$ on X such that for all $x \in X$ the family $\mathcal{U}(x)$ is the system of neighborhoods of x.*

Proof. We have to verify the characteristic properties of the neighborhood filter, cf. [5, 2.9].

(a) $U(x) \in \mathcal{U}(x), V \supseteq U(x) \Rightarrow V \in \mathcal{U}(x)$: In fact, let $V' := U \cup \{(x, y) \mid y \in V\}$. Then by 9.1.4 (i), $V' \in \mathcal{U}$ and $V'(x) = U(x) \cup V = V$.

(b) $U(x), V(x) \in \mathcal{U}(x) \Rightarrow U(x) \cap V(x) \in \mathcal{U}(x)$: Indeed, $U \cap V \in \mathcal{U}$ and $(U \cap V)(x) = U(x) \cap V(x)$.

(c) $U(x) \in \mathcal{U}(x) \Rightarrow x \in \mathcal{U}(x)$: $\Delta \subseteq U \Rightarrow \{x\} = \Delta(x) \subseteq U(x)$.

(d) $\forall U(x) \in \mathcal{U}(x)\ \exists V(x) \in \mathcal{U}(x)$ with: $U(x) \in \mathcal{U}(y)\ \forall y \in V(x)$: To see this, first note that by 9.1.4 (v) there exists some $V \in \mathcal{U}$ with $V^2 \subseteq U$. Let $y \in V(x)$. Then $V(y) \in \mathcal{U}(y)$ and $V(y) \subseteq U(x)$: Let $z \in V(y) \Rightarrow (y, z) \in V$ and: $y \in V(x) \Rightarrow (x, y) \in V \Rightarrow (x, z) \in V^2 \subseteq U \Rightarrow z \in U(x)$. Thus $U(x) \in \mathcal{U}(y)$.

□

We will always equip any uniform space $(X,\mathcal{U})$ with the topology $\mathcal{O}_{\mathcal{U}}$. All topological properties (T_i, compact, ...) therefore refer to $\mathcal{O}_{\mathcal{U}}$.

9.1.14 Examples. (i) The extreme cases: $\mathcal{U} = \{X \times X\}$ is a uniform structure such that $\mathcal{O}_{\mathcal{U}}$ is the indiscrete topology. $\mathcal{U}$ is therefore called the indiscrete uniform structure. The singleton $\mathcal{B} := \{\Delta\}$ is a fundamental system of entourages of the so-called discrete uniform structure. It induces the discrete topology on X since $\Delta(x) = \{x\}\ \forall x \in X$.

(ii) Let X be a set , Y a uniform space with fundamental system of entourages $\mathcal{B}$ and $F(X, Y) := \{f : X \to Y\}$. Let

$$W(X, B) := \{(f, g) \in F(X, Y) \times F(X, Y) \mid (f(x), g(x)) \in B\ \forall x \in X\}.$$

Then $\{W(X, B) \mid B \in \mathcal{B}\}$ is a fundamental system of entourages on $F(X, Y)$. If $\mathcal{B}$ stems from a metric d, then also this structure is induced by a (pseudo-)metric, namely by $d_*(f, g) := \sup_{x \in X} d(f(x), g(x))$, cf. 12.1.11 below.

(iii) Let G be a topological group, $\mathcal{V}$ a neighborhood basis of $e \in G$, and set for any $V \in \mathcal{U}(e)$

$$U_V := \{(x, y) \mid xy^{-1} \in V\}.$$

Then $\mathcal{B} := \{U_V \mid V \in \mathcal{V}\}$ is a fundamental system of entourages of a uniform structure on G. To see this, we verify (a) – (d) from 9.1.9 (ii):

(a) $U_{V_1} \cap U_{V_2} = U_{V_1 \cap V_2}$.

(b) $\Delta \subseteq U_V$ since $xx^{-1} = e \in V \ \forall V \in \mathcal{V}$.

(c) $U_V^{-1} = \{(x,y) \mid yx^{-1} \in V\} = \{(x,y) \mid xy^{-1} \in V^{-1}\} = U_{V^{-1}}$, where $V^{-1} := \{z^{-1} \mid z \in V\} \in \mathcal{U}_e$ because $z \mapsto z^{-1}$ is a homeomorphism. Hence there exists some $V_1 \in \mathcal{V}$ with $V_1 \subseteq V^{-1} \Rightarrow U_{V_1} \subseteq U_{V^{-1}} \subseteq U_V^{-1}$.

(d) $\cdot : G \times G \to G$ is continuous, so $\forall V \in \mathcal{V}\ \exists W_1, W_2 \in \mathcal{V}$ with $W_1 W_2 \subseteq V$. Let $W := W_1 \cap W_2 \Rightarrow W \cdot W \subseteq V$. Let $(x,y) \in U_W^2 \Rightarrow \exists z$ such that $(x,z) \in U_W \wedge (z,y) \in U_W \Rightarrow xy^{-1} = xz^{-1}zy^{-1} \in W \cdot W \subseteq V \Rightarrow (x,y) \in U_V$.

Furthermore, $U_V(x) = \{y \mid xy^{-1} \in V\} = \{y \mid y \in V^{-1}x\} = V^{-1}x$, so the neighborhoods of x are precisely the translates of the neighborhoods of e.

9.1.15 Proposition. *Let $\mathcal{B}$ be a fundamental system of entourages of the uniform structure $\mathcal{U}$ on X. Then also*

$$\mathcal{B}' := \{B^\circ \mid B \in \mathcal{B}\} \text{ and } \mathcal{B}'' = \{\bar{B} \mid B \in \mathcal{B}\}$$

are fundamental systems of entourages of $\mathcal{U}$.

Proof. Let $B \in \mathcal{B}$. Then by 9.1.8 there exists some $W \in \mathcal{U}$ with $W = W^{-1}$ and $W^3 \subseteq B$. We show that $W \subseteq B^\circ$.
In fact, let $(x,y) \in W$. Then $W(x) \times W(y)$ is a neighborhood of (x,y) in $X \times X$. Let $(z,z') \in W(x) \times W(y)$. Then $(x,z) \in W$, $(y,z') \in W$, $(x,y) \in W \Rightarrow (z,x),\ (x,y),\ (y,z') \in W \Rightarrow (z,z') \in W^3 \subseteq B \Rightarrow W(x) \times W(y) \subseteq B \Rightarrow (x,y) \in B^\circ \Rightarrow B^\circ \in \mathcal{U}\ \forall B \in \mathcal{B} \Rightarrow \mathcal{B}'$ is a fundamental system of entourages of $\mathcal{U}$.
Next we show that $\overline{W} \subseteq B$. To see this, let $(x,y) \in \overline{W}$. Since $W(x) \times W(y)$ is a neighborhood of (x,y), there exists some $(z,z') \in W \cap (W(x) \times W(y)) \Rightarrow (x,z),(y,z'),(z,z') \in W \Rightarrow (x,z),(z,z'),(z',y) \in W \Rightarrow (x,y) \in W^3 \subseteq B$. Since $\mathcal{B}$ is a fundamental system of entourages there exists some $B' \in \mathcal{B}$ with $B' \subseteq W \Rightarrow \overline{B'} \subseteq B \Rightarrow \mathcal{B}''$ is a fundamental system of entourages of $\mathcal{U}$. □

9.1.16 Proposition. *Let $(X, \mathcal{U})$ be a uniform space and $A \subseteq X$. Then*

$$\bar{A} = \bigcap_{U \in \mathcal{U}} U(A).$$

Hence $\bar{A}$ is is the intersection of all uniform neighborhoods of A.

Proof. $x \in \bar{A} \Leftrightarrow \forall U \in \mathcal{U} : U(x) \cap A \neq \emptyset \Leftrightarrow \forall U \in \mathcal{U} : x \in U^{-1}(A)$. By 9.1.4 this is equivalent to: $\forall U \in \mathcal{U} : x \in U(A)$. □

9.1.17 Theorem. *Let $(X, \mathcal{U})$ be a uniform space. Then*

(i) X is T_3.

(ii) X is T_2 $\Leftrightarrow \bigcap_{U \in \mathcal{U}} U = \Delta$.

(iii) Any uniform Hausdorff space is regular.

Proof. By 3.1.6 we have to show that for any $x \in X$ the closed neighborhoods form a neighborhood basis. Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Let $x \in X$. Then by 9.1.16, $\overline{V(x)} \subseteq V(V(x))$. Now $y \in V(V(x)) \Leftrightarrow \exists z \in V(x) :\ (z,y) \in V \Leftrightarrow y \in V^2(x)$. Hence $V(V(x)) = V^2(x) \subseteq U(x)$.

(ii) ($\Rightarrow$): Clearly $\Delta \subseteq \bigcap_{U\in\mathcal{U}} U$. Conversely, let $(x,y) \notin \Delta$, i.e., $x \neq y$. Then since X is T_2, there exist $U_1, U_2 \in \mathcal{U}$ such that $U_1(x) \cap U_2(y) = \emptyset$. Then $U := U_1 \cap U_2 \in \mathcal{U}$ and $U(x) \cap U(y) \subseteq U_1(x) \cap U_2(y) = \emptyset$. In particular, $y \notin U(x)$, i.e., $(x,y) \notin U \Rightarrow (x,y) \notin \bigcap_{U\in\mathcal{U}} U$.
($\Leftarrow$): Let $x \neq y \Rightarrow (x,y) \notin \Delta = \bigcap_{U\in\mathcal{U}} U \Rightarrow \exists U \in \mathcal{U}$ with $(x,y) \notin U$. Pick $W \in \mathcal{U}$ with $W = W^{-1}$ and $W^2 \subseteq U$. Then $W(x) \cap W(y) = \emptyset$: if there existed some $z \in W(x) \cap W(y)$, then $(x,z) \in W$, $(y,z) \in W \Rightarrow (x,z), (z,y) \in W \Rightarrow (x,y) \in W^2 \subseteq U$, a contradiction.

(iii) This is immediate from (i). □

In fact, as is easily verified, condition (ii) is also equivalent to X being T_0.

9.1.18 Proposition. *Let $(X,\mathcal{U})$ be a uniform space, $K \subseteq X$ compact and $A \subseteq X$ closed. Then*

(i) The uniform neighborhoods of K form a neighborhood basis of K.

(ii) If $K \cap A = \emptyset$, then A and K possess disjoint uniform neighborhoods.

Proof. (i) Let U be a neighborhood of K in $X \Rightarrow \forall x \in K\ \exists V_x \in \mathcal{U}$ such that $V_x(x) \subseteq U$. Let $W_x \in \mathcal{U}$ such that $W_x^2 \subseteq V_x$. Now $(W_x(x)^\circ)_{x\in K}$ is an open cover of K. Hence there exists some $L \subseteq K$ finite with $K \subseteq \bigcup_{x\in L} W_x(x)$. Let $W := \bigcap_{x\in L} W_x \Rightarrow W \in \mathcal{U}$ and $W(K) \subseteq U$: Let $y \in W(K) \Rightarrow \exists x \in K$ with $(x,y) \in W$. Moreover, $x \in K \Rightarrow \exists z \in L$ with $x \in W_z(z)$, i.e. $(z,x) \in W_z \Rightarrow (z,y) \in WW_z \subseteq W_z W_z \subseteq V_z \Rightarrow y \in V_z(z) \subseteq U$.

(ii) $K \cap A = \emptyset$, so K is contained in the open set $X \setminus A$. Thus by (i) there exists some $W \in \mathcal{U}$ with $W(K) \subseteq X \setminus A$. Let V be symmetric with $V^2 \subseteq W$. Then $V(K) \cap V(A) = \emptyset$: in fact, suppose that there exists some $x \in V(K) \cap V(A)$. Then there exist $k \in K$, $a \in A$ with $(k,x) \in V$, $(a,x) \in V$. Since $V = V^{-1}$, this gives $(k,a) \in V^2 \subseteq W$, so $a \in W(k) \subseteq W(K)$, a contradiction. □

9.2 Uniformly continuous maps

9.2.1 Definition. *Let $(X,\mathcal{U}_X), (Y,\mathcal{U}_Y)$ be uniform spaces, $f : X \to Y$. f is called uniformly continuous, if: $\forall V \in \mathcal{U}_Y\ \exists U \in \mathcal{U}_X$ with $(f \times f)(U) \subseteq V$. Here, $(f \times f)(x,y) := (f(x), f(y))$.*

Thus f is uniformly continuous if and only if $(f \times f)^{-1}(V) \in \mathcal{U}_X\ \forall V \in \mathcal{U}_Y$.

9.2.2 Proposition. *Let $f : (X,\mathcal{U}_X) \to (Y,\mathcal{U}_Y)$ uniformly continuous. Then f is continuous.*

Proof. Let $x \in X$ and $V(f(x))$ a neighborhood of $f(x)$. Let $U \in \mathcal{U}_X$ be such that $(f \times f)(U) \subseteq V$. Then $f(U(x)) \subseteq V(f(x))$: Indeed, let $y \in U(x) \Rightarrow (x,y) \in U$. Then $(f(x), f(y)) \in V$, so $f(y) \in V(f(x))$. □

9.2.3 Examples.

(i) id_X is uniformly continuous.

(ii) Any constant map is uniformly continuous: Let $f(x) \equiv y_0 \Rightarrow (f \times f)(U) = \{(y_0, y_0)\} \subseteq \Delta \subseteq V\ \forall V \in \mathcal{U}_Y\ \forall U \in \mathcal{U}_X$.

(iii) The composition of uniformly continuous maps is uniformly continuous, because $(f \circ g \times f \circ g)^{-1}(V) = (g \times g)^{-1}((f \times f)^{-1}(V))$.

(iv) $x \mapsto x^{-1}$ is not uniformly continuous on (0,1).

As we know from real analysis, if $f : [a, b] \to \mathbb{R}$ is continuous then f is also uniformly continuous. More generally, we have:

9.2.4 Theorem. *Let $(X,\mathcal{U}_X)$ be compact, $(Y,\mathcal{U}_Y)$ a uniform space and $f : X \to Y$ continuous. Then f is uniformly continuous.*

Proof. Let $W \in \mathcal{U}_Y$ and $V \in \mathcal{U}_Y$ symmetric such that $V^2 \subseteq W$. Since f is continuous, $\forall x \in X\ \exists \tilde{U}_x \in \mathcal{U}_X$ with $f(\tilde{U}_x(x)) \subseteq V(f(x))$. Choose $U_x \in \mathcal{U}_X$ symmetric with $U_x^2 \subseteq \tilde{U}_x$. Then

$$f(U_x(x)) \subseteq f(U_x^2(x)) \subseteq f(\tilde{U}_x(x)) \subseteq V(f(x)). \tag{9.2.1}$$

$(U_x(x)^\circ)_{x\in X}$ is an open cover of X, so there exists some finite set $L \subseteq X$ with $X = \bigcup_{x\in L} U_x(x)$. Let $U := \bigcap_{z\in L} U_z \ \in \mathcal{U}_X$.
To conclude the proof we show that $(f \times f)(U) \subseteq W$: Let $(x, y) \in U$. Then there exists some $z \in L$ with $x \in U_z(z)$, so (9.2.1) implies that $f(x) \in V(f(z))$, i.e., $(f(z), f(x)) \in V$. Since $(z, x) \in U_z$ and $(x, y) \in U$, $(z, y) \in UU_z \subseteq U_z^2$, so $y \in U_z^2(z)$, and (9.2.1) gives $f(y) \in V(f(z)) \Rightarrow (f(z), f(x)) \in V,\ (f(z), f(y)) \in V$. Finally, since V is symmetric, we conclude that $(f(x), f(y)) \in V^2 \subseteq W$. □

9.2.5 Definition. *Two uniform spaces X and Y are called isomorphic if there exists a bijective map $f : X \to Y$ such that both f and f^{-1} are uniformly continuous. Then also the uniform structures of X and Y are called isomorphic.*

Since by 9.2.2 both f and f^{-1} are then continuous, $f : X \to Y$ is also a homeomorphism.

9.3 Construction of uniform spaces

9.3.1 Definition. *If $\mathcal{U}_1$,$\mathcal{U}_2$ are uniform structures on a set X then $\mathcal{U}_1$ is called finer than $\mathcal{U}_2$ (and $\mathcal{U}_2$ coarser than $\mathcal{U}_1$) if $\mathcal{U}_1 \supseteq \mathcal{U}_2$ (i.e., $U \in \mathcal{U}_2 \Rightarrow U \in \mathcal{U}_1$).*

9.3.2 Proposition. *Let $\mathcal{U}_1$, $\mathcal{U}_2$ be uniform structures on a set X. Then:*

(i) $\mathcal{U}_1 \supseteq \mathcal{U}_2 \Leftrightarrow \mathrm{id} : (X,\mathcal{U}_1) \to (X,\mathcal{U}_2)$ is uniformly continuous.

(ii) $\mathcal{U}_1 \supseteq \mathcal{U}_2 \Rightarrow \mathcal{O}_{\mathcal{U}_1} \supseteq \mathcal{O}_{\mathcal{U}_2}$.

Proof. (i) is clear.
(ii) For $U \in \mathcal{U}_2$, $x \in X$, let $U(x)$ be a neighborhood of x in $\mathcal{O}_{\mathcal{U}_2}$. Then since $U \in \mathcal{U}_1$, $U(x)$ is a neighborhood of x in $\mathcal{O}_{\mathcal{U}_1}$. □

9.3.3 Theorem. *Let X be a set, $((Y_i,\mathcal{U}_i))_{i\in I}$ a family of uniform spaces, and $f_i : X \to Y_i$ $(i \in I)$ maps. Then:*

(i) $\mathcal{B} := \{\bigcap_{i\in J}(f_i \times f_i)^{-1}(V_i) \mid J \subseteq I$ finite, $V_i \in \mathcal{U}_i$ for $i \in J\}$ is a fundamental system of entourages of a uniform structure $\mathcal{U}$ on X. $\mathcal{U}$ is called the initial uniform structure on X with respect to $(f_i)_{i\in I}$.

(ii) $\mathcal{U}$ is the coarsest uniform structure on X for which all f_i are uniformly continuous.

(iii) Let Z be a uniform space and $h : Z \to (X,\mathcal{U})$. Then h is uniformly continuous if and only if $f_i \circ h$ is uniformly continuous for each $i \in I$.

(iv) $\mathcal{U}$ induces on X the initial topology with respect to $(f_i)_{i\in I}$.

Proof. (i) To show that $\mathcal{B}$ is a fundamental system of entourages, we verify the conditions from 9.1.9. First, (a) and (b) are immediate.
(c) By 9.1.3 (iv),

$$\Big(\bigcap_{i\in J}(f_i \times f_i)^{-1}(V_i)\Big)^{-1} = \bigcap_{i\in J}((f_i \times f_i)^{-1}(V_i))^{-1} = \bigcap_{i\in J}(f_i \times f_i)^{-1}(V_i^{-1}) \in \mathcal{B}.$$

(d) Let $\bigcap_{i\in J}(f_i \times f_i)^{-1}(V_i) \in \mathcal{B}$ and for $i \in J$ pick $W_i \in \mathcal{U}_i$ such that $W_i^2 \subseteq V_i$. Now let $(x,y) \in (\bigcap_{i\in J}(f_i \times f_i)^{-1}(W_i))^2$. Then $\exists z$ such that $(x,z) \in \bigcap_{i\in J}(f_i \times f_i)^{-1}(W_i) \wedge (z,y) \in \bigcap_{i\in J}(f_i \times f_i)^{-1}(W_i)$, so $(f_i(x), f_i(z)) \in W_i$ and $(f_i(z), f_i(y)) \in W_i \Rightarrow (f_i(x), f_i(y)) \in W_i^2$ $\forall i \in J$. Therefore,

$$(\bigcap_{i\in J}(f_i \times f_i)^{-1}(W_i))^2 \subseteq \bigcap_{i\in J}(f_i \times f_i)^{-1}(W_i^2) \subseteq \bigcap_{i\in J}(f_i \times f_i)^{-1}(V_i).$$

(ii) Any f_i is uniformly continuous for $\mathcal{U}$. Conversely, if $\mathcal{U}'$ is a uniform structure for which all the f_i are uniformly continuous, then $\mathcal{U}' \supseteq \mathcal{B}$, so $\mathcal{U}' \supseteq \mathcal{U}$.
(iii) $\Rightarrow$: is clear by 9.2.3 (iii).
$\Leftarrow$: Let $B := \bigcap_{i\in J}(f_i \times f_i)^{-1}(V_i) \in \mathcal{B}$. Then $(h\times h)^{-1}(B) = \bigcap_{i\in J}(f_i\circ h \times f_i\circ h)^{-1}(V_i)$ is an entourage in Z, so h is uniformly continuous.
(iv) Let $\mathcal{O}$ be the initial topology with respect to $(f_i)_{i\in I}$ and let $x \in X$. By 1.2.2, a neighborhood basis of x with respect to $\mathcal{O}$ is formed by the finite intersections of sets $f_i^{-1}(O_i)$ with $O_i \in \mathcal{O}_{\mathcal{U}_i}$ and $x \in f_i^{-1}(O_i)$, hence of finite intersections $f_i^{-1}(O_i)$ with O_i an open neighborhood of $f_i(x)$. Thus also the finite intersections of sets $f_i^{-1}(W_i)$ with W_i an element of a neighborhood basis of $f_i(x)$ form a neighborhood basis of x. By definition of $\mathcal{O}_{\mathcal{U}_i}$, we may take W_i of the form $V_i(f_i(x))$ for some $V_i \in \mathcal{U}_i$. Finally, since

$$\begin{aligned}\bigcap_{i\in J} f_i^{-1}(V_i(f_i(x))) &= \bigcap_{i\in J}\{y \mid f_i(y) \in V_i(f_i(x))\} = \bigcap_{i\in J}\{y \mid (f_i(x), f_i(y)) \in V_i\}\\ &= \bigcap_{i\in J}\{y \mid (x,y) \in (f_i \times f_i)^{-1}(V_i)\}\\ &= \{y \mid (x,y) \in \bigcap_{i\in J}(f_i \times f_i)^{-1}(V_i)\} = \Big(\bigcap_{i\in J}(f_i \times f_i)^{-1}(V_i)\Big)(x),\end{aligned}$$

x has the same neighborhoods with respect to $\mathcal{O}$ and $\mathcal{O}_{\mathcal{U}}$, so $\mathcal{O} = \mathcal{O}_{\mathcal{U}}$. □

9.3.4 Remark. By 9.3.3 (iii), the uniform structure $\mathcal{U}$ on X satisfies a universal property that results from (1.2.1) by replacing 'continuous' by 'uniformly continuous' there. The same replacement in 1.2.3 implies the transitivity of initial uniform structures.

9.3.5 Definition. *Let $(X,\mathcal{U})$ be a uniform space and let $A \subseteq X$, $j : A \hookrightarrow X$ the inclusion map. Let $\mathcal{U}_A$ be the initial uniform structure on A with respect to j. Then $(A,\mathcal{U}_A)$ is called a uniform subspace of X. We have $\mathcal{U}_A = \{U \cap (A \times A) \mid U \in \mathcal{U}\}$. By 9.3.3, $\mathcal{U}_A$ induces the trace topology on A.*

9.3.6 Example. Least upper bound of a family of uniform structures (cf. 1.2.5): Let $(\mathcal{U}_i)_{i\in I}$ be a family of uniform structures on a set X. The initial uniform structure on X with respect to $\mathrm{id} : X \to (X,\mathcal{U}_i)$ $(i \in I)$ is the coarsest uniform structure that is finer than every $\mathcal{U}_i$. It is called the supremum of $(\mathcal{U}_i)_{i\in I}$, and it induces on X exactly the supremum of the topologies $(\mathcal{O}_{\mathcal{U}_i})_{i\in I}$. A basis of $\mathcal{U}$ is formed by the finite intersections of elements from $\bigcup_{i\in I} \mathcal{U}_i$.

9.3.7 Proposition. *Let $(X,\mathcal{U})$ be a uniform space, $A \subseteq X$ dense in X. Then $\{\overline{U \cap (A \times A)} \mid U \in \mathcal{U}\}$ is a fundamental system of entourages of $\mathcal{U}$. In particular, if $V \in \mathcal{U}_A$, then $\bar{V} \in \mathcal{U}$.*

Proof. Let $U \in \mathcal{U}$. Then if $(x, y) \in U^\circ \cap \overline{A \times A}$ and if $W \in \mathcal{U}((x, y))$, it follows that $W \cap U^\circ \in \mathcal{U}((x, y))$, so $W \cap U^\circ \cap (A \times A) \neq \emptyset$. Therefore,

$$\overline{U \cap (A \times A)} \supseteq \overline{U^\circ \cap (A \times A)} \supseteq U^\circ \cap \overline{(A \times A)} = U^\circ \cap (\bar{A} \times \bar{A}) = U^\circ \in \mathcal{U}$$

by 9.1.15. Hence $\overline{U \cap (A \times A)} \in \mathcal{U}$. Now let $V \in \mathcal{U}$. Then by 9.1.15 there exists some $U \in \mathcal{U}$ with $\bar{U} \subseteq V$, so $\overline{U \cap (A \times A)} \subseteq V$. □

9.3.8 Definition. *Let $(X_i)_{i\in I}$ be uniform spaces and $p_j : \prod_{i\in I} X_i \to X_j$ the projections. The initial uniform structure on $\prod_{i\in I} X_i$ with respect to $(p_i)_{i\in I}$ is called the product of the uniform structures of the $(X_i)_{i\in I}$.*

The analogues of 1.1.13 and 1.1.14 hold (with analogous proofs) also for uniform spaces. By 9.3.3 (iv), the uniform structure on $\prod_{i\in I} X_i$ induces precisely the product topology.

9.4 Uniformization

9.4.1 Definition. *Let $(X, \mathcal{O})$ be a topological space. X is called uniformizable if there exists a uniform structure $\mathcal{U}$ on X with $\mathcal{O}_{\mathcal{U}} = \mathcal{O}$.*

Our aim is to characterize uniformizability purely in topological terms. In fact we shall see that precisely the T_{3a}-spaces are uniformizable (see 9.4.15).

9.4.2 Examples.

(i) Any metric space is uniformizable.

(ii) Any subspace of a uniformizable space is uniformizable (cf. 9.3.5).

9.4.3 Definition. *Let X be a set. A map $d : X \times X \to [0, \infty]$ is called a pseudometric if $\forall x, y, z \in X$:*

(i) $d(x, x) = 0$

(ii) $d(x, y) = d(y, x)$

(iii) $d(x, y) \leq d(x, z) + d(z, y)$

Note that, differently from metrics, even for $x \neq y$ we may have $d(x, y) = 0$.

9.4.4 Examples.

(i) Any metric is a pseudometric.

(ii) On $\mathcal{L}^1(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ measurable and } \int |f| < \infty\}$, the map $d(f,g) := \int |f-g|$ is a pseudometric but not a metric ($d(f,g) = 0 \Rightarrow f = g$ almost everywhere).

9.4.5 Definition. *Let d be a pseudometric on a set X. Then*

$$\mathcal{B}_d := \{d^{-1}([0,a]) \mid a \in \mathbb{R}, a > 0\}$$

is a fundamental system of entourages of a uniform structure $\mathcal{U}_d$ (cf. 9.1.1), called the uniform structure generated by d. If $(d_i)_{i\in I}$ is a family of pseudometrics on X then the supremum (cf. 9.3.6) of the uniform structures generated by the d_i is called the uniform structure generated by the family $(d_i)_{i\in I}$.

By 9.3.3, a fundamental system of entourages for the uniform structure generated by $(d_i)_{i\in I}$ is given by

$$\mathcal{B} = \{\bigcap_{i\in J} d_i^{-1}([0,a]) \mid J \subseteq I \text{ finite}, \ a \in (0,1]\}.$$

9.4.6 Proposition. *Let $(X,\mathcal{U})$ be a uniform space such that $\mathcal{U}$ possesses a countable fundamental system of entourages. Then there exists a pseudometric d on X that generates $\mathcal{U}$.*

Proof. Let $\{V_n \mid n \geq 1\}$ be a fundamental system of entourages for $\mathcal{U}$. For any $n \geq 1$ we pick a symmetric $U_n \in \mathcal{U}$ such that $U_1 \subseteq V_1$ and $U_{n+1}^3 \subseteq U_n \cap V_n$ (cf. 9.1.8). Since $U_{n+1} \subseteq U_{n+1}^3$, also $\{U_n \mid n \geq 1\}$ is a fundamental system of entourages for $\mathcal{U}$ and $U_{n+1} \subseteq U_n \ \forall n$. Let

$$g(x,y) := \left\{ \begin{array}{l} 1 \text{ for } (x,y) \notin U_1 \\ \inf\{2^{-k} \mid (x,y) \in U_k\} \text{ else} \end{array} \right. = \left\{ \begin{array}{ll} 1 & \text{for } (x,y) \notin U_1 \\ 0 & \text{for } (x,y) \in \bigcap_{n\geq 1} U_n \\ 2^{-k} & \text{for } (x,y) \in U_n \text{ for } 1 \leq n \leq k \\ & \text{but } (x,y) \notin U_{k+1} \end{array} \right.$$

Then $0 \leq g(x,y) \leq 1$ and $g(x,x) = 0$. Also, $g(x,y) = g(y,x)$ since all U_k are symmetric. Now set

$$d(x,y) := \inf\{\sum_{i=0}^{n-1} g(z_i, z_{i+1}) \mid n \geq 1, \ z_0, \dots, z_n \in X, \ z_0 = x, \ z_n = y\}.$$

Then $d \geq 0$ and $d(x,x) \leq g(x,x) = 0$. Since $g(z_i, z_{i+1}) = g(z_{i+1}, z_i) \Rightarrow d(x,y) = d(y,x)$. It remains to show the triangle inequality for d: Let $x, y, z \in X$ and $\varepsilon > 0 \Rightarrow \exists p \in \mathbb{N}$ and $x = z_0, z_1, \dots, z_p = y$ with $\sum_{i=0}^{p-1} g(z_i, z_{i+1}) < d(x,y) + \varepsilon$ and $\exists q \in \mathbb{N}$ and points $z_{p+1}, \dots, z_{p+q} = z$ with $\sum_{i=p}^{p+q-1} g(z_i, z_{i+1}) < d(y,z) + \varepsilon$. Thus by definition of d,

$$d(x,z) \leq \sum_{i=0}^{p+q-1} g(z_i, z_{i+1}) < d(x,y) + d(y,z) + 2\varepsilon,$$

and since ε was arbitrary, the triangle inequality for d is established.

Next we claim that

$$\frac{1}{2} g(x,y) \leq d(x,y) \leq g(x,y) \quad \forall x, y \in X. \tag{9.4.1}$$

Here, $d(x, y) \le g(x, y)$ is immediate from the definition. Also, the left hand inequality means that $\forall n \ge 1$ and $\forall z_0 = x, z_1, \dots, z_n = y$ we have:

$$\frac{1}{2} g(x, y) \le \sum_{i=0}^{n-1} g(z_i, z_{i+1}) \tag{9.4.2}$$

We show this by induction over n. The case $n = 1$ is clear. Suppose that (9.4.2) has already been established for all natural numbers $< n$. Let $z_0 = x, \dots, z_n = y$ be points in X. We set $a := \sum_{i=0}^{n-1} g(z_i, z_{i+1})$. If $a \ge \frac{1}{2}$ there is nothing to prove because $g \le 1$. Thus let $a < \frac{1}{2}$ and assume for the moment that $a > 0$. Let m be the highest index such that $\sum_{i=0}^{m-1} g(z_i, z_{i+1}) \le \frac{a}{2}$, so in particular, $m < n$.

$$\Rightarrow \sum_{i=0}^{m} g(z_i, z_{i+1}) > \frac{a}{2} \quad \Rightarrow \sum_{i=m+1}^{n-1} g(z_i, z_{i+1}) \le \frac{a}{2}.$$

As $m < n$, the induction hypothesis implies

$$\frac{1}{2} g(x, z_m) \le \sum_{i=0}^{m-1} g(z_i, z_{i+1}) \le \frac{a}{2} \quad \text{(also for } m = 0, \text{ since } g(x, z_0) = 0)$$

and: $\frac{1}{2} g(z_{m+1}, y) \le \sum_{i=m+1}^{n-1} g(z_i, z_{i+1}) \le \frac{a}{2}$. By definition of a and since $m < n$, $g(z_m, z_{m+1}) \le a$. Due to $a > 0$ we may choose k such that $2^{-k} \le a < 2^{-k+1}$. Since $a < 2^{-1} \Rightarrow k \ge 2$. Therefore, by definition of g,

$$\begin{aligned} g(x, z_m), g(z_m, z_{m+1}), g(z_{m+1}, y) < 2^{-k+1} &\Rightarrow (x, z_m), (z_m, z_{m+1}), (z_{m+1}, y) \in U_k \\ \Rightarrow (x, y) \in U_k^3 \subseteq U_{k-1} &\Rightarrow \frac{1}{2} g(x, y) \le \frac{1}{2} 2^{-(k-1)} = 2^{-k} \le a. \end{aligned}$$

On the other hand, if $a = \sum_{i=0}^{n-1} g(z_i, z_{i+1}) = 0 \Rightarrow g(z_i, z_{i+1}) = 0$ $(i = 0, \dots, n-1)$. $\Rightarrow (z_i, z_{i+1}) \in U_k \ \forall k \in \mathbb{N} \Rightarrow (x, y) = (z_0, z_n) \in U_k^n \ \forall k$. Let $n = 3p + r$ $(r \in \{0, 1, 2\})$

$$\begin{array}{lll} \Rightarrow \quad (x, y) \in U_{k-1}^p \, U_k^r & \subseteq U_{k-1}^p \, U_{k-1} \ \forall k, \dots, \Rightarrow (x, y) \in U_k \ \forall k \\ \uparrow & \uparrow \\ U_k^3 \subseteq U_{k-1} & U_k \subseteq U_k^2 \subseteq U_k^3 \subseteq U_{k-1} \end{array}$$

$$\Rightarrow g(x, y) = 0 \Rightarrow \frac{1}{2} g(x, y) \le a$$

$\Rightarrow$ (9.4.2) and thereby (9.4.1).

Finally, we show that d generates the uniform structure $\mathcal{U}$:

$\mathcal{U} \supseteq \mathcal{U}_d$: By (9.4.1), if $(x, y) \in U_k$ then $d(x, y) \le g(x, y) \le 2^{-k} \Rightarrow U_k \subseteq d^{-1}([0, 2^{-k}])$.

$\mathcal{U}_d \subseteq \mathcal{U}$: Let $(x, y) \in d^{-1}([0, 2^{-k}])$. Then again by (9.4.1), $\frac{1}{2} g(x, y) \le d(x, y) \le 2^{-k} \Rightarrow g(x, y) \le 2^{-k+1} \Rightarrow (x, y) \in U_{k-1} \Rightarrow d^{-1}([0, 2^{-k}]) \subseteq U_{k-1}$. □

9.4.7 Proposition. *Let $(X, \mathcal{U})$ be a uniform space whose uniform structure $\mathcal{U}$ is generated by a family $(d_i)_{i \in I}$ of pseudometrics. TFAE:*

(i) X is Hausdorff.

(ii) $\forall x \ne y \in X \ \exists i \in I$ such that $d_i(x, y) > 0$.

Proof. By 9.1.17 (ii), X is T_2 if and only if $\bigcap_{U \in \mathcal{U}} U = \Delta$, which by 9.4.5 means

$$\bigcap_{\substack{J \subseteq I \\ |J| < \infty}} \bigcap_{a > 0} \bigcap_{i \in J} d_i^{-1}([0, a]) = \Delta.$$

Here, $\supseteq$ is automatic, so (i) $\Leftrightarrow \forall x \neq y\ \exists J \subseteq I$ finite $\exists a > 0\ \exists i \in J : d_i(x,y) > a \Leftrightarrow \forall x \neq y\ \exists a > 0\ \exists i \in I : d_i(x,y) > a \Leftrightarrow$ (ii). $\square$

9.4.8 Definition. *A uniform space $(X,\mathcal{U})$ is called metrizable if $\mathcal{U}$ is generated by a metric.*

9.4.9 Theorem. *Let $(X,\mathcal{U})$ be a uniform space. TFAE:*

(i) $(X,\mathcal{U})$ is metrizable.

(ii) $(X,\mathcal{U})$ is Hausdorff and possesses a countable fundamental system of entourages.

Proof. (i)$\Rightarrow$(ii): Let d be a metric on X with $\mathcal{U} = \mathcal{U}_d$. Then X is T_2 and $\{d^{-1}([0,\frac{1}{n}]) \mid n \geq 1\}$ is a countable fundamental system of entourages.
(ii)$\Rightarrow$(i): By 9.4.6 there exists a pseudometric d' on X with $\mathcal{U} = \mathcal{U}_{d'}$. Since X is T_2, 9.4.7 implies that $\forall x \neq y$, $d(x,y) > 0$, so d is a metric. $\square$

9.4.10 Corollary. *Let $(X,\mathcal{U})$ be a uniform space whose uniform structure $\mathcal{U}$ is generated by countably many pseudometrics $(d_i)_{i\in I}$. If X is T_2 then X is metrizable.*

Proof. By 9.4.5, $\mathcal{B} := \{\bigcap_{i\in J} d_i^{-1}([0,\frac{1}{m})) \mid J \subseteq I$ finite, $m \geq 1\}$ is a countable fundamental system of entourages of $\mathcal{U}$. $\square$

9.4.11 Corollary. *Let I be a countable set and for $i \in I$ let $(X_i,\mathcal{U}_i)$ be a metrizable uniform space. Let $\mathcal{U}$ be the product of the uniform structures $(\mathcal{U}_i)_{i\in I}$. Then also $(\prod_{i\in I} X_i,\mathcal{U})$ is metrizable.*

Proof. By 3.2.3, $\prod_{i\in I} X_i$ is T_2, and by 9.4.9 any $\mathcal{U}_i$ possesses a countable fundamental system of entourages $\mathcal{B}_i$. Hence 9.3.3 implies that $\mathcal{B} := \{\bigcap_{i\in J}(p_i \times p_i)^{-1}(B_i) \mid J \subseteq I$ finite, $B_i \in \mathcal{B}_i\ \forall i \in J\}$ is a countable fundamental system of entourages for $\mathcal{U}$, and 9.4.9 gives the claim. $\square$

9.4.12 Theorem. *Let $(X,\mathcal{U})$ be a uniform space. Then there exists a family of pseudometrics that generates $\mathcal{U}$.*

Proof. For $V \in \mathcal{U}$ let $U_1^V \in \mathcal{U}$ be symmetric with $U_1^V \subseteq V$. Suppose that U_n^V has already been defined and choose $U_{n+1}^V \in \mathcal{U}$ symmetric with $(U_{n+1}^V)^2 \subseteq U_n^V$. Then

$$\mathcal{B}^V := \{U_n^V \mid n \geq 1\}$$

is a fundamental system of entourages of a uniform structure $\mathcal{U}^V$ on X: In fact, 9.1.9 (b), (c), (d) are clear, and for (a) note that $U_n^V \cap U_m^V \supseteq U_{\max(m,n)}^V$. Since $\mathcal{B}^V \subseteq \mathcal{U}$, $\mathcal{U}^V \subseteq \mathcal{U}\ \forall V \in \mathcal{U}$, and so $\mathcal{U} \supseteq \bigcup_{V\in\mathcal{U}} \mathcal{U}^V$. Let $\mathcal{U}'$ be a uniform structure with $\mathcal{U}' \supseteq \bigcup_{V\in\mathcal{U}} \mathcal{U}^V$. Then for every $V \in \mathcal{U}$, $\mathcal{U}' \ni U_1^V \subseteq V \Rightarrow \mathcal{U}' \supseteq \mathcal{U} \Rightarrow \mathcal{U}$ is the coarsest uniform structure finer than every $\mathcal{U}^V$ $(V \in \mathcal{U})$, i.e. it is the supremunm of the $(\mathcal{U}^V)_{V\in\mathcal{U}}$.
By 9.4.6, each $\mathcal{U}^V$ is generated by some pseudometric d^V. Hence by 9.4.5 the family $(d^V)_{V\in\mathcal{U}}$ generates the supremum of the $(\mathcal{U}^V)_{V\in\mathcal{U}}$, i.e. $\mathcal{U}$. $\square$

9.4.13 Proposition. *Let X be a completely regular space. Then X is uniformizable.*

Proof. By 3.1.11, X is homeomorphic to a subspace of $[0,1]^I$ (for a suitable I). $[0,1]^I$ is uniformizable by 9.3.8 and so is any subspace by 9.4.2 (ii). $\square$

9.4.14 Theorem. *Let $(X, \mathcal{O})$ be a compact Hausdorff space. Then:*

(i) $\mathcal{U} := \{U \subseteq X \times X \mid U$ *is a neighborhood of* $\Delta\}$ *is a uniform structure on* X *with* $\mathcal{O}_{\mathcal{U}} = \mathcal{O}$.

(ii) $\mathcal{U}$ *is the only uniform structure on* X *with* $\mathcal{O} = \mathcal{O}_{\mathcal{U}}$.

Proof. By 5.1.9, X is normal and hence (by 4.1.4) it is completely regular, so X is uniformizable by 9.4.13. Let $\mathcal{U}'$ be a uniform structure on X with $\mathcal{O}_{\mathcal{U}'} = \mathcal{O}$.

$\mathcal{U}' \subseteq \mathcal{U}$ (this even holds for *any* uniform structure on any space X): Let $V' \in \mathcal{U}'$ and pick $W' \in \mathcal{U}'$ symmetric with $W'^2 \subseteq V'$. Let $(x, x) \in \Delta$. Then $W'(x) \times W'(x)$ is a neighborhood of (x, x) and $W'(x) \times W'(x) \subseteq V'$: In fact, let $(y, z) \in W'(x) \times W'(x) \Rightarrow (x, y) \in W', (x, z) \in W' \Rightarrow (y, z) \in W'^2 \subseteq V'$, so V' is a neighborhood of Δ.

$\mathcal{U} \subseteq \mathcal{U}'$: suppose to the contrary that there exists some neighborhood V of Δ such that $V \notin \mathcal{U}'$. Then

$$\mathcal{B} := \{U' \cap ((X \times X) \setminus V) \mid U' \in \mathcal{U}'\}$$

is a filter basis of a filter $\mathcal{F}$ on $X \times X$ (see 2.2.21), and $\mathcal{F}$ is finer than $\mathcal{U}'$. $X \times X$ is compact, so $\mathcal{F}$ possesses a cluster point (x_1, x_2), which consequently is also a cluster point of $\mathcal{U}'$. Thus

$$(x_1, x_2) \in \overline{U' \cap (X \times X) \setminus V} \subseteq \overline{(X \times X) \setminus V} = (X \times X) \setminus V^\circ \subseteq (X \times X) \setminus \Delta.$$

However, $(x_1, x_2) \in \bigcap_{U' \in \mathcal{U}'} \overline{U'} = \Delta$ (by 9.1.15 and 9.1.17 (ii)), a contradiction.

Summing up, $\mathcal{U} = \mathcal{U}'$. In particular, $\mathcal{U}$ is a uniform structure, and is in fact the only one with $\mathcal{O} = \mathcal{O}_{\mathcal{U}}$. □

9.4.15 Theorem. *Let $(X, \mathcal{O})$ be a topological space. TFAE:*

(i) X is uniformizable.

(ii) X is T_{3a}.

Proof. (ii) $\Rightarrow$ (i): Let $\mathcal{A} := \{f : X \to [0, 1] \mid f$ continuous $\}$. The proof of 3.1.7 shows that X carries the initial topology with respect to $\mathcal{A}$. Let $\mathcal{U}$ be the initial uniform structure with respect to $\mathcal{A}$ on X, i.e. the coarsest uniform structure such that all $f \in \mathcal{A}$ are uniformly continuous. Then by 9.3.3 (iv), $\mathcal{O} = \mathcal{O}_{\mathcal{U}}$.

(i) $\Rightarrow$ (ii): Let $\mathcal{D}$ be a system of pseudometrics that generates the uniform structure $\mathcal{U}$ of X (cf. 9.4.12). For any $d \in \mathcal{D}$ and any $x_0 \in X$, $d_{x_0} := x \mapsto d(x, x_0)$ is continuous because $d_{x_0}(d^{-1}([0, \varepsilon))(x)) \subseteq B_\varepsilon(d_{x_0}(x))$. In fact, if $y \in d^{-1}([0, \varepsilon))(x)$, then $d(x, y) < \varepsilon \Rightarrow |d_{x_0}(y) - d_{x_0}(x)| = |d(x_0, y) - d(x_0, x)| \le d(x, y) < \varepsilon \Rightarrow y \in B_\varepsilon(d_{x_0}(x))$.

Now let $A \subseteq X$ be closed, $x_0 \in X \setminus A$. Then there exists some $V \in \mathcal{U}$ with $V(x_0) \subseteq X \setminus A$. By 9.4.5, there exist $d_1, \dots, d_n \in \mathcal{D}$ and $a > 0$ with $\bigcap_{i=1}^n d_i^{-1}([0, a]) \subseteq V$. Let $g(x, y) := \max\limits_{1 \le i \le n} d_i(x, y) \Rightarrow \bigcap_{i=1}^n d_i^{-1}([0, a]) = g^{-1}([0, a])$.

Let $f : X \to [0, 1], f(x) := \max(0, 1 - \frac{1}{a} g(x, x_0))$. Then f is continuous, $f(x_0) = 1$ and for $z \in A$, $g(z, x_0) > a$ because $g(z, x_0) \le a$ implies $z \in V(x_0) \subseteq X \setminus A$. Hence $f(x) = 0$, i.e., $f(A) \subseteq \{0\}$. It follows that X is T_{3a}. □

9.4.16 Theorem. *Let X be a topological space. TFAE:*

(i) X is completely regular.

(ii) There exists some set I such that X is homeomorphic to a subspace of $[0,1]^I$.

(iii) X is homeomorphic to a subspace of a compact T_2-space.

(iv) X is uniformizable and T_2.

Proof. (i)⇒(ii): See 3.1.11.
(ii)⇒(iii): $[0,1]^I$ is a compact T_2-space by 5.1.6, 5.1.15, and 3.2.3.
(iii)⇒(iv): Any compact T_2-space is uniformizable by 9.4.14, hence so is any subspace (see 9.4.2 (ii)).
(iv)⇒(i): See 9.4.15. □

9.4.17 Corollary. *Any locally compact space is uniformizable.*

Proof. X is completely regular by 5.2.8. Alternatively, $X \subseteq X'$ (the Alexandroff compactification of X), cf. 5.2.7, so the claim follows from 9.4.14 and 9.4.2 (ii). □

Chapter 10

Completion and compactification

10.1 Completion of uniform spaces

10.1.1 Definition. *Let $(X,\mathcal{U})$ be a uniform space, $A \subseteq X$ and $V \in \mathcal{U}$. A is called small of order V if $A \times A \subseteq V$. In particular, if (X,d) is a metric space then A is called small of order ε if $A \times A \subseteq d^{-1}([0,\varepsilon))$, i.e. if $d(x,y) < \varepsilon\ \forall x,y \in A$.*

10.1.2 Examples.

(i) Let (x_n) be a sequence in a metric space (X,d). Then (x_n) is Cauchy if and only if for any $\varepsilon > 0$ there exists a tail end of (x_n) that is small of order ε:

$$\exists n_0(\varepsilon)\ \forall p,q \geq n_0(\varepsilon):\ d(x_p,x_q) < \varepsilon.$$

(ii) Let $(X,\mathcal{U})$ be a uniform space and $V = V^{-1} \in \mathcal{U}$, $(x,y) \in V \Rightarrow V(x) \cup V(y)$ is small of order V^3: Let $v,w \in V(x) \cup V(y)$. Then there are two possibilities:

1.) $v,w \in V(x)$ (or $v,w \in V(y)$): $(x,v) \in V$, $(x,w) \in V \Rightarrow (v,w) \in V^2 \subseteq V^3$.

2.) $v \in V(x)$, $w \in V(y)$ (or $w \in V(x)$, $v \in V(y)$): $(v,x) \in V$, $(x,y) \in V$, $(y,w) \in V \Rightarrow (v,w) \in V^3$.

(iii) Let $M,N \subseteq X$ be small of order V, $M \cap N \neq \emptyset \Rightarrow M \cup N$ is small of order V^2: Let $a \in M \cap N$, $x \in M$, $y \in N \Rightarrow (x,a) \in M \times M \subseteq V$, $(a,y) \in N \times N \subseteq V \Rightarrow (x,y) \in V^2$ (and analogously for the other cases).

(iv) Let $V = V^{-1}$, M small of order V. Then $V(M)$ is small of order V^3: $V(M) = \bigcup_{m\in M} V(m)$. $x,y \in V(M) \Rightarrow \exists m_1,m_2 \in M$: $(x,m_1) \in V$, $(m_1,m_2) \in V$, $(m_2,y) \in V \Rightarrow (x,y) \in V^3$.

(v) N small of order V, $M \cap N \neq \emptyset \Rightarrow N \subseteq V(M)$: Let $z \in M \cap N$, $n \in N \Rightarrow (z,n) \in N \times N \subseteq V \Rightarrow n \in V(z) \subseteq V(M)$.

10.1.3 Definition. *A filter $\mathcal{F}$ on a uniform space $(X,\mathcal{U})$ is called Cauchy filter, if*

$$\forall V \in \mathcal{U}\ \exists F \in \mathcal{F} \text{ with } F \times F \subseteq V \ (\text{i.e.: } F \text{ small of order } V).$$

10.1.4 Example. By 10.1.2 (i), the filter associated (via 2.3.3 (i)) to a Cauchy sequence is a Cauchy filter.

10.1.5 Proposition. *Let $(X,\mathcal{U})$ be a uniform space and $\mathcal{F}$ a convergent filter on X. Then $\mathcal{F}$ is a Cauchy filter.*

Proof. Let $x \in X$ be such that $\mathcal{F} \to x$. Let $V \in \mathcal{U}$ and $U = U^{-1} \in \mathcal{U}$ with $U^2 \subseteq V$. $\mathcal{F} \to x \Rightarrow \exists F \in \mathcal{F}$ with $F \subseteq U(x)$. Let $(u,v) \in F \times F \Rightarrow (u,x),\ (x,v) \in U \Rightarrow (u,v) \in U^2 \subseteq V \Rightarrow F \times F \subseteq V$. □

The following result generalized the fact that in metric spaces, any Cauchy sequence that possesses a convergent subsequence is itself convergent:

10.1.6 Proposition. *Any Cauchy filter $\mathcal{F}$ on a uniform space $(X,\mathcal{U})$ converges to its cluster points.*

Proof. Let x be a cluster point of $\mathcal{F}$ and let $V \in \mathcal{U}$ be closed (cf. 9.1.15). Since $\mathcal{F}$ is a Cauchy filter, there exists some $F \in \mathcal{F}$ with $F \times F \subseteq V \Rightarrow \bar{F} \times \bar{F} \subseteq V,\ \bar{F} \in \mathcal{F}$. Also, x is a cluster point, so $x \in \bar{F}$. Therefore, $\bar{F} \subseteq V(x) \Rightarrow \mathcal{F} \supseteq \mathcal{U}(x)$, i.e. $\mathcal{F} \to x$. □

10.1.7 Proposition. *Let $(X,\mathcal{U})$, $(Y,\mathcal{V})$ be uniform spaces, $f : X \to Y$ uniformly continuous, and $\mathcal{F}$ a Cauchy filter on X. Then $f(\mathcal{F})$ is a Cauchy filter on Y.*

Proof. Let $V \in \mathcal{V} \Rightarrow \exists U \in \mathcal{U}$ with $(f \times f)(U) \subseteq V$. $\mathcal{F}$ is a Cauchy filter $\Rightarrow \exists F \in \mathcal{F}$ with $F \times F \subseteq U \Rightarrow f(F) \times f(F) \subseteq V$. □

10.1.8 Definition. *A Cauchy filter $\mathcal{F}$ on a uniform space X is called minimal if there is no strictly coarser Cauchy filter on X: $\mathcal{G} \subseteq \mathcal{F}$, $\mathcal{G}$ Cauchy filter $\Rightarrow \mathcal{G} = \mathcal{F}$.*

10.1.9 Theorem. *Let $(X,\mathcal{U})$ a uniform space and $\mathcal{F}$ a Cauchy filter on X. Then there exists a minimal Cauchy filter $\mathcal{F}_0$ with $\mathcal{F}_0 \subseteq \mathcal{F}$.*

Proof. Let $\mathcal{B}$ be a filter basis of $\mathcal{F}$, $\mathcal{V} := \{V \in \mathcal{U} \mid V = V^{-1}\}$ and $\mathcal{B}_0 := \{V(B) \mid V \in \mathcal{V}, B \in \mathcal{B}\}$. $\mathcal{B}_0$ is a filter basis: Let $V_1, V_2 \in \mathcal{V}$, $B_1, B_2 \in \mathcal{B}$ and $V \in \mathcal{V}$, $B \in \mathcal{B}$ such that $V \subseteq V_1 \cap V_2$, $B \subseteq B_1 \cap B_2 \Rightarrow V(B) \subseteq V_1(B_1) \cap V_2(B_2)$. Also, $V(B) \supseteq \Delta(B) = B \neq \emptyset\ \forall V, B$. Let $\mathcal{F}_0$ be the filter on X with basis $\mathcal{B}_0$.

$\mathcal{F}_0$ is a Cauchy filter: Let $V \in \mathcal{U}$ and pick $W \in \mathcal{V}$ such that $W^3 \subseteq V$ (cf. 9.1.8). Since $\mathcal{F}$ is a Cauchy filter, there exists some $B \in \mathcal{B}$ with $B \times B \subseteq W \Rightarrow W(B) \in \mathcal{B}_0$ and $W(B) \times W(B) \subseteq W^3 \subseteq V$ by 10.1.2 (iv).

$\mathcal{F}_0 \subseteq \mathcal{F}$ because $B = \Delta(B) \subseteq V(B)$.

$\mathcal{F}_0$ is minimal: Let $\mathcal{G} \subseteq \mathcal{F}$ be a Cauchy filter. We have to show that $\mathcal{F}_0 \subseteq \mathcal{G}$. Let $B \in \mathcal{B}, V \in \mathcal{V} \Rightarrow \exists G \in \mathcal{G}$ with $G \times G \subseteq V$. As $\mathcal{G} \subseteq \mathcal{F}$, $G \in \mathcal{F}$, so $G \cap B \neq \emptyset \Rightarrow G \subseteq V(B)$ by 10.1.2 (v). Hence $V(B) \in \mathcal{G} \Rightarrow \mathcal{F}_0 \subseteq \mathcal{G}$. □

10.1.10 Corollary. *Let $(X,\mathcal{U})$ be a uniform space and $x \in X$. Then $\mathcal{U}(x)$ is a minimal Cauchy filter.*

Proof. Let $\mathcal{F} := \{F \subseteq X \mid x \in F\}$. Then $\mathcal{F} \to x$, so $\mathcal{F}$ is a Cauchy filter by 10.1.5. $\mathcal{B} := \{\{x\}\}$ is a basis of $\mathcal{F}$ (hence $\mathcal{F}$ is even an ultrafilter). The proof of 10.1.9 now shows that $\mathcal{B}_0 := \{V(x) \mid V \in \mathcal{V}\}$ is a basis of a minimal Cauchy filter. Since $\mathcal{B}_0$ is a neighborhood basis, the claim follows. □

10.1.11 Definition. *A uniform space X is called complete if any Cauchy filter $\mathcal{F}$ on X converges.*

10.1.12 Example. We will show later (cf. 11.2.3) that a metric space is complete if and only if any Cauchy sequence converges. Thus $\mathbb{R}^n$, $\mathbb{C}^n$, ℓ^p, $L^p, \ldots$ are complete, but $\mathbb{Q}$ is not.

10.1.13 Lemma. *Let $\mathcal{F}$ be a minimal Cauchy filter on $(X,\mathcal{U})$ and let $F \in \mathcal{F}$. Then $F^\circ \in \mathcal{F}$. In particular, $F^\circ \neq \emptyset$.*

Proof. Let $\mathcal{V} := \{V \in \mathcal{U} \mid V = V^{-1}\}$. Then by 10.1.9, $\{V(B) \mid V \in \mathcal{V}, B \in \mathcal{F}\}$ is a basis of $\mathcal{F}$ itself since $\mathcal{F}$ is minimal. Let $F \in \mathcal{F} \Rightarrow \exists V \in \mathcal{V}, B \in \mathcal{F}$ with $V(B) \subseteq F$. By 9.1.15 there exists some $U \in \mathcal{U}$ open with $U \subseteq V$. Hence $U(B) \subseteq F$. Also

$$U(B) = \bigcup_{x \in B} U(x) = \bigcup_{x \in B} \{y \mid (x,y) \in U\}$$

is open (and $\supseteq \Delta(B) = B \neq \emptyset$). Thus $U(B) \subseteq F^\circ$. Let $V_1 \in \mathcal{V},\ V_1 \subseteq U$. Then $V_1(B) \subseteq F^\circ$, so $F^\circ \in \mathcal{F}$. □

10.1.14 Proposition. *Let $(X,\mathcal{U})$ be a uniform space, $\mathcal{F}$ a Cauchy filter on $X, A \subseteq X$, and $\mathcal{F}_A$ (cf. 2.2.21) a filter on A. Then $\mathcal{F}_A$ is a Cauchy filter on A.*

Proof. By 9.3.5, the uniform structure on A is $\{U \cap (A \times A) \mid U \in \mathcal{U}\}$. Let $U \in \mathcal{U}$ and $F \in \mathcal{F}$ such that $F \times F \subseteq U$. Then $F \cap A \in \mathcal{F}_A$ and $(F \cap A) \times (F \cap A) \subseteq U \cap (A \times A)$. □

10.1.15 Proposition. *Let $(X,\mathcal{U})$ be a uniform space, and $A \subseteq X$ dense in X. If for every Cauchy filter on A its extension to X converges, then X is complete.*

Proof. Let $i : A \hookrightarrow X$ be the inclusion map. Let $\mathcal{F}$ be a Cauchy filter on X and $\mathcal{F}_0$ the minimal Cauchy filter with $\mathcal{F}_0 \subseteq \mathcal{F}$ (see 10.1.9). Let $F \in \mathcal{F}_0$. Then by 10.1.13, $F^\circ \in \mathcal{F}_0$, so $F^\circ \cap A \neq \emptyset$. By 2.2.21, $(\mathcal{F}_0)_A := \{F \cap A \mid F \in \mathcal{F}_0\}$ is a filter on A, and 10.1.14 shows that it is in fact a Cauchy filter on A. By assumption, the filter basis $i((\mathcal{F}_0)_A)$ converges to some $x \in X$. Since $i((\mathcal{F}_0)_A) \supseteq \mathcal{F}_0$, x is a cluster point of $\mathcal{F}_0$ (by 2.3.5). Thus 10.1.6 gives $\mathcal{F}_0 \to x \Rightarrow \mathcal{F} \to x$. □

10.1.16 Proposition. *Let X be a set , $((Y_i,\mathcal{U}_i))_{i\in I}$ a family of uniform spaces and $f_i : X \to Y_i$ $(i \in I)$. Equip X with the initial uniform structure $\mathcal{U}$ with respect to $(f_i)_{i\in I}$ (cf. 9.3.3). Let $\mathcal{F}$ be a filter on X. TFAE.*

(i) $\mathcal{F}$ is a Cauchy filter.

(ii) $\forall i \in I$, $f_i(\mathcal{F})$ is a Cauchy filter.

Thus if all Y_i are complete, then so is X.

Proof. (i) $\Rightarrow$ (ii): Since all f_i are uniformly continuous, this follows from 10.1.7.
(ii) $\Rightarrow$ (i): Let $U \in \mathcal{U} \Rightarrow \exists J \subseteq I$ finite, $V_i \in \mathcal{U}_i$ $(i \in J)$ such that $\bigcap_{i\in J}(f_i \times f_i)^{-1}(V_i) \subseteq U$. Now each $f_i(\mathcal{F})$ is a Cauchy filter, so $\forall i \in J\ \exists F_i \in \mathcal{F}$ with $f_i(F_i) \times f_i(F_i) \subseteq V_i$. Let $F := \bigcap_{i\in J} F_i \in \mathcal{F}$. Then $F \times F \subseteq \bigcap_{i\in J}(f_i \times f_i)^{-1}(V_i) \subseteq U$, so $\mathcal{F}$ is a Cauchy filter. The last claim follows from 2.3.18. □

Together with 9.3.8 this gives:

10.1.17 Corollary. *Let $(X_i)_{i\in I}$ be a family of uniform spaces, and $\mathcal{F}$ a filter on $\prod_{i\in I} X_i$. TFAE:*

(i) $\mathcal{F}$ is a Cauchy filter on $\prod_{i\in I} X_i$.

(ii) $\forall i \in I$, $p_i(\mathcal{F})$ is a Cauchy filter on X.

10.1.18 Theorem. *Let $(X_i)_{i\in I}$ be a family of uniform spaces. TFAE:*

(i) $\prod_{i\in I} X_i$ *is complete.*

(ii) $\forall i \in I$, X_i *is complete.*

Proof. (i) $\Rightarrow$ (ii): Let $i_0 \in I$ and let $\mathcal{F}_{i_0}$ be a Cauchy filter on X_{i_0}. For $i \neq i_0$ let $\mathcal{F}_i$ be any Cauchy filter on X_i (e.g., a neighborhood filter of some point). Let $\mathcal{F}$ be the product of $(\mathcal{F}_i)_{i\in I}$ (cf. 2.2.26). Then $p_i(\mathcal{F}) = \mathcal{F}_i$ is a Cauchy filter $\forall i \in I \Rightarrow \mathcal{F}$ is a Cauchy filter by 10.1.17, hence convergent. Since p_{i_0} is continuous, 2.3.13 (iii) implies that $\mathcal{F}_{i_0} = p_{i_0}(\mathcal{F})$ is convergent. Consequently, X_{i_0} is complete.
(ii) $\Rightarrow$ (i): Let $\mathcal{F}$ be a Cauchy filter on $\prod_{i\in I} X_i$. Then by 10.1.17 any $p_i(\mathcal{F})$ is a Cauchy filter on X_i, hence convergent. Thus $\mathcal{F}$ converges by 2.3.19. □

10.1.19 Theorem. *Let X be complete and $A \subseteq X$ closed. Then A is complete.*

Proof. Let $\mathcal{F}$ be a Cauchy filter on A, and let $i : A \hookrightarrow X$. By 10.1.7, $i(\mathcal{F})$ is a Cauchy filter on X, hence converges to some $x \in X$. Thus x is a cluster point of $i(\mathcal{F})$ by 2.3.5. It follows that $x \in \bar{A} = A$, so $\mathcal{F} \to x$ in A (in fact, if W is a neighborhood of x in $A \Rightarrow W = U \cap A$ with $U \in \mathcal{U}(x) \Rightarrow \exists F \in \mathcal{F}$ with $F \subseteq U$. Also, $F \subseteq A$ since $\mathcal{F}$ is a filter on A. Hence $F \subseteq U \cap A = W$). □

10.1.20 Theorem. *Let X be a uniform Hausdorff space and $A \subseteq X$ complete (in the induced uniform structure). Then A is closed.*

Proof. Let $x \in \bar{A}$, then by 2.2.24 $\mathcal{U}(x)_A$ is a filter, and in fact a Cauchy filter by 10.1.14, hence converges to some $y \in A$. Also, $\mathcal{U}(x)_A$ is a basis of a filter $\mathcal{F}$ on X and $\mathcal{F}$ converges to y (if $V \in \mathcal{U}(y)$ then $V \cap A$ is a neighborhood of y in A so there is some $U \in \mathcal{U}(x)$ with $\mathcal{F} \ni U \cap A \subseteq V \cap A \subseteq V$). Moreover, due to $\mathcal{U}(x) \subseteq \mathcal{F}$ we have $\mathcal{F} \to x$. As X is T_2, 3.1.5 (ii) gives $x = y \in A$, so $A = \bar{A}$. □

10.1.21 Proposition. *Let X be a topological space, A dense in X, Y a complete T_2-space and $f : A \to Y$. TFAE:*

(i) There exists a continuous function $g : X \to Y$ with $g|_A = f$.

(ii) $\forall x \in X$, $\{f(U \cap A) \mid U \in \mathcal{U}(x)\}$ is a basis of a Cauchy filter on Y.

The function g then is uniquely determined.

Proof. Y is completely regular by 9.4.16, hence also regular. By 3.3.3 we therefore get: (i) $\Leftrightarrow \forall x \in X\ \exists \lim\limits_{\substack{a\to x\\ a\in A}} f(a) \Leftrightarrow \forall x \in X$, $\{f(U \cap A) \mid U \in \mathcal{U}(x)\}$ is a basis of a convergent filter on $Y \Leftrightarrow$ (ii) since Y is complete. Finally, g is uniquely determined by 3.3.3. □

10.1.22 Theorem. *Let $(X,\mathcal{U})$ be a uniform space, $A \subseteq X$ dense in X, Y a complete T_2-space and $f : A \to Y$ uniformly continuous. Then there is a unique continuous map $g : X \to Y$ with $g|_A = f$. Moreover, g is uniformly continuous.*

Proof. Let $x \in X = \bar{A}$. Then by 2.2.24 $\mathcal{U}(x)_A$ is a filter on A and therefore a Cauchy filter on A by 10.1.14. f is uniformly continuous, so by 10.1.7 $f(\mathcal{U}(x)_A) = \{f(U \cap A) \mid U \in \mathcal{U}(x)\}$ is a basis of a Cauchy filter on Y, and 10.1.21 shows the existence of some $g : X \to Y$ continuous with $g|_A = f$. It remains to show that g is uniformly continuous. Let V be a closed entourage in Y (cf. 9.1.15). Using 9.3.5 and the uniform continuity of f, we obtain some $U \in \mathcal{U}$ with

$$(f \times f)(U \cap (A \times A)) = (g \times g)(U \cap (A \times A)) \subseteq V,$$

i.e., $U \cap (A \times A) \subseteq (g \times g)^{-1}(V)$. Since g is continuous, so is $g \times g$, hence $(g \times g)^{-1}(V)$ is closed. Consequently, $\overline{U \cap (A \times A)} \subseteq (g \times g)^{-1}(V)$. As these sets form a fundamental system of entourages by 9.3.7, g is uniformly continuous. $\square$

We call a map f between uniform spaces an *isomorphism* if it is bijective and both f and f^{-1} are uniformly continuous.

10.1.23 Corollary. *Let X_1, X_2 be complete Hausdorff spaces, A_1 dense in X_1, A_2 dense in X_2, $f : A_1 \to A_2$ an isomorphism. Then there exists a unique isomorphism $g : X_1 \to X_2$ with $g|_{A_1} = f$.*

Proof. By 10.1.22 there exists a unique $g : X_1 \to X_2$ continuous with $g|_{A_1} = f$ and g is uniformly continuous, as well as a unique $h : X_2 \to X_1$, continuous with $h|_{A_2} = f^{-1}$, and also h is uniformly continuous. It follows that $h \circ g|_{A_1} = h \circ f|_{A_1} = \mathrm{id}_{A_1}$. Since also $\mathrm{id}_{X_1}|_{A_1} = \mathrm{id}_{A_1}$, $h \circ g = \mathrm{id}_{X_1}$, and analogously $g \circ h = \mathrm{id}_{X_1}$. $\square$

Recall from real analysis that $\mathbb{Q}$ can be completed to construct $\mathbb{R}$. In this process, real numbers are defined as equivalence classes of Cauchy sequences of rational numbers. Two such sequences are considered equivalent if their difference converges to zero. This procedure can be generalized to uniform spaces:

10.1.24 Theorem. *Let $(X,\mathcal{U})$ be a uniform space. Then there exists a complete Hausdorff space $(\tilde{X},\tilde{\mathcal{U}})$ and a uniformly continuous map $i : X \to \tilde{X}$ with the following universal property:*

If Y is a complete T_2-space and $f : X \to Y$ is uniformly continuous then there exists a unique uniformly continuous map $\tilde{f} : \tilde{X} \to Y$, such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, "\exists! \tilde{f}"] \\ & Y \end{tikzcd} \qquad (10.1.1)$$

If $(i', \tilde{X}')$ is another pair with these properties then there is a uniquely determined isomorphism $h : \tilde{X} \to \tilde{X}'$ with $i' = h \circ i$. Furthermore, $i(X)$ is dense in $\tilde{X}$ and $\mathcal{U}$ is the coarsest uniform structure on X such that i is uniformly continuous.

Proof. Let $\tilde{X} := \{\mathcal{F} \mid \mathcal{F}$ is a minimal Cauchy filter on $X\}$. Then $X \neq \emptyset$, since e.g. $\mathcal{U}(x) \in \tilde{X}\ \forall x \in X$ by 10.1.10. For $V = V^{-1} \in \mathcal{U}$ let

$$\tilde{V} := \{(\mathcal{F},\mathcal{G}) \in \tilde{X} \times \tilde{X} \mid \exists M \in \mathcal{F} \cap \mathcal{G} \text{ with } M \times M \subseteq V\}.$$

Claim 1: $\tilde{\mathcal{B}} := \{\tilde{V} \mid V \in \mathcal{U},\ V = V^{-1}\}$ is a fundamental system of entourages on $\tilde{X}$.

To see this, we have to verify 9.1.9 (a)–(d):

(a) Let V_1, V_2 be symmetric. Then $V := V_1 \cap V_2$ is symmetric and $\tilde{V} \subseteq \tilde{V}_1 \cap \tilde{V}_2$: $(\mathcal{F},\mathcal{G}) \in \tilde{V} \Rightarrow \exists M \in \mathcal{F} \cap \mathcal{G}$ with $M \times M \subseteq V \subseteq V_1 \cap V_2 \Rightarrow (\mathcal{F},\mathcal{G}) \in \tilde{V}_1 \cap \tilde{V}_2$.

(b) $\Delta_{\tilde{X}} \subseteq \tilde{V}\ \forall \tilde{V} \in \mathcal{B}$: Let $V = V^{-1} \in \mathcal{U}$ and $\mathcal{F} \in \tilde{X}$. Since $\mathcal{F}$ is a Cauchy filter, there exists some $F \in \mathcal{F} = \mathcal{F} \cap \mathcal{F}$ with $F \times F \subseteq V \Rightarrow (\mathcal{F},\mathcal{F}) \in \tilde{V}$.

(c) $\tilde{V} = \tilde{V}^{-1}$ holds by definition.

(d) Let $V = V^{-1} \in \mathcal{U}$. Choose $W = W^{-1} \in \mathcal{U}$ such that $W^2 \subseteq V$. Then $\tilde{W}^2 \subseteq \tilde{V}$: Let $(\mathcal{F},\mathcal{H}) \in \tilde{W}^2 \Rightarrow \exists \mathcal{G}$ with $(\mathcal{F},\mathcal{G}) \in \tilde{W}$, $(\mathcal{G},\mathcal{H}) \in \tilde{W} \Rightarrow \exists M \in \mathcal{F} \cap \mathcal{G}$ with $M \times M \subseteq W$, $\exists N \in \mathcal{G} \cap \mathcal{H}$ with $N \times N \subseteq W \Rightarrow M \cap N \in \mathcal{G}$. In particular, $M \cap N \neq \emptyset$, so 10.1.2 (iii) gives $(M \cup N) \times (M \cup N) \subseteq W^2 \subseteq V$. Also, $M \subseteq M \cup N \Rightarrow M \cup N \in \mathcal{F}$, $N \subseteq M \cup N \Rightarrow M \cup N \in \mathcal{H} \Rightarrow M \cup N \in \mathcal{F} \cap \mathcal{H} \Rightarrow (\mathcal{F},\mathcal{H}) \in \tilde{V}$.

Let $\tilde{\mathcal{U}}$ be the uniform structure on $\tilde{X}$ with fundamental system of entourages $\tilde{\mathcal{B}}$.

Claim 2: $(\tilde{X},\tilde{\mathcal{U}})$ is Hausdorff.

To see this, we use 9.1.17: Let $(\mathcal{F},\mathcal{G}) \in \bigcap_{\tilde{V}\in\tilde{\mathcal{B}}} \tilde{V}$. The family $\{M \cup N \mid M \in \mathcal{F}, N \in \mathcal{G}\}$ is a basis of a filter $\mathcal{H}$ with $\mathcal{H} \subseteq \mathcal{F}$ and $\mathcal{H} \subseteq \mathcal{G}$. $\mathcal{H}$ is a Cauchy filter: Let $V = V^{-1} \in \mathcal{U}$. Then since $(\mathcal{F},\mathcal{G}) \in \tilde{V}$, there exists some $P \in \mathcal{F} \cap G \subseteq \mathcal{H}$ with $P \times P \subseteq V$. Since both $\mathcal{F}$ and $\mathcal{G}$ are minimal Cauchy filters, $\mathcal{F} = \mathcal{H} = \mathcal{G}$. Consequently, $\bigcap_{\tilde{V}\in\tilde{\mathcal{B}}} \tilde{V} = \Delta_{\tilde{X}}$.

By 10.1.10, for any $x \in X$, $\mathcal{U}(x)$ is a minimal Cauchy filter. Thus we may define

$$i : X \to \tilde{X}, \quad i(x) := \mathcal{U}(x).$$

Claim 3: $i : (X,\mathcal{U}) \to (\tilde{X},\tilde{\mathcal{U}})$ ist uniformly continuous.

Let $\tilde{V} \in \tilde{\mathcal{B}}$. Then by 9.1.8 there exists some symmetric W with $W^3 \subseteq V$. Let $(x,y) \in W$, then by 10.1.2 (ii) $W(x) \cup W(y)$ is small of order $W^3 \subseteq V$. As $W(x) \cup W(y) \in \mathcal{U}(x) \cap \mathcal{U}(y) \Rightarrow (i \times i)(x,y) = (\mathcal{U}(x),\mathcal{U}(y)) \in \tilde{V}$, whence $(i \times i)(W) \subseteq \tilde{V}$.

Claim 4: $\mathcal{U}$ is the coarsest uniform structure such that $i : X \to (\tilde{X},\tilde{\mathcal{U}})$ is uniformly continuous.

To see this we first show:

$$(i \times i)^{-1}(\tilde{V}) \subseteq V \qquad \forall V = V^{-1} \in \mathcal{U}. \tag{10.1.2}$$

In fact, $(x,y) \in (i \times i)^{-1}(\tilde{V}) \Rightarrow (\mathcal{U}(x),\mathcal{U}(y)) \in \tilde{V} \Rightarrow \exists M \in \mathcal{U}(x) \cap \mathcal{U}(y)$ with $M \times M \subseteq V \Rightarrow (x,y) \in M \times M \subseteq V$.

Now suppose that $\mathcal{U}'$ is a uniform structure on X such that $i : (X,\mathcal{U}') \to (\tilde{X},\tilde{\mathcal{U}})$ is uniformly continuous. Let $V = V^{-1} \in \mathcal{U}$. Then by (10.1.2), $\mathcal{U}' \ni (i \times i)^{-1}(\tilde{V}) \subseteq V \Rightarrow V \in \mathcal{U}' \Rightarrow \mathcal{U} \subseteq \mathcal{U}'$.

Claim 5: $i(X)$ is dense in $\tilde{X}$.

Let $\mathcal{F} \in \tilde{X}$ and $\tilde{V}(\mathcal{F})$ a neighborhood of $\mathcal{F}$ in $\tilde{X}$. $\mathcal{F}$ is a Cauchy filter, so there exists some $F \in \mathcal{F}$ with $F \times F \subseteq V$. Also, 10.1.13 gives $\emptyset \neq F^\circ \in \mathcal{F}$. Let $x \in F^\circ \Rightarrow i(x) = \mathcal{U}(x) \in \tilde{V}(\mathcal{F})$, because $(\mathcal{F},\mathcal{U}(x)) \in \tilde{V}$ (in fact, $F^\circ \in \mathcal{F} \cap \mathcal{U}(x)$ and $F^\circ \times F^\circ \subseteq V$).

In addition

$$i(\mathcal{F}) \to \mathcal{F} \in \tilde{X} \tag{10.1.3}$$

because $\forall \tilde{V}(\mathcal{F})$ neighborhood of $\mathcal{F}$ $\exists F^\circ \in \mathcal{F}$ with $i(F^\circ) \subseteq \tilde{V}(\mathcal{F})$.

Claim 6: $(\tilde{X},\tilde{\mathcal{U}})$ is complete.

We use 10.1.15. Let $\mathcal{G}$ be a Cauchy filter on $i(X)$. Then $i^{-1}(\mathcal{G}) := \{i^{-1}(G) \mid G \in \mathcal{G}\}$ is a filter basis on X: Let $G \in \mathcal{G}$. Then $i^{-1}(G) \neq \emptyset$ since $\emptyset \neq G \subseteq i(X)$. Moreover, $i^{-1}(G_1) \cap i^{-1}(G_2) = i^{-1}(G_1 \cap G_2)$. $i^{-1}(\mathcal{G})$ is a basis of a Cauchy filter $\mathcal{F}'$: in fact, by 9.3.4 and Claim 4, $\mathcal{U}$ is the initial uniform structure with respect to $i : X \to i(X)$. Also, $i(i^{-1}(\mathcal{G})) = \mathcal{G}$ is a Cauchy filter on $i(X)$ and so $\mathcal{F}'$ is a Cauchy filter by 10.1.16.

10.1.9 now shows that there exists some minimal Cauchy filter $\mathcal{F} \subseteq \mathcal{F}'$ on X, and by 10.1.7 $i(\mathcal{F})$ is Cauchy filter on $i(X)$. Due to $\mathcal{F}' \supseteq \mathcal{F}$ we have $\mathcal{G} = i(i^{-1}(\mathcal{G})) = i(\mathcal{F}') \supseteq i(\mathcal{F})$. By (10.1.3), $i(\mathcal{F})$ converges in $\tilde{X}$, hence so does the finer filter $\mathcal{G}$. From this it follows using 10.1.15 and Claim 5 that indeed $(\tilde{X},\tilde{\mathcal{U}})$ is complete.

Claim 7: $(i,\tilde{X})$ possesses the universal property (10.1.1).

Let Y be complete and T_2, and let $f : X \to Y$ be uniformly continuous. Let $x, y \in X$ with $i(x) = i(y)$. Then $\mathcal{U}(x) = \mathcal{U}(y)$ and since f is continuous, $f(x) = \lim_{\mathcal{U}(x)} f = \lim_{\mathcal{U}(y)} f = f(y)$. Therefore, $\tilde{f}_0 : i(X) \to Y$, $\tilde{f}_0(i(x)) := f(x)$ is a well-defined map. $\tilde{f}_0$ is uniformly continuous: Let V' be an entourage in $Y \Rightarrow \exists V = V^{-1} \in \mathcal{U}$ with $(f \times f)(V) \subseteq V'$. We show that for the entourage $\tilde{V} \cap (i(X) \times i(X))$ in the induced uniform structure on $i(X)$ we have $(\tilde{f}_0 \times \tilde{f}_0)(\tilde{V} \cap (i(X) \times i(X))) \subseteq V'$: Let $(\mathcal{F}, \mathcal{G}) \in \tilde{V} \cap (i(X) \times i(X)) \Rightarrow \exists x, y \in X$ with $\mathcal{F} = i(x) = \mathcal{U}(x)$, $\mathcal{G} = i(y) = \mathcal{U}(y)$. $(i(x), i(y)) = (\mathcal{F}, \mathcal{G}) \in \tilde{V} \Rightarrow (x, y) \in (i \times i)^{-1}(\tilde{V}) \subseteq V$ by (10.1.2). It follows that

$$(\tilde{f}_0 \times \tilde{f}_0)(\mathcal{F}, \mathcal{G}) = (\tilde{f}_0(\mathcal{F}), \tilde{f}_0(\mathcal{G})) = (\tilde{f}_0(i(x)), \tilde{f}_0(i(y))) = (f(x), f(y)) \in V'.$$

Now by Claim 5 and 10.1.22 there exists a unique $\tilde{f} : \tilde{X} \to Y$ continuous with $\tilde{f}|_{i(X)} = \tilde{f}_0$, and $\tilde{f}$ is uniformly continuous. For $x \in X$, $\tilde{f}(i(x)) = \tilde{f}_0(i(x)) = f(x) \Rightarrow \tilde{f} \circ i = f$. Conversely, $g \circ i = f$ implies that $g|_{i(X)} = \tilde{f}|_{i(X)}$. Since $i(X)$ is dense and g, $\tilde{f}$ are continuous, $\tilde{f}$ is uniquely determined.

Claim 8: $(i, \tilde{X})$ is uniquely determined by (10.1.1).

Let $(i', \tilde{X}')$ be another pair satisfying (10.1.1) and consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i'"'] & \tilde{X} \arrow[d, shift right, "h"'] \\ & \tilde{X}' \arrow[u, shift right, "h'"'] \end{tikzcd}$$

Due to (10.1.1) for $(i, \tilde{X})$ resp. for $(i', \tilde{X}')$ there exist $h : \tilde{X} \to \tilde{X}'$ and $h' : \tilde{X}' \to \tilde{X}$ uniformly continuous with $h \circ i = i'$, $h' \circ i' = i \Rightarrow h' \circ h \circ i = h' \circ i' = i$. Hence the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i"'] & \tilde{X} \arrow[d, "h' \circ h"] \\ & \tilde{X} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i"'] & \tilde{X} \arrow[d, "\mathrm{id}_{\tilde{X}}"] \\ & \tilde{X} \end{tikzcd}$$

By (10.1.1) this implies that $h' \circ h = \mathrm{id}_{\tilde{X}}$. Analogously, $h \circ h' = \mathrm{id}_{\tilde{X}'}$, so altogether $h : \tilde{X} \to \tilde{X}'$ is an isomorphism. □

10.1.25 Definition. *The space $\tilde{X}$ is called the completion of the uniform space $(X, \mathcal{U})$. i is called the canonical map.*

10.1.26 Proposition. *Let $(X, \mathcal{U})$ be a uniform space, $i : X \to \tilde{X}$ the canonical map. Then*

(i) $\bigcap_{V \in \mathcal{U}} V = \{(x, y) \in X \times X \mid i(x) = i(y)\}$.

(ii) $\{(i \times i)(V) \mid V \in \mathcal{U}\} = \tilde{\mathcal{U}}_{i(X)}$.

(iii) $\{\overline{\tilde{V} \cap (i(X) \times i(X))} \mid \tilde{V} \in \tilde{\mathcal{U}}\}$ *is a fundamental system of entourages of* $\tilde{\mathcal{U}}$.

Proof. (i) Let $i(x) = i(y) \Rightarrow \mathcal{U}(x) = \mathcal{U}(y) \Rightarrow \forall V \in \mathcal{U} : V(x) \in \mathcal{U}(x) = \mathcal{U}(y) \Rightarrow y \in V(x) \Rightarrow (x, y) \in V\ \forall V \in \mathcal{U}$. Conversely, let $(x, y) \in V\ \forall V \in \mathcal{U}$. Then by 9.1.15, $(x, y) \in V^\circ\ \forall V$. $V^\circ(x) = \{z \mid (x, z) \in V^\circ\}$ is open, hence belongs to $\mathcal{U}(y) \Rightarrow V(x) \in \mathcal{U}(y) \Rightarrow \mathcal{U}(x) \subseteq \mathcal{U}(y)$. As $(x, y) \in V^{-1}\ \forall V \Rightarrow (y, x) \in V\ \forall V \Rightarrow \mathcal{U}(y) \subseteq \mathcal{U}(x)$, so $\mathcal{U}(x) = \mathcal{U}(y)$.

(ii) By Claim 4 in 10.1.24, $\mathcal{U}$ is the coarsest uniform structure on X such that $i : X \to \tilde{X}$ is uniformly continuous, so by 9.3.3 a basis of $\mathcal{U}$ is given by $\{(i \times i)^{-1}(\tilde{W}) \mid \tilde{W} \in \tilde{\mathcal{U}}\} =: \mathcal{B}$. Let $V \in \mathcal{U} \Rightarrow \exists \tilde{W} \in \tilde{\mathcal{U}}$ with $(i \times i)^{-1}(\tilde{W}) \subseteq V$, so

$$(i \times i)((i \times i)^{-1}(\tilde{W})) = \underbrace{\tilde{W} \cap (i(X) \times i(X))}_{\in \tilde{\mathcal{U}}_{i(X)}} \subseteq (i \times i)(V) \Rightarrow (i \times i)(V) \in \tilde{\mathcal{U}}_{i(X)}.$$

Conversely, given $W := \tilde{V} \cap (i(X) \times i(X)) \in \tilde{\mathcal{U}}_{i(X)}$, it follows that $W = (i \times i)((i \times i)^{-1}(\tilde{V}))$ and $(i \times i)^{-1}(\tilde{V}) \in \mathcal{U}$ because i is uniformly continuous.

(iii) As $i(X)$ is dense in $\tilde{X}$, this follows from 9.3.7. □

10.1.27 Corollary. *Let X be a uniform Hausdorff space. Then $i : X \to i(X)$ is an isomorphism.*

Proof. X is T_2, so 9.1.17 (ii) shows that $\Delta_X = \bigcap_{V \in \mathcal{U}} V$, which, by 10.1.26 (i) equals $\{(x, y) \mid i(x) = i(y)\}$. It follows that i is injective. Hence $i : X \to i(X)$ is bijective, and by 10.1.26 (ii), i is an isomorphism. □

By 10.1.27 any uniform Hausdorff space X can be identified with the corresponding $i(X)$. Then X is a dense subspace of $\tilde{X}$.

10.1.28 Example. Let X be a complete T_2-space, $A \subseteq X \Rightarrow \tilde{A} \cong \bar{A}$. Indeed, $\bar{A}$ is complete by 10.1.19, A is dense in $\bar{A}$ and with $i := A \hookrightarrow \bar{A}$ the pair $(i, \bar{A})$ satisfies the universal property (10.1.1) due to 10.1.22.

10.2 Compactification of completely regular spaces

10.2.1 Definition. *Let X be a topological space and $f : X \to Z$ an embedding of X in a compact space Z such that $f(X)$ is dense in Z. Then (f, Z) is called a compactification of X.*

In this section, analogous to the problem of completing a uniform space in the previous section, we analyze the question of constructing, for a given topological space X a compactification $(\beta, \beta X)$ with βX T_2 and $\beta : X \to \beta X$ possessing the following universal property:

For any compact Hausdorff space Y and any continuous $f : X \to Y$ there exists a unique continuous $f' : \beta X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[dr, "f"'] & \beta X \arrow[d, "\exists! f'"] \\ & Y \end{tikzcd} \tag{10.2.1}$$

10.2.2 Definition. *A compactification $(\beta, \beta X)$ that has the universal property* (10.2.1) *is called a Stone-Čech compactification of X.*

10.2.3 Example. The Alexandroff-compactification from 5.2.7 in general does not possess the universal property (10.2.1) (with respect to $\beta = X \hookrightarrow X'$): Let $X := (0, 1]$, then $X' \cong [0, 1]$ (X' is compact, $X' \backslash X = \{0\}$). However, the continuous function $x \mapsto \sin \frac{1}{x}$ cannot be extended continuously to X'.

10.2.4 Remark. A necessary condition for a topological space X to possess a Stone-Čech compactification is that X be completely regular: βX is compact, hence completely regular (see 5.1.9 and 4.1.4), hence so is the subspace $\beta(X)$ of βX (cf. 3.2.1).

Conversely, for any completely regular space the compactification problem is indeed solvable:

10.2.5 Theorem. *Any completely regular space X possesses a Stone-Čech compactification $(\beta, \beta X)$. If $(\beta', \beta' X)$ is another Stone-Čech compactification of X then there is a unique homeomorphism $h : \beta X \to \beta' X$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[dr, "\beta'"'] & \beta X \arrow[d, "\exists! h"] \\ & \beta' X \end{tikzcd}$$

Proof. Let $C^b(X) := \{\varphi : X \to \mathbb{R} \mid \varphi \text{ is continuous and bounded}\}$. For $\varphi \in C^b(X)$ let I_φ be the minimal closed interval containing $\varphi(X)$. Then by 3.1.10 the map

$$e : X \to \prod_{\varphi \in C^b(X)} I_\varphi, \quad x \mapsto (\varphi(x))_{\varphi \in C^b(X)}$$

is an embedding into the compact (by 5.1.15) T_2-space $\prod_{\varphi \in C^b(X)} I_\varphi$. By 5.1.8 (i),

$$\beta X := \overline{e(X)} \subseteq \prod_{\varphi \in C^b(X)} I_\varphi$$

is compact, and setting $\beta : X \to \beta X,\ x \mapsto (\varphi(x))_{\varphi \in C^b(X)}$, $\beta(X)$ is dense in βX. Therefore $(\beta, \beta X)$ is a compactification of X.

Claim: $(\beta, \beta X)$ possesses the universal property (10.2.1).

Let Y be compact and T_2, $f : X \to Y$ continuous. Since Y is completely regular, as above also for Y there exists an embedding

$$e' : Y \to \prod_{\psi \in C^b(Y)} I_\psi.$$

Now set

$$F : \prod_{\varphi \in C^b(X)} I_\varphi \to \prod_{\psi \in C^b(Y)} I_\psi, \quad p_\psi(F(t)) := t_{\psi \circ f}$$

for $t = (t_\varphi)_{\varphi \in C^b(X)}$, where $p_\psi : \prod_{\chi \in C^b(Y)} I_\chi \to I_\psi$ is the projection. Then the following diagram commutes:

$$\begin{tikzcd} \prod_{\varphi \in C^b(X)} I_\varphi \arrow[r, "F"] & \prod_{\psi \in C^b(Y)} I_\psi \\ e(X) \arrow[u, "\subseteq"] \arrow[r, "F|_{e(X)}"] & e'(Y) \arrow[u, "\subseteq"'] \\ X \arrow[u, "e"] \arrow[r, "f"] & Y \arrow[u, "e'"'] \end{tikzcd}$$

In fact, $F(e(x)) = F((\varphi(x))_{\varphi \in C^b(x)}) = (\psi \circ f(x))_{\psi \in C^b(Y)} = e'(f(x))$. Furthermore, F is continuous because $p_\psi \circ F = p_{\psi \circ f}$ is continuous for every $\psi \in C^b(Y)$. As $e' : Y \to e'(Y)$ is a homeomorphism, $e'(Y) = \overline{e'(Y)}$ (cf. 5.1.8 (ii)). It follows from the continuity of F that

$$F(\beta X) = F(\overline{e(X)}) \subseteq \overline{F(e(X))} \subseteq \overline{e'(Y)} = e'(Y)$$

Consequently, $f' := e'^{-1} \circ F|_{\beta X}$ is well-defined and continuous. Also,

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[dr, "f"'] & \beta X \arrow[d, "f'"] \\ & Y \end{tikzcd}$$

commutes because $f' \circ \beta(x) = e'^{-1} \circ F \circ e(x) = e'^{-1} \circ e' \circ f(x) = f(x)\ \forall x \in X$.
f' is unique because it is completely determined on $\beta(X) = e(X)$ by $f' \circ \beta = f$ and $\overline{e(X)} = \beta X$. Consequently, $(\beta, \beta X)$ is a Stone-Čech compactification of X.

Uniqueness: Let $(\beta', \beta' X)$ be another Stone-Čech compactification and consider:

$$
\begin{tikzcd}
X \arrow[r, "\beta"] \arrow[dr, "\beta'"'] & \beta X \arrow[d, shift right, "h"'] \\
 & \tilde{X}' \arrow[u, shift right, "h'"']
\end{tikzcd}
$$

Then for $h' \circ h$ we obtain: $h' \circ h \circ \beta = h' \circ \beta' = \beta$. Since also $\mathrm{id}_{\beta X} \circ \beta = \beta$, uniqueness in (10.2.1) gives $h' \circ h = \mathrm{id}_{\beta X}$. Analogously, $h \circ h' = \mathrm{id}_{\beta' X}$, so h is a homeomorphism. $\square$

Chapter 11

Complete, Baire-, and polish spaces

11.1 Complete spaces

In this section we investigate the relation between complete and compact spaces.

11.1.1 Definition. *Let $(X,\mathcal{U})$ be a uniform space. X is called precompact if for every $V \in \mathcal{U}$ there exists a finite cover of X whose sets are small of order V. A subset $A \subseteq X$ is called precompact if $(A,\mathcal{U}_A)$ is precompact (cf. 9.3.5). Precompact metric spaces are also called totally bounded.*

11.1.2 Theorem. *Let $(X,\mathcal{U})$ be a uniform space. TFAE:*

(i) X is precompact.

(ii) The completion $\tilde{X}$ of X is compact.

Proof. (i) $\Rightarrow$ (ii): Let $i : X \to (\tilde{X}, \tilde{U})$ be the canonical map (cf. 10.1.24). By 5.1.3 (iv) we have to show that any ultrafilter on $\tilde{X}$ converges. Thus let $\mathcal{F}$ be an ultrafilter on $\tilde{X}$. Since $\tilde{X}$ is complete it suffices to show that $\mathcal{F}$ is a Cauchy filter. Let $U \in \tilde{\mathcal{U}}$ be closed (cf. 9.1.15) and set $V := (i \times i)^{-1}(U)$. Since i is uniformly continuous, $V \in \mathcal{U}$ and so by (i) there exists a cover $\{B_1, \dots, B_n\}$ of X with $\bigcup_{j=1}^n (B_j \times B_j) \subseteq V$. Let $C_j := i(B_j)$. Then

$$(i \times i)(B_j \times B_j) = C_j \times C_j \subseteq (i \times i)(V) \subseteq U \Rightarrow \overline{C_j} \times \overline{C_j} \subseteq \bar{U} = U.$$

Now $X = \bigcup_{j=1}^n B_j \Rightarrow i(X) = \bigcup_{j=1}^n C_j \Rightarrow \tilde{X} = \overline{i(X)} = \bigcup_{j=1}^n \overline{C_j} \in \mathcal{F}$, so by 2.2.19 there exists some $j \in \{1, \dots, n\}$ with $\overline{C_j} \in \mathcal{F}$, so indeed $\mathcal{F}$ is a Cauchy filter.

(ii) $\Rightarrow$ (i): By Claim 4 from 10.1.24, $\{(i \times i)^{-1}(U) \mid U \in \tilde{\mathcal{U}}\}$ is a fundamental system of entourages of $\mathcal{U}$. Thus let $V := (i \times i)^{-1}(U) \in \mathcal{U}$. Let $U' \in \tilde{\mathcal{U}}$ be symmetric with $U'^2 \subseteq U$ (cf. 9.1.8). Now $\{U'(x)^\circ \mid x \in \tilde{X}\}$ is an open cover of $\tilde{X}$, so by (ii) there exists a finite set $F \subseteq \tilde{X}$ with $\tilde{X} = \bigcup_{x \in F} U'(x)^\circ$. Since $\bigcup_{x \in F} i^{-1}(U'(x)^\circ) = i^{-1}(\bigcup_{x \in F} U'(x)^\circ) = i^{-1}(\tilde{X}) = X$, the family $\mathcal{A} := \{i^{-1}(U'(x)^\circ) \mid x \in F\}$ is a finite cover of X.

To finish the proof we show that all elements of $\mathcal{A}$ are small of order V: Let $a, b \in i^{-1}(U'(x)^\circ) \Rightarrow i(a), i(b) \in U'(x)^\circ \subseteq U'(x) \Rightarrow (i(a), x) \in U', (x, i(b)) \in U' \Rightarrow (i(a), i(b)) \in U'^2 \subseteq U \Rightarrow (a, b) \in (i \times i)^{-1}(U) = V$. □

11.1.3 Corollary. *Let $(X, \mathcal{U})$ be a uniform Hausdorff space. TFAE:*

(i) X is compact.

(ii) X is complete and precompact.

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{F}$ be a Cauchy filter on X. Since X is compact, by 5.1.3 (ii) $\mathcal{F}$ possesses a cluster point, hence it converges by 10.1.6. It follows that X is complete. Thus $X = \tilde{X}$ (in fact, by 10.1.27, X is dense in $\tilde{X}$ and 10.1.20 gives that X is closed in $\tilde{X}$). Hence $\tilde{X}$ is compact, and a fortiori precompact by 11.1.2.

(ii) $\Rightarrow$ (i): If X is complete then as above we conclude that $X = \tilde{X}$. Consequently, $X = \tilde{X}$ is compact by 11.1.2. □

11.1.4 Corollary. *Let $(X, \mathcal{U})$ be a uniform T_2-space, $\tilde{X}$ the completion of X and $A \subseteq X$. TFAE:*

(i) A is precompact.

(ii) A is relatively compact in $\tilde{X}$.

(iii) $\bar{A}$ is compact (where $\bar{A}$ is the closure in $\tilde{X}$).

Proof. (ii) $\Leftrightarrow$ (iii): holds by 5.1.14.

(i) $\Leftrightarrow$ (iii): By 10.1.28, $\tilde{A} \cong \bar{A}$, so the claim follows from 11.1.2. □

11.1.5 Corollary. *Let $(X, \mathcal{U})$ be a uniform T_2-space. Then:*

(i) If B is precompact and $A \subseteq B \subseteq X$, then also A is precompact.

(ii) A, B precompact $\Rightarrow A \cup B$ precompact.

(iii) A precompact $\Rightarrow \bar{A}$ precompact.

Proof. (i) and (ii) are immediate from 11.1.4 (ii).

(iii) We have $\bar{A} = X \cap \bar{A}^{\tilde{X}} \subseteq \bar{A}^{\tilde{X}}$, so $\bar{A}$ is relatively compact by 11.1.4 (iii). □

11.1.6 Proposition. *Let X, Y be uniform spaces with completions $(\tilde{X}, i), (\tilde{Y}, j)$. Let $f : X \to Y$ be uniformly continuous. Then there exists a unique uniformly continuous map $\tilde{f} : \tilde{X} \to \tilde{Y}$ with $\tilde{f} \circ i = j \circ f$:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[d, "j"] \\ \tilde{X} \arrow[r, "\tilde{f}"] & \tilde{Y} \end{tikzcd}$$

Proof. $j \circ f : X \to \tilde{Y}$ is uniformly continuous, so by (10.1.1) there is a unique uniformly continuous map $\tilde{f} : \tilde{X} \to \tilde{Y}$ with $\tilde{f} \circ i = j \circ f$. □

11.1.7 Corollary. *Let X, Y be uniform T_2-spaces, $A \subseteq X$ precompact and $f : X \to Y$ uniformly continuous. Then $f(A)$ is precompact.*

Proof. By 11.1.5 there exists some $B \subseteq \tilde{X}$ compact with $A \subseteq B$. Let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be as in 11.1.6 (with $i = X \hookrightarrow \tilde{X}$, $j = Y \hookrightarrow \tilde{Y}$, cf. 10.1.27). Then $f(A) = \tilde{f}(A) \subseteq \tilde{f}(B)$, which is compact. □

11.2 Complete metric spaces

11.2.1 Examples. (i) $\mathbb{R}$ is complete: Let $\mathcal{F}$ be a Cauchy filter on $\mathbb{R}$ and let $\varepsilon > 0 \Rightarrow \exists F \in \mathcal{F}$ that is small of order $d^{-1}([0,\varepsilon])$, i.e., $|x-y| < \varepsilon\ \forall x, y \in F$. Fix $x_0 \in F$ and set $A := \{y \in \mathbb{R} \mid |x_0 - y| \le \varepsilon\}$. A is compact by 5.1.16. As $F \subseteq A$, 2.2.21 implies that $\mathcal{F}_A := \{F' \cap A \mid F' \in \mathcal{F}\}$ is a filter on A. By 5.1.3 (iii), $\mathcal{F}_A$ has a cluster point z. Hence $z \in \overline{F' \cap A}^A \subseteq \overline{F' \cap A} \subseteq \overline{F'}\ \forall F' \in \mathcal{F}$, so z is a cluster point of $\mathcal{F}$. By 10.1.6 this shows that $\mathcal{F}$ converges.

(ii) By (i) and 10.1.18, $\mathbb{R}^n$ is complete $\forall n \ge 1$.

11.2.2 Theorem. *Let (X,d) be a metric space with completion $(\tilde{X}, \tilde{\mathcal{U}})$. Then there exists a unique metric $\tilde{d}$ on $\tilde{X}$ with $\tilde{d}|_{X\times X} = d$. Moreover, $\tilde{d}$ generates the uniform structure $\tilde{\mathcal{U}}$ on $\tilde{X}$.*

Proof. *Claim 1:* $d : X \times X \to \mathbb{R}$ is uniformly continuous:

Let $V_\varepsilon := \{(x,y) \in \mathbb{R}^2 \mid |x-y| \le \varepsilon\}$ be an an entourage in $\mathbb{R}$. Let $p_i : X \times X \to X$ $(i = 1,2)$ be the projections and $V := (p_1 \times p_1)^{-1}(d^{-1}([0, \frac{\varepsilon}{2}])) \cap (p_2 \times p_2)^{-1}(d^{-1}([0, \frac{\varepsilon}{2}]))$. By 9.3.3 and 9.3.8, V is an entourage in $X \times X$.

Let $((x,y),(x',y')) \in V$. We will show that $(d \times d)((x,y),(x',y')) \in V_\varepsilon$, i.e., that $|d(x,y) - d(x',y')| \le \varepsilon$. In fact,

$d(x,y) \le d(x,x') + d(x',y') + d(y',y) \Rightarrow d(x,y) - d(x',y') \le d(x,x') + d(y,y')$, and analogously: $d(x',y') - d(x,y) \le d(x,x') + d(y,y')$, hence:

$$|d(x,y) - d(x',y')| \le d(x,x') + d(y,y').$$

Since $((x,y),(x',y')) \in (p_1 \times p_1)^{-1}(d^{-1}([0, \frac{\varepsilon}{2}]))$ it follows that $d(x,x') \le \frac{\varepsilon}{2}$, and analogously $d(y,y') \le \frac{\varepsilon}{2}$, establishing Claim 1.

$X \times X$ is dense in $\tilde{X} \times \tilde{X}$ by 2.3.20 and 10.1.27, and $\mathbb{R}$ is complete by 11.2.1 (i). Hence by 10.1.22 there exists a unique uniformly continuous extension $\tilde{d} : \tilde{X} \times \tilde{X} \to \mathbb{R}$ of d.

Claim 2. $\tilde{d}$ is a pseudometric on $\tilde{X} \times \tilde{X}$.

$\Delta_X = \{(x,x) \mid x \in X\}$ is dense in $\Delta_{\tilde{X}} = \{(y,y) \mid y \in \tilde{X}\}$ (in fact, let W be a neighborhood of (y,y). Then there exists a neighborhood U of y with $U \times U \subseteq W \Rightarrow \exists x \in X \cap U \Rightarrow (x,x) \in W$). Now $d|_{\Delta_X} = 0$ and $\tilde{d}|_{\Delta_{\tilde{X}}}$ is a continuous extension of $d|_{\Delta_X} = 0$. Thus by 3.3.1 (ii), $\tilde{d}|_{\Delta_{\tilde{X}}} = 0 \Rightarrow d(y,y) = 0\ \forall y \in \tilde{X}$.

Next, $\tilde{d}(x_1,x_2) - \tilde{d}(x_2,x_1) = d(x_1,x_2) - d(x_2,x_1) = 0\ \forall x_1, x_2 \in X$. As $X \times X$ is dense in $\tilde{X} \times \tilde{X}$, again by 3.3.1 (ii), $\tilde{d}(y_1,y_2) = \tilde{d}(y_2,y_1)\ \forall y_1, y_2 \in \tilde{X}$.

Finally, to show the triangle inequality, note that

$$D := \{(y_1,y_2,y_3) \in \tilde{X}^3 \mid \tilde{d}(y_1,y_3) \le \tilde{d}(y_1,y_2) + \tilde{d}(y_2,y_3)\}$$

is closed and contains X^3, so $\overline{X^3} = \tilde{X}^3 = D$.

Claim 3: $\tilde{d}$ generates $\tilde{\mathcal{U}}$, i.e., $\mathcal{U}_{\tilde{d}} = \tilde{\mathcal{U}}$.

$\subseteq$: By 9.4.5, $\mathcal{B} := \{\tilde{d}^{-1}([0,\varepsilon]) \mid \varepsilon > 0\}$ is a fundamental system of entourages of $\mathcal{U}_{\tilde{d}}$. Any $\tilde{d}^{-1}([0,\varepsilon])$ is closed in $\tilde{X} \times \tilde{X}$ because $\tilde{d}$ is continuous. Moreover, $U_\varepsilon := \tilde{d}^{-1}([0,\varepsilon]) \cap (X \times X) = d^{-1}([0,\varepsilon]) \in \mathcal{U}_d$, so by 9.3.7, $\bar{U}_\varepsilon \in \tilde{\mathcal{U}}$. Due to $\bar{U}_\varepsilon \subseteq \overline{\tilde{d}^{-1}([0,\varepsilon])} = \tilde{d}^{-1}([0,\varepsilon])$ we therefore get $\tilde{d}^{-1}([0,\varepsilon]) \in \tilde{\mathcal{U}}$.

$\supseteq$: Let $W \in \tilde{\mathcal{U}}$, W closed (cf. 9.1.15). Then $W \cap (X \times X) \in \mathcal{U}_d$, so there exists some $\varepsilon > 0$ such that $U_\varepsilon := d^{-1}([0,\varepsilon)) \subseteq W \cap (X \times X)$. Let $V_\varepsilon := \tilde{d}^{-1}([0,\varepsilon)) \in \mathcal{U}_{\tilde{d}}$. V_ε is open in $\tilde{X} \times \tilde{X}$ because $\tilde{d}$ is continuous and we have $U_\varepsilon = V_\varepsilon \cap (X \times X)$. Also,

$V_\varepsilon \subseteq \bar{U}_\varepsilon$: let $(z_1, z_2) \in V_\varepsilon$ and let B be a neighborhood of (z_1, z_2). Then $B \cap V_\varepsilon$ is a neighborhood of (z_1, z_2) and since $X \times X$ is dense in $\tilde{X} \times \tilde{X}$, $B \cap V_\varepsilon \cap (X \times X) \neq \emptyset$, i.e., $(z_1, z_2) \in \bar{U}_\varepsilon \Rightarrow V_\varepsilon \subseteq \bar{U}_\varepsilon \subseteq \overline{W} = W \Rightarrow \tilde{\mathcal{U}} \subseteq \mathcal{U}_{\tilde{d}}$, which proves Claim 3.

It follows that $\tilde{\mathcal{U}} = \mathcal{U}_{\tilde{d}}$. As $\tilde{\mathcal{U}}$ is T_2, 9.4.7 shows that $\tilde{d}$ is a metric. □

11.2.3 Corollary. *Let (X, d) be a metric space. TFAE:*

(i) X is complete.

(ii) Any Cauchy sequence in X is convergent.

Proof. (i) $\Rightarrow$ (ii): This is clear from 10.1.4.

(ii) $\Rightarrow$ (i): If X were not complete there would exist some $\tilde{x} \in \tilde{X} \setminus X$. As X is dense in $\tilde{X}$ and $\tilde{X}$ is a metric space by 11.2.2, there would exist a sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\tilde{x} = \lim x_n$ in $\tilde{X}$. $(x_n)_{n\in\mathbb{N}}$ is convergent, hence a Cauchy sequence, so (ii) implies $\tilde{x} \in X$, a contradiction. □

11.2.4 Definition. *Let (X, d) be a metric space, $A \subseteq X$. The diameter of A is $\delta(A) := \sup\{d(x, y) \mid x, y \in A\}$. For $A = \emptyset$ we set $\delta(A) := 0$.*

11.2.5 Examples.

(i) $\delta(B_r(x)) \leq 2r$.

(ii) $\delta(A) = 0 \Leftrightarrow A = \emptyset$ or A is a singleton.

(iii) A is small of order $\varepsilon \Leftrightarrow A \times A \subseteq d^{-1}([0, \varepsilon]) \Leftrightarrow d(x, y) \leq \varepsilon\ \forall x, y \in A \Leftrightarrow \delta(A) \leq \varepsilon$.

11.2.6 Theorem. *(Principle of nested intervals) Let (X, d) be a metric space. TFAE:*

(i) X is complete.

(ii) Let $A_n \neq \emptyset$ be closed sets ($n \in \mathbb{N}$) with $A_n \supseteq A_{n+1}$ and $\inf\limits_{n\in\mathbb{N}} \delta(A_n) = 0$. Then $\bigcap_{n=1}^\infty A_n = \{x\}$ for some $x \in X$.

Proof. (i) $\Rightarrow$ (ii): Let $a_n \in A_n$ for $n \in \mathbb{N}$. Then $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence: Let $\varepsilon > 0 \Rightarrow \exists n_0$ with $\delta(A_{n_0}) < \varepsilon$. Since A_n is decreasing, $\delta(A_n) < \varepsilon\ \forall n \geq n_0$ $\Rightarrow d(a_n, a_m) < \varepsilon\ \forall n, m \geq n_0 \Rightarrow a_n$ converges to some $a \in X$.

Let $\mathcal{F}$ be the elmentary filter belonging to $(a_n)_{n\in\mathbb{N}}$. Then $\mathcal{F} \to a$, so a is a cluster point of $\mathcal{F}$. As $A_n \in \mathcal{F}\ \forall n \Rightarrow a \in \bigcap_{n\in\mathbb{N}} \bar{A}_n = \bigcap_{n\in\mathbb{N}} A_n$. Since $\delta(\bigcap_{n\in\mathbb{N}} A_n) \leq \delta(A_n)\ \forall n \Rightarrow \delta(\bigcap_{n\in\mathbb{N}} A_n) = 0 \Rightarrow \bigcap_{n\in\mathbb{N}} A_n = \{a\}$ (cf. 11.2.5 (ii)).

(ii) $\Rightarrow$ (i): Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X and let $A_k := \overline{\{x_n \mid n \geq k\}} \Rightarrow$ $(A_k)_{k\in\mathbb{N}}$ satisfies the assumptions of (ii). Thus there exists some x with $\{x\} = \bigcap_{k\in\mathbb{N}} A_k$. Let $\varepsilon > 0 \Rightarrow \exists n_0$ with $\delta(A_{n_0}) < \varepsilon$. Since $x_n \in A_{n_0} \forall n \geq n_0$ it follows that $d(x_n, x) \leq \delta(A_{n_0}) < \varepsilon\ \forall n \geq n_0 \Rightarrow x = \lim\limits_{n\to a} x_n$. By 11.2.3, X is complete. □

It follows that, as in the special case of $\mathbb{R}$, also in general metric spaces completeness is equivalent to the principle of nested intervals.

11.2.7 Theorem. *Let X be a metric space. TFAE:*

(i) X is precompact.

(ii) $\forall \varepsilon > 0\ \exists$ *finite cover* $(U_i)_{1\le i\le n(\varepsilon)}$ *of* X *with* $\delta(U_i) \le \varepsilon\ \forall i$ *(*$(U_i)_{1\le i\le n(\varepsilon)}$ *is called an* ε*-mesh).*

(iii) Any sequence in X *possesses a Cauchy subsequence.*

Proof. (i) $\Leftrightarrow$ (ii): This follows from 11.2.5 (iii).

(ii) $\Rightarrow$ (iii): Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. For any $m \in \mathbb{N}$ there exists a finite cover $(U_{i_m})_{i_m=1}^{N_m}$ of X with $\delta(U_{i_m}) < \frac{1}{m}\ \forall i_m$. As $\{x_n \mid n \in \mathbb{N}\} \subseteq \bigcup_{i_1=1}^{N_1} U_{i_1}$, one of the U_{i_1} must contain infinitely many x_n. These form a subsequence $(x_n^{(1)})_{n\in\mathbb{N}}$. By the same token, infinitely many $x_n^{(1)}$ must lie in some U_{i_2}, so we obtain a sub-subsequence $(x_n^{(2)})_{n\in\mathbb{N}}$, etc. Let $y_n := x_n^{(n)}$. Then $\delta(\{y_n \mid n \ge k\}) \le \frac{1}{k}$, so $(y_n)_{n\in\mathbb{N}}$ is a Cauchy subsequence of $(x_n)_{n\in\mathbb{N}}$.

(iii) $\Rightarrow$ (ii): Suppose that (ii) is violated. Then there exists some $\varepsilon > 0$ such that there is no finite cover of X by sets of diameter $\le 2\varepsilon$.

We show by induction that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $d(x_i, x_n) \ge \varepsilon\ \forall i < n\ \forall n$.

$n = 0$: is clear.

$n \to n+1$: Let $x_0, \dots, x_n$ be such that $d(x_i, x_n) \ge \varepsilon\ \forall i < n$. Then $\bigcup_{i\le n} B_\varepsilon(x_i) \ne X$ by assumption, so there exists some $x_{n+1} \in X \setminus \bigcup_{i\le n} B_\varepsilon(x_i)$. Thus $d(x_i, x_{n+1}) \ge \varepsilon\ \forall i \le n$.

Obviously, $(x_n)_{n\in\mathbb{N}}$ cannot contain a Cauchy subsequence, a contradiction. □

11.2.8 Theorem. *Let* X *be a metric space. TFAE:*

(i) X *is compact.*

(ii) X *is complete and precompact.*

(iii) Any sequence in X *possesses an accumulation point.*

(iv) Any sequence in X *possesses a convergent subsequence.*

Proof. (i) $\Leftrightarrow$ (ii): See 11.1.3

(iii) $\Leftrightarrow$ (iv): See 2.3.10.

(i) $\Rightarrow$ (iii): By 5.1.3 (iii), the elementary filter of $(x_n)_{n\in\mathbb{N}}$ possesses a cluster point. By 2.3.10, this is an accumulation point of the sequence $(x_n)_{n\in\mathbb{N}}$.

(iv) $\Rightarrow$ (ii): X is precompact by 11.2.7, (iii) $\Rightarrow$ (i). Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence, hence is itself convergent. (This can be proved as in analysis. Alternatively, by 2.3.10, the elementary filter $\mathcal{F}$ of $(x_n)_{n\in\mathbb{N}}$ possesses a cluster point, so it converges by 10.1.6). Consequently, X is complete by 11.2.3. □

11.2.9 Corollary. *Let* (X, d) *be a metric space,* $A \subseteq X$*. TFAE:*

(i) A *is relatively compact.*

(ii) Any sequence in A *has an accumulation point in* X*.*

Proof. (i) $\Rightarrow$ (ii): By 5.1.14, $\bar{A}$ is compact, so by 11.2.8 $(x_n)_n$ possesses an accumulation point in $\bar{A} \subseteq X$.

(ii) $\Rightarrow$ (i): We use 11.2.8 (iv) to show that $\bar{A}$ is compact. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\bar{A}$. Then for each $n \ge 1$, $B_{\frac{1}{n}}(x_n) \cap A \ne \emptyset$. Let $y_n \in B_{\frac{1}{n}}(x_n) \cap A$. Then by (ii), (y_n) has an accumulation point y in X, and 2.3.10 implies that there is a subsequence $(y_{n_k})_{k\ge1}$ with $y_{n_k} \to y$ $(k \to \infty)$. Since $d(x_{n_k}, y) \le d(x_{n_k}, y_{n_k}) + d(y_{n_k}, y) \to 0$ it follows that $y = \lim_{k\to\infty} x_{n_k} \in \bar{A}$, so A is compact. □

11.2.10 Proposition. *Any precompact metric space X is separable.*

Proof. The completion $\tilde{X}$ of X is compact by 11.1.2 and metrizable by 11.2.2. Also, 7.2.6 shows that $\tilde{X}$ and thereby also X has a countable basis. By 3.1.3, X is normal and consequently also regular. Finally, 7.2.9 shows separability of X. $\square$

11.2.11 Proposition. *Let X be a metrizable and separable topological space. Then X possesses a metric d that induces the toplogy on X such that (X, d) is precompact.*

Proof. By 7.2.9, we may without loss of generality suppose that $X \subseteq [0,1]^{\mathbb{N}}$. Now $[0,1]^{\mathbb{N}}$ is complete by 10.1.18 and metrizable by 7.2.9 (or 9.4.11). By 10.1.28, $\tilde{X} \cong \bar{X}$. Since $[0,1]^{\mathbb{N}}$ is compact, $\bar{X}$ is compact by 5.1.8. Thus by 11.1.2, X is precompact as a subspace of the metric space $[0,1]^{\mathbb{N}}$. $\square$

11.3 Polish spaces

11.3.1 Definition. *A topological space X is called completely metrizable if there is a metric d on X such that d induces the topology of X and (X, d) is complete.*

The following class of topological spaces plays an important rôle in measure theory:

11.3.2 Definition. *A topological space X is called polish if it is completely metrizable and has a countable basis.*

11.3.3 Examples.

(i) $\mathbb{R}^n$ is polish.

(ii) $[0,1]^{\mathbb{N}}$ is polish by 7.2.9.

11.3.4 Theorem.

(i) Any closed subspace of a polish space is polish.

(ii) Any open subspace of a polish space is polish.

(iii) Any countable product of polish spaces is polish.

Proof. (i) Let X be polish and $A \subseteq X$ closed. Then also A has a countable basis and is complete by 10.1.19.

(iii) Let X_n be polish ($n \in \mathbb{N}$). Then by 1.1.10, $X = \prod_{n=0}^{\infty} X_n$ has a countable basis and is completely metrizable by 9.4.11 and 10.1.18.

(ii) Let X be polish, and let d be a metric on X such that (X, d) is complete. Let $U \subseteq X$ be open, $U \neq X$. Then

$$V := \{(t, x) \in \mathbb{R} \times X \mid t \cdot d(x, X \setminus U) = 1\}$$

is closed because $(t, x) \mapsto t \cdot d(x, X \setminus U)$ is continuous. By (i) and (iii), V is polish. Let $p_2 : \mathbb{R} \times X \to X$ be the projection and $f := p_2|_V$.

Claim: $f : V \to U$ is a homeomorphism.

Let $(t, x) \in V \Rightarrow d(x, X \setminus U) \neq 0 \Rightarrow x \notin X \setminus U \Rightarrow f : V \to U$.

f is surjective: Let $x \in U$, $\alpha := d(x, X \setminus U) > 0 \Rightarrow \exists t \in \mathbb{R}$ with $\alpha = \frac{1}{t} \Rightarrow td(x, X \setminus U) = 1 \Rightarrow (t, x) \in V$ and $f(t, x) = x$.

f is injective: Let $(t_1, x_1), (t_1, x_2) \in V$ with $f(t_1, x_1) = x_1 = f(t_2, x_2) = x_2 \Rightarrow x_1 = x_2$ and: $d(x_1, X \setminus U) = \frac{1}{t_1} = d(x_2, X \setminus U) = \frac{1}{t_2} \Rightarrow t_1 = t_2$.

f is continuous, being a restriction of p_2. Finally, $f^{-1} = x \mapsto (\frac{1}{d(x, X\setminus U)}, x)$ is continuous on U.

Summing up, U is homeomorphic to a polish space, hence is itself polish. □

11.3.5 Proposition. *Any σ-compact metrizable space X is polish.*

Proof. By 7.2.7, the Alexandroff-compactification X' of X is metrizable. Since X has a countable basis, the same is true of X' (∞ has a countable neighborhood basis). X' is compact, so 9.4.14 shows that X' is uniformizable by a unique uniform structure, and in this structure (which by uniqueness has to be the one induced by the metric on X') X' is complete by 11.1.3. Finally, 11.3.4 (ii) shows that X is polish. □

11.3.6 Proposition. *Let X be a T_2-space and let $(A_n)_{n\in\mathbb{N}}$ be a family of polish subspaces of X. Then also $A := \bigcap_{n\in\mathbb{N}} A_n$ is polish.*

Proof. Let $f : A \to X^{\mathbb{N}}$, $f(x) := (x)_{n\in\mathbb{N}}$. Then f is continuous and injective and $f(A) = \{(x_n)_{n\in\mathbb{N}} \mid \exists x \in A : \; x_n = x \; \forall n \in \mathbb{N}\} \subseteq \prod_{n\in\mathbb{N}} A_n$. $p_0|_{f(A)}$ is a continuous inverse of f, so $f : A \to f(A)$ is a homeomorphism. Let $\tilde{p}_i : \prod_{n\in\mathbb{N}} A_n \to A_i$ be the projection. Then

$$f(A) = \bigcap_{i,j\in\mathbb{N}} \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n \mid \tilde{p}_i((x_n)_n) = \tilde{p}_j((x_n)_n)\}$$

is a closed subspace of $\prod_{n\in\mathbb{N}} A_n$ (any set in this intersection is closed by 3.3.1 (i)). Altogether, A is homeomorphic to a closed subspace of $\prod_{n\in\mathbb{N}} A_n$, hence is polish by 11.3.4 (i) and (iii). □

11.3.7 Corollary. *$I = \mathbb{R} \setminus \mathbb{Q}$ is a polish subspace of $\mathbb{R}$.*

Proof. Let $\mathbb{Q} = \{r_n \mid n \in \mathbb{N}\}$. Then $\mathbb{R} \setminus \mathbb{Q} = \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} r_n = \bigcap_{n\in\mathbb{N}} \mathbb{R} \setminus \{r_n\}$ is polish by 11.3.6. □

11.3.8 Theorem. *(Mazurkiewicz). Let X be polish, $A \subseteq X$. TFAE:*

(i) A is polish.

(ii) A is a G_δ-set.

Proof. (i) $\Rightarrow$ (ii): Let d, d_A be metrics such that (X, d), (A, d_A) are complete. Letting δ_A denote the diameter in A, set

$$A_n := \{x \in \bar{A} \mid \exists U \text{ open in } X, \text{ s.t. } x \in U \text{ and } \delta_A(U \cap A) < \frac{1}{n}\}.$$

Then A_n is open in $\bar{A}$ and $A \subseteq A_n$ (for $x \in A$ pick U open in X such that $U \cap A = B^{d_A}_{1/(3n)}(x) \Rightarrow x \in A_n$). Hence $A \subseteq \bigcap_{n\geq 1} A_n$.

Conversely, let $x \in \bigcap_{n\geq 1} A_n \Rightarrow x \in \bar{A}$, so by 2.2.24 $\mathcal{U}(x)_A = \{U \cap A \mid U \in \mathcal{U}(x)\}$ is a filter on A and thereby a filter basis on X that is finer than $\mathcal{U}(x)$, hence converges also in X to x. In addition, $\mathcal{U}(x)_A$ is a Cauchy filter on A: Let $n \geq 1 \Rightarrow \exists U \in \mathcal{U}(x)$ such that $U \cap A$ is small of order $\frac{1}{n}$ with respect to d_A. Since (A, d_A) is complete, $\mathcal{U}(x)_A$ converges in A to some $a \in A$. As $j : A \hookrightarrow X$ is continuous, $\mathcal{U}(x)_A \to a$ in $X \Rightarrow x = a \in A$. Summing up, $A = \bigcap_{n\geq 1} A_n$.

For any $n \geq 1$, choose U_n open in X with $U_n \cap \bar{A} = A_n$. X is metrizable and $\bar{A}$ is closed, so by 7.2.3 there exists some V_n open in X with $\bar{A} = \bigcap_{n\geq 1} V_n$. Thus, finally,

$$A = \bigcap_{n\geq 1} A_n = \bar{A} \cap \bigcap_{n\geq 1} U_n = \bigcap_{n\geq 1} (U_n \cap V_n)$$

is a G_δ-set.

(ii) $\Rightarrow$ (i): This is immediate from 11.3.4 (ii) and 11.3.6. $\square$

11.3.9 Theorem. *Let X be a topological space. TFAE:*

(i) X is polish.

(ii) X is homeomorphic to a G_δ-set in $[0,1]^{\mathbb{N}}$.

Proof. (i) $\Rightarrow$ (ii): X is completely metrizable and possesses a countable basis, so the proof of 7.2.9 (i)$\Leftrightarrow$(ii) shows that X is metrizable and separable. Then 7.2.9 implies that there exists some $Y \subseteq [0,1]^{\mathbb{N}}$ such that X is homeomorphic to Y. It follows that Y is polish, and by 11.3.8 and 11.3.3 (ii), Y is G_δ.

(ii) $\Rightarrow$ (i): $[0,1]^{\mathbb{N}}$ is polish by 11.3.3 (ii). Thus the claim follows from 11.3.8. $\square$

11.4 Baire spaces

11.4.1 Definition. *Let X be a topological space. Then $A \subseteq X$ is called*

- *nowhere dense, if $(\bar{A})^\circ = \emptyset$.*
- *meager (resp. of first category) if A is a countable union of nowhere dense sets.*
- *of second category, if A is not meager.*

11.4.2 Proposition. *Let X be a topological space. Then:*

(i) If A is meager and $B \subseteq A$, then B is meager.

(ii) If $(A_n)_{n\in\mathbb{N}}$ is a family of meager sets, then also $A := \bigcup_{n\in\mathbb{N}} A_n$ is meager.

(iii) Let $A \subseteq B \subseteq X$, A meager in X. Then A is also meager in B.

Proof. (i) Let $A = \bigcup_{n\in\mathbb{N}} A_n$ with $(\overline{A_n})^\circ = \emptyset \ \forall n \Rightarrow B = \bigcup_{n\in\mathbb{N}} (A_n \cap B)$ and $\left(\overline{(A_n \cap B)}\right)^\circ \subseteq (\overline{A_n})^\circ = \emptyset$.

(ii) is clear.

(iii) Let $A = \bigcup_{n\in\mathbb{N}} A_n$ with $(\overline{A_n})^\circ = \emptyset$ for all n. Then $A = \bigcup_{n\in\mathbb{N}} (A_n \cap B)$ and $(\overline{A_n \cap B}^B)^{\circ_B} \subseteq (\overline{A_n}^B)^\circ \subseteq (\overline{A_n})^\circ = \emptyset$, so A is meager in B. $\square$

11.4.3 Examples. (i) $\mathbb{Q}$ is meager in $\mathbb{R}$ because $\mathbb{Q} = \bigcup_{n\in\mathbb{N}} \{r_n\}$ and $\overline{\{r_n\}}^\circ = \emptyset$.

(ii) Let V be a subspace of $\mathbb{R}^n$ with $\dim V < n$. Then V is nowhere dense in $\mathbb{R}^n$. In fact, there exists some linear map f on $\mathbb{R}^n$ with $V = f^{-1}(\{0\}) = \ker f$. Since f is continuous, V is closed, and $V^\circ = \emptyset$: suppose there exists some $x \in V$ and some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq V \Rightarrow B_\varepsilon(0) = B_\varepsilon(x) - x \subseteq V \Rightarrow \frac{\varepsilon}{2} e_1, \ldots, \frac{\varepsilon}{2} e_n \in V \Rightarrow V = \mathbb{R}^n$, a contradiction.

11.4.4 Theorem. *Let X be a topological space. TFAE:*

(i) If $A = \bigcup_{n\in\mathbb{N}} A_n$, A_n closed, $A_n^\circ = \emptyset$, then $A^\circ = \emptyset$.

(ii) If U_n is open and dense for every $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X.

(iii) If $U \subseteq X$ is open and $\neq \emptyset$, then U is not meager.

(iv) If A is meager in X, then $X \setminus A$ is dense in X.

Proof. (i) $\Rightarrow$ (ii): Any $X \setminus U_n$ is closed and $(X \setminus U_n)^\circ = X \setminus \overline{U_n} = X \setminus X = \emptyset$. Let $A := \bigcup_{n\in\mathbb{N}} X \setminus U_n = X \setminus \bigcap_{n\in\mathbb{N}} U_n$. Then by (i), $A^\circ = \emptyset$. Since $A^\circ = (X \setminus \bigcap_{n\in\mathbb{N}} U_n)^\circ = X \setminus (\overline{\bigcap_{n\in\mathbb{N}} U_n})$, (ii) follows.
(ii) $\Rightarrow$ (iii): Suppose that U is meager $\Rightarrow \exists A_n$ with $(\overline{A_n})^\circ = \emptyset$ such that $U = \bigcup_{n\in\mathbb{N}} A_n$. Then $X \setminus \overline{A_n}$ is open and $(\overline{X \setminus \overline{A_n}}) = X \setminus (\overline{A_n})^\circ = X \setminus \emptyset = X$, so $X \setminus \overline{A_n}$ is dense. Hence $\bigcap_{n\in\mathbb{N}} X \setminus \overline{A_n} = X \setminus \bigcup_{n\in\mathbb{N}} \overline{A_n}$ is dense in X, i.e. $\overline{X \setminus \bigcup_n \overline{A_n}} = X \setminus (\bigcup_n \overline{A_n})^\circ = X \Rightarrow (\bigcup_{n\in\mathbb{N}} \overline{A_n})^\circ = \emptyset$. But $\emptyset \neq U \subseteq \bigcup_{n\in\mathbb{N}} \overline{A_n}$, a contradiction.
(iii) $\Rightarrow$ (iv): Suppose that $X \neq \overline{X \setminus A} = X \setminus A^\circ$. Then $A^\circ \neq \emptyset \Rightarrow \exists U$ open, $\neq \emptyset$ with $U \subseteq A$. By 11.4.2 (i), this contradicts (iii).
(iv) $\Rightarrow$ (i): A is meager $\Rightarrow \overline{X \setminus A} = X \setminus A^\circ = X \Rightarrow A^\circ = \emptyset$. $\square$

11.4.5 Definition. *A topological space satisfying the equivalent conditions from 11.4.4 is called a Baire space.*

11.4.6 Corollary. *Let $X \neq \emptyset$ be a Baire space. Then:*

(i) X is not meager, hence is of second category.

(ii) If $X = \bigcup_{n\in\mathbb{N}} A_n$, A_n closed $\forall n$, then there exists some $n \in \mathbb{N}$ with $A_n^\circ \neq \emptyset$.

Proof. (i) Suppose that X is meager $\Rightarrow X = \bigcup_{n\in\mathbb{N}} A_n$ with $(\overline{A_n})^\circ = \emptyset\ \forall n \Rightarrow X = \bigcup_{n\in\mathbb{N}} \overline{A_n} \Rightarrow X^\circ = \emptyset$, a contradiction to $X = X^\circ \neq \emptyset$.
(ii) See 11.4.4 (i). $\square$

11.4.7 Example. $\mathbb{Q}$ is not a Baire space because $\mathbb{Q} = \bigcup_{r\in\mathbb{Q}}\{r\}$, $\{r\}$ is closed, and $\{r\}^\circ = \emptyset$.

11.4.8 Proposition. *Let X be a Baire space, $U \subseteq X$ open, $U \neq \emptyset$. Then U is a Baire space.*

Proof. We first show that if A is nowhere dense in U, then it is also nowhere dense in X: Let $B := (\bar{A})^\circ \Rightarrow U \cap B$ is open in U, $U \cap B \subseteq \bar{A}$. Also, $U \cap B \subseteq \bar{A}^U = \bar{A} \cap U$. Since $(\bar{A}^U)^{\circ_U} = \emptyset \Rightarrow U \cap B = \emptyset$, i.e. $U \subseteq X \setminus B \Rightarrow \bar{A} \subseteq \bar{U} \subseteq X \setminus B$ and due to $B \subseteq \bar{A}$ we get $B = \emptyset$.

It follows that any set that is meager with respect to U is also meager in X. Moreover, any open subset of U is also open in X. Thus if $V \subseteq U$ is open and $\neq \emptyset$ then by 11.4.4 (iii) V is not meager in X, and therefore V is not meager in U. Consequently, U is a Baire space. $\square$

11.4.9 Proposition. *Let $X \neq \emptyset$ be a Baire space, A meager in X. Then $X \setminus A$ is a Baire space, hence not meager (of second category) in X.*

Proof. By 11.4.4 (iv), $\Rightarrow A^c := X \setminus A$ is dense in X.

We first show that if B is nowhere dense in A^c, then it also is in X.

In fact, suppose that $\bar{B}^\circ \neq \emptyset \Rightarrow \exists \emptyset \neq U$ open in X with $U \subseteq \bar{B} \Rightarrow U \cap A^c \neq \emptyset$ and $U \cap A^c \subseteq \bar{B} \cap A^c = \overline{B}^{A^c} \Rightarrow$ The interior of $\overline{B}^{A^c}$ in A^c is non-empty, so B is not nowhere dense in A^c, a contradiction.

Hence any set M that is meager in $X \setminus A$ is also meager in X. By 11.4.2 (ii), $A \cup M$ is meager in X. Now 11.4.4 (iv) implies that $(X \setminus A) \setminus M = X \setminus (A \cup M)$ is dense in X, hence also dense in $X \setminus A$. Therefore, 11.4.4 (iv) shows that $X \setminus A$ is a Baire space. The final claim follows from 11.4.6 (i) and 11.4.2 (iii). □

11.4.10 Theorem. *(Baire)*

(i) Any completely metrizable space is a Baire space.

(ii) Any locally compact space is a Baire space.

Proof. Using 11.4.4 (ii) we show both statements in parallel.

Let U_n be open and dense in X $(n \geq 1)$. Let $G \neq \emptyset$ open. We have to show that $G \cap \bigcap_{n \geq 1} U_n \neq \emptyset$. If X is locally compact then by 5.2.4 we may without loss of generality suppose that $\bar{G}$ is compact. Let $G_1 := G$ and suppose that $G_n \neq \emptyset$ open has already been constructed. X ist regular (cf. 3.1.3). Also U_n is dense, so the open set $U_n \cap G_n$ is non-empty, and so by 3.1.6 there exists some $\emptyset \neq G_{n+1}$ open with $\overline{G_{n+1}} \subseteq U_n \cap G_n$.

In case (i) we may additionally have $\delta(G_{n+1}) \leq \frac{1}{n}\delta(G_n)$ (This is clear if $\delta(G_n) = 0$. Otherwise choose $x \in G_{n+1}$ and $r > 0$ such that $B_r(x) \subseteq G_{n+1}$ and $2r \leq \frac{1}{n}\delta(G_n)$ and set $G_{n+1}^{\text{new}} := B_r(x)$). Then $\lim_{n\to\infty} \delta(G_n) = 0$.

For X locally compact, $\bigcap_{n=1}^m \overline{G_n} = \overline{G_m} \neq \emptyset$ and $\overline{G_n} \subseteq \overline{G}$ is compact for all n. Hence by 5.1.3 (ii), $\bigcap_{n \geq 1} \overline{G_n} \neq \emptyset$.

For X completely metrizable it follows from 11.2.6 that $\bigcap_{n \geq 1} \overline{G_n} \neq \emptyset$.

Thus in both cases

$$\emptyset \neq \bigcap_{n=2}^{\infty} \overline{G_n} = \bigcap_{n=1}^{\infty} \overline{G_{n+1}} \subseteq \bigcap_{n=1}^{\infty} (G_n \cap U_n) \subseteq \bigcap_{n=1}^{\infty} (G \cap U_n) = G \cap \bigcap_{n=1}^{\infty} U_n.$$

□

As an application of Baire's theorem we show:

11.4.11 Theorem. *(Banach) There exists a continuous real-valued function f on $[0,1]$ that is not differentiable in any $x \in (0,1)$.*

Proof. Let $I := [0,1]$, $C(I) := \{f : [0,1] \to \mathbb{R} \mid f \text{ continuous}\}$ with the metric:

$$d(f,g) := \sup\{|f(x) - g(x)| \ \mid x \in I\}.$$

Then $C(I)$ is a complete metric space, hence a Baire space by 11.4.10.

Claim: $A := \{f \in C(I) \mid \exists x \in I^\circ \text{ s.t. } f'(x) \text{ exists}\}$ ist meager in $C(I)$.

Let $A_n := \{f \in C(I) \mid \exists x \in [0, 1 - \frac{1}{n}] \text{ with } \left|\frac{f(x+h)-f(x)}{h}\right| \leq n \ \forall h \in (0, \frac{1}{n}]\}$

If $f \in A \Rightarrow \exists n \geq 1$ with $f \in A_n \Rightarrow A \subseteq \bigcup_{n \geq 1} A_n$.

1.) $A_n^\circ = \emptyset \ \forall n \geq 1$:

Let $f \in A_n$ and let $\varepsilon > 0$. We show: $\exists g \in C(I)$ with $d(f,g) < \varepsilon$ and $g \notin A_n$, i.e.:

$$\forall x \in \left[0, 1 - \frac{1}{n}\right] \exists h \in \left(0, \frac{1}{n}\right] \text{ with } \left|\frac{g(x+h) - g(x)}{h}\right| > n.$$

By 6.1.8 there exists a polynomial $p \in C(I)$ satisfying $d(f,p) < \frac{\varepsilon}{2}$. Let $M := \max_{x\in I} |p'(x)|$ and let s be a piecewise linear function such that any straight piece in s has inclination $\pm(M+n+1)$ and such that $0 \leq s(x) < \frac{\varepsilon}{2}$. Let $g(x) := p(x) + s(x),\ x \in I$. Then:

$$d(f,g) \leq d(f,p) + d(p,g) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and:

$$\begin{aligned}\left|\frac{g(x+h)-g(x)}{h}\right| &= \left|\frac{p(x+h)+s(x+h)-p(x)-s(x)}{h}\right| \\ &\geq \Big|\underbrace{\frac{s(x+h)-s(x)}{h}}_{(1)}\Big| - \Big|\underbrace{\frac{p(x+h)-p(x)}{h}}_{(2)}\Big| =: (*)\end{aligned}$$

Now by construction, for any $x \in [0, 1-\frac{1}{n}]$ there exists some $h \in (0, \frac{1}{n}]$ such that $(1) = (M+n+1)$ and $(2) \leq \sup|p'(y)| = M$. Consequently, $(*) \geq M+n+1-M = n+1$ for this $h \Rightarrow g \notin A_n$.

2.) A_n is closed:

Let $f_m \in A_n,\ f_m \to f$ in $C(I) \Rightarrow \forall m\ \exists x_m \in [0, 1-\frac{1}{n}] : \left|\frac{f_m(x_m+h)-f_m(x_m)}{h}\right| \leq n\ \forall h \in (0, \frac{1}{n}]$. Now $[0, 1-\frac{1}{n}]$ is compact $\Rightarrow x_m$ possesses a convergent subsequence, so without loss of generality $x_m \to x \in [0, 1-\frac{1}{n}]$. Therefore,

$$|f_m(x_m) - f(x)| \leq |\underbrace{f_m(x_m) - f(x_m)}_{\leq d(f_m,f)}| + |\underbrace{f(x_m) - f(x)}_{\to 0}| \to 0$$

Analogously, $|f_m(x_m+h) - f(x+h)| \to 0$.

$$\Rightarrow \left|\frac{f(x+h)-f(x)}{h}\right| = \lim_{m\to\infty}\left|\frac{f_m(x_m+h)-f_m(x_m)}{h}\right| \leq n\ \forall h \in \left(0, \frac{1}{n}\right] \Rightarrow f \in A_n.$$

Summing up, $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$, A_n closed, $A_n^\circ = \emptyset$, so A is meager. By 11.4.4, $C(I) \setminus A$ is dense in $C(I)$, so in particular $C(I) \setminus A \neq \emptyset$, which gives the claim. $\square$

Chapter 12

Function spaces

In this final chapter we study topologies and uniform structures on function spaces.

12.1 Uniform structures on spaces of functions

12.1.1 Definition. *For any sets X, Y, denote by $F(X,Y) = Y^X = \{f \mid f : X \to Y\}$ the set of all maps from X to Y.*

12.1.2 Examples.

(i) Let X be a set , Y a topological space. Then $F(X,Y) = Y^X = \prod_{x\in X} Y$ can be endowed with the product topology. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $F(X,Y)$. Then f_n converges to $f \in F(X,Y)$ in this topology if and only if $f_n \to f$ pointwise. In fact, by 2.3.19, $f_n \to f \Leftrightarrow f_n(x) = p_x \circ f_n \to p_x \circ f = f(x)\ \forall x \in X$, where $p_x : \prod_{x\in X} Y \to Y, f \mapsto f(x)$ is the projection.

(ii) Uniform convergence: Let Y be a metric space. Then $f_n \in Y^X$ converges uniformly to f if $\forall \varepsilon > 0\ \exists n_0$ such that $d(f_n(x), f(x)) < \varepsilon\ \forall n \geq n_0$, i.e.: $(f_n, f) \in d^{-1}([0,\varepsilon]\ \forall n \geq n_0$.

We want to generalize these observations:

12.1.3 Lemma. *Let X be a set and $(Y, \mathcal{V})$ a uniform space. For $V \in \mathcal{V}$ let*

$$W(V) := \{(f,g) \in F(X,Y) \times F(X,Y) \mid (f(x), g(x)) \in V\ \forall x \in X\}$$

If $\mathcal{B}$ is a fundamental system of entourages of $\mathcal{V}$, then $\tilde{\mathcal{B}} := \{W(V) \mid V \in \mathcal{B}\}$ is a fundamental system of entourages for a uniform structure on $F(X,Y)$.

Proof. We have to verify 9.1.9 (ii), (a)–(d):

(a) Let $W(V_1), W(V_2) \in \tilde{\mathcal{B}} \Rightarrow \exists V_3 \in \mathcal{B}$ with $V_3 \subseteq V_1 \cap V_2 \Rightarrow W(V_3) \subseteq W(V_1) \cap W(V_2)$.

(b) $\Delta_{F(X,Y)} \subseteq W(V)\ \forall V$.

(c) Let $V \in \mathcal{B}$ and $V' \in \mathcal{B}$ such that $V'^{-1} \subseteq V$. then $W(V')^{-1} = W(V'^{-1}) \subseteq W(V)$.

(d) Let $V \in \mathcal{B}$ and $V_1 \in \mathcal{B}$ such that $V_1^2 \subseteq V \Rightarrow W(V_1)^2 \subseteq W(V_1^2) \subseteq W(V)$: In fact, $(f,g) \in W(V_1)^2 \Rightarrow \exists h$ such that $(f,h) \in W(V_1) \wedge (h,g) \in W(V_1) \Rightarrow (f(x), h(x)), (h(x), g(x)) \in V_1\ \forall x \Rightarrow (f,g) \in W(V_1^2)$. $\square$

12.1.4 Definition. *The uniform structure defined by the fundamental system of entourages from 12.1.3 is called the uniform structure of uniform convergence. When equipped with this structure, $F(X,Y)$ is denoted by $F_u(X,Y)$. The topology on $F(X,Y)$ induced by this unifrom structure is called the topology of uniform convergence on $F(X,Y)$.*

If X is a topological space, then $C(X,Y) := \{f : X \to Y \mid f \text{ continuous } \}$, equipped with the trace topology of $F_u(X,Y)$ is denoted by $C_u(X,Y)$.

12.1.5 Example. Let X be a set and (Y,d) a metric space. A fundamental system of entourages in $F_u(X,Y)$ is then given by $(\{(f,g) \mid (f(x),g(x)) \in d^{-1}([0,\varepsilon]) \ \forall x \in X\})_{\varepsilon>0}$. Hence this uniform structure is induced by the pseudometric $d_u(f,g) := \sup_{x\in X} d(f(x),g(x))$.

12.1.6 Lemma. *Let X be a topological space, $(Y,\mathcal{V})$ a uniform space and $\mathcal{F}$ a filter on X. Then $A := \{f \in F(X,Y) \mid f(\mathcal{F}) \text{ is a Cauchy filter } \}$ is closed in $F_u(X,Y)$.*

Proof. Let $g \in \bar{A}$. If $U \in \mathcal{V}$ and V is symmetric with $V^3 \subseteq U$ (cf. 9.1.8), then $(W(V))(g)$ is a neighborhood of g in $F_u(X,Y)$, hence $(W(V))(g) \cap A \neq \emptyset$. Therefore there exists some $f \in A$ with $(g,f) \in W(V)$, i.e. $(g(x),f(x)) \in V \ \forall x \in X$. $f(\mathcal{F})$ is a Cauchy filter $\Rightarrow \exists F \in \mathcal{F}$ with $f(F) \times f(F) \subseteq V$. For $x_1,x_2 \in F$ we then obtain:

$$(g(x_1),f(x_1)),\ (f(x_1),f(x_2)),\ (f(x_2),g(x_2)) \in V \Rightarrow g(F) \times g(F) \subseteq V^3 \subseteq U$$

We conclude that $g(\mathcal{F})$ is a Cauchy filter $\Rightarrow g \in A \Rightarrow A = \bar{A}$ □

12.1.7 Theorem. *Let X be a topological space, and Y a uniform space. Then $C(X,Y)$ is closed in $F_u(X,Y)$.*

Proof. *Claim:* $f \in F(X,Y)$ is continuous in $x \Leftrightarrow f(\mathcal{U}(x))$ is a Cauchy filter.

$\Rightarrow$: By 2.3.13, $f(\mathcal{U}(x))$ converges to $f(x)$, so in particular it is a Cauchy filter.

$\Leftarrow$: $f(x)$ is a cluster point of $f(\mathcal{U}(x))$: in fact, for every $U \in \mathcal{U}(x)$, $f(x) \in f(U) \subseteq \overline{f(U)}$. $f(\mathcal{U}(x))$ is a Cauchy filter and therefore converges to $f(x)$ by 10.1.6. It follows that $f(\mathcal{U}(x)) \supseteq \mathcal{U}(f(x))$, so f is continuous by 2.3.13.

Now let $A_x := \{f : X \to Y \mid f \text{ continuous in } x\} = \{f : X \to Y \mid f(\mathcal{U}(x)) \text{ is a Cauchy filter}\}$. Then A_x is closed in $F_u(X,Y)$ by 12.1.6 $\Rightarrow C(X,Y) = \bigcap_{x\in X} A_x$ is closed in $F_u(X,Y)$. □

12.1.8 Remark. 12.1.7 in particular shows that uniform limits of continuous functions are continuous.

12.1.9 Theorem. *Let X be a set and $(Y,\mathcal{V})$ a complete uniform space. Then also $F_u(X,Y)$ is complete.*

Proof. Let $\mathcal{F}$ be a Cauchy filter on $F_u(X,Y)$. For $A \in \mathcal{F}$ and $x \in X$ let $A(x) := \{f(x) \mid f \in A\}$. Then $\mathcal{F}(x) := \{A(x) \mid A \in \mathcal{F}\}$ is a filter on Y: $A_1(x) \cap A_2(x) = (A_1 \cap A_2)(x)$. Furthermore, $\mathcal{F}(x)$ is a Cauchy filter: Let $V \in \mathcal{V} \Rightarrow \exists A \in \mathcal{F}$ with $A \times A \subseteq W(V) \Rightarrow A(x) \times A(x) \subseteq V$. Y is complete $\Rightarrow \exists y \in Y$ with $\mathcal{F}(x) \to y$. Choose some such y and set $f(x) := y$. Then $f : X \to Y,\ x \mapsto f(x) \in F(X,Y)$.

Claim: $\mathcal{F} \to f$ in $F_u(X,Y)$.

Let $V \in \mathcal{V}$ be closed (9.1.15). $\mathcal{F}$ is a Cauchy filter $\Rightarrow \exists A \in \mathcal{F}$ with $A \times A \subseteq W(V)$, i.e. $A(x) \times A(x) \subseteq V \ \forall x \in X \Rightarrow \overline{A(x)} \times \overline{A(x)} \subseteq V$. Let $x \in X \Rightarrow f(x)$ is a limit, hence also a cluster point of $\mathcal{F}(x) \Rightarrow f(x) \in \overline{A(x)}$. Therefore,

$$(f(x),g(x)) \in V,\ \forall x \in X\ \forall g \in A, \text{ i.e. } g \in W(V)(f)\ \forall g \in A.$$

For $V \in \{V \in \mathcal{V} \mid V \text{ closed }\}$, the sets $W(V)$ provide a fundamental system of entourages in $F_u(X,Y)$ by 12.1.3. Hence $W(V)(f)$ describes a neighborhood basis of f in $F_u(X,Y)$. Any neighborhood of f therefore contains some $A \in \mathcal{F}$, i.e. $\mathcal{F} \to f$ in $F_u(X,Y)$. □

In applications it is often important to study uniform convergence on subsets of X, e.g. on finite or compact subsets. We analyze this question at once for arbitrary subsets:

12.1.10 Definition. *Let $\mathcal{S}$ be a system of subsets of a set X ($\mathcal{S} \subseteq \mathcal{P}(X)$) and let Y be a uniform space. For $S \in \mathcal{S}$ denote by*

$$\begin{aligned} R_S : F(X,Y) &\to F_u(S,Y) \\ f &\mapsto f|_S \end{aligned}$$

the restriction map. The coarsest uniform structure on $F(X,Y)$ with respect to which all R_S are uniformly continuous (i.e., the initial uniform structure with respect to $(R_S)_{S\in\mathcal{S}}$, cf. 9.3.3) is called the uniform structure of uniform convergence on the sets of $\mathcal{S}$, or the uniform structure of $\mathcal{S}$-convergence. When endowed with this structure, $F(X,Y)$ is denoted by $F_{\mathcal{S}}(X,Y)$. The topology induced by this uniform structure is called the topology of $\mathcal{S}$-convergence.

12.1.11 Proposition. *(Properties of $F_{\mathcal{S}}(X,Y)$). Under the assumptions of 12.1.10 we have:*

(i) If $\mathcal{B}$ is a fundamental system of entourages of Y and if, for $S \in \mathcal{S}$ and $V \in \mathcal{B}$ we set

$$W(S,V) := \{(f,g) \in F(X,Y) \times F(X,Y) \mid (f(x),g(x)) \in V \ \forall x \in S\},$$

then the finite intersections of the $W(S,V)$ form a fundamental system of entourages in $F_{\mathcal{S}}(X,Y)$.

(ii) The topology of $\mathcal{S}$-convergence is the coarsest topology on $F(X,Y)$ for which the restriction maps $f \mapsto f|_S : F(X,Y) \to F_u(S,Y)$ ($S \in \mathcal{S}$) are continuous.

(iii) Let $\mathcal{F}$ be a filter on $F(X,Y)$ and $f \in F(X,Y)$. Then:

$$\mathcal{F} \to f \text{ in } F_{\mathcal{S}}(X,Y) \Leftrightarrow R_S(\mathcal{F}) \to R_S(f) = f|_S \text{ in } F_u(S,Y) \ \forall S \in \mathcal{S}.$$

(iv) For $H \subseteq F(X,Y)$, $x \in X$ let $H(x) := \{h(x) \mid h \in H\}$. Let

$$\begin{aligned} ev_x : F_{\mathcal{S}}(X,Y) &\to Y \\ f &\mapsto f(x). \end{aligned}$$

Then ev_x is uniformly continuous $\forall x \in \bigcup_{S\in\mathcal{S}} S$. Thus $\bar{H}(x) \subseteq \overline{H(x)}$ $\forall x \in \bigcup_{S\in\mathcal{S}} S$ (where $\bar{H}$ denotes the closure of H in $F_{\mathcal{S}}(X,Y)$).

Proof. (i) According to 12.1.3, for any $S \in \mathcal{S}$,

$$(\{(h,k) \in F(S,Y) \times F(S,Y) \mid (h(x),k(x)) \in V \ \forall x \in S\})_{V\in\mathcal{B}} =: (W_S(V))_{V\in\mathcal{B}}$$

is a fundamental system of entourages in $F_u(S,Y)$. Now $(R_S \times R_S)^{-1}(W_S(V)) = W(S,V)$, so the claim follows from 9.3.3 (i).

(ii) See 9.3.3 (iv) and 1.2.2.

(iii) This follows from 2.3.18.

(iv) Let $V \in \mathcal{V},\ x \in \bigcup_{S\in\mathcal{S}} S$. Choose $S \in \mathcal{S}$ such that $x \in S$. Then

$$(ev_x \ \times \ ev_x)(W(S,V)) \subseteq V.$$

In fact, $(f,g) \in W(S,V) \Rightarrow (f(y),g(y)) \in V\ \forall y \in S$, hence in particular for $y = x$. Thus ev_x is uniformly continuous and therefore continuous. Consequently, $\bar{H}(x) = ev_x(\bar{H}) \subseteq \overline{ev_x(H)} = \overline{H(x)}$. □

12.1.12 Remark. Some special cases:

(i) If $A \subseteq X,\ \mathcal{S} = \{A\}$, then the uniform structure on $F_{\mathcal{S}}(X,Y)$ is called the uniform structure of uniform convergence on A. For $A = X$ we have $F_{\mathcal{S}}(X,Y) = F_u(X,Y)$. E.g., if E, F are normed spaces and $A = B_1(0)$ then $L(E,F)$ carries the uniform structure with respect to $A = B_1(0)$, so $L(E,F) \subseteq F_{\{A\}}(E,F)$.

(ii) If $A \subseteq X$ and $\mathcal{S}$ is the family of all finite subsets of A, then the uniform structure of $F_{\mathcal{S}}(X,Y)$ is called the uniform structure of pointwise convergence on A. Let $\mathcal{F}$ be a filter on $F_{\mathcal{S}}(X,Y)$. Then

$$\mathcal{F} \to g \in F(X,Y) \Leftrightarrow \mathcal{F}(a) \to g(a)\ \forall a \in A:$$

$\Rightarrow$: Let $V \in \mathcal{V},\ a \in A,\ S := \{a\} \Rightarrow \exists B \in \mathcal{F}$ with $B \subseteq W(S,V)(g) \Rightarrow B(a) \subseteq V(g(a))$: Let $f \in B \Rightarrow (g,f) \in W(S,V) \Rightarrow (g(a),f(a)) \in V \Rightarrow f(a) \in V(g(a)) \Rightarrow \mathcal{F}(a) \to g(a)$.

$\Leftarrow$: Let $S = \{a_1,\dots,a_n\} \subseteq A,\ V \in \mathcal{V} \Rightarrow \forall k \in \{1,\dots,n\}\ \exists B_k \in \mathcal{F}$ with $B_k(a_k) \subseteq V(g(a_k))$. Let $B := \bigcap_{k=1}^n B_k,\ f \in B \Rightarrow (g(a_k),f(a_k)) \in V\ \forall a_k \in S \Rightarrow f \in W(S,V)(g) \Rightarrow \mathcal{F} \to g$.

For $A = X,\ F_{\mathcal{S}}(X,Y)$ is denoted by $F_s(X,Y)$ (s stands for 'simple'). The topology induced by this uniform structure is the coarsest topology for which all $f \mapsto f|_S$ are continuous, with S finite, i.e. the coarsest topology such that all $f \mapsto f(x)$ are continuous, i.e. the product topology on Y^X. (Indeed, $W(S,V) = \bigcap_{a\in S} W(\{a\},V)$, hence $\mathcal{S}$ and $\mathcal{S}_1 := \{\{a\} \mid a \in A\}$ induce the same uniform structure, hence also the same topology). See also 12.1.2 (i).

(iii) Let X be a topological space and $\mathcal{S} := \{K \subseteq X \mid K \text{ compact}\}$. Then the uniform structure of $F_{\mathcal{S}}(X,Y)$ is called the uniform structure of compact convergence and we write $F_c(X,Y)$ for $F_{\mathcal{S}}(X,Y)$.

(iv) Let $\mathcal{U}_u,\ \mathcal{U}_c,\ \mathcal{U}_s$ be the uniform structures of uniform, compact, and simple convergence, respectively. Then $\mathcal{U}_u \supseteq \mathcal{U}_c \supseteq \mathcal{U}_s$: $\forall S$ finite, S is compact, so $W(S,V)$ is an entourage in $\mathcal{U}_c$. Also $W(V) = W(X,V) \subseteq W(K,V)\ \forall K$ compact in X. If K is compact then $\mathcal{U}_u = \mathcal{U}_c$, since in this case $W(V) = W(X,V) \in \mathcal{U}_c$.

12.1.13 Example. Let $X = Y = [0,1],\ f_n : t \mapsto t^n,\ f := \begin{cases} 0 & t < 1 \\ 1 & t = 1 \end{cases}$
Then $f_n \to f$ in $F_s(X,Y)$, but $f_n \nrightarrow f$ in $F_c(X,Y)$ and in $F_u(X,Y)$.

12.1.14 Proposition.

(i) Let $(Y,\mathcal{V})$ be a uniform T_2-space, X a set, $\mathcal{S} \subseteq \mathcal{P}(X),\ X = \bigcup_{S\in\mathcal{S}} S$. Then $F_{\mathcal{S}}(X,Y)$ is T_2.

(ii) If Y is a uniform T_k-space for $k \in \{1,2,3,3a\}$, then also $F_s(X,Y)$ is T_k.

Proof. (i) Let $f, g \in F(X,Y)$, $(f,g) \in \bigcap_{\substack{S\in\mathcal{S}\\ V\in\mathcal{V}}} W(S,V)$. Then for all $x \in S$ and all $S \in \mathcal{S}$, $(f(x), g(x)) \in \bigcap_{V\in\mathcal{V}} V = \Delta_Y$ (by 9.1.17). As any x lies in some S, $(f(x), g(x)) \in \Delta_Y\ \forall x \in X \Rightarrow f(x) = g(x)\ \forall x \Rightarrow f = g \Rightarrow \bigcap_{\substack{S\in\mathcal{S}\\ V\in\mathcal{V}}} W(S,V) = \Delta_{F(X,Y)}$, so the claim follows from 9.1.17.

(ii) By 12.1.12 (ii), $F(X,Y)$ carries the product topology, so 3.2.3 gives the result. □

12.1.15 Theorem. *Let X be a set, $\mathcal{S} \subseteq \mathcal{P}(X)$, and Y a complete uniform space. Then also $F_{\mathcal{S}}(X,Y)$ is complete.*

Proof. By 12.1.10, $F_{\mathcal{S}}(X,Y)$ carries the initial uniform structure with respect to the maps $R_S : F(X,Y) \to F_u(S,Y)$ $(S \in \mathcal{S})$. By 12.1.9, all $F_u(S,Y)$ are complete, so 10.1.16 implies that $F_{\mathcal{S}}(X,Y)$ is complete. □

Compare this result with 12.1.12 (i): $L(E,F)$ is complete if F is complete. In particular, setting $F = \mathbb{R}$ it follows that E^* is always complete.

12.1.16 Theorem. *Let X be a topological space, Y a uniform space, $\mathcal{S} \subseteq \mathcal{P}(X)$, $X = \bigcup_{S\in\mathcal{S}} S^\circ$. Then $C(X,Y)$ is closed in $F_{\mathcal{S}}(X,Y)$. (For $\mathcal{S} = \{X\}$ this reduces to 12.1.7.)*

Proof. Again let $R_S : F_{\mathcal{S}}(X,Y) \to F_u(S,Y)$, $f \mapsto f|_S$, and

$$C'(S,Y) := R_S^{-1}(C(S,Y)) = \{f \in F(X,Y) \mid f|_S : S \to Y \text{ continuous}\}.$$

Since $X = \bigcup_{S\in\mathcal{S}} S^\circ$, $f : X \to Y$ is continuous $\Leftrightarrow$ $f|_S$ is continuous $\forall S \in \mathcal{S} \Rightarrow C(X,Y) = \bigcap_{S\in\mathcal{S}} C'(S,Y)$. Therefore it is enough to show that any $C'(S,Y)$ is closed in $F_{\mathcal{S}}(X,Y)$. Now $C(S,Y)$ is closed in $F_u(S,Y)$ by 12.1.7. Also, by 12.1.10 all R_S are (uniformly) continuous $\Rightarrow C'(S,Y) = R_S^{-1}(C(S,Y))$ is closed. □

Note that 12.1.16 is not applicable to $F_s(X,Y)$ since $\{x\}^\circ = \emptyset$ in general (unless the topology of X is discrete). In fact pointwise limits of continuous functions need not be continuous.

12.1.17 Definition. *$C(X,Y)$, equipped with the uniform structure induced by $F_{\mathcal{S}}(X,Y)$ is denoted by $C_{\mathcal{S}}(X,Y)$. In particular: $C_s(X,Y) \subseteq F_s(X,Y)$, $C_c(X,Y) \subseteq F_c(X,Y)$, $C_u(X,Y) \subseteq F_u(X,Y)$.*

12.2 The compact-open topology

By 12.1.12 (ii), $F_s(X,Y)$ carries the product topology. Thus the topology on $F_s(X,Y)$ is already determined by the topology of Y. A similar observation is true for $C_c(X,Y)$:

12.2.1 Theorem. *Let X be a topological space, and $(Y,\mathcal{V})$ a uniform space. For $K \subseteq X$ compact and $U \subseteq Y$ open let $(K,U) := \{f : X \to Y \mid f \text{ continuous and } f(K) \subseteq U\}$. Then $\mathcal{B} := \{(K,U) \mid K \text{ compact in } X, U \text{ open in } Y\}$ is a subbasis of the topology of $C_c(X,Y)$.*

Proof. For $K \subseteq X$ compact, $V \in \mathcal{V}$, $W(K,V) = \{(f,g) \in F(X,Y) \times F(X,Y) \mid (f(x), g(x)) \in V\ \forall x \in K\}$ is a typical element of the uniform structure on $F_c(X,Y)$. Therefore the sets $\widehat{W}(K,V) = W(K,V) \cap (C(X,Y) \times C(X,Y)) = \{(f,g) \in C(X,Y) \times$

$C(X,Y) \mid (f(x),g(x)) \in V \ \forall x \in K\}$ form a fundamental system of entourages in $C_c(X,Y)$ (cf. 9.3.4). For $f \in C(X,Y)$ we therefore obtain that $\{\widehat{W}(K,V)(f) \mid K \subseteq X$ compact, $V \in \mathcal{V}\}$ is a neighborhood basis in $C_c(X,Y)$. Let $\mathcal{O}_1$ be the topology generated by $\mathcal{B}$, and $\mathcal{O}_2$ the topology of $C_c(X,Y)$.

$\mathcal{O}_1 \subseteq \mathcal{O}_2$: We show that any (K,U) lies in $\mathcal{O}_2$.

Let $f \in (K,U)$. Then $f(K)$ is compact and $\subseteq U$. Thus for all $y \in f(K)$ there exists some $T_y \in \mathcal{V}$ with $T_y(y) \subseteq U$. Let $V_y \in \mathcal{V}$ such that $V_y^2 \subseteq T_y$. $f(K)$ is compact $\Rightarrow \exists L \subseteq f(K)$ finite with

$$f(K) \subseteq \bigcup_{z \in L} V_z(z) \subseteq U.$$

Let $V := \bigcap_{z \in L} V_z$. If $y \in f(K) \Rightarrow \exists z \in L$ with $y \in V_z(z)$. Therefore,

$$V(y) \subseteq V(V_z(z)) \subseteq V_z(V_z(z)) = V_z^2(z) \subseteq T_z(z) \subseteq U,$$

so

$$V(f(K)) = \bigcup_{y \in f(K)} V(y) \subseteq U.$$

Let $g \in \widehat{W}(K,V)(f) \Rightarrow (f(x),g(x)) \in V \ \forall x \in K \Rightarrow g(x) \in V(f(x)) \subseteq V(f(K)) \subseteq U \ \forall x \in K$, i.e.: $g \in (K,U) \Rightarrow \widehat{W}(K,V)(f) \subseteq (K,U) \Rightarrow (K,U) \in \mathcal{O}_2$.

$\mathcal{O}_2 \subseteq \mathcal{O}_1$: We show that any $\widehat{W}(K,V)(f)$ contains a finite intersection of elements of $\mathcal{B}$:

Let $T \in \mathcal{V}$ be closed and symmetric, $T^3 \subseteq V$ (cf. 9.1.8 and 9.1.15). $f(K)$ is compact $\Rightarrow \exists x_1,\dots,x_n \in K$ with $f(K) \subseteq \bigcup_{k=1}^n T(f(x_k))$. Let $K_i := K \cap f^{-1}(T(f(x_i)))$, $U_i := (T^2(f(x_i)))^\circ$ $(i = 1,\dots,n) \Rightarrow K_i$ compact and $f(K_i) \subseteq U_i$: $f(K_i) \subseteq T(f(x_i)) \subseteq (T^2(f(x_i)))^\circ = U_i$ (in fact, $z \in T(f(x_i)) \Rightarrow \mathcal{U}(z) \ni T(z) \subseteq T^2(f(x_i))$: $y \in T(z) \Rightarrow (z,y) \in T, (f(x_i),z) \in T \Rightarrow (f(x_i),y) \in T^2 \Rightarrow y \in T^2(f(x_i)) \Rightarrow z \in T^2(f(x_i))^\circ$).
Moreover, $K = K \cap f^{-1}(f(K)) \subseteq \bigcup_{i=1}^n K \cap f^{-1}(T(f(x_i))) = \bigcup_{i=1}^n K_i$.

Claim: $\bigcap_{i=1}^n (K_i,U_i) \subseteq \widehat{W}(K,V)(f)$

In fact, let $g \in \bigcap_{i=1}^n (K_i,U_i)$ and $x \in K \Rightarrow \exists i \in \{1,\dots,n\}$ with $x \in K_i$, hence also $f(x) \in T(f(x_i))$. Then $g(x) \in g(K_i) \subseteq U_i \subseteq T^2(f(x_i)) \Rightarrow (f(x),f(x_i)) \in T, (f(x_i),g(x)) \in T^2 \Rightarrow (f(x),g(x)) \in T^3 \subseteq V \Rightarrow (f(x),g(x)) \in V \ \forall x \in K \Rightarrow g \in \widehat{W}(K,V)(f)$, proving the claim.

Hence $\mathcal{O}_2 \subseteq \mathcal{O}_1$, and thereby $\mathcal{O}_1 = \mathcal{O}_2$. □

Since $\mathcal{B}$ as defined in 12.2.1 depends exclusively on the topologies of X and Y, we can generalize the construction as follows:

12.2.2 Definition. *Let X, Y be topological spaces. Then the topology defined by the subbasis*

$$\{(K,U) \mid K \subseteq X \text{ compact, } U \subseteq Y \text{ open}\}$$

on $C(X,Y)$ is called the compact-open topology. When equipped with this topology, $C(X,Y)$ is denoted by $C_c(X,Y)$.

In the following result, $f(x,.)$ denotes the map $y \mapsto f(x,y)$.

12.2.3 Theorem. *Let X,Y,Z be topological spaces and $f : X \times Y \to Z$. Then:*

(i) If f is continuous, then so is

$$\begin{aligned} \bar{f} : X &\to C_c(Y,Z) \\ x &\mapsto f(x,.) \end{aligned}$$

(ii) If Y is locally compact and $\bar{f}$ is continuous, then so is f.

Proof. (i) Let $x \in X$ and let (K, U) be an open neighborhood of $\bar{f}(x)$ in $C_c(Y, Z)$. Since $\bar{f}(x) \in (K, U)$, $\bar{f}(x)(K) \subseteq U$, i.e. $f(\{x\} \times K) \subseteq U$, so $\{x\} \times K \subseteq f^{-1}(U)$, which is open. Hence there exists some $V \in \mathcal{U}(x)$ with $V \times K \subseteq f^{-1}(U) \Rightarrow f(V \times K) = \bar{f}(V)(K) \subseteq U$, so $\bar{f}(V) \subseteq (K, U) \Rightarrow \bar{f}$ continuous in x. As x was arbitrary, this shows continuity of $\bar{f}$.

(ii) Let $(x_0, y_0) \in X \times Y$, $V \in \mathcal{U}(f(x_0, y_0))$ open in Z. Since $f(x_0, \cdot) = \bar{f}(x_0) \in C_c(X, Y)$, $y \mapsto f(x_0, y)$ is continuous: $Y \to Z$. Y is locally compact, so 5.2.4 and the continuity of $f(x_0, .)$ imply that there exists some $W \in \mathcal{U}(y_0)$, W compact, such that $f(\{x_0\} \times W) \subseteq V$. Let $U := \{x \in X \mid \bar{f}(x) \in (W, V)\} \Rightarrow x_0 \in U$. U is open since $\bar{f}$ is continuous. Also, $\bar{f}(U) \subseteq (W, V)$, i.e. $\bar{f}(U)(W) = f(U \times W) \subseteq V$. As $(x_0, y_0) \in U \times W \Rightarrow U \times W \in \mathcal{U}(x_0, y_0) \Rightarrow f$ is continuous at (x_0, y_0). It follows that f is continuous. □

12.2.4 Corollary. *Let X be locally compact. Then the compact-open topology on $C(X, Y)$ is the coarsest topology for which the map*

$$\begin{aligned} e : C(X, Y) \times X &\to Y \\ (f, x) &\mapsto f(x) \end{aligned}$$

is continuous.

Proof. By 12.2.3, e is continuous $\Leftrightarrow \bar{e} : C(X, Y) \to C_c(X, Y)$, $f \mapsto e(f, .) = f$ is continuous. Hence e is continuous for a topology on $C(X, Y)$, if and only if this topology is finer than the compact-open topology. □

12.3 Equicontinuity and the Arzela-Ascoli theorem

12.3.1 Definition. *Let X be a topological space $(Y, \mathcal{V})$ a uniform space and $H \subseteq F(X, Y)$. H is called equicontinuous in $x \in X$, if:*

$$\forall V \in \mathcal{V}\ \exists U \in \mathcal{U}(x) \text{ s.t. } f(U) \subseteq V(f(x))\ \forall f \in H.$$

If H is equicontinuous in each $x \in X$, then H is called equicontinuous.

12.3.2 Examples.

(i) Let $(M, d), (M', d')$ be metric spaces, $x, y \in M$. Let $k, \alpha > 0$. Then

$$H := \{f : M \to M' \mid d'(f(x), f(y)) \leq k\ d(x, y)^\alpha\}$$

is equicontinuous.

(ii) Let $H := \{f : [a, b] \to \mathbb{R} \text{ differentiable} \mid |f'(x)| \leq k\ \forall x \in [a, b]\}$. Then H is equicontinuous.

12.3.3 Proposition. *Let X be a topological space, $(Y, \mathcal{V})$ a uniform space and $H \subseteq F(X, Y)$. TFAE:*

(i) H is equicontinuous in x_0.

(ii) The closure $\overline{H}^{F_s(X,Y)}$ of H in $F_s(X, Y)$ is equicontinuous in x_0.

Proof. (i) $\Rightarrow$ (ii): Let $V \in \mathcal{V}$ be closed (cf. 9.1.15). Then there exists some $U \in \mathcal{U}(x_0)$ with $f(x) \in V(f(x_0))$ $\forall f \in H$ $\forall x \in U$. Let $M := \{g \in F(X,Y) \mid (g(x_0), g(x)) \in V \ \forall x \in U\}$. Since $\forall x \in X$ the map $\psi_x : F_s(X,Y) \to Y \times Y$ ($\cong F_u(\{x_0\}, Y) \times F_u(\{x\}, Y)$), $\psi_x(g) := (R_{x_0}(g), R_x(g)) = (g(x_0), g(x))$ is continuous, $M = \bigcap_{x \in U} \psi_x^{-1}(V)$ is closed in $F_s(X,Y)$. Since $H \subseteq M$, also $\bar{H} \subseteq M$. Consequently, $h(x) \in V(h(x_0))$ $\forall x \in U$ $\forall h \in \bar{H}$, i.e. $\bar{H}$ is equicontinuous in x_0.

(ii) $\Rightarrow$ (i): is clear. □

12.3.4 Proposition. *Let X be a topological space, $(Y, \mathcal{V})$ a uniform space and $H \subseteq C(X,Y)$ equicontinuous. Let $\mathcal{U}_c$ and $\mathcal{U}_s$ as in 12.1.12 (iv). Then $\mathcal{U}_c|_H = \mathcal{U}_s|_H$.*

Proof. By 12.1.12 (iv), $\mathcal{U}_c \supseteq \mathcal{U}_s$, so $\mathcal{U}_c|_H \supseteq \mathcal{U}_s|_H$.

Conversely, we have to show: $\forall V \in \mathcal{V}$ $\forall K$ compact in X $\exists T \in \mathcal{V}$ and $\exists S \subseteq X$ finite with $\widehat{W}(S,T) := W(S,T) \cap (H \times H) \subseteq \widehat{W}(K,V) := W(K,V) \cap (H \times H)$.

Choose $T \in \mathcal{V}$ symmetric with $T^5 \subseteq V$ (cf. 9.1.8). H is equicontinuous $\Rightarrow \forall x \in X$ $\exists U_x \in \mathcal{U}(x)$ with $(h(x), h(y)) \in T$ $\forall y \in U_x$ $\forall h \in H$. Let $x', x'' \in U_x \Rightarrow (h(x), h(x')) \in T, (h(x''), h(x)) \in T \Rightarrow (h(x'), h(x'')) \in T^2$ $\forall h \in H$.

K compact $\Rightarrow \exists x_1, \dots, x_n \in K$ with $K \subseteq \bigcup_{i=1}^n U_{x_i}$. In each U_{x_i}, pick some a_i and set $S := \{a_1, \dots, a_n\}$. Let $x \in K \Rightarrow \exists i \in \{1, \dots, n\}$ with $x, a_i \in U_{x_i} \Rightarrow (h(x), h(a_i)) \in T^2$ $\forall h \in H$. Now let $(g,h) \in \widehat{W}(S,T) \Rightarrow (h(a_i), g(a_i)) \in T$ and $(g(a_i), g(x)) \in T^2$. Thus, finally,

$$(h(x), g(x)) \in T^5 \subseteq V \ \forall (h,g) \in \widehat{W}(S,T) \ \forall x \in K$$

$\Rightarrow \widehat{W}(S,T) \subseteq \widehat{W}(K,V)$, as claimed. □

12.3.5 Corollary. *Under the assumptions of 12.3.4 we have: $\overline{H}^{C_c(X,Y)} = \overline{H}^{F_s(X,Y)}$.*

Proof. By 12.3.3, $\overline{H}^{F_s(X,Y)}$ is equicontinuous, so in particular

$$\overline{H}^{F_s(X,Y)} \subseteq C(X,Y). \tag{12.3.1}$$

Let $\mathcal{O}_c$ be the topology induced by $\mathcal{U}_c$, and $\mathcal{O}_s$ the one induced by $\mathcal{U}_s$ on $F(X,Y)$. Since $\mathcal{U}_c \supseteq \mathcal{U}_s \Rightarrow \mathcal{O}_c \supseteq \mathcal{O}_s \Rightarrow \overline{H}^{\mathcal{O}_c} \subseteq \overline{H}^{\mathcal{O}_s}$. By (12.3.1) and 12.3.4, $\mathcal{U}_s|_{\overline{H}^{\mathcal{O}_s}} = \mathcal{U}_c|_{\overline{H}^{\mathcal{O}_s}}$, hence also $\mathcal{O}_s|_{\overline{H}^{\mathcal{O}_s}} = \mathcal{O}_c|_{\overline{H}^{\mathcal{O}_s}} \Rightarrow \overline{H}^{\mathcal{O}_c} = \overline{H}^{\mathcal{O}_c} \cap \overline{H}^{\mathcal{O}_s} = \overline{H}^{\mathcal{O}_c|_{\overline{H}^{\mathcal{O}_s}}} = \overline{H}^{\mathcal{O}_s|_{\overline{H}^{\mathcal{O}_s}}} = \overline{H}^{\mathcal{O}_s} \cap \overline{H}^{\mathcal{O}_s} = \overline{H}^{\mathcal{O}_s} \Rightarrow \overline{H}^{C_c(X,Y)} = \overline{H}^{\mathcal{O}_c} = \overline{H}^{\mathcal{O}_s} = \overline{H}^{F_s(X,Y)}$. □

12.3.6 Theorem. *(Arzela-Ascoli) Let X be locally compact, $(Y, \mathcal{V})$ a uniform T_2-space and $H \subseteq C(X,Y)$. TFAE:*

(i) H is relatively compact in $C_c(X,Y)$.

(ii) H is equicontinuous and $H(x)$ is relatively compact in Y $\forall x \in X$.

Proof. (i) $\Rightarrow$ (ii): By 12.1.14, $C_c(X,Y)$ is T_2. Therefore $\bar{H}$ is compact in $C_c(X,Y)$. Now 12.1.11 (iv) implies that $ev_x : C_c(X,Y) \to Y$ is continuous, so 5.1.10 gives that $ev_x(\bar{H})$ is compact in $Y \Rightarrow \bar{H}(x)$ compact $\Rightarrow H(x) \subseteq \bar{H}(x)$ ist relatively compact.

Let $x_0 \in X$ and let K be a compact neighborhood of x_0. Let $V' \in \mathcal{V}$ and $V \in \mathcal{V}$ such that $V = V^{-1}$ and $V^3 \subseteq V'$ (cf. 9.1.8). 11.1.3 $\Rightarrow \bar{H}$ is precompact in $C_c(X,Y) \Rightarrow H$ is precompact in $C_c(X,Y)$ (see 11.1.5 (i)). Thus there exists a finite cover of H with sets that are small of order $W(K,V)$, i.e.: $\exists M_1, \dots, M_n \subseteq C(X,Y)$ with $M_i \times M_i \subseteq W(K,V)$ and $H \subseteq \bigcup_{i=1}^n M_i$. Fix some $f_i \in M_i$ for $1 \leq i \leq n$. If

$f \in H \Rightarrow \exists i \in \{1, \dots, n\}$ with $f \in M_i \Rightarrow (f, f_i) \in W(K, V)$, i.e. $(f(x), f_i(x)) \in V \ \forall x \in K$.

f_i continuous $\Rightarrow \exists$ neighborhood U_i of x_0 such that $f_i(x) \in V(f_i(x_0))$, i.e. $(f_i(x), f_i(x_0)) \in V \ \forall x \in U_i$. Let $U := \bigcap_{i=1}^n U_i \cap K$, $x \in U$ and $f \in H \Rightarrow \exists i \in \{1, \dots, n\}$ with $(f, f_i) \in W(K, V)$. Then since $x, x_0 \in K$ and $x \in U_i$ we have

$$(f(x), f_i(x)) \in V, \quad (f_i(x), f_i(x_0)) \in V, \quad (f_i(x_0), f(x_0)) \in V.$$

Consequently, $(f(x_0), f(x)) \in V^3 \subseteq V' \Rightarrow f(x) \in V'(f(x_0)) \ \forall x \in U \Rightarrow f(U) \subseteq V'(f(x_0)) \ \forall f \in H$, i.e.: H is equicontinuous in $x_0 \Rightarrow H$ is equicontinuous.

(ii) $\Rightarrow$ (i): H is equicontinuous, so by 12.3.3 also $\overline{H}^{\mathcal{O}_s}$ is equicontinuous, and 12.3.4 implies that $\mathcal{U}_c|_{\overline{H}^{\mathcal{O}_s}} = \mathcal{U}_s|_{\overline{H}^{\mathcal{O}_s}} \Rightarrow \mathcal{O}_c|_{\overline{H}^{\mathcal{O}_s}} = \mathcal{O}_s|_{\overline{H}^{\mathcal{O}_s}} \Rightarrow$ we may view H as a topological subspace of $\overline{H}^{\mathcal{O}_s}$ and thereby of $\prod_{x \in X} Y = Y^X$. Then 12.1.11 (iv) gives:

$$\bar{H} \subseteq \prod_{x \in X} p_x(\bar{H}) = \prod_{x \in X} \bar{H}(x) \subseteq \prod_{x \in X} \overline{H(x)},$$

which is compact by 5.1.14 and 5.1.15. It follows that $\overline{H}^{\mathcal{O}_s}$ is compact with respect to $\mathcal{O}_s$, so $\overline{H}^{\mathcal{O}_s}$ is compact with respect to $\mathcal{O}_c$, and thereby H is relatively compact in $C_c(X, Y)$. □

For applications in Analysis and Functional Analysis the case where $C_c(X, Y)$ is metrizable is of the greatest importance. The following result provides a sufficient condition for this:

12.3.7 Proposition. *Let X be σ-compact and let $(Y, \mathcal{V})$ be a metrizable uniform space. Then $F_c(X, Y)$ and therefore also $C_c(X, Y)$ are metrizable.*

Proof. By 5.2.13, there exist K_n compact in X such that $X = \bigcup_{n \geq 1} K_n^\circ$, $K_n \subseteq K_{n+1}$. Let d be a metric on Y that induces the uniform structure $\mathcal{V}$ of Y. For $m \geq 1$, let

$$V_m := \{(y_1, y_2) \in Y \times Y \mid d(y_1, y_2) < \frac{1}{m}\} = d^{-1}\left(\left[0, \frac{1}{m}\right)\right).$$

Then $(V_m)_{m \geq 1}$ is a fundamental system of entourages of $\mathcal{V}$, so $W(K_n, V_m) \in \mathcal{U}_c$ (the uniform structure of $F_c(X, Y)$).

Then $\{W(K_n, V_m) \mid n, m \geq 1\}$ is a fundamental system of entourages for $\mathcal{U}_c$. Indeed, let $V \in \mathcal{V}$, $K \subseteq X$ compact $\Rightarrow \exists m$ with $V_m \subseteq V$, $\exists n$ with $K \subseteq K_n$. Hence $W(K_n, V_m) \subseteq W(K, V)$.

Finally, $F_c(X, Y)$ is T_2 by 12.1.14. Hence by 9.4.9 $F_c(X, Y)$ is metrizable. □

12.3.8 Corollary. *Let X be σ-compact, $(Y, \mathcal{V})$ a metrizable uniform space and $H \subseteq C(X, Y)$. TFAE:*

(i) Any sequence in H possesses a subsequence that converges uniformly on compact subsets of X.

(ii) H is equicontinuous and $H(x)$ is relatively compact in Y $\forall x \in X$.

The limit of the sequence from (i) then is continuous, i.e. $\in C(X, Y)$.

Proof. $C_c(X, Y)$ is metrizable by 12.3.7 and by 12.1.11 (iii) we have (i) $\Leftrightarrow$ any sequence in H possesses a subsequence converging in $F_c(X, Y)$. By 12.1.16 this is the case if and only if any sequence in H possesses a subsequence converging in $C_c(X, Y)$, which, by 11.2.9 holds if and only if H is relatively compact in $C_c(X, Y)$. By 12.3.6, this is equivalent to (ii). The final claim is immediate from 12.1.16. □

12.3.9 Remark. For $Y = \mathbb{R}^n$, 'relatively compact' can be replaced by 'bounded' in 12.3.8 (ii). (cf. 5.1.16).

Finally we consider two typical applications:

12.3.10 Example. (Peano's theorem) Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous and bounded. Then there exists at least one solution of the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)) & (t \in [0,1]) \\ y(0) = y_0 \end{cases}$$

Proof. By the fundamental theorem of calculus, it suffices to find a continuous solution of the integral equation

$$y(t) = y_0 + \int_0^t f(s, y(s))\, ds \qquad (t \in [0,1]). \tag{12.3.2}$$

Let $\alpha > 0$ and set

$$y_\alpha(t) := \begin{cases} y_0 & \text{for } t \le 0 \\ y_0 + \int_0^t f(s, y_\alpha(s-\alpha))ds & \text{for } 0 < t \le 1 \end{cases} \tag{12.3.3}$$

This is well-defined: For $0 \le t \le \alpha$ we have $y_\alpha(t-\alpha) = y_0$, so the integral can be calculated. Hence y_α can be determined for $\alpha \le t \le 2\alpha$, then for $2\alpha \le t \le 3\alpha$, etc. Also, $y_\alpha|_{[0,1]}$ is continuous.

Claim: $H := \{y_\alpha \mid \alpha > 0\}$ is equicontinuous.

Let K be such that $|f(x,y)| \le K \ \forall x \in [0,1] \ \forall y \in \mathbb{R} \Rightarrow |y'_\alpha(t)| \le K \ \forall t \in [0,1]$ and therefore:

$$|y_\alpha(t_1) - y_\alpha(t_2)| \le K|t_1 - t_2| \quad (t_1, t_2 \in [0,1]).$$

Since f is bounded, $H(t)$ is relatively compact in $\mathbb{R} \ \forall t \in [0,1]$. Thus by 12.3.6 H is relatively compact in $C_c([0,1], \mathbb{R})$ $(= C_u([0,1], \mathbb{R})$, because $[0,1]$ is compact). By 12.3.8, $(y_{\frac{1}{n}})_{n \ge 1}$ possesses a subsequence $y_k := y_{\alpha_k}$ (with $\alpha_k \to 0$) that converges in $C_c([0,1], \mathbb{R})$. Let $y := \lim_{k\to\infty} y_{\alpha_k}$. Then:

$$|y_k(t - \alpha_k) - y(t)| \le |y_k(t-\alpha_k) - y_k(t)| + |y_k(t) - y(t)| \le K\alpha_k + |y_k(t) - y(t)|.$$

Hence $y_k(. - \alpha_k) \to y$ in $C_u([0,1], \mathbb{R})$. Letting $k \to \infty$ in (12.3.3) it follows that y solves (12.3.2).

12.3.11 Example. (Montel's theorem) Let $(f_j)_{j\in\mathbb{N}}$ be a locally bounded sequence of holomophic functions on an open and connected domain $U \subseteq \mathbb{C}$. Then $(f_j)_{j\in\mathbb{N}}$ has a locally uniformly convergent subsequence.

Proof. As $\mathbb{C}$ is locally compact, the word 'locally' can be replaced here by 'on any compact set'.

We first show that $H := (f_j)_{j\in\mathbb{N}}$ is locally equicontinuous. To this end, let $a \in U$, $\overline{B_r(a)} \subseteq U$ and $|f(x)| \le C \ \forall x \in \overline{B_r(a)} \ \forall f \in H$. Set $D := \overline{B_{\frac{r}{2}}(a)}$ and let $z_1, z_2 \in D$, $f \in H$. Then

$$|f(z_2) - f(z_1)| = \Big| \int_{[z_1,z_2]} f'(z) d_z \Big| \le |z_2 - z_1| \max_{x\in D} |f'(z)|.$$

Furthermore,

$$|f'(z)| = \Big| \frac{1}{2\pi i} \int_{\partial B_r(a)} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \Big| \le r \frac{C}{(r/2)^2} = \frac{4C}{r} \text{ for } z \in D.$$

Now let $\varepsilon > 0$ and $\delta := \frac{r\varepsilon}{4C} \Rightarrow$ if $z_1, z_2 \in D,\ |z_2 - z_1| < \delta$ then $|f(z_2) - f(z_1)| < \varepsilon \Rightarrow \{f|_D \mid f \in H\}$ is equicontinuous, i.e.: H is locally equicontinuous. By 12.3.8, for any $K \subseteq U$ compact, the sequence $f_j|_K$ possesses a uniformly convergent subsequence. Since U is a countable union compact sets $(K_m)_{m \in \mathbb{N}}$ (cf. 5.2.11 (ii)) we may extract from the sequences that converge uniformly on the K_m a diagonal sequence that converges locally uniformly on U.

Bibliography

[1] Bourbaki, N., General Topology, Hermann, Paris.

[2] Engelking, R., General Topology, Heldermann.

[3] Haller, S., Differentialtopologie, Skriptum, Universität Wien, 2006.

[4] Hatcher, A., Algebraic Topology, Cambridge University Press.

[5] Querenburg, B. v., Mengentheoretische Topologie, Springer.

[6] Rudin, M.E., Two nonmetrizable manifolds, Topology Appl. 35 (1990), 137–152.

[7] Steen, L.A., Seebach, J.A., Counterexamples in Topology, Springer 1978.

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