Analysis on Manifolds

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Preface

These are lecture notes for an introductory course on analysis on manifolds. The underlying intention is to provide the fundamental notions and results of modern global analysis in a concise and rigorous way. The topics included here were chosen with a view to their applicability in the many fields of mathematics and mathematical physics where the theory of manifolds forms the underpinning and common language on which everything else depends. For one such field, symplectic geometry, the final chapter provides a first introduction, mainly to demonstrate the usefulness of the tools developed throughout the course.

The requirements for successfully participating in this course are a solid working knowledge of analysis on $\mathbb{R}^n$, some linear algebra, some set-theoretic topology, and a basic understanding of the theory of ordinary differential equations. Given this, I have tried to give complete and readable proofs of all results.

Michael Kunzinger, summer term 2022

Contents

Chapter 1

Differentiable Manifolds

The notion of a differentiable manifold is one of the central concepts of modern mathematics. Among others it finds applications in analysis, differential geometry, topology, the theory of Lie groups, ordinary and partial differential equations, as well as in numerous branches of physics, e.g. in mechanics or general relativity.

We start out by studying the special case of submanifolds of $\mathbb{R}^n$, a direct generalization of the notion of surface in $\mathbb{R}^3$ which already displays all the essential characteristics of the concept of abstract manifolds.

1.1 Submanifolds of $\mathbb{R}^n$

To begin with we recall some notions and results from analysis. For simplicity, from now on we will assume all maps to be C^∞.

1.1.1 Theorem. (Inverse Function Theorem) *Let $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ C^∞, $x_0 \in U$, $y_0 := f(x_0)$ and $Df(x_0)$ invertible ($\det Df(x_0) \neq 0$). Then locally around x_0, f is a diffeomorphism, i.e., there exist $U_1 \subseteq U$ an open neighborhood of x_0, and V_1 an open neighborhood of y_0, such that $f : U_1 \to V_1$ is bijective and $f^{-1} : V_1 \to U_1$ is C^∞.*

1.1.2 Theorem. (Implicit Function Theorem) *Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ open, $f : U \times V \to \mathbb{R}^m$ C^∞, $(x_0, y_0) \in U \times V$, $f(x_0, y_0) = 0$ and let $\frac{\partial f}{\partial y}(x_0, y_0) : \mathbb{R}^m \to \mathbb{R}^m$ be invertible ($\det \frac{\partial f}{\partial y}(x_0, y_0) \neq 0$). Then there exist open neighborhoods $U_1 \subseteq U$ of x_0, $V_1 \subseteq V$ of y_0, such that: $\forall x \in U_1$ $\exists!$ $y = y(x) \in V_1$ with $f(x, y(x)) = 0$. The map $x \mapsto y(x)$ is C^∞.*

1.1.3 Definition. *Let $U \subseteq \mathbb{R}^k$ be open and $\varphi : U \to \mathbb{R}^n$ C^∞. φ is called* regular *if for all $x \in U$ the rank of the Jacobian $D\varphi(x)$ is maximal, hence equal to $\min(k, n)$. Then for the rank $\mathrm{rk}(D\varphi)$ of $D\varphi$ (also called the rank of φ) we have*

$$\mathrm{rk}(D\varphi(x)) = \dim \mathrm{im}(D\varphi(x)) = \dim(\mathbb{R}^k) - \dim(\ker D\varphi(x)).$$

Thus if $k \leq n$ then $\ker D\varphi(x) = \{0\}$ and $D\varphi(x)$ is injective for all x. In this case φ is called an immersion. *For $k \geq n$, $D\varphi(x)$ is surjective for all x and φ is called a* submersion.

Hence 1.1.1 says that a regular map $f : U \to V$ with U, $V \subseteq \mathbb{R}^n$ open is a local diffeomorphism.

1.1.4 Remark. (Properties of immersions). Let $U \subseteq \mathbb{R}^k$ open and $\varphi : U \to \mathbb{R}^n$ an immersion.

(i) $\mathrm{rk}(D\varphi(x_0)) = k$ means that $\{\frac{\partial \varphi}{\partial x_1}(x_0), \dots, \frac{\partial \varphi}{\partial x_k}(x_0)\}$ is linearly independent in $\mathbb{R}^n$.

(ii) Equivalently, there exist indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ such that

$$\det \frac{\partial(\varphi_{i_1}, \dots, \varphi_{i_k})}{\partial(x_1, \dots, x_k)}(x_0) \neq 0$$

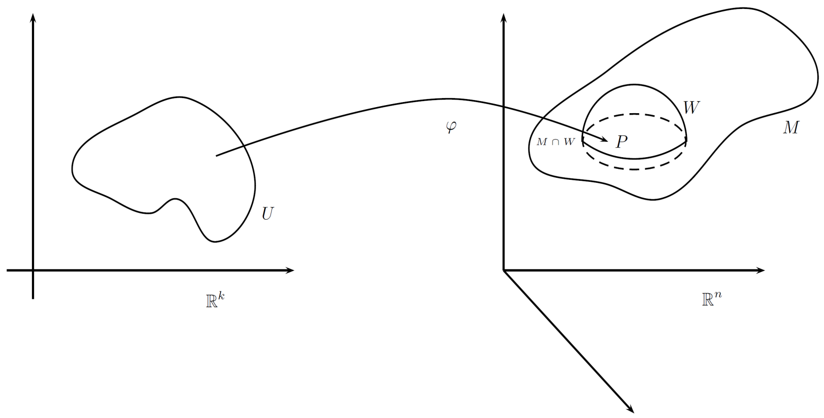
Since det is continuous it follows that $\mathrm{rk}(D\varphi(x)) = k$ in a neighborhood of x_0.

(iii) In particular for $k = 1$, $\varphi : U \subseteq \mathbb{R} \to \mathbb{R}^n$ is an immersion if $\varphi'(t) \neq 0$ $\forall t$, i.e., if φ is a regular curve.

1.1.5 Definition. *A subset M of $\mathbb{R}^n$ is called a k-dimensional submanifold of $\mathbb{R}^n$ ($k \le n$) if*

$$(P) \begin{cases} \text{For each } p \in M \text{ there exists an open neighborhood } W \text{ of } p \text{ in } \mathbb{R}^n, \\ \text{an open subset } U \text{ of } \mathbb{R}^k \text{ and an immersion } \varphi : U \to \mathbb{R}^n \text{ such that} \\ \varphi : U \to \varphi(U) \text{ is a homeomorphism and } \varphi(U) = M \cap W. \end{cases}$$

Such a φ is called a local parametrization *of M.*



Thus φ is regular and identifies U and $\varphi(U) = M \cap W$ topologically ($\varphi(U) = M \cap W$ carries the trace topology of $\mathbb{R}^n$). The following result gives an alternative criterion which is sometimes used in the definition of submanifolds of $\mathbb{R}^n$.

1.1.6 Proposition. *For each $M \subseteq \mathbb{R}^n$, property (P) is equivalent to*

$$(P') \begin{cases} \text{For each } p \in M \text{ there exists a smooth map } \varphi : U \to \mathbb{R}^n, \text{ where } U \\ \text{is an open neighborhood of } 0 \text{ in } \mathbb{R}^k, \varphi(0) = p \text{ and } \varphi \text{ is regular at } 0 \\ \text{(i.e., } D\varphi(0) \text{ is injective) and such that for any open neighborhood} \\ U_1 \subseteq U \text{ of } 0 \text{ there exists an open neighborhood } W_1 \text{ of } p \text{ in } \mathbb{R}^n \text{ with} \\ \varphi(U_1) = W_1 \cap M. \end{cases}$$

Proof. Obviously (P) implies (P'). Conversely, we first note that if φ is regular at 0 then in fact it is regular in a neighborhood of 0 (the rank of $D\varphi$ cannot decrease locally by continuity: the determinant of a suitable sub-matrix of the Jacobian of φ is non-zero in 0, hence in a neighborhood of 0). By assumption, φ is continuous and (P') secures that it is an open map (maps open sets in U to open sets in the trace topology of $\mathbb{R}^n$ on M). To establish (P) we will show that there exists an open neighborhood U_1 of 0 in U such that $\varphi|_{U_1}$ is a homeomorphism onto its image. To do this, by the above it suffices to show that φ is injective if we restrict it to a suitable open subset U_1 of U.

Since $D\varphi(0)$ is injective there exists a left inverse linear map $A : \mathbb{R}^n \to \mathbb{R}^k$, i.e., $\mathrm{id}_{\mathbb{R}^k} = A \cdot D\varphi(0) = D(A \cdot \varphi)(0)$. [Let $B := D\varphi(0)$, then $B : \mathbb{R}^k \to \mathrm{im}(B)$ is bijective. Call $\tilde{A}$ the inverse of this map. Then we may take $A := \tilde{A} \circ \mathrm{pr}_{\mathrm{im}(B)}$.] By 1.1.1 the map $x \mapsto A \cdot \varphi(x)$ is a local diffeomorphism on $\mathbb{R}^k$, so there exist open neighborhoods $U_1 \subseteq U$ of 0 and U_2 of $A(p)$ such that $h := (A \circ \varphi)^{-1} : U_2 \to U_1$ is smooth.

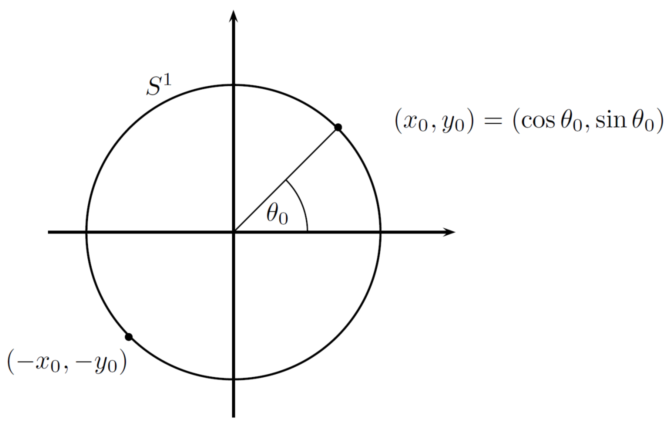
Now set $\psi := h \circ A : A^{-1}(U_2) \to U_1$. Then ψ is smooth and

$$\psi \circ \varphi(x) = (A \circ \varphi)^{-1} \circ A \circ \varphi(x) = x \qquad \forall x \in U_1 ,$$

so ψ is a left-inverse of $\varphi|_{U_1}$. In particular, $\varphi|_{U_1}$ is injective. □

1.1.7 Examples.

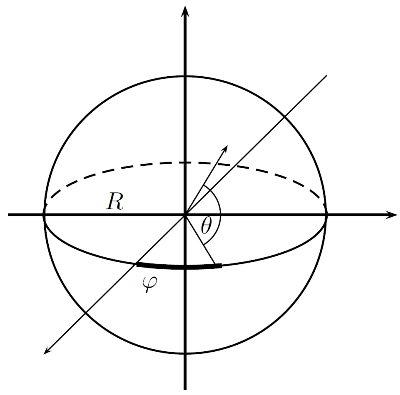
(i) The unit circle S^1.
Let $\varphi : \theta \mapsto (\cos\theta, \sin\theta)$. Then for all $(x_0, y_0) = (\cos\theta_0, \sin\theta_0)$, $\varphi : (\theta_0 - \pi, \theta_0 + \pi) \to \mathbb{R}^2$ is a parametrization of S^1 around (x_0, y_0). Here W can be taken, e.g., as $\mathbb{R}^2 \backslash \{(-x_0, -y_0)\}$. Hence S^1 is a 1-dimensional submanifold of $\mathbb{R}^2$. Note that no single parametrization can be used for all of S^1! (There is no homeomorphism from some open subset of $\mathbb{R}$ onto S^1 since S^1 is compact).



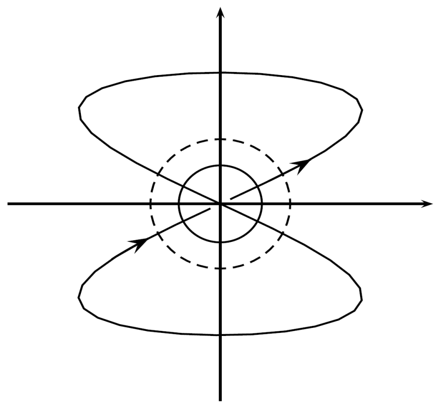
(ii) The 2-sphere S^2 in $\mathbb{R}^3$.
Let $\varphi(\phi, \theta) = (\cos\phi\cos\theta, \sin\phi\cos\theta, \sin\theta)$. Then

$$D\varphi = \begin{pmatrix} -\sin\phi\cos\theta & -\cos\phi\sin\theta \\ \cos\phi\cos\theta & -\sin\phi\sin\theta \\ 0 & \cos\theta \end{pmatrix}$$

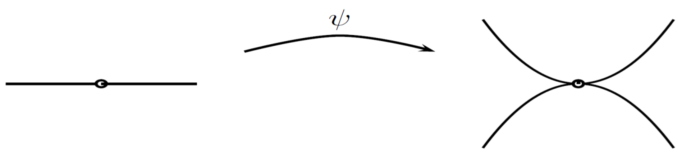
φ is a parametrization of S^2 e.g. on $(0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})$. In fact, on this domain φ is injective and $\mathrm{rk}(D\varphi) = 2$, since $\cos\theta \neq 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. Again, more than one parametrization is needed to cover S^2.



(iii) Figure eight manifold.
Let $M := \{(\sin 2s, \sin s) | s \in (0, 2\pi)\}$. The map $\varphi : s \mapsto (\sin 2s, \sin s)$ is an injective immersion: indeed, $D\varphi(s) = \varphi'(s) = (2\cos 2s, \cos s) \neq (0, 0)$ on $(0, 2\pi)$.

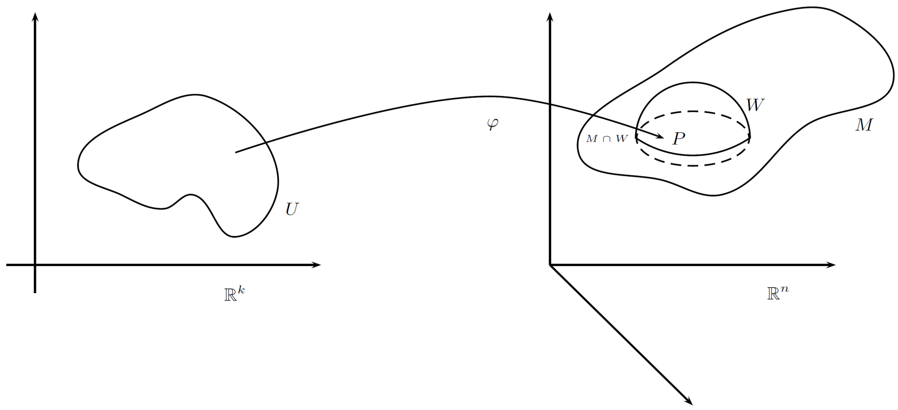


However, M is not a submanifold of $\mathbb{R}^2$! In fact, suppose that there exists a parametrization $\psi : (-\varepsilon, \varepsilon) \to B_{\frac{1}{2}}(0, 0)$ of M around $p = (0, 0)$ such that $\psi : (-\varepsilon, \varepsilon) \to B_{\frac{1}{2}}(0, 0) \cap M$ is a homeomorphism. Then $(-\varepsilon, \varepsilon) \setminus \{0\}$ has two connected components, while $(M \cap B_{\frac{1}{2}}(0, 0)) \setminus (0, 0)$ has four, a contradiction. M is what is usually called an *immersive submanifold* of $\mathbb{R}^2$. In what follows, we will restrict our attention to submanifolds in the sense of 1.1.5.



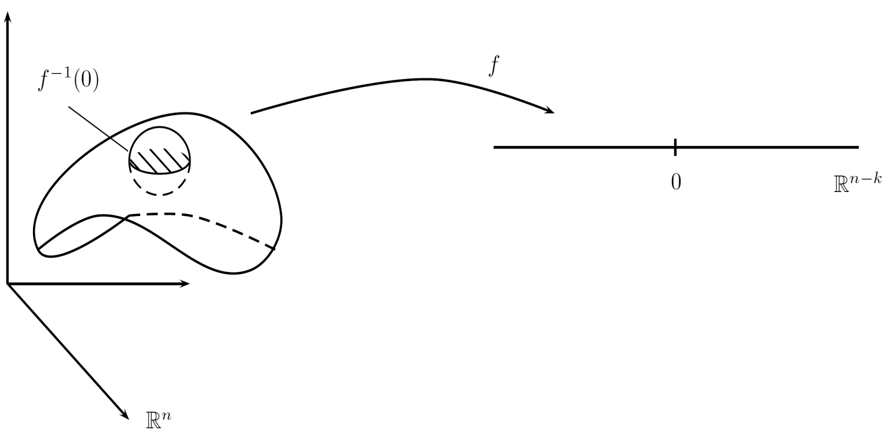
1.1.8 Theorem. *Let $M \subseteq \mathbb{R}^n$. The following are equivalent:*

(P) (Local Parametrization) M is a k-dimensional submanifold of $\mathbb{R}^n$.



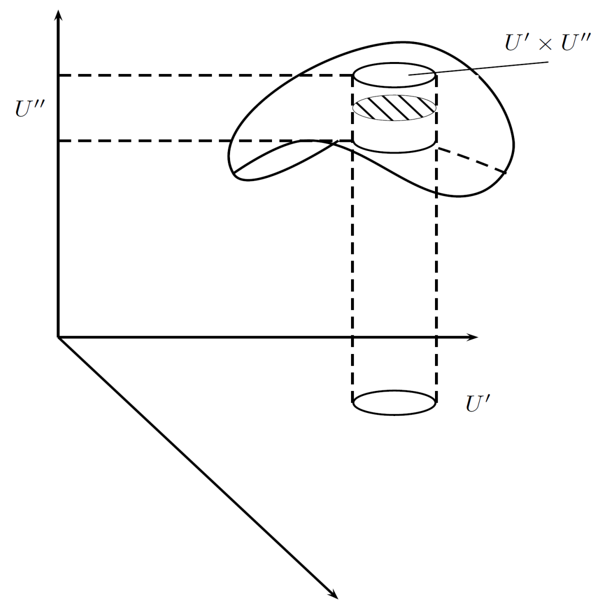
(Z) (Local Zero Set) For every $p \in M$ there exist an open neighborhood W of p in $\mathbb{R}^n$ and a C^∞-map $f : W \to \mathbb{R}^{n-k}$ which is regular (i.e., $\mathrm{rk} Df(q) = n - k$ for all $q \in W$) satisfying

$$M \cap W = f^{-1}(0) = \{x \in W \mid f(x) = 0\}.$$



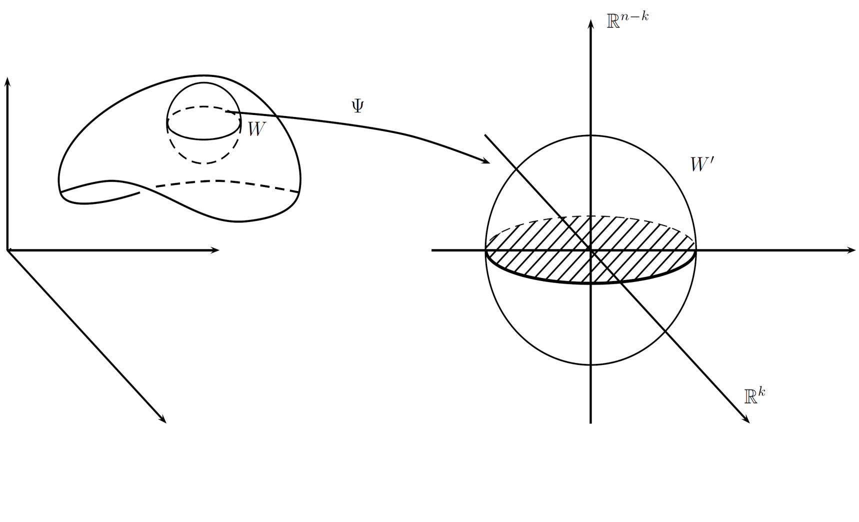
(G) (Local Graph) For each $p \in M$ there exist (after re-numbering the coordinates if necessary) open neighborhoods $U' \subseteq \mathbb{R}^k$ of $p' := (p_1, \dots, p_k)$ and $U'' \subseteq \mathbb{R}^{n-k}$ of $p'' := (p_{k+1}, \dots, p_n)$ and a C^∞-map $g : U' \to U''$ such that

$$M \cap (U' \times U'') = \{(x', x'') \in U' \times U'' | x'' = g(x')\} = \operatorname{graph}(g)$$



(T) (Local Trivialization) For each $p \in M$ there exist an open neighborhood W of p in $\mathbb{R}^n$, an open set W' in $\mathbb{R}^n \cong \mathbb{R}^k \times \mathbb{R}^{n-k}$ and a diffeomorphism $\Psi : W \to W'$ such that

$$\Psi(M \cap W) = W' \cap (\mathbb{R}^k \times \{0\}) \subseteq \mathbb{R}^k \times \{0\} \cong \mathbb{R}^k.$$



Proof. $(P) \Rightarrow (G)$:
Without loss of generality we may suppose that $\varphi(0) = p$ and $\det \frac{\partial(\varphi_1, \dots, \varphi_k)}{\partial(x_1, \dots, x_k)}(0) \neq 0$. By 1.1.1 there exists some open neighborhood $U_1 \subseteq U$ of 0 and some open $V_1 \subseteq \mathbb{R}^k$

such that $\varphi' := (\varphi_1, \ldots, \varphi_k)$ is a diffeomorphism. Let $\psi : V_1 \to U_1$ be the inverse of φ' and $G := \varphi \circ \psi : V_1 \to \mathbb{R}^n$. Then with $\varphi'' := (\varphi_{k+1}, \ldots, \varphi_n)$ we have

$$G(x_1, \ldots, x_k) := (\underbrace{\varphi' \circ \psi(x_1, \ldots, x_k)}_{=\underbrace{(x_1, \ldots, x_k)}_{x'}}, \underbrace{\varphi'' \circ \psi}_{=:g}(x_1, \ldots, x_k)) = (x', g(x'))$$

with $g : V_1 \to \mathbb{R}^{n-k}$ smooth. Since φ is a homeomorphism, $\varphi(U_1)$ is open in M, i.e., there exists some W_1 open in $\mathbb{R}^n$ such that $\varphi(U_1) = M \cap W_1$. Hence

$$M \cap W_1 = \varphi(\underbrace{\psi(V_1)}_{U_1}) = G(V_1) = \{(x', g(x'))|x' \in V_1\}.$$

We now choose open sets $U' \subseteq V_1$ and $U'' \subseteq \mathbb{R}^{n-k}$ such that $p \in U' \times U'' \subseteq W_1$. Then

$$\begin{aligned} M \cap (U' \times U'') &= M \cap W_1 \cap (U' \times U'') = \{(x', g(x'))|x' \in V_1\} \cap (U' \times U'') \\ &= \{(x', x'') \in U' \times U''|g(x') = x''\} \end{aligned}$$

$(G) \Rightarrow (Z)$:
Set $W := U' \times U''$ and $f : W \to \mathbb{R}^{n-k}$,

$$f_j(x_1, \ldots, x_n) := x_{k+j} - g_j(x_1, \ldots, x_k) \qquad (1 \le j \le n-k)$$

Then $f \in \mathcal{C}^\infty$ and $\frac{\partial(f_1, \ldots, f_{n-k})}{\partial(x_{k+1}, \ldots, x_n)} = I_{n-k}$, so f is regular. Moreover

$$f^{-1}(0) = \{(x', x'') \in U' \times U''|g(x') = x''\} = M \cap (U' \times U'') = M \cap W.$$

$(Z) \Rightarrow (T)$:
Without loss of generality we may suppose that $\det \frac{\partial(f_1, \ldots, f_{n-k})}{\partial(x_{k+1}, \ldots, x_n)}(p) \neq 0$. Let $\Psi(x) := (x', f(x)) = (x_1, \ldots, x_k, f_1(x), \ldots, f_{n-k}(x))$. Then

$$D\Psi(p) = \begin{pmatrix} I_k & 0 \\ * & \frac{\partial(f_1, \ldots, f_{n-k})}{\partial(x_{k+1}, \ldots, x_n)}(p) \end{pmatrix}$$

is invertible.
By 1.1.1, there exists an open neighborhood $W_1 \subseteq W$ of p in $\mathbb{R}^n$, and some W' open in $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, such that $\Psi : W_1 \to W'$ is a diffeomorphism. We show that $\Psi(M \cap W_1) = (\mathbb{R}^k \times \{0\}) \cap W'$:

$\subseteq$: $\Psi(M \cap W_1) \subseteq \Psi(W_1) = W'$ and $x \in M \cap W_1 \Rightarrow f(x) = 0$
$\Rightarrow \Psi(x) = (x', f(x)) = (x', 0) \in \mathbb{R}^k \times \{0\}$.

$\supseteq$:

$$\left.\begin{array}{l} y \in W' \Rightarrow y = \Psi(x) = (x', f(x)) \text{ with } x \in W_1 \\ f(x) = 0 \Rightarrow x \in f^{-1}(0) = W \cap M \end{array}\right\} \Rightarrow x \in W_1 \cap M$$
$\Rightarrow y = \Psi(x) \in \Psi(M \cap W_1)$.

(Moreover, $\psi := \Psi|_{W_1 \cap M} : W_1 \cap M \to W' \cap (\mathbb{R}^k \times \{0\})$ is a homeomorphism: it is clearly continuous and bijective, and $\psi^{-1} = \Psi^{-1}|_{(W' \cap (\mathbb{R}^k \times \{0\}))}$ is continuous.)
$(T) \Rightarrow (P)$:
Let $\Phi : W' \to W$ be the inverse of Ψ and denote by $\varphi : (\mathbb{R}^k \times \{0\}) \cap W' =: U \subseteq \mathbb{R}^k \times \{0\} \cong \mathbb{R}^k \to \mathbb{R}^n$ the map $(x_1, \ldots, x_k) \mapsto \Phi(x_1, \ldots, x_k, 0, \ldots, 0)$, i.e., $\varphi = \Phi \circ i$

with $i : \mathbb{R}^k \hookrightarrow \mathbb{R}^n$. Then φ is an immersion since $D\varphi = D\Phi \circ Di$ is injective. Moreover,

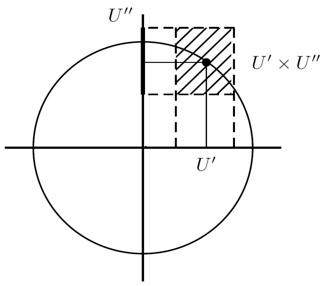
$$\varphi(U) = \Phi((\mathbb{R}^k \times \{0\}) \cap W') = \Psi^{-1}((\mathbb{R}^k \times \{0\}) \cap W') = M \cap W.$$

Finally, $\varphi : (\mathbb{R}^k \times \{0\}) \cap W' \to M \cap W$ is a homeomorphism, since it is bijective, continuous, and: $\varphi^{-1} = \Psi|_{M\cap W}$ is continuous. $\square$

1.1.9 Examples. (cf. 1.1.7!)

(i) Circle

- *Local Zero Set*: $W := \mathbb{R}^2 \setminus \{(0,0)\},\ f : W \to \mathbb{R},\ f(x,y) = x^2 + y^2 - R^2,\ S^1 \cap W = f^{-1}(0)$.
- *Local Graph*: $S^1 \cap (U' \times U'') = \mathrm{graph}(g),\ g : x \mapsto \sqrt{R^2 - x^2}$.



- *Local Trivialization*: $\Psi : (x,y) = (r\cos\varphi, r\sin\varphi) \mapsto (\varphi, r - R)$. Then locally $\psi := \Psi|_{W\cap S^1} = (R\cos\varphi, R\sin\varphi) \mapsto (\varphi, 0)$ (with suitable W).

(ii) Sphere in $\mathbb{R}^3$

- *Local Zero Set*: $x^2 + y^2 + z^2 = R^2$.
- *Local Graph*: $(x,y) \mapsto \sqrt{R^2 - x^2 - y^2}$
- *Local Trivialization*: Inverse spherical coordinates (with fixed radius).

(iii) Let $U \subseteq \mathbb{R}^n$ be open. Then U is a submanifold of $\mathbb{R}^n$ with local parametrization $\mathrm{id} : U \to U$.

For example, $\mathrm{GL}(n,\mathbb{R}) = \{A \in \mathbb{R}^{n^2} \,|\, \det A \neq 0\}$ is open in $\mathbb{R}^{n^2}$ since $\det : \mathbb{R}^{n^2} \to \mathbb{R}$ is continuous (even $\mathcal{C}^\infty$) $\Rightarrow \mathrm{GL}(n,\mathbb{R})$ is an n^2-dimensional submanifold of $\mathbb{R}^{n^2}$.

(iv) An example of a matrix group as a submanifold.

Let $\mathrm{SL}(n,\mathbb{R}) := \{A \in \mathbb{R}^{n^2} \,|\, \det A = 1\} \subseteq \mathrm{GL}(n,\mathbb{R})$. Hence $\mathrm{SL}(n,\mathbb{R})$ is given as the zero set of the smooth map $f(A) = \det A - 1$. By 1.1.8 (Z) it therefore remains to show that f is regular in any $A \in \mathrm{SL}(n,\mathbb{R})$ (note that if a map is regular in one point then it is regular in a whole neighborhood of that point since a sub-determinant of the Jacobian is nonzero in the point, hence in a neighborhood by continuity). Thus let $A \in \mathrm{SL}(n,\mathbb{R})$. Then

$$\begin{aligned} Df(A)\cdot A &= \frac{d}{dt}\Big|_0 f((1+t)A) = \frac{d}{dt}\Big|_0 (\det(1+t)A - 1) \\ &= n(1+t)^{n-1}\det A\Big|_{t=0} = n\det A \neq 0, \end{aligned}$$

so for all $r \in \mathbb{R}$ we have $Df(A)(\frac{r}{n \det A}A) = r$, i.e., f is regular near A. By 1.1.8, $\mathrm{SL}(n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{n^2}$ of dimension $n^2 - 1$ (in fact $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n; \mathbb{R})$ are examples of *Lie groups*).

Our next aim is to do analysis on submanifolds of $\mathbb{R}^n$. We begin by introducing the notion of smooth map on submanifolds:

1.1.10 Definition. *Let $M \subseteq \mathbb{R}^m$ and $N \subseteq \mathbb{R}^n$ be submanifolds. A map $f : M \to N$ is called smooth (or $\mathcal{C}^\infty$), if for all $p \in M$ there exists some open neighborhood U_p of p in $\mathbb{R}^m$ and some smooth map $\tilde{f} : U_p \to \mathbb{R}^n$ with $\tilde{f}|_{M \cap U_p} = f|_{M \cap U_p}$.*
If f is bijective and both f and f^{-1} are smooth, then f is called diffeomorphism.

1.1.11 Remark.

(i) The case where M is an open subset of $\mathbb{R}^m$ and $N = \mathbb{R}^n$ is included as a special case of the above definition.

(ii) The composition of smooth maps is smooth: Let $f_1 : M_1 \to M_2, f_2 : M_2 \to M_3$ be smooth, $p \in M_1$, and $\tilde{f}_1 : U_p \to \mathbb{R}^{m_2}$, $\tilde{f}_2 : U_{f_1(p)} \to \mathbb{R}^{m_3}$ smooth extensions. Then (since $\tilde{f}_1$ is smooth, hence continuous): $\tilde{f}_1^{-1}(U_{f_1(p)}) \cap U_p$ is an open neighborhood of p and $\tilde{f}_2 \circ \tilde{f}_1 : \tilde{f}_1^{-1}(U_{f_1(p)}) \cap U_p \to \mathbb{R}^{m_3}$ is a smooth extension of $f_2 \circ f_1$.

1.1.12 Definition. *Let M be a k-dimensional submanifold of $\mathbb{R}^n$. A chart (ψ, V) of M is a diffeomorphism of an open set $V \subseteq M$ onto an open subset of $\mathbb{R}^k$.*

Charts are the inverses of local parametrizations in the following sense:

1.1.13 Proposition. *Let M be a k-dimensional submanifold of $\mathbb{R}^n$.*

(i) Let $\varphi : U \subseteq \mathbb{R}^k \to \mathbb{R}^n$ (U open) be a local parametrization of M, $\varphi(U) = W \cap M$ ($W \subseteq \mathbb{R}^n$ open). Then $\psi := \varphi^{-1} : W \cap M \to U$ is a chart of M.

(ii) Conversely, if $\psi : V \to U \subseteq \mathbb{R}^k$ is a chart of M, then $\varphi := \mathrm{id}_{M \hookrightarrow \mathbb{R}^n} \circ \psi^{-1} : U \to \mathbb{R}^n$ is a local parametrization of M.

Proof.

(i) By 1.1.10, φ is a smooth map from U to $W \cap M$. Also, φ is bijective. It remains to prove that $\psi = \varphi^{-1} : W \cap M \to U$ is smooth in the sense of 1.1.10, i.e., possesses a smooth extension to some neighborhood of any given point of $W \cap M$.

Let $p \in W \cap M$ and set $x_0' := \psi(p) \in U$. Here we employ the notations of 1.1.8: $x' := (x_1, \ldots, x_k)$, $x'' := (x_{k+1}, \ldots, x_n)$, $\varphi' := (\varphi_1, \ldots, \varphi_k)$, $\varphi'' := (\varphi_{k+1}, \ldots, \varphi_n)$. φ is an immersion, so without loss of generality we may suppose that $\frac{\partial(\varphi_1, \ldots, \varphi_k)}{\partial(x_1, \ldots, x_k)}(x_0')$ is invertible.

Let $\Phi : U \times \mathbb{R}^{n-k} \to \mathbb{R}^n, \Phi(x', x'') := (\varphi'(x'), \varphi''(x') + x'') = \varphi(x') + (0, x'')$. In particular: $\Phi(x', 0) = \varphi(x')$. Then

$$D\Phi(x_0', 0) = \begin{pmatrix} D\varphi'(x_0') & 0 \\ D\varphi''(x_0') & I_{n-k} \end{pmatrix}$$

is invertible. By 1.1.1, Φ is a local diffeomorphism of $U_1 \times U_2$ onto some W_1, where U_1, U_2 are open neighborhoods of x'_0 in U respectively of 0 in $\mathbb{R}^{n-k}$. Since $p = \Phi(x'_0, 0) \in W_1$ we may w.l.o.g. suppose that $W_1 \subseteq W$.

We have $\varphi(U_1) = \Phi(U_1 \times \{0\}) \subseteq W_1 \subseteq W$. Since φ is a homeomorphism there exists some open subset W_2 of $\mathbb{R}^n$ with $\varphi(U_1) = W_2 \cap M$. W.l.o.g. we may suppose that $W_2 \subseteq W_1$ (otherwise replace W_2 by $W_2 \cap W_1$). Let $\Psi : W_1 \to U_1 \times U_2$ be the inverse of Φ.

Then for $q \in W_2 \cap M$ we have $q = \varphi(x') = \Phi(x', 0)$ for some $x' \in U_1$. Since $(x', 0) \in U_1 \times U_2$ we get $\psi(q) = \varphi^{-1}(q) = x' = \mathrm{pr}_1 \circ \Psi(q)$. Hence $\mathrm{pr}_1 \circ \Psi$ is a smooth extension of ψ to the neighborhood W_2 of p, so ψ is smooth at p, as claimed.

(ii) Let $\psi : V \to U \subseteq \mathbb{R}^k$ be a chart, and set $\varphi := \mathrm{id}_{M \hookrightarrow \mathbb{R}^n} \circ \psi^{-1} : U \to \mathbb{R}^n$. Then φ is smooth and $\varphi : U \to V$ is a homeomorphism (since $\psi : V \to U$ is).

Finally, φ is an immersion: let $\tilde{\psi}$ be a smooth extension of ψ (to some open neighborhood), then $\tilde{\psi} \circ \varphi = \psi \circ \varphi = \mathrm{id}_U$, so $D\tilde{\psi}(\varphi(x)) \cdot D\varphi(x) = \mathrm{id}_U \ \forall x \in U$, implying that $D\varphi(x)$ is injective.

□

1.1.14 Remark. If Ψ is a trivialization as in 1.1.8 (T), $\Psi : W \to W'$, $\Psi(W \cap M) = W' \cap (\mathbb{R}^k \times \{0\})$, then $\psi := \Psi|_{M \cap W}$ is a chart of M (cf. the proof of 1.1.8, (T)⇒(P) and 1.1.13 (i)).

If M is a k-dimensional submanifold of $\mathbb{R}^n$ and (ψ, V) is a chart of M, then for $p \in V$ we may write $\psi(p) = (\psi_1(p), \dots, \psi_k(p)) = (x_1, \dots, x_k)$. The smooth functions $\psi_i = \mathrm{pr}_i \circ \psi$ are called *local coordinate functions*, the x_i are called *local coordinates* of p.

Let M^m, N^n be submanifolds[1], $f : M \to N$, $p \in M$, φ a chart of M around p and ψ a chart of N around $f(p)$. Then $\psi \circ f \circ \varphi^{-1}$ is called local representation of f. We have

$$\psi \circ f \circ \varphi^{-1} : (x_1, \dots, x_m) \mapsto (\underbrace{\psi_1(f(\varphi^{-1}}_{=:f_1}(x)), \dots, \underbrace{\psi_n(f(\varphi^{-1}}_{=:f_n}(x)))).$$

The f_i are called coordinate functions of f with respect to φ, ψ.

By means of charts, smoothness of maps can be characterized without resorting to the surrounding Euclidean space, hence intrinsically:

1.1.15 Proposition. *Let $M^m \subseteq \mathbb{R}^s$, $N^n \subseteq \mathbb{R}^t$ be submanifolds and $f : M \to N$. TFAE:*

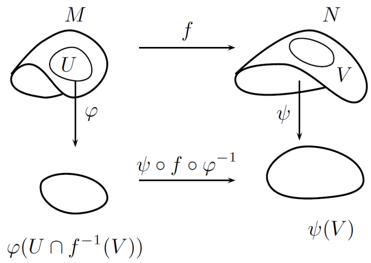
(i) f is smooth.

(ii) For all $p \in M$ there exist charts (φ, U) of M at p, (ψ, V) of N at $f(p)$ such that the domain $\varphi(U \cap f^{-1}(V))$ of the local representation $\psi \circ f \circ \varphi^{-1}$ is open and $\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$ is smooth.

(iii) f is continuous and for all $p \in M$ there exist charts (φ, U) of M at p, (ψ, V) of N at $f(p)$ such that the local representation $\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$ is smooth.

[1] The superscripts m, n here signify the dimension of M resp. N

(iv) f *is continuous and for all* $p \in M$*, all charts* (φ, U) *of* M *at* p *and all charts* (ψ, V) *of* N *at* $f(p)$*, the local representation* $\psi \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}(V)) \to \psi(V)$ *is smooth.*



Proof. (i)⇒(iv): f is continuous since around any point it is the restriction of a continuous map. Hence $f^{-1}(V)$ and therefore also $\varphi(U \cap f^{-1}(V))$ is open. By 1.1.11 (ii), the map $\psi \circ f \circ \varphi^{-1}$ (whose domain of definition is $\varphi(U \cap f^{-1}(V))$) is smooth.

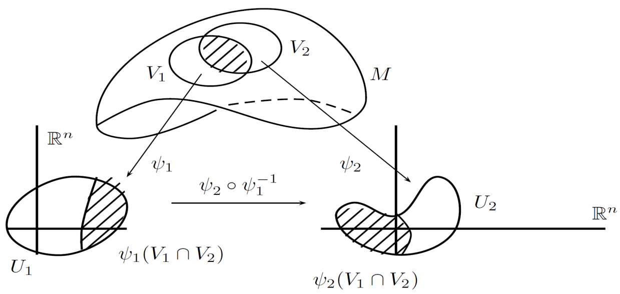
(iv)⇒(iii), and (iii)⇒(ii) are clear.

(ii)⇒(i): On the open neighborhood $U \cap f^{-1}(V)$ of p we have $f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$, so f is smooth by 1.1.11 (ii). □

1.2 Abstract Manifolds

In what follows we want to extend the concept of differentiable manifolds to sets which a priori are not realized as subsets of some $\mathbb{R}^n$. The key to this generalization of the notion of submanifold of $\mathbb{R}^n$ is the formulation of the properties we derived in the previous section in terms of charts. These will allow us to dispense with the surrounding Euclidean space.

1.2.1 Definition. *Let* M *be a set. A* chart (ψ, V) *of* M *is a bijective map* ψ *of* $V \subseteq M$ *onto an open subset* U *of* $\mathbb{R}^n$*,* $\psi : V \to U$*. Two charts* (ψ_1, V_1)*,* (ψ_2, V_2) *are called (*$\mathcal{C}^\infty$*-)* compatible *if* $\psi_1(V_1 \cap V_2)$ *and* $\psi_2(V_1 \cap V_2)$ *are open in* $\mathbb{R}^n$ *and the* chart transition function $\psi_2 \circ \psi_1^{-1} : \psi_1(V_1 \cap V_2) \to \psi_2(V_1 \cap V_2)$ *is a* $\mathcal{C}^\infty$*-diffeomorphism (note that this condition is symmetric in* ψ_1, ψ_2*).*

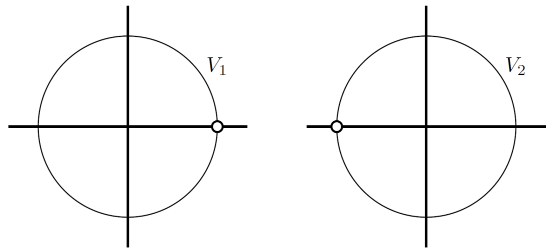


A $\mathcal{C}^\infty$-atlas *of* M *is a family* $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ *of pairwise compatible charts such that* $M = \bigcup_{\alpha \in A} V_\alpha$*. Two atlasses* $\mathcal{A}_1, \mathcal{A}_2$ *are called equivalent if* $\mathcal{A}_1 \cup \mathcal{A}_2$

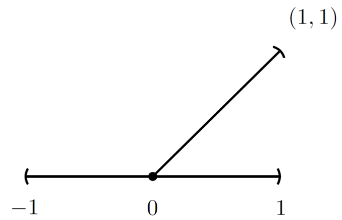
itself is an atlas of M, i.e., if all charts of $\mathcal{A}_1 \cup \mathcal{A}_2$ are compatible. An (abstract) differentiable manifold *is a set M together with an equivalence class of atlasses. Such an equivalence structure is called a differentiable (or $\mathcal{C}^\infty$-)structure on M. The n from above is called the* dimension *of M.*

1.2.2 Examples.

(i) Let $S^1 = \{(x, y) \mid x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$ and set $V_1 := \{(\cos\varphi, \sin\varphi) \mid 0 < \varphi < 2\pi\}$ and $\psi_1 : V_1 \to (0, 2\pi)$, $(\cos\varphi, \sin\varphi) \mapsto \varphi$. Let $V_2 := \{(\cos\varphi, \sin\varphi) \mid -\pi < \varphi < \pi\}$, $\psi_2 : V_2 \to (-\pi, \pi)$, $(\cos\varphi, \sin\varphi) \mapsto \varphi$. Then (ψ_1, V_1) and (ψ_2, V_2) are charts for S^1 and $S^1 = V_1 \cup V_2$. Moreover, ψ_1 and ψ_2 are compatible. In fact, $\psi_1(V_1 \cap V_2) = (0, \pi) \cup (\pi, 2\pi)$ and $\psi_2 \circ \psi_1^{-1}\big|_{(0,\pi)} = \varphi \mapsto \varphi$. We have $\psi_2 \circ \psi_1^{-1}|_{(\pi,2\pi)} = \varphi \mapsto \varphi - 2\pi$, so the chart transition function $\psi_2 \circ \psi_1^{-1} : \psi_1(V_1 \cap V_2) \to \psi_2(V_1 \cap V_2)$ is a diffeomorphism. Hence $\mathcal{A} := \{(\psi_1, V_1), (\psi_2, V_2)\}$ is an atlas of S^1.



(ii) Let M be the subset of $\mathbb{R}^n$ depicted below. Let $V_1 := \{(s, 0) | -1 < s < 1\}$, $\psi_1 : V_1 \to (-1, 1)$, $\psi_1(s, 0) = s$. Further, let $V_2 := \{(s, 0) | -1 < s \leq 0\} \cup \{(s, s) | 0 < s < 1\}$, $\psi_2 : V_2 \to (-1, 1)$, $\psi_2(s, 0) = s$, $\psi_2(s, s) = s$.



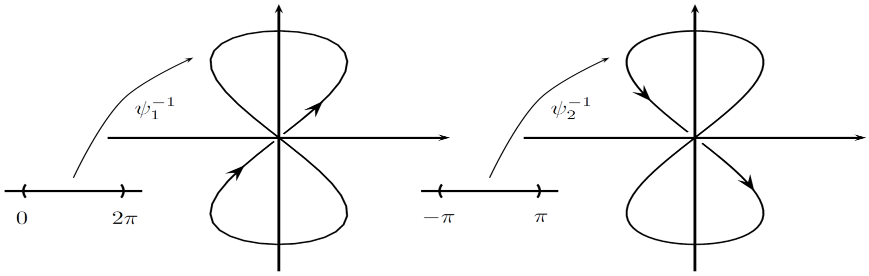
Then ψ_1, ψ_2 are bijective, hence charts, and $\psi_2 \circ \psi_1^{-1} = s \mapsto s$.

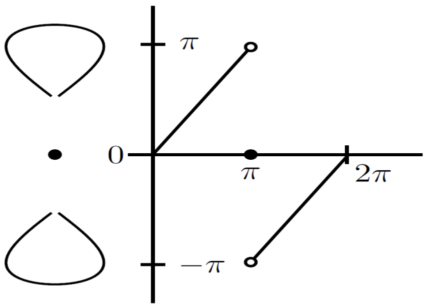
However, $\psi_1(V_1 \cap V_2) = (-1, 0]$ is not open, so ψ_1, ψ_2 are *not* compatible. In fact M also can't be a submanifold of $\mathbb{R}^n$ (same argument as in 1.1.7(iii)).

(iii) As in 1.1.7 (iii) let $M := \{(\sin 2s, \sin s) | s \in \mathbb{R}\}$ be the figure eight manifold. Let $V_1 = M$, $\psi_1 : V_1 \to (0, 2\pi)$, $\psi(\sin 2s, \sin s) = s$. Then ψ_1 is a chart and $\mathcal{A}_1 := \{(\psi_1, V_1)\}$ is an atlas defining a $\mathcal{C}^\infty$-structure on M.

On the other hand, let $V_2 = M$, $\psi_2 : V_2 \to (-\pi, \pi)$, $\psi_2(\sin 2s, \sin s) = s$. Then also $\mathcal{A}_2 := \{(\psi_2, V_2)\}$ is an atlas. However, $\mathcal{A}_1$ and $\mathcal{A}_2$ are *not* equivalent: $\psi_2 \circ \psi_1^{-1} : (0, 2\pi) \to (-\pi, \pi)$,

$$\psi_2 \circ \psi_1^{-1}(s) = \begin{cases} s & 0 < s < \pi & \text{upper loop} \\ s - \pi & s = \pi & \text{origin} \\ s - 2\pi & \pi < s < 2\pi & \text{lower loop} \end{cases}$$





Hence $\psi_2 \circ \psi_1^{-1}$ is not even continuous.

Thus M can be endowed with different $\mathcal{C}^\infty$-structures. With any such structure, M is an example of a $\mathcal{C}^\infty$-manifold that is not a submanifold of $\mathbb{R}^2$ (cf. 1.1.7 (iii)!).

(iv) One can show that for $n \neq 4$, up to diffeomorphism there is precisely one $\mathcal{C}^\infty$-structure on $\mathbb{R}^n$. On $\mathbb{R}^4$ however, there are uncountably many inequivalent (so-called *exotic*) smooth structures!

An atlas for a manifold is called maximal if it is not contained in any strictly larger atlas.

1.2.3 Proposition. *Let M be a $\mathcal{C}^\infty$-manifold with atlas $\mathcal{A}$. Then there is a unique maximal atlas on M that contains $\mathcal{A}$.*

Proof. Let $\tilde{\mathcal{A}} := \{\varphi | \varphi$ is a chart of M and φ is compatible with every $\psi \in \mathcal{A}\}$. Then $\tilde{\mathcal{A}} \supseteq \mathcal{A}$ and we show that $\tilde{\mathcal{A}}$ itself is an atlas.

Let $(\varphi_1, W_1),\ (\varphi_2, W_2) \in \tilde{\mathcal{A}}$ with $W_1 \cap W_2 \neq \emptyset$. Then since φ_1, φ_2 are bijective, so is $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(W_1 \cap W_2) \to \varphi_2(W_1 \cap W_2)$. It remains to show that $\varphi_2 \circ \varphi_1^{-1}$ is a diffeomorphism whose domain $\varphi_1(W_1 \cap W_2)$ is open. Let $x \in \varphi_1(W_1 \cap W_2)$ and (ψ, V) a chart in $\mathcal{A}$ with $\varphi_1^{-1}(x) \in V$. By definition of $\tilde{\mathcal{A}}$, $\varphi_2 \circ \psi^{-1} : \psi(W_2 \cap V) \to \varphi_2(W_2 \cap V)$ and $\psi \circ \varphi_1^{-1} : \varphi_1(W_1 \cap V) \to \psi(W_1 \cap V)$ are diffeomorphisms between open subsets of $\mathbb{R}^n$. Therefore, $(\varphi_2 \circ \psi^{-1}) \circ (\psi \circ \varphi_1^{-1})$ is a diffeomorphism with domain $(\psi \circ \varphi_1^{-1})^{-1}(\psi(W_2 \cap V)) = \varphi_1(W_1 \cap W_2 \cap V)$.

Note that

$$\varphi_1(W_1 \cap W_2 \cap V) = \varphi_1 \circ \psi^{-1}(\psi(V \cap W_1 \cap W_2)) = \varphi_1 \circ \psi^{-1}(\psi(V \cap W_1) \cap \psi(V \cap W_2))$$

is open. Summing up, for all $x \in \varphi_1(W_1 \cap W_2)$ there exists an open neighborhood $\varphi_1(W_1 \cap W_2 \cap V) \subseteq \varphi_1(W_1 \cap W_2)$, on which $\varphi_2 \circ \varphi_1^{-1}$ is a diffeomorphism. Moreover,

$\varphi_2\circ\varphi_1^{-1}$ is bijective on the open set $\varphi_1(W_1\cap W_2)$. Thus $\varphi_2\circ\varphi_1^{-1}$ is a diffeomorphism, so φ_1 and φ_2 are compatible.

Maximality and uniqueness of $\tilde{\mathcal{A}}$ are clear. $\square$

From now on, whenever a smooth manifold M is given, by a chart of M we mean an element of the maximal atlas of M.

Next we want to equip any smooth manifold with a natural topology induced by its charts. We will make use of the following auxilliary result:

1.2.4 Lemma. *Let M be a smooth manifold, (ψ, V) a chart of M and $W \subseteq V$ such that $\psi(W)$ is open in $\mathbb{R}^n$. Then also $(\psi|_W, W)$ is a chart of M.*

Proof. $\psi|_W : W \to \psi(W)$ is bijective. Let (φ, U) be another chart of M. We have to show that $\psi|_W$ and φ are compatible. Now $\psi|_W \circ \varphi^{-1} : \varphi(U\cap W) \to \psi(U\cap W)$ is bijective and is the restriction of the diffeomorphism $\psi\circ\varphi^{-1}$ to $\varphi(U\cap W)$. Also,

$$\varphi(U\cap W) = \varphi\circ\psi^{-1}(\psi(U\cap W)) = \varphi\circ\psi^{-1}(\psi(U\cap V)\cap\psi(W))$$

is open. Thus $\psi|_W\circ\varphi^{-1}$ itself is a diffeomorphism, so $\psi|_W \in \mathcal{A}$. $\square$

1.2.5 Proposition. *Let M be a manifold with maximal atlas $\mathcal{A} = \{(\psi_\alpha, V_\alpha)|\alpha \in A\}$. Then $\mathcal{B} := \{V_\alpha|\alpha\in A\}$ is the basis of a topology, the so-called* natural *or* manifold topology *of M.*

Proof. Clearly $\bigcup_{\alpha\in A} V_\alpha = M$. For $\alpha,\beta\in A$, $\psi_\alpha(V_\alpha\cap V_\beta)$ is open in $\mathbb{R}^n$ (since ψ_α and ψ_β are compatible), hence by 1.2.4, $(\psi_\alpha|_{V_\alpha\cap V_\beta}, V_\alpha\cap V_\beta)$ itself is an element of $\mathcal{A}$. Therefore, $V_\alpha\cap V_\beta\in\mathcal{B}$ and so $\mathcal{B}$ is the basis of a uniquely defined topology. $\square$

1.2.6 Proposition. *With respect to the manifold topology of M, any chart (ψ, V) is a homeomorphisms of the open subset V of M onto the open subset $\psi(V)$ of $\mathbb{R}^n$.*

Proof. Let $\psi : V \to U$ be a chart M. Then by 1.2.5, V is open in M. We first show that ψ is continuous. Let $U_1 \subseteq U$ be open and $W_1 := \psi^{-1}(U_1)$. By 1.2.4, $(\psi|_{W_1}, W_1)$ is a chart of M, so $W_1\in\mathcal{B}$, hence open in M. It remains to show that ψ is open (so that ψ^{-1} is continuous). To this end it suffices to show that ψ maps any $W\in\mathcal{B}$ with $W\subseteq V$ to an open subset of $\mathbb{R}^n$.

By 1.2.5 there exists a chart φ with domain W. Hence $\varphi\circ\psi^{-1} : \psi(W\cap V) \to \varphi(W\cap V)$ is a diffeomorphism. In particular, $\psi(W\cap V) = \psi(W)$ is open. $\square$

1.2.7 Lemma. *Let M be a set, $\mathcal{A}$ a $\mathcal{C}^\infty$-atlas of M, τ the manifold topology induced by $\mathcal{A}$ and τ' another topology on M. TFAE:*

(i) $\tau = \tau'$

(ii) If $(\psi, V)\in\mathcal{A}$, then $V\in\tau'$ and $\psi : V \to \psi(V)$ is a homeomorphism with respect to τ'.

Proof. $(i) \Rightarrow (ii)$ is immediate from 1.2.6.
$(ii) \Rightarrow (i)$: Let $p \in M$, $(\psi, V) \in \mathcal{A}$ with $p \in V$. Let $\mathcal{U}$ be a basis of neighborhoods of $\psi(p)$ in $\psi(V) \subseteq \mathbb{R}^n$. Then $(\psi^{-1}(U))_{U \in \mathcal{U}}$ is a neighborhood basis of p with respect to τ and also with respect to τ'. It follows that every $p \in M$ has the same neighborhoods with respect to τ and τ', so $\tau = \tau'$. □

After these preparations we are now in a position to completely clarify the relationship between submanifolds of $\mathbb{R}^n$ and abstract manifolds.

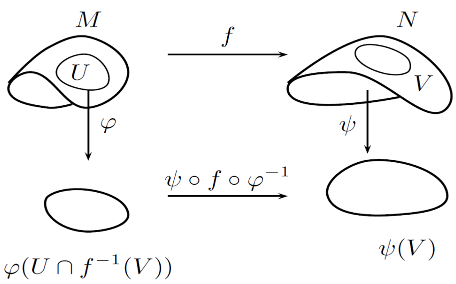
1.2.8 Theorem. *Let M be an m-dimensional submanifold of $\mathbb{R}^n$. Then M is an m-dimensional $\mathcal{C}^\infty$-manifold in the sense of 1.2.1. The manifold topology of M coincides with the trace topology of $\mathbb{R}^n$ on M.*

Proof. As an atlas of M we pick the family of all $\psi = \varphi^{-1}$, where φ is a local parametrization. By 1.1.13 these are precisely the charts in the sense of 1.1.12. By 1.1.15 (iii) (with $f = \mathrm{id}_M$, which is smooth) all chart transition functions are diffeomorphisms, so M is a smooth manifold in the sense of 1.2.1. According to 1.1.5, every φ is a homeomorphism with respect to the trace topology of $\mathbb{R}^n$ on M. Hence by 1.2.7 the trace topology of $\mathbb{R}^n$ is precisely the manifold topology. □

From 1.1.15 we may distill an appropriate definition of smoothness for mappings between abstract manifolds:

1.2.9 Definition. *Let M, N be $\mathcal{C}^\infty$-manifolds and $f : M \to N$ a map. f is called smooth ($\mathcal{C}^\infty$) if it is continuous and for all $p \in M$ there exists a chart φ of M around p and a chart ψ of N around $f(p)$ such that $\psi \circ f \circ \varphi^{-1}$ is smooth. f is called a* diffeomorphism *if it is bijective and f and f^{-1} are smooth.*

1.2.10 Remark.



(i) Let (φ, U), (ψ, V) be charts as above. Then the domain of definition of $\psi \circ f \circ \varphi^{-1}$ is $\varphi(U \cap f^{-1}(V))$. This set is open since f is continuous and φ is a homeomorphism.

Conversely, if $f : M \to N$ is some map such that for all $p \in M$ there exists a chart φ of M around p and a chart ψ of N around $f(p)$ such that $\varphi(U \cap f^{-1}(V))$ is open and $\psi \circ f \circ \varphi^{-1}$ is smooth, then f is smooth. In fact, f is continuous since $f = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$ on the open set $U \cap f^{-1}(V)$ (cf. also 1.1.15(ii)).

(ii) If $(\tilde{\varphi}, \tilde{U})$, $(\tilde{\psi}, \tilde{V})$ are further charts around p resp. $f(p)$, then also $\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1}$ is smooth: near p we have

$$\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1} = \underbrace{(\tilde{\psi} \circ \psi^{-1})}_{\mathcal{C}^\infty} \circ \underbrace{(\psi \circ f \circ \varphi^{-1})}_{\mathcal{C}^\infty} \circ \underbrace{(\varphi \circ \tilde{\varphi}^{-1})}_{\mathcal{C}^\infty}.$$

Since p was arbitrary, $\tilde{\psi} \circ f \circ \tilde{\varphi}^{-1}$ is smooth on its entire domain of definition.

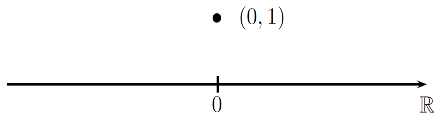
(iii) Obviously the composition of smooth maps is smooth.

1.3 Topological Properties of Manifolds

1.3.1 Proposition. *Every manifold M satisfies the separation axiom T_1.*

Proof. Let $p_1 \neq p_2 \in M$. If there exists a chart (ψ, V) with $p_1,\ p_2 \in V$ then there exist U_1, U_2 open in $\psi(V)$ such that $\psi(p_1) \in U_1,\ \psi(p_2) \in U_2,\ U_1 \cap U_2 = \emptyset$. Hence $\psi^{-1}(U_1)$ and $\psi^{-1}(U_2)$ are disjoint neighborhoods of p_1 resp. p_2. Otherwise there exists a chart (ψ_1, V_1) with $p_1 \in V_1$ and $p_2 \notin V_1$ and vice versa. □

1.3.2 Example. The natural topology of a manifold is *not* automatically T_2 (Hausdorff): Let M be the following set:



Let $V_1 = \{(s, 0) | s \in \mathbb{R}\},\ V_2 := \{(s, 0) | s \neq 0\} \cup \{(0, 1)\},\ \psi_1 : V_1 \to \mathbb{R},\ \psi_1(s, 0) = s,\ \psi_2 : V_2 \to \mathbb{R},\ \psi_2(s, 0) = s\ (s \neq 0),\ \psi_2(0, 1) = 0$. Then $\psi_2 \circ \psi_1^{-1} : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\},\ s \mapsto s$. Therefore $\mathcal{A} := \{\psi_1, \psi_2\}$ is a $\mathcal{C}^\infty$-atlas for M. However, M is not T_2 since $(0, 0)$ and $(0, 1)$ cannot be separated by open sets in M. In fact, let $V,\ W$ be open in M, $(0, 0) \in V,\ (0, 1) \in W$. Then $\psi_1(V_1 \cap V),\ \psi_2(V_2 \cap W)$ are open in $\mathbb{R}$ and contain 0. Hence they contain some $a \neq 0$, so $\psi_1^{-1}(a) = (a, 0) = \psi_2^{-1}(a) \in V_1 \cap V \cap V_2 \cap W \subseteq V \cap W$. Thus $V \cap W \neq \emptyset$, so M is not Hausdorff.

1.3.3 Proposition. *Every manifold satisfies the first axiom of countability, i.e., each of its points possesses a countable basis of neighborhoods.*

Proof. Let $p \in M$, and (ψ, V) a chart around p. Then there exists a countable basis of neighborhoods $(U_m)_{m \in \mathbb{N}}$ of $\psi(p)$ in $\psi(V)$. Hence $(\psi^{-1}(U_m))_{m \in \mathbb{N}}$ is a countable basis of neighborhoods of p in M. □

1.3.4 Proposition. *Every manifold is locally pathwise connected.*

Proof. Let $p \in M$ and (ψ, V) a chart around p such that $\psi(V)$ is pathwise connected (e.g., $\psi(V)$ a ball in $\mathbb{R}^n$, cf. 1.2.4). For $q \in V$ there exists a continuous map $c : [0, 1] \to \psi(V)$ with $c(0) = \psi(p),\ c(1) = \psi(q)$, hence $\tilde{c} := \psi^{-1} \circ c : [0, 1] \to M,\ \tilde{c}(0) = p,\ \tilde{c}(1) = q$. □

1.3.5 Corollary. *Every connected manifold is pathwise connected.*

1.3.6 Proposition. *Every Hausdorff manifold is locally compact.*[2]

Proof. Let $p \in M$ and let (ψ, V) be a chart around p. Let B be a closed ball with center $\psi(p)$ in $\mathbb{R}^n$ and $B \subseteq \psi(V)$. Then since ψ is a homeomorphism, $\psi^{-1}(B)$ is a compact neighborhood of p in M. □

[2] With the understanding that *locally compact* means Hausdorff and that every point has a compact neighborhood. In non-Hausdorff spaces, compact sets need not be closed.

1.3.7 Proposition. *Let M be a manifold. TFAE:*

(i) M satisfies the second axiom of countability (i.e., M possesses a countable basis of its topology, or: M is second countable).

(ii) M possesses a countable atlas.

Proof. (i)$\Rightarrow$(ii): Let $\mathcal{B}$ be a countable basis of the topology of M and let $\mathcal{A} = \{(\psi_\alpha, V_\alpha)|\alpha \in A\}$ be an atlas of M. Then by 1.2.4, $\tilde{\mathcal{A}} := \{(\psi_\alpha|_B, B)|B \in \mathcal{B},\ B \subseteq V_\alpha$ for some $\alpha \in A\}$ is a countable atlas of M.
(ii)$\Rightarrow$(i): Let $\mathcal{A} = \{(\psi_\alpha, V_\alpha)|\alpha \in \mathbb{N}\}$ be a countable atlas of M. Every $U_\alpha = \psi_\alpha(V_\alpha)$ is open in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is second countable there are open sets U_{α_i} ($i \in \mathbb{N}$) in $\mathbb{R}^n$ such that $\{U_{\alpha_i}|i \in \mathbb{N}\}$ is a basis of U_α. Hence every open subset V of V_α is the union of certain $\psi_\alpha^{-1}(U_{\alpha_i})$. Since any open $W \subseteq M$ is the union of certain $W \cap V_\alpha$, $\{V_{\alpha_i}|\alpha \in \mathbb{N}, i \in \mathbb{N}\}$ is a countable basis of the manifold topology of M. □

1.3.8 Corollary. *Every compact manifold is second countable.*

Proof. We may even select a finite atlas from any given atlas. □

In differential geometry and analysis on manifolds one frequently encounters problems that can easily be solved locally (in a chart domain). To obtain global statements, one has to 'patch together' these local constructions. The most important tool in this context are the so-called *partitions of unity*:

1.3.9 Definition. *Let M be a manifold. The* support *of any $f : M \to \mathbb{R}$ is defined as the set* $\mathrm{supp}(f) := \overline{\{p \in M|f(p) \neq 0\}}$. *A family $\mathcal{V}$ of subsets of M is called* locally finite *if every $p \in M$ possesses a neighborhood which intersects only finitely many $V \in \mathcal{V}$. Let $\mathcal{U}$ be an open cover of M. A* partition of unity *subordinate to $\mathcal{U}$ is a family $\{\chi_\alpha|\alpha \in A\}$ of smooth maps $\chi_\alpha : M \to \mathbb{R}^+$ such that:*

(i) $\{\mathrm{supp}\chi_\alpha|\alpha \in A\}$ *is locally finite.*

(ii) For all $\alpha \in A$ there exists some $U \in \mathcal{U}$ such that $\mathrm{supp}(\chi_\alpha) \subseteq U$.

(iii) For all $p \in M$, $\sum_{\alpha \in A} \chi_\alpha(p) = 1$

Note that by (i) the sum in (iii) is finite for any $p \in M$.

Our next goal is to prove the following result:

1.3.10 Theorem. *Let M be a second countable Hausdorff manifold. Then for any open cover $\mathcal{U}$ of M there exists a partition of unity $\{\chi_j|j \in \mathbb{N}\}$ subordinate to $\mathcal{U}$ such that, for all j,* $\mathrm{supp}\chi_j$ *is compact and contained in a chart domain.*

To prepare the proof we need several auxilliary results. To begin with, we show that there exist smooth functions on $\mathbb{R}$ of arbitrarily small support:

1.3.11 Lemma. *Let $f : \mathbb{R} \to \mathbb{R}$,*

$$f(x) := \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$$

Then f is smooth.

Proof. By induction we obtain that

$$f^{(n)}(x) := \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}} P_n(\frac{1}{x}) & x > 0 \end{cases}$$

where P_n is a polynomial. Hence $\lim_{x \nearrow 0} f^{(n)}(x) = \lim_{x \searrow 0} f^{(n)}(x) = 0$ for all n. $\Box$

1.3.12 Lemma. *Let M be a Hausdorff manifold, U an open subset of M and $p \in U$. Then there exists a chart neighborhood V of p and a $\mathcal{C}^\infty$-function $\chi : M \to \mathbb{R}^+$ such that $\overline{V}$ is compact, $\overline{V} \subseteq U$, $\chi > 0$ on V and $\chi \equiv 0$ on $M \backslash V$.*

Proof. Choose a chart (ψ, W) around p such that $W \subseteq U$ and $\psi(p) = 0$. Let $r > 0$ such that for the open ball $B_r(0)$ around 0 we have $\overline{B_r(0)} \subseteq \psi(W)$. Then $V := \psi^{-1}(B_r(0))$ is a neighborhood of p, and $\overline{V} = \psi^{-1}(\overline{B_r(0)})$ is a compact subset of W. Choose f as in 1.3.11 and let $g : \mathbb{R}^n \to \mathbb{R}^+$, $g(x) := f(r^2 - |x|^2)$. Then g is smooth, $g > 0$ on $B_r(0)$, and $g = 0$ on $\mathbb{R}^n \setminus B_r(0)$. Now let

$$\chi(q) := \begin{cases} g \circ \psi(q) & q \in W \\ 0 & q \in M \setminus \overline{V} \end{cases}$$

Now W and $M \setminus \overline{V}$ are open, cover M and χ is smooth on both sets, hence on M. It follows that χ has the desired properties. $\Box$

1.3.13 Lemma. *Let M be a second countable Hausdorff manifold. Then M possesses an exhaustion by compact sets: $\exists (K_j)_{j \in \mathbb{N}}$, $K_j \Subset M$, $K_j \subseteq K_{j+1}^\circ$ $\forall j$ and $M = \bigcup_{j \in \mathbb{N}} K_j$.*

Proof. Since M is locally compact, there exists a cover $\mathcal{V}$ of M consisting of open sets whose closure is compact. By second countability, we may extract from this a countable cover $(V_j)_{j \in \mathbb{N}}$ of M. (Let $\mathcal{B}$ be a countable basis of the topology and $\mathcal{B}' := \{B \in \mathcal{B} | \exists V_B \in \mathcal{V} \text{ with } B \subseteq V_B\}$. Then $\{V_B | B \in \mathcal{B}'\}$ fulfills this purpose.)

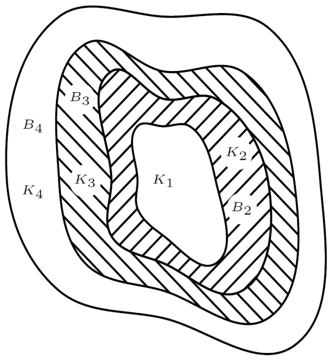
Let $K_1 := \overline{V_1} \Subset M$. Choose $r_2 > 1$ such that $K_1 \subseteq \bigcup_{i=1}^{r_2} V_i$ (possible since K_1 is compact). Let $W_2 := \bigcup_{i=1}^{r_2} V_i$ and $K_2 = \overline{W_2} = \bigcup_{i=1}^{r_2} \overline{V_i} \Subset M$. Then K_2 is compact and $K_1 \subseteq K_2^\circ$. For $j \geq 2$, suppose that $K_j = \overline{W_j}$ has already been defined. Denote by r_{j+1} the first index with $K_j \subseteq \bigcup_{i=1}^{r_{j+1}} V_i$ and set $W_{j+1} = \bigcup_{i=1}^{\max(r_{j+1}, j+1)} V_i$, $K_{j+1} := \overline{W_{j+1}} = \bigcup_{i=1}^{\max(r_{j+1}, j+1)} \overline{V_i}$. Then $K_{j+1} \Subset M$, $K_j \subseteq K_{j+1}^\circ$ and $\bigcup_{j=1}^\infty K_j \supseteq \bigcup_{j=1}^\infty V_j = M$. $\Box$

Proof of 1.3.10 Let $(K_i)_{i \in \mathbb{N}}$ be as in 1.3.13.

Set $K_{-1} = K_0 = \emptyset$ and $B_j := K_j \setminus K_{j-1}^\circ$, so $B_j \Subset M$. For each $p \in B_j$ there exists a $U \in \mathcal{U}$ with $p \in U$ and (by 1.3.12) a chart neighborhood V of p with $\overline{V}$ compact, $\overline{V} \subseteq U \cap M \setminus K_{j-2} = U \setminus K_{j-2}$. Moreover, there exists $\tilde{\chi} \in \mathcal{C}^\infty(M)$ with $\tilde{\chi} > 0$ on V and $\tilde{\chi} \equiv 0$ on $M \setminus V$.

Since B_j is compact it is contained in a finite union of such V. Carrying out this construction for each $j \in \mathbb{N}$ we obtain a countable cover $(V_k)_{k \in \mathbb{N}}$ of M with corresponding $\mathcal{C}^\infty$-functions $(\chi_j)_{j \in \mathbb{N}}$. The family $(\overline{V_k})_{k \in \mathbb{N}}$ is locally finite. In fact, those $\overline{V}_k$ coming from the cover of B_j are disjoint from K_{j-2}, hence disjoint from K_l for $l \leq j-2$. Hence every $p \in M$ possesses an open neighborhood K_l° which intersects only finitely many $\overline{V_k}$. Now let $\chi_j : M \to \mathbb{R}$,

$$\chi_j := \frac{\tilde{\chi}_j}{\sum_{i \in \mathbb{N}} \tilde{\chi}_i}.$$



Then χ_j is well-defined since $\sum_{i\in\mathbb{N}} \tilde{\chi}_i > 0$ (the $(V_j)_{j\in\mathbb{N}}$ form a cover of M, and $\tilde{\chi}_j|_{V_j} > 0$). Summing up, $\chi_j \in \mathcal{C}^\infty(M, \mathbb{R}^+)$, and $\sum_{j\in\mathbb{N}} \chi_j = \frac{\sum_{j\in\mathbb{N}} \tilde{\chi}_j}{\sum_{i\in\mathbb{N}} \tilde{\chi}_i} = 1$, so $(\chi_j)_{j\in\mathbb{N}}$ is the desired partition of unity subordinate to $\mathcal{U}$. □

1.3.14 Corollary. *Let M be a second countable Hausdorff manifold and $\mathcal{U} = \{U_\alpha | \alpha \in A\}$ an open cover of M. Then there exists a partition of unity $\{\chi_\alpha | \alpha \in A\}$ with* $\mathrm{supp}\chi_\alpha \subseteq U_\alpha\ \forall \alpha \in A$. *(The χ_α will not have compact support in general).*

Proof. Choose $\{\chi_j | j \in \mathbb{N}\}$ as in 1.3.10, subordinate to $\mathcal{U}$. Then $\forall j \in \mathbb{N}\ \exists\ \alpha_j$ with $\mathrm{supp}\chi_j \subseteq U_{\alpha_j}$. Let $\chi_\alpha = \sum_{\{j|\alpha_j=\alpha\}} \chi_j$. Then by 1.3.9 (i),

$$\mathrm{supp}\chi_\alpha = \overline{\{p \mid \chi_\alpha(p) \neq 0\}} \subseteq \overline{\bigcup_{\alpha_j=\alpha} \mathrm{supp}\chi_j} = \bigcup_{\alpha_j=\alpha} \overline{\mathrm{supp}\chi_j} = \bigcup_{\alpha_j=\alpha} \mathrm{supp}\chi_j \subseteq U_\alpha.$$

□

1.3.15 Remark. More generally, one can show (cf., e.g., [4, Ch. 8]): for any manifold M, the following are equivalent:

(a) For each open cover $\mathcal{U}$, M possesses a partition of unity subordinate to $\mathcal{U}$.

(b) M is Hausdorff and every connected component of M is second countable.

(c) M is metrizable.

(d) M is Hausdorff and paracompact.

Convention: From now on, by a smooth manifold we will always mean a manifold (in the above sense) whose natural topology is Hausdorff and second countable.

Note that, in particular, every submanifold of $\mathbb{R}^n$ is a smooth manifold in this sense (by 1.2.8 it carries the trace topology of $\mathbb{R}^n$, hence is Hausdorff and second countable).

Chapter 2

Differentiation

2.1 Tangent space and tangent map

After the topological interlude of the previous section we now turn to a study of analysis on manifolds. From 1.2.9 and 1.2.10 we know what smooth maps between manifolds are. However, so far we have not given a definition of the derivative of a smooth map. In $\mathbb{R}^n$, the derivative of a map is the optimal linear approximation to the map. This terminology only makes sense in the vector space setting. Manifolds, on the other hand, in general do not carry a vector space structure. Differentiation on manifolds therefore can be viewed (heuristically) as a two-step approximation process: first, in any given point the manifold is approximated by a vector space (the tangent space, corresponding to the tangent plane of a surface). The derivative itself is then defined as a linear map on this tangent space. To motivate this general procedure we first have a look at the special case of submanifolds of $\mathbb{R}^n$.

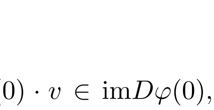
2.1.1 Theorem. *Let M be a submanifold of $\mathbb{R}^n$ and $p \in M$. Then the following subsets of $\mathbb{R}^n$ coincide:*

(i) $\mathrm{im} D\varphi(0)$ *where φ is a local parametrization of M with $\varphi(0) = p$.*

(ii) $\{c'(0) \mid c : I \to M\ \mathcal{C}^\infty, I \subseteq \mathbb{R}$ *an interval*$, c(0) = p\}$

(iii) $\ker Df(p)$*, where, locally around p, M is the zero set of the regular map $f : \mathbb{R}^n \to \mathbb{R}^{n-k}$ (with $k = \dim M$).*

(iv) $\mathrm{graph}(Dg(p'))$*, where, locally around p, M is the graph of the smooth map g and $p = (p', g(p'))$.*



Proof. $(i) \subseteq (ii)$: Given $D\varphi(0) \cdot v \in \mathrm{im} D\varphi(0)$, let $c(t) := \varphi(t \cdot v)$. Then for a suitable interval I, $c : I \to M$ is smooth, $c(0) = \varphi(0) = p$ and $c'(0) = \frac{d}{dt}\big|_0 \varphi(t \cdot v) = D\varphi(0)v \in (ii)$.
$(ii) \subseteq (iii)$: Let $c'(0) \in (ii)$, $c : I \to M$ and f as in (iii). Then locally around 0 we

have $f \circ c(t) \equiv 0$. Hence

$$0 = \frac{d}{dt}\Big|_0 f(c(t)) = Df(\underbrace{c(0)}_{=p})c'(0) \Rightarrow c'(0) \in \ker Df(p)$$

$(iii) \subseteq (i)$: Since $(i) \subseteq (iii)$ it suffices to prove that $\dim(\mathrm{im} D\varphi(0)) = \dim\ker Df(p)$. Since φ is an immersion, $\dim(\mathrm{im} D\varphi(0)) = k = \dim M$. Moreover, $\dim(\mathrm{im} Df(p)) = n-k$, so $\dim\ker Df(p) = n-(n-k) = k$.
$(iii) = (iv)$: Let g as in (iv) (cf. 1.1.8, (Gr)$\Rightarrow$(Z)), and $f_j(x_1,\dots,x_n) := x_{k+j} - g_j(x')$ $(j = 1,\dots,n-k)$. Then locally around p, M is the zero set of f and $\ker(Df(p)) = \ker(q \mapsto q'' - Dg(p')q') = \{(q', Dg(p')q') | q' \in \mathbb{R}^k\} = \mathrm{graph}(Dg(p'))$.
$\square$

2.1.2 Definition. *Let M be a submanifold of $\mathbb{R}^n$ and $p \in M$. The linear subspace of $\mathbb{R}^n$ characterized in 2.1.1 is called the tangent space of M at p and is denoted by T_pM ($\dim T_pM = k = \dim M$). The elements of T_pM are called tangent vectors of M at p.*

If N is a submanifold of $\mathbb{R}^{n'}$ and $f : M \to N$ is smooth, then let $T_pf : T_pM \to T_{f(p)}N$, $c'(0) \mapsto (f \circ c)'(0)$. T_pf is called the tangent map of f at p.

T_pf is well-defined: let $c_1, c_2 : I \to M$, $c_1(0) = p = c_2(0)$ be smooth with $c_1'(0) = c_2'(0)$. Since f is smooth, locally around p there exists some $\tilde f : U \to \mathbb{R}^{n'}$ (U open in $\mathbb{R}^n$) with $\tilde f|_{U\cap M} = f|_{U\cap M}$. Then $\tilde f \circ c_i = f \circ c_i$ $(i = 1,2)$, so

$$(f \circ c_1)'(0) = (\tilde f \circ c_1)'(0) = D\tilde f(p)c_1'(0) = D\tilde f(p)c_2'(0) = \cdots = (f \circ c_2)'(0).$$

Moreover, we conclude that $T_pf(c'(0)) = D\tilde f(p)c'(0)$, so T_pf is linear.

2.1.3 Lemma. *(Chain Rule) Let M, N, P be submanifolds, $f : M \to N$, $g : N \to P$ $\mathcal{C}^\infty, p \in M$. Then*

$$T_p(g \circ f) = T_{f(p)}g \circ T_pf$$

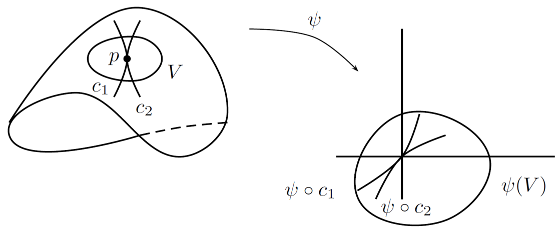
Proof. Let $\tilde g$ and $\tilde f$ be smooth extensions of g and f. Then $\tilde g \circ \tilde f$ is a smooth extension of $g \circ f$ and

$$\begin{aligned} T_p(g \circ f)(c'(0)) &= (\tilde g \circ \tilde f \circ c)'(0) = D\tilde g(f \circ c(0))((\tilde f \circ c)'(0)) = \\ &= T_{f(p)}g(D\tilde f(p)c'(0)) = T_{f(p)}g \circ T_pf(c'(0)) \end{aligned}$$

$\square$

Next we want to extend the concept of tangent space also to abstract manifolds. However, for M an abstract manifold and $c : I \to M$ smooth, the derivative $c'(0)$ at the moment does not make sense due to the lack of a surrounding Euclidean space. Instead, we will resort to charts:

2.1.4 Definition. *Let M be a manifold, $p \in M$ and (ψ, V) a chart around p. Two $\mathcal{C}^\infty$-curves c_1, $c_2 : I \to M$ with $c_1(0) = p = c_2(0)$ are called tangential at p with respect to ψ if $(\psi \circ c_1)'(0) = (\psi \circ c_2)'(0)$.*



2.1.5 Lemma. *The notion of being tangent at a point is independent of the chart used in 2.1.4*

Proof. Let $c_1,\ c_2$ be smooth curves at p with c_1 tangent to c_2 with respect to the chart ψ_1. Let ψ_2 be another chart around p. Then locally around 0 we have $\psi_2 \circ c_i = (\psi_2 \circ \psi_1^{-1}) \circ (\psi_1 \circ c_i)$ $(i = 1, 2)$, so

$$(\psi_2 \circ c_1)'(0) = D(\psi_2 \circ \psi_1^{-1})(\psi_1(p)) \underbrace{(\psi_1 \circ c_1)'(0)}_{=(\psi_1 \circ c_2)'(0)} = (\psi_2 \circ c_2)'(0).$$

$\square$

On the space of smooth curves at p we define an equivalence relation by $c_1 \sim c_2 :\Leftrightarrow$ c_1 tangential to c_2 at p with respect to one (hence any) chart. For $c : I \to M,\ c(0) = p$ we denote by $[c]_p$ the equivalence class of c with respect to $\sim$. Then $[c]_p$ is called a tangent vector at p.

2.1.6 Definition. *The* tangent space *of a manifold M at $p \in M$ is $T_pM = \{[c]_p \mid c : I \to M\ \mathcal{C}^\infty,\ I$ interval in $\mathbb{R},\ c(0) = p\}$.*

We first note that for submanifolds of $\mathbb{R}^n$ this definition reduces to 2.1.2 since in this case the map $c'(0) \mapsto [c]_p$ gives a bijection between 'old' and 'new' tangent space. In fact, picking a chart ψ around p as in 1.1.14, we have

$$[c_1]_p = [c_2]_p \Leftrightarrow \underbrace{(\psi \circ c_1)'(0)}_{D\Psi(p)c_1'(0)} = \underbrace{(\psi \circ c_2)'(0)}_{D\Psi(p)c_2'(0)} \Leftrightarrow c_1'(0) = c_2'(0)$$

since $D\Psi(p)$ is bijective. Hence the map $c'(0) \mapsto [c]_p$ is well-defined and injective. Also, it obviously is surjective.

2.1.7 Definition. *Let M, N be manifolds and $f : M \to N$ a smooth map. Then we call*

$$\begin{aligned} T_pf : T_pM &\to T_{f(p)}N \\ [c]_p &\mapsto [f \circ c]_{f(p)} \end{aligned}$$

the tangent map *of f at p.*

2.1.8 Remark.

(i) T_pf is well-defined:
Let φ be a chart of M at p, ψ a chart of N at $f(p)$, $c_1,\ c_2 : I \to M$ curves through p with $c_1 \sim c_2$. Then

$$\begin{aligned} (\psi \circ f \circ c_1)'(0) &= ((\psi \circ f \circ \varphi^{-1}) \circ (\varphi \circ c_1))'(0) \\ &= D(\psi \circ f \circ \varphi^{-1})(\varphi(p)) \underbrace{(\varphi \circ c_1)'(0)}_{=(\varphi \circ c_2)'(0)} \\ &= \cdots = (\psi \circ f \circ c_2)'(0), \end{aligned}$$

so $f \circ c_1 \sim_{f(p)} f \circ c_2$, i.e., $[f \circ c_1]_{f(p)} = [f \circ c_2]_{f(p)}$.

(ii) In the particular case where $M,\ N$ are submanifolds, $T_p f$ is precisely the map from 2.1.2 in the sense of the above identification ($c'(0) \leftrightarrow [c]_p$).

$$\underbrace{c'(0)}_{\substack{\updownarrow \\ [c]_p}} \mapsto \underbrace{(f \circ c)'(0)}_{\substack{\updownarrow \\ [f\circ c]_{f(p)}}}$$

2.1.9 Proposition. *(Chain Rule) Let M, N, P be manifolds, $f : M \to N$ and $g : N \to P$ smooth, and $p \in M$. Then*

$$T_p(g \circ f) = T_{f(p)}g \circ T_p f$$

Moreover, since $T_p(\mathrm{id}_M) = \mathrm{id}_{T_pM}$, for any diffeomorphism $f : M \to N$, $T_p f$ is bijective and $(T_p f)^{-1} = T_{f(p)} f^{-1}$.

Proof. Let c be a curve through p. Then

$$T_p(g \circ f)([c]_p) = [(g \circ f) \circ c]_{g(f(p))} = T_{f(p)}g([f \circ c]_{f(p)}) = T_{f(p)}g \circ T_p f([c]_p).$$

$\square$

So far we did not endow T_pM with a vector space structure. In order to do this we first analyze the local situation in more detail.

2.1.10 Lemma. *Let $U \subseteq \mathbb{R}^n$ be open and $p \in U$. Then $i : T_pU \to \mathbb{R}^n,\ i([c]_p) := c'(0)$ is bijective, so T_pU can be identified with $\mathbb{R}^n$. In terms of this identification, for any smooth map $f : U \to V$ with $V \subseteq \mathbb{R}^m$ open we have $T_p f = Df(p)$.*

Proof. The map i is well-defined (choose the chart $\psi = \mathrm{id}_U$) and injective ($c_1'(0) = c_2'(0) \Rightarrow (\psi \circ c_1)'(0) = (\psi \circ c_2)'(0)$ for any chart ψ). Also, i is surjetive: Let $v \in \mathbb{R}^n$ and $c : t \mapsto p + t \cdot v$. Then $c'(0) = v$. Now let $f : U \to V$ be smooth and consider

$$\begin{array}{ccc} T_pU & \xrightarrow{T_p f} & T_{f(p)}V \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ \mathbb{R}^n & \xrightarrow{Df(p)} & \mathbb{R}^m \end{array}$$

The diagram commutes since

$$i \circ T_p f([c]_p) = i([f \circ c]_{f(p)}) = (f \circ c)'(0) = Df(p) \cdot c'(0) = Df(p) \circ i([c]_p).$$

$\square$

2.1.11 Proposition. *Let M be a manifold, $p \in M$, and (ψ, V) a chart around p. The vector space structure induced on T_pM by the bijection $T_p\psi : T_pM \to T_{\psi(p)}\psi(V) \cong \mathbb{R}^n$ is independent of the chosen chart (ψ, V).*

Proof. By definition, $T_pV = T_pM$, so $T_p\psi : T_pM \to T_{\psi(p)}\psi(V) \cong \mathbb{R}^n$ (by 2.1.10). Also, $T_p\psi$ is bijective by 2.1.9. Let $[c_1]_p, [c_2]_p \in T_pM,\ \alpha, \beta \in \mathbb{R}$ and φ another chart at p, w.l.o.g. with the same domain V. Then

$$\begin{aligned}
\alpha[c_1]_p + \beta[c_2]_p &:= (T_p\psi)^{-1}(\alpha T_p\psi([c_1]_p) + \beta T_p\psi([c_2]_p)) \\
&\overset{2.1.10}{=} (T_p\psi)^{-1}(\alpha(\psi\circ c_1)'(0) + \beta(\psi\circ c_2)'(0)) \\
&= (T_p\psi)^{-1}(\alpha(\psi\circ\varphi^{-1}\circ\varphi\circ c_1)'(0) + \beta(\psi\circ\varphi^{-1}\circ\varphi\circ c_2)'(0)) \\
&= (T_p\psi)^{-1}(D(\psi\circ\varphi^{-1})(\varphi(p))(\alpha(\varphi\circ c_1)'(0) + \beta(\varphi\circ c_2)'(0))) \\
&\overset{2.1.10}{=} (T_p\psi)^{-1}(T_{\varphi(p)}(\psi\circ\varphi^{-1}))(\alpha(\varphi\circ c_1)'(0) + \beta(\varphi\circ c_2)'(0)) \\
&\overset{2.1.9}{=} (T_p\varphi)^{-1}(\alpha T_p\varphi([c_1]_p) + \beta T_p\varphi([c_2]_p)),
\end{aligned}$$

which establishes our claim. □

In this way, T_pM is endowed with an intrinsic (chart independent) vector space structure. Moreover, if $f : M \to N$ is smooth, then $T_pf : T_pM \to T_{f(p)}N$ is linear with respect to the corresponding vector space structures on T_pM, $T_{f(p)}N$: it suffices to show that $T_{f(p)}\psi \circ T_pf \circ T_{\varphi(p)}\varphi^{-1}$ is linear for any charts φ of M at p and ψ of N at $f(p)$. This map is given by

$$T_{\varphi(p)}(\psi\circ f\circ\varphi^{-1}) \overset{2.1.10}{=} D(\psi\circ f\circ\varphi^{-1})(\varphi(p)),$$

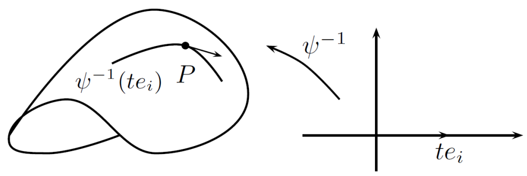
hence is indeed linear.

Any chart of M allows one to pick a particular basis of T_pM: Let (ψ, V) be a chart of M at p, and let $\psi(p) = (x^1(p), \dots, x^n(p))$ (the x^i are called coordinate functions of ψ). For $1 \le i \le n$ let e_i denote the i-th standard unit vector of $\mathbb{R}^n$. Let $\psi(p) = 0$. Then we set

$$\left.\frac{\partial}{\partial x^i}\right|_p := (T_p\psi)^{-1}(e_i) \in T_pM.$$

More precisely, in the sense of 2.1.10 we have

$$\left.\frac{\partial}{\partial x^i}\right|_p = (T_p\psi)^{-1}([t \mapsto te_i]_0) = [t \mapsto \psi^{-1}(te_i)]_p.$$



Hence $\left.\frac{\partial}{\partial x^i}\right|_p$ results from transporting the tangent vector of the coordinate line $t \mapsto te_i$ to M via the chart ψ. Since $T_p\psi$ is a linear isomorphism, $\{\left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^n}\right|_p\}$ indeed forms a basis of T_pM.

If, in particular, M is a submanifold of $\mathbb{R}^n$, and φ is a local parametrization of p (with $\varphi(0) = p$), then $\psi = \varphi^{-1}$ is a chart at p (cf. 1.1.13(i)) and we have

$$\left.\frac{\partial}{\partial x^i}\right|_p = T_0\varphi(e_i) = (\varphi\circ(t\cdot e_i))'(0) = D\varphi(0)e_i.$$

Thus $\left.\frac{\partial}{\partial x^i}\right|_p$ is precisely the i-th column of the Jacobian of φ at $\psi(p) = 0$.

The notation $\frac{\partial}{\partial x^i}\big|_p$ already suggests another interpretation of tangent vectors, namely as directional derivatives. In fact, any tangent vector can be viewed as a directional derivative in the following sense:

Let $v = [c]_p \in T_pM$. Let $f \in \mathcal{C}^\infty(M, \mathbb{R})$ (or $\mathcal{C}^\infty(M)$, for short), the space of smooth maps from M to $\mathbb{R}$. Then define $\partial_v : \mathcal{C}^\infty(M, \mathbb{R}) \to \mathbb{R}$ by $\partial_v f := T_pf(v)$. Since we use the identification 2.1.10 we have:

$$\partial_v(f) = T_pf(v) = T_pf([c]_p) = [f \circ c]_{f(p)} = (f \circ c)'(0), \tag{2.1.1}$$

which corresponds to differentiation in the direction v.

In particular, for $v = \frac{\partial}{\partial x^i}\big|_p$ we have (writing v instead of ∂_v):

$$\frac{\partial}{\partial x^i}\bigg|_p (f) = (f \circ \psi^{-1}(t \mapsto te_i))'(0) = D_i(f \circ \psi^{-1})(\psi(p)), \tag{2.1.2}$$

so $\frac{\partial}{\partial x^i}\big|_p$ corresponds to partial differentiation in the chart ψ.

2.1.12 Definition. *A map $\partial : \mathcal{C}^\infty(M) \to \mathbb{R}$ is called a* derivation *at $p \in M$ if ∂ is linear and satisfies the Leibniz-rule:*

(i) $\partial(f + \alpha g) = \partial f + \alpha \partial g$

(ii) $\partial(f \cdot g) = \partial f \cdot g(p) + f(p) \cdot \partial g$

for all $f, g \in \mathcal{C}^\infty(M)$ and all $\alpha \in \mathbb{R}$. The vector space of all derivations at p is denoted by $\mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R})$.

The following theorem shows that in fact, the tangent space T_pM can be identified with the space $\mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R})$ of derivations at p.

2.1.13 Theorem. *The map*

$$\begin{aligned} A : T_pM &\to \mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R}) \\ v &\mapsto \partial_v \end{aligned}$$

is a linear isomorphism.

Proof. To begin with we show that any ∂_v is a derivation at p: Linearity is obvious from (2.1.1) ($\partial_v(f + \alpha g) = T_p(f + \alpha g)(v) = (T_pf + \alpha T_pg)(v)$) and letting $v = [c]_p$ we have

$$\begin{aligned} \partial_v(f \cdot g) &= ((f \cdot g) \circ c)'(0) = ((f \circ c) \cdot (g \circ c))'(0) \\ &= f(c(0)) \cdot (g \circ c)'(0) + g(c(0)) \cdot (f \circ c)'(0) \\ &= f(p)\partial_v(g) + \partial_v(f)g(p) \end{aligned}$$

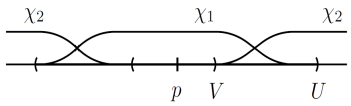
A is linear:

$$(A(v_1 + \alpha v_2))(f) = T_pf(v_1 + \alpha v_2) = T_pf(v_1) + \alpha T_pf(v_2) = (A(v_1) + \alpha A(v_2))(f).$$

A is injective:
We first show that any derivation ∂ at p only 'feels' values of f near p. More precisely, if U is an open neighborhood of p and $f_1,\ f_2 \in \mathcal{C}^\infty(M)$ are such that $f_1|_U = f_2|_U$, then $\partial(f_1) = \partial(f_2)$. In fact, let $f := f_1 - f_2$. Then $f|_U = 0$ and we want to show that $\partial(f)|_U = 0$.

Choose a neighborhood V of p such that $V \Subset U$, i.e., $\overline{V}$ is compact and $\overline{V} \subseteq U$ (cf. 1.3.6). Then by 1.3.14 there is a partition of unity $\{\chi_1,\ \chi_2\}$ subordinate to $\{U, M \setminus \overline{V}\}$. Then

$$0 = \partial(0) = \partial(\chi_1 \cdot f) = \underbrace{\chi_1(p)}_{=1} \cdot \partial(f) + \partial(\chi_1) \underbrace{f(p)}_{=0} = \partial(f).$$



Since in this way any $\mathcal{C}^\infty$-function defined locally at p can be extended to M it follows that in fact any derivation at p is a map from all local $\mathcal{C}^\infty$-functions at p (the so called *germs* of smooth functions at p) into $\mathbb{R}$.

Suppose that $A(v) = 0$, where $v = [c]_p$, i.e., $\partial_v f = 0$ for all smooth functions f locally defined at p. Let ψ be a chart at p with $\psi(p) = 0$ and set $f := x^i$ (where $\psi = (x^1, \ldots, x^n)$). Then $0 = \partial_v f = T_p f(v) = T_p f([c]_p) = (x^i \circ c)'(0)$, so $(\psi \circ c)'(0) = 0$. By 2.1.10, then, $i(T_p\psi(v)) = (\psi \circ c)'(0) = 0$ and therefore $v = 0$ since $T_p\psi$ is a linear isomorphism by 2.1.11.

A is surjective:
Let $\partial \in \mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R})$. We first note that ∂ vanishes on any constant function $f \equiv k$:

$$\partial(k) = \partial(1 \cdot k) = 1 \cdot \partial(k) + k \cdot \partial(1) = 2\partial(k) \Rightarrow \partial(k) = 0.$$

Let $\psi : V \to U$ be a chart of M at p, $\psi(p) = 0$, $\psi = (x^1, \ldots, x^n)$ and $B_1(0) \subseteq U$. Let $f \in \mathcal{C}^\infty(M)$ and $g := f \circ \psi^{-1}$. Then for $x \in B_1(0)$ we have:

$$\begin{aligned} g(x) - g(0) &= \int_0^1 \frac{d}{dt} g(tx)dt = \int_0^1 Dg(tx)xdt = \int_0^1 \sum_{i=1}^n D_i g(tx) \cdot x^i dt \\ &= \sum_{i=1}^n x^i \underbrace{\int_0^1 D_i g(tx)dt}_{=:h_i(x)}. \end{aligned}$$

Hence, on $\psi^{-1}(B_1(0))$,

$$f(q) = g(\psi(q)) = g(0) + \sum_{i=1}^n \psi^i(q) \underbrace{h_i(\psi(q))}_{=:\tilde{h}_i(q)}.$$

Since ∂ acts locally, we conclude:

$$\partial(f) = 0 + \sum_{i=1}^n [\partial(\psi^i)\tilde{h}_i(p) + \underbrace{\psi^i(p)}_{=0} \partial(\tilde{h}_i)].$$

Now

$$\tilde{h}_i(p) = h_i(0) = \int_0^1 D_i g(0)dt = D_i g(0) = D_i(f \circ \psi^{-1})(\psi(p)) = \left.\frac{\partial}{\partial x^i}\right|_p (f)$$

Summing up, we get

$$\partial(f) = \partial_v(f) \qquad \forall f \in \mathcal{C}^\infty(M)$$

where $v = \sum_{i=1}^n \partial(\psi^i) \left.\frac{\partial}{\partial x^i}\right|_p$, establishing that A is surjective. $\square$

Due to this result we will henceforth identify T_pM and $\mathrm{Der}_p(\mathcal{C}^\infty(M),\mathbb{R})$. In fact, in the literature it is quite common to *define* T_pM as $\mathrm{Der}_p(\mathcal{C}^\infty(M),\mathbb{R})$. One of the reasons for this approach is that formal manipulations become particularly simple: let $\partial \in \mathrm{Der}_p(\mathcal{C}^\infty(M),\mathbb{R})$, $f \in \mathcal{C}^\infty(M)$. Then $\partial = \partial_v$ for some $v \in T_pM$. Therefore,

$$T_pf(\partial) = T_pf(\partial_v) \overset{(2.1.1)}{=} \partial_v(f) = \partial(f),$$

and we obtain:

$$T_pf(\partial) = \partial(f) \tag{2.1.3}$$

Now let $f \in \mathcal{C}^\infty(M,N)$. Then the tangent map of f in the derivation picture is computed as follows:

$$\begin{aligned} T_pf : \mathrm{Der}_p(\mathcal{C}^\infty(M),\mathbb{R}) &\to \mathrm{Der}_{f(p)}(\mathcal{C}^\infty(N),\mathbb{R}) \\ \partial &\mapsto (g \mapsto \partial(g\circ f)) \end{aligned} \tag{2.1.4}$$

In fact, by (2.1.3) we have

$$(T_pf(\partial))(g) \overset{(2.1.3)}{=} T_{f(p)}g(T_pf(\partial)) \overset{2.1.9}{=} T_p(g\circ f)(\partial) \overset{(2.1.3)}{=} \partial(g\circ f)$$

2.1.14 Proposition. *Let M^m, N^n be $\mathcal{C}^\infty$-manifolds, $f \in \mathcal{C}^\infty(M,N)$, $p \in M$, $\varphi = (x^1,\dots,x^m)$ a chart of M around p, $\psi = (y^1,\dots,y^n)$ a chart of N around $f(p)$. Then the matrix representation of the linear map $T_pf : T_pM \to T_{f(p)}N$ with respect to the bases $\mathcal{B}_{T_pM} = \{\frac{\partial}{\partial x^1}\big|_p,\dots,\frac{\partial}{\partial x^m}\big|_p\}$ and $\mathcal{B}_{T_{f(p)}N} = \{\frac{\partial}{\partial y^1}\Big|_{f(p)},\dots,\frac{\partial}{\partial y^n}\Big|_{f(p)}\}$ is precisely the Jacobian of the local representation $f_{\psi\varphi} := \psi\circ f\circ\varphi^{-1}$ of f. Thus,*

$$T_pf\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) = \sum_{k=1}^n D_i(\psi^k\circ f\circ\varphi^{-1})(\varphi(p))\,\frac{\partial}{\partial y^k}\Big|_{f(p)} = \sum_{k=1}^n \frac{\partial f^k_{\psi\varphi}}{\partial x^i}\frac{\partial}{\partial y^k} \tag{2.1.5}$$

Proof. The i-th column of $[T_pf]_{\mathcal{B}_{T_pM},\mathcal{B}_{T_{f(p)}N}}$ is $[T_pf(\frac{\partial}{\partial x^i}\big|_p)]_{\mathcal{B}_{T_{f(p)}N}}$. Hence we want to write $T_pf(\frac{\partial}{\partial x^i}\big|_p)$ in the basis $\{\frac{\partial}{\partial y^1}\Big|_{f(p)},\dots,\frac{\partial}{\partial y^n}\Big|_{f(p)}\}$. We have

$$Df_{\psi\varphi}(\varphi(p)) \overset{2.1.10}{=} T_{\varphi(p)}(\psi\circ f\circ\varphi^{-1}) \overset{2.1.9}{=} T_{f(p)}\psi\circ T_pf\circ(T_p\varphi)^{-1}.$$

Let $J_{ki} := D_i(f^k_{\psi\varphi})(\varphi(p)) = D_i(\psi^k\circ f\circ\varphi^{-1})(\varphi(p))$. Then

$$\begin{aligned} T_pf\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) &= T_pf((T_p\varphi)^{-1}(e^i)) = (T_{f(p)}\psi)^{-1}(Df_{\psi\varphi}(\varphi(p))e^i) = \\ &= \sum_{k=1}^n J_{ki}(T_{f(p)}\psi)^{-1}(e^k) = \sum_{k=1}^n J_{ki}\,\frac{\partial}{\partial y^k}\Big|_{f(p)} \end{aligned}$$

□

2.1.15 Corollary. *Let M^n be a manifold, $p \in M$ and let $\varphi = (x^1,\dots,x^n)$ and $\psi = (y^1,\dots,y^n)$ be charts around p. Then*

$$\frac{\partial}{\partial x^i}\Big|_p = \sum_{k=1}^n D_i(\psi^k\circ\varphi^{-1})(\varphi(p))\,\frac{\partial}{\partial y^k}\Big|_p = \sum_{k=1}^n \frac{\partial y^k}{\partial x^i}\frac{\partial}{\partial y^k} \tag{2.1.6}$$

Proof. Set $f = \mathrm{id}_M$ in 2.1.14. □

2.2 Tangent bundle, vector fields, vector bundles

A vector field on an open subset U of $\mathbb{R}^n$ is an assignment $p \mapsto X_p$ of a vector $X_p \in \mathbb{R}^n \cong T_pU$ to each $p \in U$. To analyze, e.g., differential equations with right hand side X (i.e., $c'(t) = X(c(t))$) one will typically assume X to be smooth (at least C^1). We want to extend such notions to the manifold setting. Thus we are looking for maps X mapping points in a manifold M to vectors in T_pM. At the moment, however, we do not have a concept of smoothness for such maps: the individual tangent spaces are not yet bundled together into one manifold. Our first aim therefore is to remedy this deficiency.

2.2.1 Definition. *Let M be a smooth manifold. The tangent bundle (or tangent space) of M is defined as the disjoint union of the vector spaces T_pM ($p \in M$):*

$$TM := \bigsqcup_{p\in M} T_pM := \bigcup_{p\in M} \{p\} \times T_pM$$

The map $\pi_M : TM \to M$, $(p, v) \mapsto p$ is called the canonical projection. If $f : M \to N$ is smooth, then the tangent map Tf of f is defined as $Tf(p, v) = (f(p), T_pf(v))$.

2.2.2 Remark. Depending on whether one wants to make the representation of TM as disjoint union $\bigsqcup_{p\in M} T_pM$ explicit (as above, i.e., $TM = \bigcup_{p\in M}\{p\}\times T_pM$) or not one obtains slightly different notations for the relation between Tf and T_pf. In the first case, one simply considers $T_pM \subseteq TM$ and writes $T_pf = Tf|_{T_pM} : T_pM \to T_{f(p)}N \subseteq TN$. In the second, one writes more explicitly $T_pf = \mathrm{pr}_2 \circ Tf|_{T_pM}$. We will usually prefer the first type of notation.

2.2.3 Lemma. *(Chain Rule) Let $f : M \to N$, $g : N \to P$ be smooth. Then $T(g \circ f) = Tg \circ Tf$. Moreover, $T(\mathrm{id}_M) = \mathrm{id}_{TM}$, so for any diffeomorphism $f : M \to N$ we have $(Tf)^{-1} = T(f^{-1})$.*

Proof. By 2.1.9,

$$\begin{aligned} T(g\circ f)(p,v) &= (g(f(p)), T_p(g\circ f)(v)) = (g(f(p)), T_{f(p)}g \circ T_pf(v))) \\ &= Tg(f(p), T_pf(v)) = (Tg\circ Tf)(p,v) \end{aligned}$$

and

$$T(\mathrm{id}_M)(p,v) = (p, T_p\mathrm{id}_M(v)) = (p,v) = \mathrm{id}_{TM}(p,v)$$

□

In order to turn TM into a smooth manifold we have to endow it with a $\mathcal{C}^\infty$-atlas. Natural candidates for the charts of TM are the tangent maps $T\psi$ of charts (ψ, V) of M:

$$T\psi : TV = \bigcup_{p\in V} \{p\} \times T_pV = \bigcup_{p\in V} \{p\} \times T_pM =: TM|_V \to T(\psi(V)) = \psi(V) \times \mathbb{R}^n$$

Here, $T(\psi(V)) = \bigcup_{x\in\psi(V)}\{x\} \times \underbrace{T_x(\psi(V))}_{=\mathbb{R}^n} = \psi(V) \times \mathbb{R}^n$. Any such $T\psi$ is bijective.

2.2.4 Proposition. *Let M^n be a smooth manifold with atlas $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$. Then $\tilde{\mathcal{A}} := \{(T\psi_\alpha, TM|_{V_\alpha}) \mid \alpha \in A\}$ is a $\mathcal{C}^\infty$-atlas for TM. The natural manifold topology of TM is Hausdorff and second countable, hence TM is a smooth manifold of dimension $2n$.*

Proof. The TV_α cover TM and any $T\psi_\alpha : TV_\alpha \to \psi_\alpha(V_\alpha) \times \mathbb{R}^n$ is bijective. Let $TM|_{V_\alpha} \cap TM|_{V_\beta} \neq \emptyset$, i.e., $V_\alpha \cap V_\beta \neq \emptyset$. Then:

$$\begin{aligned}
T\psi_\beta \circ (T\psi_\alpha)^{-1} &= T(\psi_\beta \circ \psi_\alpha^{-1}) : \underbrace{T(\psi_\alpha(V_\alpha \cap V_\beta))}_{=\psi_\alpha(V_\alpha \cap V_\beta)\times\mathbb{R}^n} \to \underbrace{T(\psi_\beta(V_\alpha \cap V_\beta))}_{=\psi_\beta(V_\alpha \cap V\beta)\times\mathbb{R}^n} \\
T(\psi_\beta \circ \psi_\alpha^{-1})(x,w) &= (\psi_\beta \circ \psi_\alpha^{-1}(x), T_x(\psi_\beta \circ \psi_\alpha^{-1}) \cdot w) \\
&\overset{2.1.10}{=} (\psi_\beta \circ \psi_\alpha^{-1}(x), D(\psi_\beta \circ \psi_\alpha^{-1})(x) \cdot w), \qquad (2.2.1)
\end{aligned}$$

Since any such map is smooth, TM is a $\mathcal{C}^\infty$-manifold of dimension $2n$ if we additionally verify that it is Hausdorff and second countable.

TM is Hausdorff: Let $(p_1, v_1) \neq (p_2, v_2) \in TM$. Then there are two possibilities.

1.) $p_1 \neq p_2$. Then since M is Hausdorff there exist chart neighborhoods V_1, V_2 of p_1, p_2 with $V_1 \cap V_2 = \emptyset$. Then TV_1, TV_2 are neighborhoods of (p_1, v_1), (p_2, v_2) in the natural manifold topology of TM with $TV_1 \cap TV_2 = \emptyset$.

2.) $p_1 = p_2$: Choose a chart (ψ, V) around p and separate $T\psi(p_1, v_1)$, $T\psi(p_2, v_2)$ in $T\psi(TV) = \psi(V) \times \mathbb{R}^n$. Since $T\psi$ is a homeomorphism this gives the desired separation in TM.

TM is second countable: By 1.3.7 there exists a countable atlas $\{(\psi_m, V_m) \mid m \in \mathbb{N}\}$ of M. Then $\{(T\psi_m, TV_m) \mid m \in \mathbb{N}\}$ is a countable atlas of TM, so, again by 1.3.7, the claim follows. □

2.2.5 Remark.

(i) If $f : M^m \to N^n$ is smooth, then so is $Tf : TM \to TN$. In fact, for (ψ, V) a chart of N, and (φ, U) a chart of M we have

$$\begin{aligned}
T\psi \circ Tf \circ T\varphi^{-1}(x,w) &= T(\psi \circ f \circ \varphi^{-1})(x,w) \\
&= (\psi \circ f \circ \varphi^{-1}(x), D(\psi \circ f \circ \varphi^{-1})(x) \cdot w),
\end{aligned}$$

which is smooth on its open domain $\varphi(U \cap f^{-1}(V)) \times \mathbb{R}^m = T(\varphi(U \cap f^{-1}(V)))$. This gives the result by 1.2.10 (ii).

(ii) $\pi_M : TM \to M$ is smooth. In fact, locally π_M is a projection: let (ψ, V) be a chart of M^n. Then

$$\begin{array}{ccc}
TM|_V & \xrightarrow{\pi_M} & V \subseteq M \\
{\scriptstyle T\psi}\big\downarrow & & \big\downarrow{\scriptstyle \psi} \\
T(\psi(V)) = \psi(V) \times \mathbb{R}^n & \xrightarrow{\mathrm{pr}_1} & \psi(V)
\end{array}$$

$$\begin{aligned}
\psi \circ \pi_M \circ T\psi^{-1}(x,w) &= \psi \circ \pi_M(\psi^{-1}(x), T_x\psi^{-1}(w)) \\
&= \psi(\psi^{-1}(x)) = x = \mathrm{pr}_1(x,w).
\end{aligned}$$

On closer examination it turns out that TM in fact has more structure than a 'pure' manifold: the images of the charts $T\psi_\alpha(TV_\alpha) = \psi_\alpha(V_\alpha) \times \mathbb{R}^n$ are cartesian products of open subsets of $\mathbb{R}^n$ with vector spaces. The chart transitions (2.2.1) respect this structure, as they are of the form $(x, w) \mapsto (\varphi_1(x), \varphi_2(x) \cdot w)$ with $\varphi_2(x)$ a linear map for each x. Thus TM furnishes our first example of a *vector bundle* in the sense of the following definition.

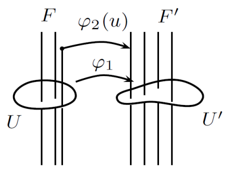
2.2.6 Definition.

(i) Local vector bundles*: Let E, F be (finite dimensional, real) vector spaces, and $U \subseteq E$ open. Then $U \times F$ is called a local vector bundle with base U. We identify U with $U \times \{0\}$. For $u \in U$ we call $\{u\} \times F$ the* fiber *over u. The fiber is equipped with the vector space structure of F. The map $\pi : U \times F \to U,\ (u, f) \mapsto u$ is called the projection of $U \times F$. Then the fiber over u is precisely $\pi^{-1}(u)$.*

A map $\varphi : U \times F \to U' \times F'$ of local vector bundles is called a local vector bundle homomorphism (resp. a local vector bundle isomorphism) if φ is smooth (resp. a diffeomorphism) and has the form

$$\varphi(u, f) = (\varphi_1(u), \varphi_2(u) \cdot f),$$

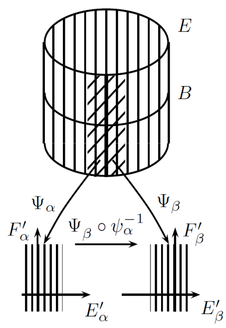
where $\varphi_2(u)$ is linear (resp. a linear isomorphism) from F to (resp. onto) F' for each $u \in U$.



(ii) Vector bundles*: Let E be a set. A local vector bundle chart (or vb-chart) of E is a pair (Ψ, W), where $W \subseteq E$ and $\Psi : W \to W' \times F'$ is a bijection onto a local vector bundle $W' \times F'$ (with W', F' depending on Ψ). A vector bundle atlas is a family $\mathcal{A} = \{(\Psi_\alpha, W_\alpha) \mid \alpha \in A\}$ of local vector bundle charts such that the W_α cover E and any two vector bundle charts (Ψ_α, W_α), (Ψ_β, W_β) in $\mathcal{A}$ with $W_\alpha \cap W_\beta \neq \emptyset$ are compatible in the sense that*

$$\Psi_\beta \circ \Psi_\alpha^{-1} : \Psi_\alpha(W_\alpha \cap W_\beta) \to \Psi_\beta(W_\alpha \cap W_\beta)$$

is a local vector bundle isomorphism (in particular, $\Psi_\alpha(W_\alpha \cap W_\beta)$, $\Psi_\beta(W_\alpha \cap W_\beta)$ are supposed to be local vector bundles).



Two vector bundle atlasses $\mathcal{A}_1$, $\mathcal{A}_2$ are called equivalent if $\mathcal{A}_1 \cup \mathcal{A}_2$ is again a vector bundle atlas. A vector bundle structure $\mathcal{V}$ is an equivalence class of vector bundle atlasses. A vector bundle is a set E together with a vector bundle structure. Since any vector bundle atlas is, in particular, a $\mathcal{C}^\infty$-atlas, E is automatically a $\mathcal{C}^\infty$-manifold. Again we require that the natural manifold topology of E is Hausdorff and second countable.

2.2.7 Remark.

(i) In any vector bundle E there exists a distinguished subset B, the basis of E, defined by:

$$B := \{e \in E \mid \exists \text{ vb-chart } (\Psi, W) \text{ s.t. } e = \Psi^{-1}(w', 0) \text{ for some } w' \in W'\}.$$

B is independent of the vector bundle charts used in the definition since any transition of vector bundle charts is linear in the second component (so 0 is mapped to 0). If $\mathcal{A} = \{(\Psi_\alpha, W_\alpha) \mid \alpha \in A\}$ is a vector bundle atlas for E, then $\mathcal{A}' = \{(\Psi_\alpha|_{W_\alpha \cap B}, W_\alpha \cap B) \mid \alpha \in A\}$ is a $\mathcal{C}^\infty$-atlas for B. Thus B is a smooth manifold. In fact, if $\Psi_\beta \circ \Psi_\alpha^{-1}(w', f') = (\psi^{(1)}_{\beta\alpha}(w'), \psi^{(2)}_{\beta\alpha}(w') \cdot f')$, then $\Psi_\beta|_{W_\beta \cap B} \circ (\Psi_\alpha|_{W_\alpha \cap B})^{-1}(w', 0) = (\psi^{(1)}_{\beta\alpha}(w'), 0)$, which is smooth. Thus the chart transitions in B are exactly the $\psi^{(1)}_{\beta\alpha}$, if we identify $W' \times \{0\}$ with W'.

There is a well-defined projection $\pi : E \to B$: let $e \in E$, Ψ_α a vector bundle chart around e and $\Psi_\alpha(e) = (w', f')$ ($\Psi_\alpha : W \to W' \times F'$). Then let $\pi(e) := \Psi_\alpha^{-1}(w', 0)$. This definition is independent of Ψ_α: Let (Ψ_β, W_β) be another vector bundle chart around e, $\Psi_\beta(e) = (w'', f'')$. Then

$$\Psi_\beta \circ \Psi_\alpha^{-1}(w', f') = (\psi^{(1)}_{\beta\alpha}(w'), \psi^{(2)}_{\beta\alpha}(w') \cdot f') = (w'', f''),$$

so $w'' = \psi^{(1)}_{\beta\alpha}(w')$ and therefore $\Psi_\beta \circ \Psi_\alpha^{-1}(w', 0) = (w'', 0)$. Hence $\pi(e) = \Psi_\alpha^{-1}(w', 0) = \Psi_\beta^{-1}(w'', 0)$. Obviously, π is surjective. Moreover, π is smooth:

$$\begin{array}{ccc} E & \xrightarrow{\pi} & B \\ {\scriptstyle \Psi_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \Psi_\alpha|_B} \\ W' \times F' & \xrightarrow{\mathrm{pr}_1(\times 0)} & W' \times \{0\} \end{array}$$

Since pr_1 is smooth, so is π.

For $b \in B$ we call $\pi^{-1}(b)$ the fiber over b. It carries a vector space structure induced by the vector bundle charts: Let $e_1, e_2 \in \pi^{-1}(b)$, Ψ_α a vector bundle chart around b, $\Psi_\alpha(e_i) = (w', f'_i)$ ($i = 1, 2$). Then let $e_1 + \lambda e_2 := \Psi_\alpha^{-1}(w', f'_1 + \lambda f'_2)$. This is independent of the chosen vector bundle chart: Let Ψ_β be another vector bundle chart, $\Psi_\beta(e_i) = (v', g'_i)$ ($i = 1, 2$). Then $\Psi_\beta \circ \Psi_\alpha^{-1}(w', f'_i) = (\psi^{(1)}_{\beta\alpha}(w'), \psi^{(2)}_{\beta\alpha}(w') f'_i) = (v', g'_i)$, so $\Psi_\beta \circ \Psi_\alpha^{-1}(w', f'_1 + \lambda f'_2) = (\psi^{(1)}_{\beta\alpha}(w'), \psi^{(2)}_{\beta\alpha}(w') \cdot f'_1 + \lambda \psi^{(2)}_{\beta\alpha}(v') \cdot f'_2) = (v', g'_1 + \lambda g'_2)$. Thus $e_1 + \lambda e_2 = \Psi_\alpha^{-1}(w'_1, f'_1 + \lambda f'_2) = \Psi_\beta^{-1}(w', g'_1 + \lambda g'_2)$.

For $U \subseteq B$ open let $E|_U := \bigcup_{b \in U} \{b\} \times E_b$.

(ii) In the literature the following alternative definition of vector bundles is very common:

A vector bundle is a triple (E, B, π) consisting of two $\mathcal{C}^\infty$-manifolds E, B and a smooth surjection $\pi : E \to B$ such that for some fixed vector space F' and all $b \in B$ we have:

- The fiber $\pi^{-1}(b) =: E_b$ is a vector space.
- There exists an open neighborhood V of b in B and a diffeomorphism $\tilde{\Psi} : W := \pi^{-1}(V) \to V \times F'$, which is fiberwise linear (i.e., $\tilde{\Psi}|_{\pi^{-1}(b)}$ is

linear $\forall b \in V$) and such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(V) \arrow[rr, "\tilde{\Psi}"] \arrow[dr, "\pi"'] & & V \times F' \arrow[dl, "\mathrm{pr}_1"] \\ & V & \end{tikzcd} \tag{2.2.2}$$

To see that this definition is equivalent to 2.2.6, suppose first that E is a vector bundle as in 2.2.6, and let π and B be as in (i) above. Then for any vb-chart (Ψ_α, W_α) we have, setting $\psi_\alpha := \Psi_\alpha|_B : W_\alpha \cap B \to W'_\alpha \times \{0\} \cong W'_\alpha$,

$$\begin{tikzcd} W_\alpha = \pi^{-1}(W_\alpha \cap B) \arrow[r, "\Psi_\alpha"] \arrow[dr, "\pi"'] & W'_\alpha \times F' \arrow[r, "\psi_\alpha^{-1}\times \mathrm{id}"] & (W_\alpha \cap B) \times F' \arrow[dl, "\mathrm{pr}_1"] \\ & W_\alpha \cap B & \end{tikzcd}$$

Thus we may set $\tilde{\Psi}_\alpha := ((\Psi_\alpha|_B)^{-1} \times \mathrm{id}_{F'}) \circ \Psi_\alpha$ to obtain (2.2.2).
Conversely, having a vector bundle in the above sense, choose a covering of B by charts $(\varphi_\alpha, V_\alpha)$ such that for each α we have (2.2.2) with diffeomorphism $\tilde{\Psi}_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times F'$. Now set $\Psi_\alpha := (\varphi_\alpha \times \mathrm{id}) \circ \tilde{\Psi}_\alpha : \pi^{-1}(V_\alpha) \to \varphi_\alpha(U_\alpha) \times F'$. Then

$$\Psi_\beta \circ \Psi_\alpha^{-1} = (\varphi_\beta \times \mathrm{id}) \circ \tilde{\Psi}_\beta \circ \tilde{\Psi}_\alpha^{-1} \circ (\varphi_\alpha^{-1} \times \mathrm{id}). \tag{2.2.3}$$

By (2.2.2) we have $\mathrm{pr}_1 \circ \tilde{\Psi}_\beta \circ \tilde{\Psi}_\alpha^{-1} = \mathrm{pr}_1$ on $(V_\alpha \cap V_\beta) \times F'$, so $\mathrm{pr}_1 \circ \Psi_\beta \circ \Psi_\alpha^{-1}(v, f') = \varphi_\beta \circ \varphi_\alpha^{-1}(v)$ for any $(v, f') \in (V_\alpha \cap V_\beta) \times F'$. Moreover, all maps in (2.2.3) are fiber-linear, so we can write

$$\Psi_\beta \circ \Psi_\alpha^{-1}(v, f') = (\varphi_\beta \circ \varphi_\alpha^{-1}(v), \psi^{(2)}_{\beta\alpha}(v) \cdot f'),$$

where $\psi^{(2)}_{\beta\alpha}$ is linear and depends smoothly on v, i.e., $\Psi_\beta \circ \Psi_\alpha^{-1}$ is a local vector bundle isomorphism.

2.2.8 Example. (TM, M, π_M) is a vector bundle:

Let $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ be an atlas of M. By 2.2.4, with $\Psi_\alpha := T\psi_\alpha$, $W_\alpha := TV_\alpha$ the family $\mathcal{A}' := \{(\Psi_\alpha, W_\alpha) \mid \alpha \in A\}$ is a vector bundle atlas of TM. By 2.1.11, the fibers $\pi_M^{-1}(p) = \{p\} \times T_pM \cong T_pM$ carry the vector space structure induced by Ψ_α. Hence, locally TM has a product structure: $T\psi_\alpha : TM|_{V_\alpha} = TV_\alpha \to \psi_\alpha(V_\alpha) \times \mathbb{R}^n$ and we obtain the following commutative diagram:

$$\begin{tikzcd} TV_\alpha = \pi^{-1}(V_\alpha) \arrow[r, "T\psi_\alpha"] \arrow[d, "\pi"'] & \psi_\alpha(V_\alpha) \times \mathbb{R}^n \arrow[r, "\psi_\alpha^{-1}\times \mathrm{id}"] \arrow[d, "\mathrm{pr}_1"] & V_\alpha \times \mathbb{R}^n \arrow[d, "\mathrm{pr}_1"] \\ V_\alpha \arrow[r, "\psi_\alpha"] & \psi_\alpha(V_\alpha) \arrow[r, "\psi_\alpha^{-1}"] & V_\alpha \end{tikzcd}$$

After this clarification of the underlying structures we return to our original task of defining vector fields on manifolds. Thus we are looking for maps which smoothly assign to each $p \in M$ an element $X_p = X(p)$ of T_pM.

2.2.9 Definition. *Let (E, B, π) be a vector bundle. A map $X : B \to E$ is called a* section *of E (more precisely: of $\pi : E \to B$), if $\pi \circ X = \mathrm{id}_B$. The set of all smooth sections of E is denoted by $\Gamma(B, E)$ (or $\Gamma(E)$).*

Thus a vector field is a section of TM ($\pi(X_p) = p\ \forall p$). If $(\psi, V), \psi = (x^1, \ldots, x^n)$ is a chart of M then for any $p \in V$ the $\frac{\partial}{\partial x^i}\big|_p$ form a basis of T_pM. Since $X_p \in T_pM$, for each p there exist uniquely determined $X^i(p) \in \mathbb{R}$ such that $X_p = \sum_{i=1}^n X^i(p)\, \frac{\partial}{\partial x^i}\big|_p$. This is called the local representation of X on V.

2.2.10 Proposition. *Let X be a vector field on a manifold M. TFAE:*

(i) $X : M \to TM$ *is smooth, i.e.,* $X \in \Gamma(TM)$.

(ii) For every $f \in \mathcal{C}^\infty(M)$, $p \mapsto X_p(f) : M \to \mathbb{R}$ *is smooth.*

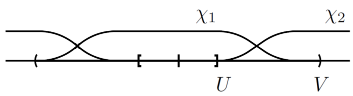
(iii) For every chart (ψ, V) *of* M, $\psi = (x^1, \ldots, x^n)$ *we have: in the local representation*

$$X(p) = \sum_{i=1}^n X^i(p)\, \frac{\partial}{\partial x^i}\bigg|_p\,,$$

$X^i \in \mathcal{C}^\infty(V, \mathbb{R})$ *for all* $i = 1, \ldots, n$.

Proof. (i)⇒(ii): $X : M \to TM$ is smooth by assumption. Also, if $f \in \mathcal{C}^\infty(M)$, then $Tf : TM \to T\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ is smooth by 2.2.5(i). Hence $p \mapsto Tf(X_p) = (f(p), T_pf(X_p)) = (f(p), X_p(f))$, and therefore also $p \mapsto X_p(f)$ is smooth by 2.1.3.

(ii)⇒(iii): Let $p_0 \in V$ and let U be an open neighborhood of p_0 such that $\overline{U}$ is compact and $\overline{U} \subseteq V$. By 1.3.14 we may choose a partition of unity $\{\chi_1, \chi_2\}$ subordinate to $\{V, M \setminus \overline{U}\}$.



Let $1 \le j \le n$ and set $f := \chi_1 x^j$ (extended by 0 outside of V). Then $f \in \mathcal{C}^\infty(M)$ and $f|_U = x^j|_U$. For $p \in U$ we obtain:

$$\begin{aligned} X_p(f) &= \sum_{i=1}^n X^i(p)\, \frac{\partial}{\partial x^i}\bigg|_p (x^j) = \\ &= \sum_{i=1}^n X^i(p) D_i(\underbrace{x^j \circ \psi^{-1}}_{=\mathrm{pr}_j \circ \psi \circ \psi^{-1}})(\psi(p)) = \\ &= \sum_{i=1}^n X^i(p) \delta_{i,j} = X^j(p) \end{aligned}$$

Therefore, each $X^j\big|_U$ is smooth. Since p_0 was an arbitrary point in V, each X^j is smooth on V ($1 \le j \le n$).

(iii)⇒ (i): Let (ψ, V) be a chart at $p \in M$. By 1.2.10 (i), it suffices to show that $T\psi \circ X \circ \psi^{-1}$ is smooth (on its open domain $\psi(V)$). Now

$$\begin{aligned} T\psi \circ X(p) &= T\psi(\sum_{i=1}^n X^i(p)\, \frac{\partial}{\partial x^i}\bigg|_p) = T\psi(\sum_{i=1}^n X^i(p)(T_p\psi)^{-1}(e_i)) \\ &= (\psi(p), T_p\psi(\sum_{i=1}^n X^i(p)(T_p\psi)^{-1}(e_i)) = (\psi(p), \sum_{i=1}^n X^i(p)e_i), \end{aligned}$$

so, finally,

$$T\psi \circ X \circ \psi^{-1}(x) = (x, \sum_{i=1}^{n} X^i(\psi^{-1}(x))e_i) \tag{2.2.4}$$

is smooth, as claimed. □

2.2.11 Definition. *The space of smooth vector fields on M is denoted by $\mathfrak{X}(M)$.*

2.2.12 Examples.

(i) Vector fields on $\mathbb{R}^n$:

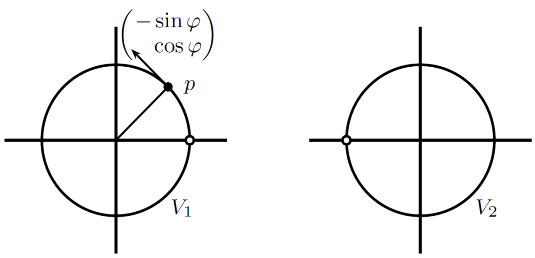
Let $U \subseteq \mathbb{R}^n$ be open. From our analysis course we know: a vector field is a $\mathcal{C}^\infty$-map $X : U \to \mathbb{R}^n$, $X(p) = (X^1(p), \dots, X^n(p)) = \sum_{i=1}^n X^i(p)e_i$. How does this fit into the above framework?

U is a manifold with the single chart $\psi = \mathrm{id}_U$ and the corresponding atlas $\mathcal{A} = \{(id_U, U)\}$. By 2.1.10 we have $T_p\psi = D\psi(p) = \mathrm{id}$, so $\frac{\partial}{\partial x^i}\big|_p = (T_p\psi)^{-1}(e_i) = e_i$. As a derivation, according to (2.1.2), $\frac{\partial}{\partial x^i}\big|_p$ acts as follows:

$$\frac{\partial}{\partial x^i}\bigg|_p (f) = D_i(f \circ \mathrm{id}^{-1})(\mathrm{id}(p)) = D_i f(p) = \frac{\partial f}{\partial x^i}(p).$$

Hence $X_p = \sum_{i=1}^n X^i(p)e_i$ resp. $X_p = \sum_{i=1}^n X^i(p)\,\frac{\partial}{\partial x^i}\big|_p$ correspond to viewing X as a vector or as a differential operator (directional derivative in the direction $(X^1(p), \dots, X^n(p))$), respectively.

(ii) As in 1.2.2, let $M = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, and set $V_1 = \{(\cos\varphi, \sin\varphi) \mid \varphi \in (0, 2\pi)\}$, $\psi_1 : V_1 \to (0, 2\pi)$, $\psi_1(\cos\varphi, \sin\varphi) = \varphi$, and $V_2 = \{(\cos\tilde\varphi, \sin\tilde\varphi) \mid \tilde\varphi \in (-\pi, \pi)\}$, $\psi_2 : V_2 \to (-\pi, \pi)$, $\psi_2(\cos\tilde\varphi, \sin\tilde\varphi) = \tilde\varphi$.



With respect to the chart ψ_1, at $p = (\cos\varphi, \sin\varphi)$ the vector field $\frac{\partial}{\partial\varphi}$ is given by

$$\frac{\partial}{\partial\varphi}\bigg|_p = (T_p\psi_1)^{-1}(e_1) = T_\varphi\psi_1^{-1}(1) = D\psi_1^{-1}(\varphi)\cdot 1 = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$$

Analogously, with respect to ψ_2 we have:

$$\frac{\partial}{\partial\tilde\varphi}\bigg|_p = \begin{pmatrix}-\sin\tilde\varphi\\ \cos\tilde\varphi\end{pmatrix}$$

at $p = (\cos\tilde\varphi, \sin\tilde\varphi)$. By (2.1.6), on $V_1 \cap V_2$ we have

$$\frac{\partial}{\partial\varphi}\bigg|_p = \frac{\partial\tilde\varphi}{\partial\varphi}\,\frac{\partial}{\partial\tilde\varphi}\bigg|_p$$

and

$$\frac{\partial\tilde{\varphi}}{\partial\varphi} = D(\psi_2 \circ \psi_1^{-1})(\psi_1(p)) = 1$$

since

$$\psi_2 \circ \psi_1^{-1} = \varphi \mapsto \begin{cases} \varphi & \varphi \in (0, \pi) \\ \varphi - 2\pi & \varphi \in (\pi, 2\pi) \end{cases}$$

Therefore, $\frac{\partial}{\partial\varphi} = \frac{\partial}{\partial\tilde{\varphi}}$ on $V_1 \cap V_2$ and we conclude that

$$X := \begin{cases} \frac{\partial}{\partial\varphi} & \text{on } V_1 \\ \frac{\partial}{\partial\tilde{\varphi}} & \text{on } V_2 \end{cases}$$

is a well-defined vector field on S^1. Often one simply writes $X = \frac{\partial}{\partial\varphi}$.

Let $f : S^1 \to \mathbb{R}$ be a smooth function. By (2.1.1) we have:

$$\begin{aligned}
(Xf)(p) &= \left.\frac{\partial}{\partial\varphi}\right|_p (f) = D(f \circ \psi_1^{-1})(\underbrace{\psi_1(p)}_{=\varphi}) = \frac{\partial}{\partial\varphi} f(\cos\varphi, \sin\varphi) = \\
&= \frac{\partial f}{\partial x}(\cos\varphi, \sin\varphi) \cdot (-\sin\varphi) + \frac{\partial f}{\partial y}(\cos\varphi, \sin\varphi) \cdot \cos\varphi = \\
&= (-\sin\varphi \cdot \frac{\partial}{\partial x} + \cos\varphi \cdot \frac{\partial}{\partial y}) f.
\end{aligned}$$

It follows that $\frac{\partial}{\partial\varphi} = -\sin\varphi \cdot \frac{\partial}{\partial x} + \cos\varphi \cdot \frac{\partial}{\partial y}$ in the basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\} \cong \{e_1, e_2\}$ of $\mathbb{R}^2$.

2.2.13 Remark. In the local representation (2.2.4) of a vector field in terms of a chart ψ we *pushed forward* $X|_U$ via ψ. More generally, if $F : M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(N)$, we define the *pullback* of X under F by $F^*X := TF^{-1} \circ X \circ F$:

$$\begin{tikzcd}
TM \arrow[r, "TF"] & TN \\
M \arrow[u, "F^*X"] \arrow[r, "F"'] & N \arrow[u, "X"']
\end{tikzcd}$$

Then $F^*X \in \mathfrak{X}(M)$. Also, if $X \in \mathfrak{X}(M)$ we write F_*X for the vector field $(F^{-1})^*X \in \mathfrak{X}(N)$ and call it the *push-forward* of X under F. Due to the chain rule we have the useful composition properties (for $F : M \to N, G : N \to P$ diffeomorphisms, $X \in \mathfrak{X}(P)$, $Y \in \mathfrak{X}(M)$):

$$(G \circ F)^*X = F^*(G^*X) \qquad (G \circ F)_*Y = G_*(F_*Y).$$

In 2.1.13 we identified T_pM with the space of derivations $\mathrm{Der}_p(\mathcal{C}^\infty(M), \mathbb{R})$ at p. Thus for any $X \in \mathfrak{X}(M)$ and any $p \in M$, X_p is a derivation at p. The map $\mathcal{C}^\infty(M) \ni f \mapsto X(f)$, where $X(f) := p \mapsto X_p(f)$ is linear and satisfies

$$X(f \cdot g) = X(f) \cdot g + f \cdot X(g) \tag{2.2.5}$$

In fact, for $p \in M$ we have:

$$\begin{aligned}
(X(f \cdot g))(p) &= X_p(f \cdot g) \\
&= f(p)X_p(g) + g(p)X_p(f) \\
&= (f \cdot X(g) + g \cdot X(f))(p).
\end{aligned}$$

Consequently, X is a derivation in the following sense:

2.2.14 Definition. *An $\mathbb{R}$-linear map $D : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is called a derivation of the algebra $\mathcal{C}^\infty(M)$ if it satisfies the following product rule:*

$$D(f \cdot g) = f \cdot D(g) + g \cdot D(f).$$

The space of derivations on $\mathcal{C}^\infty(M)$ is denoted by $\mathrm{Der}(\mathcal{C}^\infty(M))$.

2.2.15 Theorem. *The derivations on $\mathcal{C}^\infty(M)$ are precisely the smooth vector fields on M: $\mathrm{Der}(\mathcal{C}^\infty(M)) = \mathfrak{X}(M)$. More precisely, every smooth vector field is a derivation on $\mathcal{C}^\infty(M)$, and, conversely, every derivation on $\mathcal{C}^\infty(M)$ is given by the action of a smooth vector field.*

Proof. $\mathfrak{X}(M) \subseteq \mathrm{Der}(\mathcal{C}^\infty(M))$ by 2.2.10 (ii) and the above considerations. Conversely, let $D \in \mathrm{Der}(\mathcal{C}^\infty(M))$. Then for any $p \in M$ the map $\mathcal{C}^\infty(M) \ni f \mapsto (D(f))(p)$ is a derivation at p:

$$\begin{aligned}(D(f \cdot g))(p) &= (D(f) \cdot g + f \cdot D(g))(p) = \\ &= (D(f))(p) \cdot g(p) + f(p) \cdot D(g)(p).\end{aligned}$$

By 2.1.13 it follows that there exists a unique $X_p \in T_pM$ with $X_p(f) = (D(f))(p)$. Hence $p \mapsto X_p$ is a vector field on M with $X(f) = D(f)\ \forall f \in \mathcal{C}^\infty(M)$. X is smooth by 2.2.10 (ii). □

2.2.16 Definition. *Let $X,\ Y \in \mathfrak{X}(M)$. The* Lie bracket *of X and Y is defined as*

$$[X, Y](f) := X(Yf) - Y(Xf) \qquad (f \in \mathcal{C}^\infty(M))$$

It follows that $[X, Y] : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is linear and satisfies the product rule, so by 2.2.15, $[X, Y] \in \mathfrak{X}(M)$.

2.2.17 Proposition. *(Properties of the Lie bracket) Let $X,\ Y,\ Z \in \mathfrak{X}(M)$, $f,\ g \in \mathcal{C}^\infty(M)$. Then:*

(i) $(X, Y) \mapsto [X, Y]$ *is $\mathbb{R}$-bilinear.*

(ii) $[X, Y] = -[Y, X]$ ($[\ ,\ ]$ *is skew-symmetric*).

(iii) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *(Jacobi-identity).*

(iv) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

(v) $[\ ,\ ]$ *is local: If V is open in M, then $[X, Y]|_V = [X|_V\, , Y|_V]$.*

(vi) Local representation: If (ψ, V) is a chart, $\psi = (x^1, \dots, x^n)$, $X|_V = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$, $Y|_V = \sum_{i=1}^n Y^i \frac{\partial}{\partial x^i}$, then:

$$[X, Y]|_V = \sum_{i=1}^n (\sum_{k=1}^n (X^k \frac{\partial Y^i}{\partial x^k} - Y^k \frac{\partial X^i}{\partial x^k})) \frac{\partial}{\partial x^i}$$

Proof. (i), (ii) are immediate from the definition.

(iii) We calculate:

$$\begin{aligned}[X, [Y, Z]]f &= X(Y(Zf) - X(Z(Yf)) - [Y, Z](Xf) = \\ &= X(Y(Zf)) - X(Z(Yf)) - Y(Z(Xf)) + Z(Y(Xf)) \\ [Y, [Z, X]]f &= Y(Z(Xf)) - Y(X(Zf)) - Z(X(Yf)) + X(Z(Yf)) \\ [Z, [X, Y]]f &= Z(X(Yf)) - Z(Y(Xf)) - X(Y(Zf)) + Y(X(Zf)),\end{aligned}$$

which sums to 0.

(iv) Let $h \in \mathcal{C}^\infty(M)$. Then

$$\begin{aligned}
[fX, gY]h &= (fX)(gY(h)) - (gY)(fX(h)) = \\
&= fX(g) \cdot Y(h) + \underbrace{f \cdot g \cdot X(Y(h)) - f \cdot g \cdot Y(X(h))}_{=fg[X,Y](h)} - gY(f)X(h).
\end{aligned}$$

(v) Let $f \in \mathcal{C}^\infty(V)$. Then $X_p(f)$ is well-defined for all $p \in V$ (cf. the proof of 2.1.13). Thus the map $p \mapsto X_p(f)$ is defined on V and coincides with $X|_V(f)$. An analogous statement holds for Y. For $p \in V$ we therefore have:

$$\begin{aligned}
[X, Y]_p(f) &= X_p(Yf) - Y_p(Xf) = X_p(Y|_V(f)) - Y_p(X|_V(f)) = \\
&= (X|_V)_p(Y|_V(f)) - (Y|_V)_p(X|_V(f)) = \\
&= [X|_V, Y|_V]_p(f).
\end{aligned}$$

(vi) Let $f \in \mathcal{C}^\infty(V, \mathbb{R})$. Then:

$$[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]_p f = \frac{\partial}{\partial x^i}\Big|_p (\frac{\partial}{\partial x^j} f) - \frac{\partial}{\partial x^j}\Big|_p (\frac{\partial}{\partial x^i} f)$$

Now

$$\begin{aligned}
\frac{\partial}{\partial x^i}\Big|_p (\frac{\partial f}{\partial x^j}) &\overset{(2.1.2)}{=} \frac{\partial}{\partial x^i}\Big|_p (q \mapsto \underbrace{D_j(f \circ \psi^{-1})(\psi(q))}_{=:g_j(q)}) = \\
&\overset{(2.1.2)}{=} \underbrace{D_i(g_j \circ \psi^{-1})}_{D_i D_j(f \circ \psi^{-1})}(\psi(p)) = \\
&= D_j D_i(f \circ \psi^{-1})(\psi(p)) = \\
&= \frac{\partial}{\partial x^j}\Big|_p (\frac{\partial}{\partial x^i} f),
\end{aligned}$$

so $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0 \ \forall i, j$. Hence

$$\begin{aligned}
[X, Y]|_V &\overset{(v)}{=} [X|_V, Y|_V] = \\
&= [\sum_{i=1}^n X^i \frac{\partial}{\partial x^i}, \sum_{k=1}^n Y^k \frac{\partial}{\partial x^k}] = \\
&\overset{(i),(iv)}{=} \sum_{i,k=1}^n (X^i Y^k \underbrace{[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}]}_{=0} + X^i \frac{\partial Y^k}{\partial x^i} \frac{\partial}{\partial x^k} - Y^k \frac{\partial X^i}{\partial x^k} \frac{\partial}{\partial x^i}) = \\
&= \sum_{i=1}^n (\sum_{k=1}^n (X^k \frac{\partial Y^i}{\partial x^k} - Y^k \frac{\partial X^i}{\partial x^k})) \frac{\partial}{\partial x^i}
\end{aligned}$$

□

2.2.18 Remark. A vector space that is equipped with a bracket operation satisfying (i)–(iii) from the previous result is called a Lie algebra. Thus $(\mathfrak{X}(M), [\,.\,,\,.\,])$ forms a Lie algebra (of infinite dimension).

2.3 Ordinary differential equations, flows of vector fields

In the theory of dynamical systems one analyzes solutions of autonomous ODEs $c'(t) = X(c(t))$, where X is a vector field. In applications, X is often not defined on an open subset of $\mathbb{R}^n$. For example, c might be subject to certain 'constraints', i.e., be constrained by some regular equation. By 1.1.8 this means that X is in fact defined on some differentiable manifold M. Thus we are interested in the ODE

$$c'(t) = X(c(t)) \tag{2.3.1}$$

with $X \in \mathfrak{X}(M)$.

To begin with we have to clarify what we mean by $c'(t)$. For $c \in \mathcal{C}^\infty(I, \mathbb{R}^n)$, $c'(t)$ is given by the vector $Dc(t) \cdot 1$ (where $1 = e_1 \in \mathbb{R}$). For $c \in \mathcal{C}^\infty(I, M)$ we analogously set

$$c'(t) = T_t c(1) \overset{2.2.12}{=} T_t c\Big(\frac{\partial}{\partial t}\Big|_t \Big).$$

2.3.1 Remark. This definition of c' provides a convenient interpretation of tangent vectors as derivatives of curves analogous to the setting of submanifolds of $\mathbb{R}^n$ from 2.1.1 (ii). Since $\mathbb{R}$, as a manifold, is equipped with the trivial atlas $\{\mathrm{id}\}$, we may identify $\frac{\partial}{\partial t}\big|_t$ with $[\mathrm{id}]_t$ (from 2.1.6). Therefore, using 2.1.7 we obtain for any smooth curve $c : I \to M$ with $c(0) = p$

$$c'(0) = T_0 c\Big(\frac{\partial}{\partial t}\Big|_0 \Big) = T_0 c([\mathrm{id}]_0) = [c \circ \mathrm{id}]_{c(0)} = [c]_p. \tag{2.3.2}$$

Since differentiation is a local operation we may write (2.3.1) in local coordinates: let (ψ, V) be a chart in M. The local representation of X with respect to $\psi = (x^1, \ldots, x^n)$ is the push-forward $\psi_* X := T\psi \circ X \circ \psi^{-1}$ of X under ψ:

$$\begin{array}{ccc} TM & \xrightarrow{T\psi} & \psi(V) \times \mathbb{R}^n \\ {\scriptstyle X}\uparrow & & \uparrow{\scriptstyle \psi_* X} \\ M \supseteq V & \xrightarrow{\psi} & \psi(V) \end{array}$$

By (2.2.4), $\psi_* X$ is the map

$$x \mapsto (x, \sum_{i=1}^n X^i(\psi^{-1}(x)) e_i)$$

(for $X|_V = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$). One often drops the first component in this formula. Hence locally X is a vector field with components $(X^1 \circ \psi^{-1}, \ldots, X^n \circ \psi^{-1})$. We also localize c', i.e., we write $c'(t)$ in the chart ψ: c' is the second component of Tc, applied to 1 ($\cong \frac{\partial}{\partial t}$). An application of $T\psi$ gives

$$(T\psi \circ Tc)(t, 1) = T(\psi \circ c)(t, 1) = (\psi \circ c(t), D(\psi \circ c)(t) \cdot 1).$$

Now $D(\psi \circ c)(t) \cdot 1 = (\psi \circ c)'(t)$. Thus with respect to the chart ψ, (2.3.1) reads:

$$(\psi \circ c)'(t) = (\psi_* X)(\psi \circ c(t)), \tag{2.3.3}$$

so locally we obtain the autonomous ODE

$$(x^i \circ c)'(t) = (X^i \circ \psi^{-1})(\psi \circ c(t)) \qquad (1 \le i \le n) \tag{2.3.4}$$

or, with $\tilde{c}^i = x^i \circ c$, $\tilde{X}^i = X^i \circ \psi^{-1}$

$$(\tilde{c}^i)'(t) = \tilde{X}^i(\tilde{c}(t)).$$

To study the global behavior of the solutions of (2.3.1) (the so-called *integral curves* of X) we will need the following fundamental existence and uniqueness result for ODEs:

2.3.2 Theorem. *Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be smooth. Then there exists an open interval I around $0 \in \mathbb{R}$ and an open ball U around $0 \in \mathbb{R}^n$ such that for each $x \in U$ there is a unique solution $c_x : I \to \mathbb{R}^n$ of the initial value problem*

$$\begin{aligned} c_x'(t) &= F(t, c_x(t)) \\ c_x(0) &= x \end{aligned}$$

The map $(t,x) \mapsto c_x(t)$, $I \times U \to \mathbb{R}^n$ is smooth.

Proof. See your favorite ODE-course or Dieudonne, Vol. 1, 10.8.1, 10.8.2. □

Based on this result we establish the following fundamental theorem on ODEs on manifolds.

2.3.3 Theorem. *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Then*

(i) Any $p \in M$ is contained in a unique maximal integral curve of X, i.e., there is a unique smooth solution c_p of (2.3.1) with $c_p(0) = p$ and maximal domain of definition (t_-^p, t_+^p).

(ii) If $t_+^p < \infty$, then $\lim_{t \to t_+^p} c_p(t) = \infty$. That is to say, for $t \to t_+^p$, the curve $c_p(t)$ leaves every compact subset of M (and analogously for $t_-^p > -\infty$).

(iii) The set $U = \{(t,p) \mid p \in M,\ t_-^p < t < t_+^p\}$ is an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$. The flow of X, defined by $\mathrm{Fl}^X : U \to M$, $(t,p) \mapsto c_p(t)$ is smooth and U is the maximal domain of definition of Fl^X. For every $p \in M$ the map $t \mapsto \mathrm{Fl}^X(t,p) \equiv \mathrm{Fl}^X_t(p)$ satisfies the following semi-group property:

$$\mathrm{Fl}^X_{t+s}(p) = \mathrm{Fl}^X_t(\mathrm{Fl}^X_s(p))$$

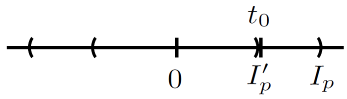
whenever the right hand side of this equation exists.

Proof. (i) As we have seen in (2.3.3), in every chart domain (2.3.1) can be transformed into a local autonomous ODE. Thus 2.3.2 implies the existence of smooth solutions of (2.3.1), i.e., of integral curves of X. Moreover, again by 2.3.2 these solutions are locally unique, i.e., if two solutions coincide in a t-value t_0 then they in fact coincide on a neighborhood of t_0.

Let $p \in M$, and $c_1 : I_1 \to M$, $c_2 : I_2 \to M$ two integral curves of X with $c_1(0) = p = c_2(0)$. Then $J := \{t \in I_1 \cap I_2 \mid c_1(t) = c_2(t)\}$ is nonempty (since $0 \in J$) and closed in $I_1 \cap I_2$. By the above J is also open in $I_1 \cap I_2$, so $J = I_1 \cap I_2$. Thus c_1 and c_2 can be combined into a single integral curve on $I_1 \cup I_2$. The maximal integral curve c_p through p therefore is defined on $(t_-^p, t_+^p) = \bigcup\{I \mid \exists \text{ integral curve } c : I \to M \text{ with } c(0) = p\}$.

(iii) Since $0 \in (t_-^p, t_+^p)$ for all $p \in M$ it follows that $\{0\} \times M \subseteq U$. Moreover, $\mathrm{Fl}^X(0,p) = c_p(0) = p$. Suppose that $\mathrm{Fl}^X_t(\mathrm{Fl}^X_s(p)))$ exists, i.e., $t \mapsto \mathrm{Fl}^X_t(\mathrm{Fl}^X_s(p))$ is the maximal integral curve of X through $\mathrm{Fl}^X_s(p)$. Since also $t \mapsto \mathrm{Fl}^X_{s+t}(p)$ is an integral curve of X with initial value $\mathrm{Fl}^X_s(p)$, it follows that $\mathrm{Fl}^X_{s+t}(p) = \mathrm{Fl}^X_t(\mathrm{Fl}^X_s(p))$. By

2.3.2, Fl^X is defined and smooth on a neighborhood of $\{0\} \times M$. For $p \in M$ let $I_p := (t_-^p, t_+^p)$ and $I'_p := \{t \in \mathbb{R} \mid \mathrm{Fl}^X$ is defined and smooth on a neighborhood of $[0,t] \times \{p\}$ (for $t \geq 0$) resp. of $[t, 0] \times \{p\}$ (for $t < 0$)$\}$
Then $I'_p \subseteq I_p$, $0 \in I'_p$ and I'_p is an open interval. We will show that $I'_p = I_p$. Suppose to the contrary that $I'_p \subsetneq I_p$.



Without loss of generality we may suppose that $t_0 := \inf\{t > 0 \mid t \in I_p \setminus I'_p\} > 0$. Note that $t_0 \notin I'_p$ since I'_p is open.

We know that Fl^X is defined and smooth on a neighborhood W of $(0, \mathrm{Fl}^X_{t_0}(p)) \in \mathbb{R} \times M$. We choose some δ with $0 < \delta < t_0$, and a neighborhood V of p in M such that

$$(-\delta, 2\delta) \times \mathrm{Fl}^X_{t_0 - \delta}(V) \subseteq W$$

(which is possible since $(s, q) \mapsto \mathrm{Fl}^X_s(q)$ is continuous) and such that $q \mapsto \mathrm{Fl}^X_{t_0-\delta}(q)$ is smooth on V. Then the map

$$(s, q) \mapsto \mathrm{Fl}^X_s(\mathrm{Fl}^X_{t_0-\delta}(q)) = \mathrm{Fl}^X_{s+t_0-\delta}(q)$$

is smooth on the neighborhood $(-\delta, 2\delta) \times V$ of $[0, \delta] \times \{p\}$, so Fl^X is smooth on the neighborhood $(t_0 - 2\delta, t_0 + \delta) \times V$ of $[t_0 - \delta, t_0] \times \{p\}$. Moreover (by definition of t_0) $t_0 - \delta \in I'_p$, so Fl^X is smooth on a neighborhood of $[0, t_0 - \delta] \times \{p\}$. Summing up, Fl^X is smooth on a neighborhood of $([0, t_0 - \delta] \cup [t_0 - \delta, t_0]) \times \{p\} = [0, t_0] \times \{p\}$. But according to the definition of I'_p this means that $t_0 \in I'_p$, contradicting the definition of t_0, which establishes $I_p = I'_p$.

Hence $U = \{(t, p) \mid t \in I_p\} = \{(t, p) \mid t \in I'_p\}$ is open and Fl^X is smooth on U (both according to the definition of I'_p).

(ii) Let $p \in M$, $t_+^p < \infty$ and K a compact subset of M. We want to show that $c_p(t) \notin K$ for t sufficiently close to t_+^p. Suppose to the contrary that there exists a sequence (t_n) with $t_n \nearrow t_+^p$ and $c_p(t_n) \in K$ for all n. Since K is compact, a subsequence of $(c_p(t_n))$, hence w.l.o.g. $(c_p(t_n))$ itself converges to some $p' \in K$.

There exists some $\varepsilon > 0$ and some neighborhood V of p' such that Fl^X is smooth on $(-\varepsilon, \varepsilon) \times V$. Choose n_0 such that $c_p(t_n) \in V$ $\forall n \geq n_0$. Since

$$\mathrm{Fl}^X_t(c_p(t_n)) = \mathrm{Fl}^X_t(\mathrm{Fl}^X_{t_n}(p)) = \mathrm{Fl}^X_{t+t_n}(p) = c_p(t + t_n),$$

$c_p(t + t_n)$ exists for all $|t| < \varepsilon$ and all $n \geq n_0$. Thus $c_p(s)$ is defined for $s \in (t_n - \varepsilon, t_n + \varepsilon)$ $\forall n \geq n_0$. Choose $n \geq n_0$ such that $t_n > t_+^p - \frac{\varepsilon}{2}$. Then $c_p(s)$ is exists up to $t_+^p - \frac{\varepsilon}{2} + \varepsilon = t_+^p + \frac{\varepsilon}{2} > t_+^p$, contradicting the definition of t_+^p. $\square$

From this main theorem we will be able to conclude the existence of *flow boxes* in the following sense:

2.3.4 Definition. *Let X be a smooth vector field on a manifold M and let $p \in M$. A* flow box *of X at p is a triple (V, a, Fl^X) with the following properties:*

(i) V is an open neighborhood of p in M and $a \in (0, \infty]$.

(ii) Setting $I_a := (-a, a)$, $\mathrm{Fl}^X : I_a \times V \to M$ is smooth.

(iii) *For each* $m \in V$, $t \mapsto \mathrm{Fl}^X_t(m)$, $I_a \to M$ *is an integral curve of* X *through* m.

(iv) *For each* $t \in I_a$, $\mathrm{Fl}^X_t : V \to \mathrm{Fl}^X_t(V)$ *is a diffeomorphism between open sets of* M.

2.3.5 Corollary. *Let* X *be a smooth vector field on a manifold* M. *Then at any* $p \in M$ *there exists a flow box of* X

Proof. With U as in 2.3.3, pick an open neighborhood V of p in M and $a \in (0, \infty]$ such that $I_{2a} \times V \subseteq U$. Then (i)–(iii) are immediate from 2.3.3. Concerning (iv), by our choice of a, for any $m \in V$ we have that $I_a \times \mathrm{Fl}^X_t(m) \in U$ for every $t \in I_a$, i.e., $I_a \times \mathrm{Fl}^X_t(V) \subseteq U$. Then 2.3.3 (iii) shows that, for any $t \in I_a$, $\mathrm{Fl}^X_t : V \to \mathrm{Fl}^X_t(V)$ is a diffeomorphism with inverse Fl^X_{-t}. Finally, any diffeomorphism is an open map by the inverse function theorem. □

2.3.6 Definition. *Let* M *be a manifold and let* $X \in \mathfrak{X}(M)$. X *is called* complete, *if* Fl^X *is defined on all of* $\mathbb{R} \times M$ *(i.e.,* $U = \mathbb{R} \times M$*).*

Completeness of X therefore means that each integral curve of X exists for all times. From 2.3.3 (ii) we conclude:

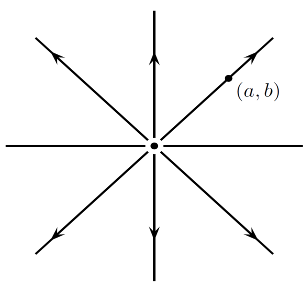
2.3.7 Corollary. *Every vector field on a compact manifold is complete.*

2.3.8 Examples.

(*i*) Let $M = \mathbb{R}^2$, and $X = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2}$. To determine the integral curves of X we have to solve the ODE $c'(t) = X(c(t))$. Applying (2.3.4) with $\psi = \mathrm{id}_{\mathbb{R}^2}$ gives: $\tilde{c} = c$, $\tilde{X} = X$. Hence we consider

$$\begin{aligned} (c^1)'(t) &= x^1(c(t)) = c^1(t) \\ (c^2)'(t) &= x^2(c(t)) = c^2(t) \\ c(0) &= (a, b) \in \mathbb{R}^2 \end{aligned}$$

Thus $c(t) = (ae^t, be^t) = \mathrm{Fl}^X_t(a, b)$. Obviously, $\mathrm{Fl}^X_{s+t}(a, b) = \mathrm{Fl}^X_s(\mathrm{Fl}^X_t(a, b))$. For $(a, b) = (0, 0)$ it follows that $c(t) \equiv 0$ since $X(0, 0) = (0, 0)$. $(0, 0)$ is called a *critical point* of X (i.e., zero of X).



Every integral curve of X is defined on all of $\mathbb{R}$, so X is complete.

(ii) Let $M = \mathbb{R}^2$, and $X = e^{-x^1} \frac{\partial}{\partial x^1}$. Using the chart $\psi = \mathrm{id}_{\mathbb{R}^2}$ we obtain

$$\begin{aligned} (c^1)'(t) &= e^{-c^1(t)} \\ (c^2)'(t) &= 0 \\ c(0) &= (a, b) \end{aligned}$$

Thus $c(t) = (\log(t + \exp a), b) = \mathrm{Fl}_t^X(a, b)$ (it is easily verified that the flow property $\mathrm{Fl}_{t+s}^X(a, b) = \mathrm{Fl}_t^X(\mathrm{Fl}_s^X(a, b))$ holds). c is defined on $(-e^a, \infty) \subsetneq \mathbb{R}$, so X is not complete.

(iii) (cf. 1.1.7 (ii)).
Let $\tilde{M} = S^2$, $\psi : (x, y, z) = (\cos\phi\cos\theta, \sin\phi\cos\theta, \sin\theta) \mapsto (\phi, \theta) = (\psi^1, \psi^2)$, and $M := \psi^{-1}((0, 2\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2})) \stackrel{\text{open}}{\subseteq} \tilde{M}$. Let X on M be given, with respect to ψ, by

$$X = \phi\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta}$$

In (2.3.4) we have $\tilde{X}^1(\phi, \theta) = \phi$, $\tilde{X}^2(\phi, \theta) = 1$, $\tilde{c}(t) = (\phi(t), \theta(t))$. (Note that X cannot be extended smoothly to S^2 since ϕ has a jump.) Hence (2.3.4) reads:

$$\begin{aligned} \phi'(t) &= \phi(t) \\ \theta'(t) &= 1 \\ (\phi(0), \theta(0)) &= (\phi_0, \theta_0) \end{aligned}$$

Thus $\tilde{c}(t) = (\phi(t), \theta(t)) = (\phi_0 e^t, t + \theta_0)$, so

$$c(t) = \psi^{-1} \circ \tilde{c}(t) = (\cos(\phi_0 e^t)\cos(t + \theta_0), \sin(\phi_0 e^t)\cos(t + \theta_0), \sin(t + \theta_0)).$$

$X \in \mathfrak{X}(M)$ is not complete.

2.3.9 Remark. Let M^k be a k-dimensional submanifold of $\mathbb{R}^n$. Then $\mathfrak{X}(M) = \{X : M \to \mathbb{R}^n \mid X\ \ \mathcal{C}^\infty \text{ and } X_p \in T_pM\ \forall p \in M\}$.

Proof. Let $X \in \mathfrak{X}(M)$. Then $X_p \in T_pM\ \forall p$. Locally (with respect to a parametrization (φ, U)), X is given by

$$\begin{aligned} X(\varphi(x)) &= \sum_{i=1}^k X^i(\varphi(x)) \left.\frac{\partial}{\partial x^i}\right|_{\varphi(x)} \\ &= \sum_{i=1}^k X^i(\varphi(x)) D_i\varphi(x) \end{aligned}$$

(cf. the remark preceding 2.1.12). Hence $X \circ \varphi$ is smooth since X^i and φ are. But then X is smooth by 1.1.13 (i) and 1.1.15.

Conversely, let $X : M \to \mathbb{R}^n$ be smooth and suppose that $X_p \in T_pM$ for all $p \in M$. Then X is a section of TM and it remains to show that X is smooth. To this end we employ 2.2.10 (ii): let $f \in \mathcal{C}^\infty(M)$ with local smooth extension $\tilde{f}$. Then $X(f)$ is locally given by $p \mapsto X_p(f) = T_pf(X_p) = D\tilde{f}(p)X_p$, which clearly is smooth on M. □

Caution: Note that the $X^1, \dots, X^k$ should not be confused with the n components of X as a vector in $\mathbb{R}^n$!

Next, we want to study a further interpretation of vector fields, namely as differential operators on functions and vector fields, in the shape of *Lie derivatives.*

2.3.10 Definition. *Let $X \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M)$. The Lie derivative of f in direction X is*

$$L_X f(p) := \left.\frac{d}{dt}\right|_0 f(\mathrm{Fl}_t^X(p)) = \left.\frac{d}{dt}\right|_0 [(\mathrm{Fl}_t^X)^* f](p).$$

2.3.11 Lemma. $L_X(f) = X(f)$.

Proof.

$$\frac{d}{dt}\Big|_0 f \circ \mathrm{Fl}_t^X(p) = T_p f\Big(\frac{d}{dt}\Big|_0 \mathrm{Fl}_t^X(p)\Big) = T_p f(X(p)) = X(f)(p).$$

□

2.3.12 Definition. *Let $X, Y \in \mathfrak{X}(M)$. The Lie derivative of Y along X is the vector field*

$$L_X Y(p) := \frac{d}{dt}\Big|_0 ((\mathrm{Fl}_t^X)^* Y)(p) \qquad ((\mathrm{Fl}_t^X)^* Y = T\mathrm{Fl}_{-t}^X \circ Y \circ \mathrm{Fl}_t^X).$$

2.3.13 Proposition. *Let $X, Y \in \mathfrak{X}(M)$. Then*

(i) $L_X Y = [X, Y]$.

(ii) $\frac{d}{dt}(\mathrm{Fl}_t^X)^* Y = (\mathrm{Fl}_t^X)^* L_X Y$.

Proof. (i) Let $f \in \mathcal{C}^\infty(M)$, $p \in M$, and set $\alpha(t,s) := Y(\mathrm{Fl}_t^X(p))(f \circ \mathrm{Fl}_s^X)$. Then $\alpha : I \times I \to \mathbb{R}$ for some interval I around 0. We have $\alpha(t,0) = Y(\mathrm{Fl}_t^X(p))(f)$ and $\alpha(0,s) = Y(p)(f \circ \mathrm{Fl}_s^X)$. Now

$$\begin{aligned}
\frac{\partial \alpha}{\partial t}(0,0) &= \frac{\partial}{\partial t}\Big|_0 Y(\mathrm{Fl}_t^X(p))(f) = \frac{\partial}{\partial t}\Big|_0 (Yf)(\mathrm{Fl}_t^X(p)) \underset{2.3.10}{=} L_X(Yf)(p) \underset{2.3.11}{=} X(Yf)(p)\\
\frac{\partial \alpha}{\partial s}(0,0) &= \frac{\partial}{\partial s}\Big|_0 Y(p)(f \circ \mathrm{Fl}_s^X) = \frac{\partial}{\partial s}\Big|_0 T_p(f \circ \mathrm{Fl}_s^X)(Y(p)) = T_p\Big(\frac{\partial}{\partial s}\Big|_0 f \circ \mathrm{Fl}_s^X\Big)(Y(p))\\
&\underset{2.3.10}{=} T_p(L_X f)(Y(p)) = Y(L_X f)(p) = Y(X(f))(p).
\end{aligned}$$

Moreover, using $(T_p g Y_p)(h) = Y_p(h \circ g)$ (see (2.1.4)),

$$\begin{aligned}
\frac{\partial}{\partial u}\Big|_0 \alpha(u,-u) &= \frac{\partial}{\partial u}\Big|_0 Y(\mathrm{Fl}_u^X(p))(f \circ \mathrm{Fl}_{-u}^X) = \frac{\partial}{\partial u}\Big|_0 \big(T_{\mathrm{Fl}_u^X(p)} \mathrm{Fl}_{-u}^X(Y(\mathrm{Fl}_u^X(p))(f))\big)\\
&= \frac{\partial}{\partial u}\Big|_0 (T\mathrm{Fl}_{-u}^X \circ Y \circ \mathrm{Fl}_u^X(p)(f)) = \frac{\partial}{\partial u}\Big|_0 (\mathrm{Fl}_u^X)^* Y(p)(f)\\
&\underset{2.3.12}{=} L_X Y(p)(f) = L_X Y(f)(p).
\end{aligned}$$

On the other hand, by what we calculated above,

$$\frac{\partial}{\partial u}\Big|_0 \alpha(u,-u) = \frac{\partial \alpha}{\partial t}(0,0) - \frac{\partial \alpha}{\partial s}(0,0) = (X(Yf) - Y(Xf))(p) = [X,Y](f)(p).$$

(ii)

$$\begin{aligned}
\frac{d}{dt}(\mathrm{Fl}_t^X)^* Y &= \frac{d}{ds}\Big|_0 (\mathrm{Fl}_{t+s}^X)^* Y = \frac{\partial}{\partial s}\Big|_0 (T\mathrm{Fl}_{-(s+t)}^X \circ Y \circ \mathrm{Fl}_{s+t}^X)\\
&= \frac{\partial}{\partial s}\Big|_0 (T\mathrm{Fl}_{-t}^X \circ T\mathrm{Fl}_{-s}^X \circ Y \circ \mathrm{Fl}_s^X \circ \mathrm{Fl}_t^X)\\
&= T\mathrm{Fl}_{-t}^X \circ \Big(\frac{\partial}{\partial s}\Big|_0 (\mathrm{Fl}_s^X)^* Y\Big) \circ \mathrm{Fl}_t^X \underset{2.3.12}{=} (\mathrm{Fl}_t^X)^*(L_X Y).
\end{aligned}$$

□

2.3.14 Definition. *Let $f : M \to N$ be a smooth map between two manifolds. Then vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called f-related, denoted by $X \sim_f Y$ if $T_pf(X_p) = Y_{f(p)}$ for each $p \in M$.*

$$\begin{tikzcd} TM \arrow[r, "Tf"] & TN \\ M \arrow[u, "X"] \arrow[r, "f"] & N \arrow[u, "Y"'] \end{tikzcd}$$

2.3.15 Lemma. *Smooth vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are f-related if and only if for each $g \in \mathcal{C}^\infty(N)$ we have*

$$X(g \circ f) = Y(g) \circ f.$$

Proof.

$$\begin{aligned} X(g \circ f) = Y(g) \circ f \ \forall g \Leftrightarrow X_p(g \circ f) = Y_{f(p)}(g) \ \forall g \ \forall p \\ \underset{(2.1.4)}{\Leftrightarrow} T_pf(X_p)(g) = Y_{f(p)}(g) \ \forall g \ \forall p \Leftrightarrow X \sim_f Y. \end{aligned}$$

□

2.3.16 Lemma. *Let $X_1, X_2 \in \mathfrak{X}(M)$, $Y_1, Y_2 \in \mathfrak{X}(N)$, $f \in \mathcal{C}^\infty(M,N)$ and suppose that $X_1 \sim_f Y_1$ and $X_2 \sim_f Y_2$. Then also $[X_1, X_2] \sim_f [Y_1, Y_2]$.*

Proof. Using 2.3.15 we calculate:

$$\begin{aligned} [X_1, X_2](g \circ f) &= X_1(X_2(g \circ f)) - X_2(X_1(g \circ f)) = X_1(Y_2(g) \circ f) - X_2(Y_1(g) \circ f) \\ &= Y_1(Y_2(g)) \circ f - Y_2(Y_1(g)) \circ f = [Y_1, Y_2](g) \circ f. \end{aligned}$$

□

In particular, if $f : M \to N$ is a diffeomorphism, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, then $X \sim_f Y$ if and only if $X = f^*Y$. From 2.3.16 it then follows that $f^*[Y_1, Y_2] = [f^*Y_1, f^*Y_2]$.

2.3.17 Lemma. *Let M, N be manifolds, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$, $f \in \mathcal{C}^\infty(M,N)$ and $X \sim_f Y$. Then $f \circ \mathrm{Fl}^X_t = \mathrm{Fl}^Y_t \circ f$ for all t such that the left-hand side of this equality is defined.*

Proof. Let $c : I \to M$ be an integral curve of X, so $c'(t) = X(c(t))$ for all $t \in I$. Then

$$(f \circ c)'(t) = T_{c(t)}f(c'(t)) = T_{c(t)}f(X(c(t))) = Y_{f(c(t))},$$

showing that $f \circ c$ is an integral curve of Y. Thus, for all $t \in I$, $f \circ \mathrm{Fl}^X_t = \mathrm{Fl}^Y_t \circ f$.

□

From these preparations we can conclude that the Lie bracket of two vector fields can be seen as an obstruction to the commuting of the corresponding flows. To be precise, we say that the flows of two vector fields $X, Y \in X(M)$ commute if for any $p \in M$ we have: whenever I and J are open intervals containing 0 such that one of the expressions $\mathrm{Fl}^X_t \circ \mathrm{Fl}^Y_s$ or $\mathrm{Fl}^Y_s \circ \mathrm{Fl}^X_t$ is defined for all $(s,t) \in I \times J$, then both are defined and are equal. With this understanding, we have:

2.3.18 Theorem. *Let $X, Y \in \mathfrak{X}(M)$. TFAE:*

(i) $L_X Y = [X, Y] = 0$.

(ii) $(\mathrm{Fl}^X_t)^* Y = Y$*, wherever the left hand side exists.*

(iii) The flows of X and Y commute.

Proof. (i)⇒(ii): By 2.3.13 (ii), $L_X Y = 0$ implies $\frac{d}{dt}(\mathrm{Fl}^X_t)^* Y = 0$, so $t \mapsto (\mathrm{Fl}^X_t)^* Y$ is constant, namely equal to $(\mathrm{Fl}^X_0)^* Y = Y$.

(ii)⇒(i): immediate from 2.3.12.

(ii)⇒(iii): Fix $p \in M$ and let I and J be intervals as described above. In the proof of 2.3.17, replace X by $\tilde{X} := (\mathrm{Fl}^X_t)^* Y$ and f by Fl^X_t. By (ii), $\tilde{X} = Y$, so the assumption on I and J secures that for any $s \in I$, the curve $c(s) := \mathrm{Fl}^{\tilde{X}}_s(p)$ lies in the domain of f, and the proof works although f is not globally defined. Without this assumption it could happen that the integral curve c of $\tilde{X}$ leaves and re-enters the domain of f, leading to undefined intermediate expressions $f \circ c(s)$.[1] Now (the proof of) 2.3.17 yields that for $(s, t) \in I \times J$ we have

$$\mathrm{Fl}^Y_s \circ \mathrm{Fl}^X_t(p) = \mathrm{Fl}^X_t \circ \mathrm{Fl}^{(\mathrm{Fl}^X_t)^* Y}_s(p) = \mathrm{Fl}^X_t \circ \mathrm{Fl}^Y_s(p), \tag{2.3.5}$$

giving (iii).

(iii)⇒(i): Given $p \in M$ and I, J as above, fix any $t \in J$. We first note that for s near 0, the integral curve c of $\tilde{X} := (\mathrm{Fl}^X_t)^* Y$ through p remains inside the domain of Fl^X_t. Thus as in the previous step of the proof we have

$$\mathrm{Fl}^Y_s \circ \mathrm{Fl}^X_t(p) = \mathrm{Fl}^X_t \circ \mathrm{Fl}^{(\mathrm{Fl}^X_t)^* Y}_s(p)$$

By (iii) we can now switch the order of the flows on the left hand side of this equality, which results in

$$\mathrm{Fl}^Y_s(p) = \mathrm{Fl}^{(\mathrm{Fl}^X_t)^* Y}_s(p)$$

for all $s \in I$ sufficiently small. But then differentiating at $s = 0$ gives $Y(p) = (\mathrm{Fl}^X_t)^* Y(p)$ for all $t \in J$. Finally, differentiating with respect to t at $t = 0$ gives (i). □

To conclude this chapter, we note that the Lie bracket of two vector fields can be calculated by differentiating along an infinitesimal prallelepiped traced out by the flows of the vector fields:

2.3.19 Theorem. *Let X, Y be smooth local vector fields on M. Then for any p in the intersection of the domains of X, Y we have:*

$$[X, Y](p) = \frac{d}{dt}\Big|_0 \mathrm{Fl}^Y_{-\sqrt{t}}(\mathrm{Fl}^X_{-\sqrt{t}}(\mathrm{Fl}^Y_{\sqrt{t}}(\mathrm{Fl}^X_{\sqrt{t}}(p))))$$

Proof. For any local smooth function f we have

$$\frac{d}{dt}(\mathrm{Fl}^X_t)^* f(p) = \frac{d}{ds}\Big|_0 (\mathrm{Fl}^X_{t+s})^* f(p) = \frac{d}{ds}\Big|_0 (\mathrm{Fl}^X_t)^* (\mathrm{Fl}^X_s)^* f(p) = (\mathrm{Fl}^X_t)^* (L_X f). \tag{2.3.6}$$

[1] This can lead to actual problems: It can happen that although X, Y commute there are specific values of p, s, and t where both $\mathrm{Fl}^X_t \circ \mathrm{Fl}^Y_s(p)$ and $\mathrm{Fl}^Y_s \circ \mathrm{Fl}^X_t(p)$ exist but are not equal, cf. the discussion preceding Th. 9.44 in [5].

Since

$$\left.\frac{d}{dt}\right|_0 f(\mathrm{Fl}^Y_{-\sqrt{t}}(\mathrm{Fl}^X_{-\sqrt{t}}(\mathrm{Fl}^Y_{\sqrt{t}}(\mathrm{Fl}^X_{\sqrt{t}}(p))))) = Tf\left(\left.\frac{d}{dt}\right|_0 \mathrm{Fl}^Y_{-\sqrt{t}}(\mathrm{Fl}^X_{-\sqrt{t}}(\mathrm{Fl}^Y_{\sqrt{t}}(\mathrm{Fl}^X_{\sqrt{t}}(p))))\right)$$

and $Tf([X,Y])(p) = ([X,Y](f))(p)$, the claim will follow if we can show that for any such f we have

$$\left.\frac{d}{dt}\right|_0 \Big((\mathrm{Fl}^X_{\sqrt{t}})^*(\mathrm{Fl}^Y_{\sqrt{t}})^*(\mathrm{Fl}^X_{-\sqrt{t}})^*(\mathrm{Fl}^Y_{-\sqrt{t}})^* f\Big)(p) = ([X,Y](f))(p).$$

Using (2.3.6) we obtain

$$\begin{aligned}
&\frac{d}{dt}\Big((\mathrm{Fl}^X_{\sqrt{t}})^*(\mathrm{Fl}^Y_{\sqrt{t}})^*(\mathrm{Fl}^X_{-\sqrt{t}})^*(\mathrm{Fl}^Y_{-\sqrt{t}})^* f\Big)\\
&\qquad\qquad =\Big((\mathrm{Fl}^X_{\sqrt{t}})^* L_X((\mathrm{Fl}^Y_{\sqrt{t}})^*(\mathrm{Fl}^X_{-\sqrt{t}})^*(\mathrm{Fl}^Y_{-\sqrt{t}})^* f)\\
&\qquad\qquad\quad + (\mathrm{Fl}^X_{\sqrt{t}})^*(\mathrm{Fl}^Y_{\sqrt{t}})^* L_Y((\mathrm{Fl}^X_{-\sqrt{t}})^*(\mathrm{Fl}^Y_{-\sqrt{t}})^* f)\\
&\qquad\qquad\quad - (\mathrm{Fl}^X_{\sqrt{t}})^*(\mathrm{Fl}^Y_{\sqrt{t}})^*(\mathrm{Fl}^X_{-\sqrt{t}})^* L_X((\mathrm{Fl}^Y_{-\sqrt{t}})^* f)\\
&\qquad\qquad\quad - (\mathrm{Fl}^X_{\sqrt{t}})^*(\mathrm{Fl}^Y_{\sqrt{t}})^*(\mathrm{Fl}^X_{-\sqrt{t}})^*(\mathrm{Fl}^Y_{-\sqrt{t}})^*(L_Y f)\Big)\cdot\frac{1}{2\sqrt{t}} =: \frac{g(\sqrt{t})}{2\sqrt{t}}
\end{aligned}$$

We need to calculate the limit as $t \searrow 0$ of this expression. Now since $g(0) = L_X f + L_Y f - L_X f - L_Y f = 0$, it follows that $\lim_{t\searrow 0}\frac{g(\sqrt{t})}{2\sqrt{t}} = \frac{1}{2}g'(0)$. Again using (2.3.6) we calculate:

$$\begin{aligned}
g'(0) =&\big(L_XL_X + L_X(L_Y - L_X - L_Y) + (L_X + L_Y)L_Y + L_Y(-L_X - L_Y)\\
&- (L_X + L_Y - L_X)L_X + L_XL_Y - L_XL_Y - L_YL_Y + L_XL_Y + L_YL_Y\big)f\\
&= 2(L_XL_Y - L_YL_X)f = 2[X,Y]f,
\end{aligned}$$

giving the claim. $\square$

Chapter 3

Products and Submanifolds

3.1 Products of manifolds

Let $M_1, \dots, M_k$ be smooth manifolds of dimensions $n_1, \dots, n_k$, respectively. Then for any choice of charts (U_i, φ_i), (V_i, ψ_i) in M_i $(1 \le i \le k)$, the transition function

$$(\psi_1 \times \cdots \times \psi_k) \circ (\varphi_1 \times \dots \varphi_k)^{-1} = (\psi_1 \circ \varphi_1^{-1}) \times \cdots \times (\psi_k \circ \varphi_k^{-1})$$

is a smooth diffeomorphism between open sets of $\mathbb{R}^n$, where $n := n_1 + \cdots + n_k$. We thereby obtain an atlas that turns $M_1 \times \dots M_k$ into a smooth manifold, the *product* of the manifolds $M_1, \dots, M_k$. The natural manifold topology on $M_1 \times \dots M_k$ is the product topology of the individual manifold topologies. In particular, it is Hausdorff or second countable if this is true for each M_i.

3.1.1 Lemma. *Let* $\mathrm{pr}_i : M_1 \times \dots M_k \to M_i$ *be the projection map and let* N *be a smooth manifold. Then*

(i) pr_i *is smooth.*

(ii) A map $f : N \to M_1 \times \dots M_k$ *is smooth if and only if* $\mathrm{pr}_i \circ f : N \to M_i$ *is smooth for each* $1 \le i \le k$.

Proof. Both properties are immediate by the definition of the atlas of $M_1 \times \dots M_k$ and 1.2.9. □

It should not come as a surprise that for product manifolds also the corresponding tangent spaces exhibit a natural product structure. To ease notation we formulate the following results for products of two manifolds, but of course they hold analogously for arbitrarily many factors.

3.1.2 Proposition. *Let* $p_i \in M_i$ $(i = 1, 2)$. *The map*

$$\begin{aligned} \Phi : T_{(p_1,p_2)}(M_1 \times M_2) &\to T_{p_1}M_1 \times T_{p_2}M_2 \\ v &\mapsto (T_{(p_1,p_2)}\mathrm{pr}_1(v), T_{(p_1,p_2)}\mathrm{pr}_2(v)) \end{aligned}$$

is a linear isomorphism that canonically identifies its domain and target space.

Proof. Since Φ is a linear map between vector spaces of equal dimension it suffices to show that it is surjective. Thus let $(v_1, v_2) \in T_{p_1}M_1 \times T_{p_2}M_2$. Then by 2.1.6

there exist smooth curves $c_i : I \to M_i$ from some interval I in $\mathbb{R}$ into M_i with $v_i = [c_i]_{p_i}$ (i=1,2). The curve $c := t \mapsto (c_1(t), c_2(t))$ is smooth into $M_1 \times M_2$ by 3.1.1, so $v := [c]_{(p_1,p_2)} \in T_{(p_1,p_2)}(M_1 \times M_2)$. Finally, 2.1.7 shows that

$$\Phi([c]_{(p_1,p_2)}) = (T_{(p_1,p_2)}\mathrm{pr}_1(v), T_{(p_1,p_2)}\mathrm{pr}_2(v)) = ([c_1]_{p_1}, [c_2]_{p_2}) = (v_1, v_2),$$

giving the claim. □

Finally, we note the following Leibnitz rule:

3.1.3 Proposition. *Let $f : M_1 \times M_2 \to N$ be smooth, $p_i \in M_i$, and denote by $f_{p_1} : M_2 \to N$, $p \mapsto f(p_1, p)$, and $f_{p_2} : M_1 \to N$, $p \mapsto f(p, p_2)$ the corresponding partial maps. Then for $(v_1, v_2) \in T_{(p_1,p_2)}(M_1 \times M_2) \cong T_{p_1}M_1 \times T_{p_2}M_2$ we have*

$$T_{(p_1,p_2)}f(v_1, v_2) = T_{p_1}f_{p_2}(v_1) + T_{p_2}f_{p_1}(v_2).$$

Proof. We have

$$T_{(p_1,p_2)}f(v_1, v_2) = T_{(p_1,p_2)}f(v_1, 0) + T_{(p_1,p_2)}f(0, v_2).$$

By 2.1.6 there is a smooth curve $c_1 : I \to M_1$ with $v_1 = [c_1]_{p_1}$, and we set $c : I \to M_1 \times M_2$, $c(t) := (c_1(t), p_2)$. Then c is smooth and $[c]_{(p_1,p_2)} = ([c_1]_{p_1}, 0) = (v_1, 0)$, so again by 2.1.7 we obtain

$$\begin{aligned} T_{(p_1,p_2)}f(v_1, 0) = T_{(p_1,p_2)}f([c]_{(p_1,p_2)}) &= [f \circ c]_{f(p_1,p_2)} = [f_{p_2} \circ c_1]_{f_{p_2}(p_1)} \\ &= (T_{p_1}f_{p_2})([c_1]_{p_1}) = T_{p_1}f_{p_2}(v_1), \end{aligned}$$

and analogously for the second summand. □

3.2 Application: Time-dependent vector fields

In Section 2.3 we considered global versions of *autonomous* ODEs, of the form $c'(t) = X(c(t))$, where $X \in \mathfrak{X}(M)$. In ODE theory and many applications, one is also interested in understanding *non-autonomous* ODEs of the form

$$\begin{aligned} c'(t) &= X(t, c(t)) \\ c(0) &= p. \end{aligned}$$

Here we want to derive the basic properties of such systems in the manifold setting, basically by reducing the problem to an autonomous ODE on a higher dimensional manifold, namely on the product $\mathbb{R} \times M$.

Let M be a smooth manifold and let $I \subseteq \mathbb{R}$ be an open interval. A *time-dependent vector field* is a smooth map $X : I \times M \to TM$ such that $X(t, p) \in T_pM$ for each $(t, p) \in I \times M$. Thus for each $t \in I$ the map $X_t : M \to TM$, $X_t(p) := X(t, p)$ is a smooth vector field on M, i.e., belongs to $\mathfrak{X}(M)$. Given a time-dependent vector field X on M, an *integral curve* of X is a smooth curve $c : I_0 \to M$, where I_0 is an open interval contained in I such that

$$c'(t) = X(t, c(t)) \qquad (t \in I_0).$$

Any $X \in \mathfrak{X}(M)$ induces a time-dependent vector field on M by simply setting $X(t, p) := X_p$.

We want to develop an appropriate notion of flow for time-dependent vector fields. This requires some care because in the current situation two integral curves that meet in one point but do so at different times need not coincide. Nonetheless we can employ 2.3.3 to obtain a satisfactory solution to our problem:

3.2.1 Theorem. *Let $X : I \times M \to TM$ be a time-dependent vector field on M. Then there exists an open subset $W \subseteq I \times I \times M$ and a smooth map $\Psi : W \to M$, called the* time-dependent flow *of X satisfying:*

(i) For each $t_0 \in I$ and each $p \in M$ the set $W^{(t_0,p)} := \{t \in I \mid (t, t_0, p) \in W\}$ is an open interval around t_0 and the smooth curve $\Psi^{(t_0,p)} : W^{(t_0,p)} \to M$, $\Psi^{(t_0,p)}(t) := \Psi(t, t_0, p)$ is the unique maximal integral curve of X with initial condition $\Psi^{(t_0,p)}(t_0) = p$.

(ii) If $t_1 \in W^{(t_0,p)}$ and $q = \Psi^{(t_0,p)}(t_1)$, then $W^{(t_1,q)} = W^{(t_0,p)}$ and $\Psi^{(t_1,q)} = \Psi^{(t_0,p)}$.

(iii) For each $(t_1, t_0) \in I \times I$ the set $M_{t_1,t_0} := \{p \in M \mid (t_1, t_0, p) \in W\}$ is open in M, and the map

$$\begin{aligned} \Psi_{t_1,t_0} : M_{t_1,t_0} &\to M \\ p &\mapsto \Psi(t_1, t_0, p) \end{aligned}$$

is a diffeomorphism from M_{t_1,t_0} onto M_{t_0,t_1} with inverse Ψ_{t_0,t_1}.

(iv) If $p \in M_{t_1,t_0}$ and $\Psi_{t_1,t_0}(p) \in M_{t_2,t_1}$, then $p \in M_{t_2,t_0}$ and

$$\Psi_{t_2,t_1} \circ \Psi_{t_1,t_0}(p) = \Psi_{t_2,t_0}(p). \tag{3.2.1}$$

Proof. Consider the smooth vector field $\tilde{X}$ on $I \times M$ defined by

$$\tilde{X}_{(s,p)} := \Big(\frac{\partial}{\partial s}\Big|_s, X(s, p)\Big),$$

where s is the standard (identity) coordinate function on $I \subseteq \mathbb{R}$ and we use 3.1.2 to identify $T_{(s,p)}(I \times M)$ with $T_sI \times T_pM$. Since $\tilde{X} \in \mathfrak{X}(I \times M)$, by 2.3.3 it possesses a smooth flow $\mathrm{Fl}^{\tilde{X}} : \tilde{U} \to I \times M$, where the open subset $\tilde{U}$ of $I \times I \times M$ is the maximal domain of $\mathrm{Fl}^{\tilde{X}}$ from 2.3.3. Let us write $\mathrm{Fl}^{\tilde{X}}$ in components:

$$\mathrm{Fl}^{\tilde{X}}(t, (s, p)) =: (\alpha(t, (s, p)), \beta(t, (s, p))) \in I \times M.$$

Then $\alpha : \tilde{U} \to I$ and $\beta : \tilde{U} \to M$ satisfy the initial value problem

$$\begin{aligned} &\frac{\partial\alpha}{\partial t}(t, (s, p)) = 1, \qquad \alpha(0, (s, p)) = s \\ &\frac{\partial\beta}{\partial t}(t, (s, p)) = X(\alpha(t, (s, p)), \beta(t, (s, p))), \qquad \beta(0, (s, p)) = p. \end{aligned}$$

Here, we simply write $1 \in \mathbb{R} \cong T_{\alpha(t,(s,p))}\mathbb{R}$ instead of $\frac{\partial}{\partial s}\big|_{\alpha(t,(s,p))}$.

Unique solvability of this system dictates that $\alpha(t, (s, p)) = t + s$, which in turn implies that

$$\frac{\partial\beta}{\partial t}(t, (s, p)) = X(t + s, \beta(t, (s, p))). \tag{3.2.2}$$

Now we set

$$W := \{(t, t_0, p) \mid (t - t_0, (t_0, p)) \in \tilde{U}\},$$

which is open in $I \times I \times M$ as the inverse image of the open set $\tilde{U}$ under a continuous map. Since $\alpha(\tilde{U}) \subseteq I$, if $(t, t_0, p) \in W$ then $t = \alpha(t - t_0, (t_0, p)) \in I$, hence $W \subseteq I \times I \times M$. Also, openness of M_{t_1,t_0} follows from that of W. We now set

$$\begin{aligned} \Psi &: W \to M \\ \Psi(t, t_0, p) &:= \beta(t - t_0, (t_0, p)). \end{aligned}$$

Then Ψ is smooth and (3.2.2) precisely says that $\Psi^{(t_0,p)} = t \mapsto \Psi(t,t_0,p)$ is an integral curve of X with initial value $\Psi^{(t_0,p)}(t_0) = p$. To see uniqueness, suppose that $\gamma : I_0 \to M$ is another integral curve of X defined on an open interval $I_0 \subseteq I$ with $t_0 \in I_0$ such that $\gamma(t_0) = p$. Then let $\tilde{\gamma} : I_0 \to I \times M$ be the smooth curve $\tilde{\gamma}(t) := (t, \gamma(t))$. This curve is an integral curve of $\tilde{X}$ with initial value $\tilde{\gamma}(t_0) = (t_0, p)$, hence by uniqueness and maximality of the integral curves of $\tilde{X}$ (see 2.3.3) we conclude that

$$(t, \gamma(t)) = \tilde{\gamma}(t) = \mathrm{Fl}^{\tilde{X}}_{t-t_0}((t_0,p)) = (\alpha(t-t_0,(t_0,p)), \beta(t-t_0,(t_0,p))) = (t, \Psi^{(t_0,p)}(t)),$$

on its entire domain. Consequently, I_0 is contained in the domain of $\Psi^{(t_0,p)}$ and $\gamma = \Psi^{(t_0,p)}$ on I_0. This shows that $\Psi^{(t_0,p)}$ is indeed the unique maximal integral curve of X passing through p at $t = t_0$, finishing the proof of (i).

(ii) Let $t_1 \in W^{(t_0,p)}$ and set $q := \Psi^{(t_0,p)}(t_1)$. Then by (i), both $\Psi^{(t_1,q)}$ and $\Psi^{(t_0,p)}$ are integral curves of X that pass through q at $t = t_1$. Uniqueness and maximality then imply that they are defined on the same domain and are equal there.

(iv) Let $p \in M_{(t_1,t_0)}$ and suppose that $\Psi_{t_1,t_0}(p) \in M_{t_2,t_1}$. Set $q := \Psi_{t_1,t_0}(p) = \Psi^{(t_0,p)}(t_1)$. By assumption, $q = \Psi_{t_1,t_0}(p) \in M_{t_2,t_1}$, so $(t_2,t_1,q) \in W$ and hence $t_2 \in W^{(t_1,q)}$. By (ii) we have $W^{(t_1,q)} = W^{(t_0,p)}$ and $\Psi^{(t_1,q)}(t_2) = \Psi^{(t_0,p)}(t_2)$. Consequently, $t_2 \in W^{(t_0,p)}$, i.e., $(t_2,t_0,p) \in W$, i.e., $p \in M_{t_2,t_0}$. Hence we can insert (t_2,t_0,p) into Ψ and calculate as follows:

$$\begin{aligned}\Psi_{t_2,t_0}(p) &= \Psi(t_2,t_0,p) = \Psi^{(t_0,p)}(t_2) = \Psi^{(t_1,q)}(t_2) = \Psi(t_2,t_1,q) \\ &= \Psi_{t_2,t_1}(q) = \Psi_{t_2,t_1}(\Psi_{t_1,t_0}(p)).\end{aligned}$$

(iii) Let $(t_1,t_0) \in I \times I$, $p \in M_{t_1,t_0}$ and set $q := \Psi_{t_1,t_0}(p)$. By (ii) we know that $t_0 \in W^{(t_0,p)} = W^{(t_1,q)}$, i.e., $(t_0,t_1,q) \in W$, i.e., $q \in M_{t_0,t_1}$. This shows that $\Psi_{t_1,t_0}(M_{t_1,t_0}) \subseteq M_{t_0,t_1}$. But this argument is symmetric in t_0, t_1, so we also get $\Psi_{t_0,t_1}(M_{t_0,t_1}) \subseteq M_{t_1,t_0}$. Hence by (iv) we conclude that

$$\Psi_{t_1,t_0} \circ \Psi_{t_0,t_1}(p) = \Psi_{t_1,t_1}(p) = p$$

for all $p \in M_{t_0,t_1}$. Again by symmetry we also obtain $\Psi_{t_0,t_1} \circ \Psi_{t_1,t_0} = \mathrm{id}_{M_{t_1,t_0}}$. □

3.3 Submanifolds

In Section 1.1 we studied submanifolds of $\mathbb{R}^n$: $M \subseteq \mathbb{R}^n$ is called a submanifold of dimension k if for every $p \in M$ there exists an open neighborhood W of p in $\mathbb{R}^n$, an open subset U of $\mathbb{R}^k$ and an immersion $\varphi : U \to \mathbb{R}^n$ such that $\varphi : U \to \varphi(U)$ is a homeomorphism and $\varphi(U) = W \cap M$. Then φ is called a local parametrization of M. By 1.1.8, any such M is an abstract manifold whose natural manifold topology is precisely the trace topology of $\mathbb{R}^n$ on M. We now want to introduce appropriate notions of submanifolds for abstract manifolds in general. To this end we first need a few results on smooth maps between manifolds.

3.3.1 Definition. *Let M, N be manifolds and let $f : M \to N$ be smooth. The rank $\mathrm{rk}_p(f)$ of f at $p \in M$ is the rank of the linear map $T_pf : T_pM \to T_{f(p)}N$. If $\varphi = (x^1, \ldots, x^m)$ is a chart of M at p and $\psi := (y^1, \ldots, y^n)$ a chart of N at $f(p)$, then the matrix of $T_pf : T_pM \to T_{f(p)}N$ with respect to the bases $(\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p)$ of T_pM and $(\frac{\partial}{\partial y^1}\Big|_{f(p)}, \ldots, \frac{\partial}{\partial y^n}\Big|_{f(p)})$ of $T_{f(p)}N$ is the Jacobi matrix of $\psi \circ f \circ \varphi^{-1}$ at $\varphi(p)$ (see (2.1.5)). Thus $\mathrm{rk}_p(f) = \mathrm{rk}_{\varphi(p)} \psi \circ f \circ \varphi^{-1}$.*

3.3.2 Definition. *Let $f : M \to N$ be smooth. f is called immersion (submersion) if $T_p f$ is injective (surjective) for every $p \in M$.*

If $\dim(M) = m$ and $\dim(N) = n$ (which, as before, we will indicate by writing M^m and N^n, respectively) then f is an immersion (resp. submersion) if and only if $\mathrm{rk}_p(f) = m$ (resp. $= n$) for all $p \in M$. The following result shows that maps of constant rank locally always are of a particularly simple form.

3.3.3 Theorem. *(Rank Theorem) Let M^m, N^n be manifolds and let $f : M \to N$ be smooth. Let $p \in M$ and suppose that $\mathrm{rk}_q(f) = k$ for q in a neighborhood of p. Then there exist charts (φ, U) of M at p and (ψ, V) of N at $f(p)$ such that $\varphi(p) = 0 \in \mathbb{R}^m$, $\psi(f(p)) = 0 \in \mathbb{R}^n$ and*

$$\psi \circ f \circ \varphi^{-1}(x^1, \dots, x^m) = (x^1, \dots, x^k, 0, \dots, 0).$$

Proof. By the above, the rank of f is independent of the chosen charts, so without loss of generality we may assume that $f : W \to W'$, where W is open in $\mathbb{R}^m$ and W' is open in $\mathbb{R}^n$, $p = 0$, $f(p) = 0$ and $\mathrm{rk}(f) \equiv k$ on W. Since $\mathrm{rk}(Df(0)) = k$ there exists an invertible $k \times k$ submatrix of $Df(0)$ and without loss of generality we may assume that this matrix is given by $(\frac{\partial f^i}{\partial x^j})_{i,j=1}^k$. Now consider the smooth map $\varphi : W \to \mathbb{R}^m$,

$$\varphi(x^1, \dots, x^m) = (f^1(x^1, \dots, x^m), \dots, f^k(x^1, \dots, x^m), x^{k+1}, \dots, x^m).$$

Then $\varphi(0) = 0$ and

$$D\varphi(0) = \begin{pmatrix} \left(\frac{\partial f^i}{\partial x^j}\right)_{i,j=1}^k & * \\ 0 & I_{m-k} \end{pmatrix}$$

is invertible. By the inverse function theorem φ thereby is a diffeomorphism from some open neighborhood $W_1 \subseteq W$ of 0 onto some open neighborhood U_1 of 0 in $\mathbb{R}^m$. Then on U_1 we have

$$f \circ \varphi^{-1}(x) = f \circ \varphi^{-1}(x^1, \dots, x^k, x^{k+1}, \dots, x^m) = (x^1, \dots, x^k, \bar{f}^{k+1}(x), \dots, \bar{f}^n(x))$$

for suitable smooth functions $\bar{f}^{k+1}, \dots, \bar{f}^n$. Consequently,

$$D(f \circ \varphi^{-1})(x) = \begin{pmatrix} I_k & 0 \\ * & \left(\frac{\partial \bar{f}^r}{\partial x^s}\right)_{\substack{r=k+1,\dots,n \\ s=k+1,\dots,m}} \end{pmatrix}.$$

Since $D(f \circ \varphi^{-1}) = Df \circ D\varphi^{-1}$ and $D\varphi^{-1}$ is bijective it follows that $\mathrm{rk}(D(f \circ \varphi^{-1})) = \mathrm{rk}(Df) \equiv k$ on U_1. Then necessarily $\frac{\partial \bar{f}^r}{\partial x^s} = 0$ for $r = k+1, \dots, n$ and $s = k+1, \dots, m$, i.e., $\bar{f}^{k+1}, \dots, \bar{f}^n$ depend only on $x^1, \dots, x^k$. Now set

$$T(y^1, \dots, y^k, y^{k+1}, \dots, y^n) := \\ \left(y^1, \dots, y^k, y^{k+1} + \bar{f}^{k+1}(y^1, \dots, y^k), \dots, y^n + \bar{f}^n(y^1, \dots, y^k)\right).$$

Then $T(0) = 0$ and

$$DT(y) = \begin{pmatrix} I_k & 0 \\ * & I_{n-k} \end{pmatrix},$$

so T is a diffeomorphism from some open neighborhood $\tilde{V}$ of 0 in $\mathbb{R}^n$ onto some open $0 \in V \subseteq W'$. Choose $\tilde{U} \subseteq U_1$ open such that $f \circ \varphi^{-1}(\tilde{U}) \subseteq V$ and let $U := \varphi^{-1}(\tilde{U})$. Let $\psi := T^{-1}$, then

$$\tilde{U} \xrightarrow{\varphi^{-1}} U \xrightarrow{f} V \xrightarrow{\psi} \tilde{V}$$

and

$$\psi \circ f \circ \varphi^{-1}(x^1, \dots, x^k, x^{k+1}, \dots, x^m) = \\ \psi(x^1, \dots, x^k, \bar{f}^{k+1}(x^1, \dots, x^k), \dots, \bar{f}^n(x^1, \dots, x^k)) = (x^1, \dots, x^k, 0, \dots, 0)$$

on $\tilde{U}$. □

3.3.4 Lemma. *Let $f : M^m \to N^n$ be smooth, let $p \in M$ and suppose that $\mathrm{rk}_p(f) = k$. Then there exists a neighborhood U of p in M such that $\mathrm{rk}_q(f) \geq k$ for all $q \in U$. In particular, if $k = \min(m, n)$ then $\mathrm{rk}_q(f) = k$ for all $q \in U$.*

Proof. Picking charts φ around p and ψ around $f(p)$, $\mathrm{rk}_p(f) = k$ if and only if there exists a $k \times k$-submatrix of $D(\psi \circ f \circ \varphi^{-1})(\varphi(p))$ with nonzero determinant. By continuity, the same is then true on an entire neighborhood of p. This means that the rank cannot drop locally. If $k = \min(m, n)$ then it also cannot increase. □

3.3.5 Theorem. (Inverse function theorem) *Let $f : M^m \to N^m$ be smooth, let $p \in M$ and suppose that $T_p f : T_p M \to T_{f(p)} N$ is bijective. Then there exist open neighborhoods U of p in M and V of $f(p)$ in N such that $f : U \to V$ is a diffeomorphism.*

Proof. For charts φ of M at p, and ψ at $f(p)$ in N the map $D(\psi \circ f \circ \varphi^{-1})(\varphi(p)) = T_{f(p)}\psi \circ T_p f \circ T_{\varphi(p)}\varphi^{-1}$ is invertible. Hence by the classical inverse function theory, $\psi \circ f \circ \varphi^{-1}$ is a diffeomorphism around $\varphi(p)$ and the claim follows. □

3.3.6 Proposition. (Local characterization of immersions) *Let $f : M^m \to N^n$ be smooth and let $p \in M$. TFAE:*

(i) *$T_p f$ is injective.*

(ii) $\mathrm{rk}_p(f) = m$.

(iii) *If $\psi = (\psi^1, \dots, \psi^n)$ is a chart at $f(p)$ in N then there exist $1 \leq i_1 < \dots < i_m \leq n$ such that $(\psi^{i_1} \circ f, \dots, \psi^{i_m} \circ f)$ is a chart at p in M.*

Proof. Clearly, (i)⇔ (ii).
(ii)⇒(iii): Let φ be a chart at p in M. Then $\mathrm{rk}(D(\psi \circ f \circ \varphi^{-1})(\varphi(p))) = m$, hence there exist $1 \leq i_1 < \dots < i_m \leq n$ with $\det D((\psi^{i_1}, \dots, \psi^{i_m}) \circ f \circ \varphi^{-1})(\varphi(p)) \neq 0$. By 3.3.5, then, $(\psi^{i_1} \circ f, \dots, \psi^{i_m} \circ f)$ is a diffeomorphism locally around p, hence a chart.
(iii)⇒(ii): The linear map $D((\psi^{i_1}, \dots, \psi^{i_m}) \circ f \circ \varphi^{-1})(\varphi(p))$ is bijective, so $\mathrm{rk}(D(\psi \circ f \circ \varphi^{-1})(\varphi(p))) = m$. □

3.3.7 Proposition. (Local characterization of submersions) *Let $f : M^m \to N^n$ be smooth and let $p \in M$. TFAE:*

(i) *$T_p f$ is surjective.*

(ii) $\mathrm{rk}_p(f) = n$.

(iii) If $\psi = (\psi^1, \dots, \psi^n)$ is any chart at $f(p)$ in N then there exists a chart φ of M at p such that $(\psi^1 \circ f, \dots \psi^n \circ f, \varphi^{n+1}, \dots, \varphi^m)$ is a chart at p in M.

Proof. Again, (i)⇔(ii) is obvious.
(ii)⇒(iii): Let $\tilde\varphi$ and ψ be charts at p and $f(p)$, respectively. Since $\mathrm{rk}(D(\psi \circ f \circ \tilde\varphi^{-1})(\tilde\varphi(p))) = n$, the Jacobi matrix $D(\psi \circ f \circ \tilde\varphi^{-1})(\tilde\varphi(p))$ possesses n linearly independent columns. By permuting the coordinates of $\tilde\varphi$ we obtain a chart φ such that the first n columns of $D(\psi \circ f \circ \varphi^{-1})(\varphi(p))$ are linearly independent. Now set $\chi := (\psi^1 \circ f, \dots, \psi^n \circ f, \varphi^{n+1}, \dots, \varphi^m)$. Then

$$D(\chi \circ \varphi^{-1})(\varphi(p)) = \begin{pmatrix} \left(\frac{\partial \psi^i \circ f \circ \varphi^{-1}}{\partial x^j}(\varphi(p))\right)_{i,j=1}^n & * \\ 0 & I_{m-n} \end{pmatrix} \tag{3.3.1}$$

Hence, by 3.3.5, $\chi \circ \varphi^{-1}$ is a diffeomorphism around $\varphi(p)$, and so χ is a chart at p.
(iii)⇒(ii): Since $\mathrm{rk}(D(\chi \circ \varphi^{-1})(\varphi(p))) = m$, (3.3.1) implies that $\mathrm{rk}(D(\psi \circ f \circ \varphi^{-1})(\varphi(p))) = n$. □

3.3.8 Proposition. *Let M^m, N^n, R^r be manifolds, $f : M \to N$ continuous and $g : N \to R$ an immersion. If $g \circ f$ is smooth then so is f.*

Proof. Given $p \in M$, by 3.3.3 we may choose charts (φ, U) around $f(p)$ in N, and (ψ, V) around $g(f(p))$ in R such that

$$g_{\psi\varphi} := \psi \circ g \circ \varphi^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0). \tag{3.3.2}$$

Let and (χ, W) be a chart in M around p and set $f_{\varphi\chi} := \varphi \circ f \circ \chi^{-1}$.

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{g} & R \\ \chi\downarrow & & \downarrow\varphi & & \downarrow\psi \\ \mathbb{R}^m & \xrightarrow{f_{\varphi\chi}} & \mathbb{R}^n & \xrightarrow{g_{\psi\varphi}} & \mathbb{R}^r \end{array}$$

Then $\psi\circ(g\circ f)\circ\chi^{-1}$ is defined on $\chi((g\circ f)^{-1}(V)\cap W)$, $f_{\varphi\chi}$ is defined on $\chi(f^{-1}(U)\cap W)$, and $g_{\psi\varphi}$ is defined on $\varphi(g^{-1}(V) \cap U)$. It follows that $g_{\psi\varphi} \circ f_{\varphi\chi}$ is defined on

$$\begin{aligned} \chi(f^{-1}(U) \cap W) \cap f_{\varphi\chi}^{-1}(\varphi(g^{-1}(V) \cap U)) &= \chi(f^{-1}(U) \cap W) \cap \chi(f^{-1}(g^{-1}(V) \cap U)) \\ &= \chi(f^{-1}(g^{-1}(V)) \cap f^{-1}(U) \cap W) \end{aligned}$$

Since f is continuous, this shows that $g_{\psi\varphi} \circ f_{\varphi\chi}$ is a restriction of $\psi \circ (g \circ f) \circ \chi^{-1}$ to an open set, hence is smooth. By (3.3.2), $(g_{\psi\varphi} \circ f_{\varphi\chi})^i = f^i_{\varphi\chi}$ for $1 \le i \le n$, hence $f_{\varphi\chi}$ is smooth. Thus, finally, f is smooth. □

3.3.9 Proposition. *Let M^m, N^n, R^r be manifolds, $f : M \to N$ a surjective submersion and $g : N \to R$ arbitrary. If $g \circ f$ is smooth then so is g.*

Proof. Using the same notations as in the proof of 3.3.8, by 3.3.3 we may choose the charts (χ, W) around p and (φ, U) around $f(p)$ in such a way that $f_{\varphi\chi} = \varphi\circ f\circ\chi^{-1} = (x^1, \dots, x^m) \mapsto (x^1, \dots, x^n)$. As in the proof of 3.3.8, $g_{\psi\varphi} \circ f_{\varphi\chi}$ is a restriction of $\psi\circ(g\circ f)\circ\chi^{-1}$ to an open set, hence is smooth. Thus $(x^1, \dots, x^m) \mapsto g_{\psi\varphi}(x^1, \dots, x^n)$ and thereby $g_{\psi\varphi}$ itself is smooth, which implies smoothness of g. □

After these preparations we are now ready to introduce the notion of submanifold of an abstract manifold.

3.3.10 Definition. *Let M^m and N^n be manifolds with $N \subseteq M$ and denote by $j : N \hookrightarrow M$ the inclusion map. N is called an* immersive submanifold *of M if j is an immersion. N is called a* submanifold *(or a* regular submanifold*, or an* embedded *submanifold), if it is an immersive submanifold and in addition N is a topological subspace of M, i.e., if the natural manifold topology of N is the trace topology of the natural manifold topology on M.*

This definition is a natural generalization of the notion of submanifold of $\mathbb{R}^n$, cf. 1.1.5. The figure-eight manifold from 1.1.5 (with atlas $\{N, j^{-1}\}$) is an example of an immersive submanifold that is not a regular submanifold.

3.3.11 Remark. If N is a submanifold of M then for each $p \in N$, the map $T_pj : T_pN \to T_pM$ is injective. Hence $T_pj(T_pN)$ is a subspace of T_pM that is isomorphic to T_pN. We will therefore henceforth identify $T_pj(T_pN)$ with T_pN and notationally suppress the map T_pj, i.e., we will consider T_pN directly as a subspace of T_pM.

3.3.12 Theorem. *Let N^n be an immersive submanifold of M^m. TFAE:*

(i) N is a submanifold of M (i.e., N carries the trace topology of M).

(ii) Around any $p \in N$ there exists an adapted coordinate system*, i.e., for every $p \in N$ there exists a chart (φ, U) around p in M such that $\varphi(p) = 0$, $\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\})$ (with $0 \in \mathbb{R}^{m-n}$) and such that $\varphi|_{U \cap N}$ is a chart of N around p.*

(iii) Every $p \in N$ possesses a neighborhood basis $\mathcal{U}$ in M such that $U \cap N$ is connected in N for every $U \in \mathcal{U}$.

Proof. (i)⇒(ii): Let $p \in N$. By assumption, $j : N \hookrightarrow M$ is an immersion. Thus by 3.3.3 there exist charts (ψ, V) around p in N and $(\varphi, \tilde{U})$ around $j(p) = p$ in M, with $\varphi(p) = 0$, such that

$$\varphi \circ j \circ \psi^{-1} = (x^1, \dots, x^n) \mapsto (x^1, \dots, x^n, 0, \dots, 0).$$

The domain of $\varphi \circ j \circ \psi^{-1}$ is $\psi(V \cap j^{-1}(\tilde{U}))$. Since j is continuous, $j^{-1}(\tilde{U})$ is open in N. Shrinking V to $V \cap j^{-1}(\tilde{U})$ if necessary, we can assume w.l.o.g. that $V \subseteq j^{-1}(\tilde{U}) (= \tilde{U} \cap N)$. The domain of definition of $\varphi \circ j \circ \psi^{-1}$ then is $\psi(V)$. By (i) there exists some open subset W of M such that $V = W \cap N$ and without loss of generality we may assume that $W = \tilde{U}$ (otherwise replace both $\tilde{U}$ and W by $\tilde{U} \cap W$). Then $V = \tilde{U} \cap N$.

Denote by $\mathrm{pr}_1 : \mathbb{R}^m \to \mathbb{R}^n$ the projection map. We have

$$\varphi(V) = \varphi(j(V)) = \varphi \circ j \circ \psi^{-1}(\psi(V)) = \psi(V) \times \{0\},$$

so $\mathrm{pr}_1(\varphi(V)) = \psi(V)$, which is open in $\mathbb{R}^n$. Hence the set

$$U := \varphi^{-1}((\mathrm{pr}_1(\varphi(V)) \times \mathbb{R}^{m-n}) \cap \varphi(\tilde{U}))$$

is open in M and contains p. It follows that (φ, U) is a chart of M around p and we claim that $\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\})$.

To see '$\subseteq$', note that obviously $\varphi(U \cap N) \subseteq \varphi(U)$ and $U \cap N \subseteq \tilde{U} \cap N = V$, so $\varphi(U \cap N) \subseteq \varphi(V) \subseteq \mathbb{R}^n \times \{0\}$. Conversely,

$$\varphi(U) \cap (\mathbb{R}^n \times \{0\}) = (\mathrm{pr}_1(\varphi(V)) \times \{0\}) \cap \varphi(\tilde{U}) = (\psi(V) \times \{0\}) \cap \varphi(\tilde{U})$$

Now let $\varphi(u) \in \varphi(U) \cap (\mathbb{R}^n \times \{0\})$. Then for some $v \in V$ we have

$$\varphi(u) = (\psi(v), 0) = \varphi \circ j \circ \psi^{-1}(\psi(v)) = \varphi(j(v)) = \varphi(v),$$

so $u = v \in V \subseteq N$ and thereby $\varphi(u) \in \varphi(U \cap N)$.

Finally, $\varphi|_{U\cap N}$ is a chart of N around p since $U \cap N = j^{-1}(U)$ is an open neighborhood of p in N and

$$\begin{aligned}\varphi|_{U\cap N} \circ \psi^{-1} = \varphi|_{U\cap N} \circ j \circ \psi^{-1} &= \varphi \circ j \circ (\psi|_{U\cap N})^{-1} \\ &= (x^1, \dots, x^n) \mapsto (x^1, \dots, x^n, 0, \dots, 0).\end{aligned}$$

Identifying $\mathbb{R}^n \times \{0\}$ with $\mathbb{R}^n$, this latter map is the identity on $\mathbb{R}^n$, so $\varphi|_{U\cap N} = \psi|_{U\cap N}$, hence it is a chart.

(ii)⇒(iii): Let (φ, U) be a chart as in (ii). Pick $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(0) \subseteq \varphi(U)$ and let $U_\varepsilon := \varphi^{-1}(B_\varepsilon(0))$ for $\varepsilon < \varepsilon_0$. Then $\mathcal{U} := \{U_\varepsilon \mid \varepsilon < \varepsilon_0\}$ is a neighborhood basis of p in M and

$$\varphi(U_\varepsilon \cap N) = \varphi(U_\varepsilon \cap U \cap N) = B_\varepsilon(0) \cap \varphi(U \cap N) = B_\varepsilon(0) \cap (\mathbb{R}^n \times \{0\})$$

is connected in $\mathbb{R}^n$. Thus $\mathcal{U}$ serves the desired purpose.

(iii)⇒(i): Denote by $\mathcal{T}_M$ and $\mathcal{T}_N$ the topologies on M and N, respectively. Since $j : N \hookrightarrow M$ is continuous, for every $W \in \mathcal{T}_M$ we get $j^{-1}(W) = W \cap N \in \mathcal{T}_N$, so $\mathcal{T}_M|_N \leq \mathcal{T}_N$. Conversely we will show that any $\mathcal{T}_N$-neighborhood of any $p \in N$ is also a $\mathcal{T}_M|_N$-neighborhood of p. To this end let $p \in N$ and let U be a neighborhood of p in N that is homeomorphic to a closed ball in $\mathbb{R}^n$ (e.g. the inverse image of such a ball under a chart). Then ∂U is compact in N, so also $j(\partial U) = \partial U$ is compact in M (since j is continuous). Since $p \in U^\circ$, $p \notin \partial U$ and so by (iii) there exists some $V \in \mathcal{U}$ with $V \cap \partial U = \emptyset$. If we can show that $V \cap N \subseteq U$ then we are done since $V \cap N$ is a neighborhood of p in $\mathcal{T}_M|_N$. Assume, therefore, that $V \cap N \not\subseteq U$. This means that $(V\cap N)\cap(N\setminus U) \neq \emptyset$. Thus $V\cap N$ is connected and $(p \in)(V\cap N)\cap U \neq \emptyset$ as well as $(V \cap N) \cap (N \setminus U) \neq \emptyset$. But this implies $(V \cap N) \cap \partial U \neq \emptyset$ and thereby $V \cap \partial U \neq \emptyset$, a contradiction. □

3.3.13 Remark. (i) For $M = \mathbb{R}^m$, condition (ii) from 3.3.12 is precisely (T) from 1.1.8 (local trivialization). Therefore, submanifolds of $\mathbb{R}^m$ in the sense of Section 1.1 are exactly submanifolds of $\mathbb{R}^m$ in the sense of 3.3.10.

(ii) Consider the subset N of $\mathbb{R}^2$ that consists of the interval $(-1, 1)$ on the y-axis, plus the graph of $\sin(1/x)$ between $x = 0$ and $x = 1$. Then N is an immersive submanifold of $\mathbb{R}^2$ that is not a submanifold due to 3.3.12 (iii): in fact, any ball around $(0, 0)$ of radius less than 1 intersects N in a non-connected set.

3.3.14 Proposition. *Let N be a submanifold of M and let $f : P \to M$ be smooth and such that $f(P) \subseteq N$. Then also $f : P \to N$ is smooth.*

Proof. Since N carries the trace topology of M and $f : P \to M$ is continuous, also $f : P \to N$ is continuous. Also, $j : N \hookrightarrow M$ is an immersion and by assumption $j \circ f$ is smooth. The claim therefore follows from 3.3.8. □

3.3.15 Corollary. *Let M be a manifold and let N be a subset of M. Then N can be endowed with the structure of a submanifold of M in at most one way.*

Proof. By definition, N has to carry the trace topology of M. Suppose that there are two differentiable structures that make N a submanifold of M and denote N with these structures by N_1, N_2. Since $j : N_i \to M$ is smooth for $i = 1, 2$, 3.3.14 shows that both $\mathrm{id} : N_1 \to N_2$ and $\mathrm{id} : N_2 \to N_1$ are smooth. Hence $\mathrm{id} : N_1 \to N_2$ is a diffeomorphism and so the differentiable structures on N coincide. □

3.3.16 Definition. *Let M, N be manifolds. A smooth map $i : N \to M$ is called an* embedding *if i is an injective immersion and if i is a homeomorphism from N onto $(i(N), \mathcal{T}_M|_{i(N)})$.*

3.3.17 Remark. If $i : N \to M$ is an embedding then $i(N)$ can be turned into a submanifold of M by declaring i to be a diffeomorphism. The charts of $i(N)$ then are the $\psi \circ i^{-1}$, where ψ is any chart of N. This manifold $i(N)$ then is a submanifold of M: Let $j : i(N) \hookrightarrow M$ be the inclusion map. Then $i = j \circ i$ is an immersion and i is a diffeomorphism by definition, so j is an immersion. Also, $i(N)$ carries the trace topology by assumption. By 3.3.15 this manifold structure on $i(N)$ is the only one possible.

Next we want to check how to tell whether a given subset N of M can be made into a submanifold of M. We first generalized the condition from 3.3.12 (ii):

3.3.18 Definition. *Let M^m be a manifold and let N be a subset of M. We say that N possesses the submanifold-property of dimension n if for every $p \in N$ there exists a chart (φ, U) of p in M such that $\varphi(p) = 0$ and $\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\})$. (φ, U) then is called an adapted coordinate system.*

3.3.19 Theorem. *Let M^m be a manifold and let N be a subset of M possessing the submanifold-property of dimension n. Then N can be equipped in a unique way with a differentiable structure such that it becomes an n-dimensional submanifold of M. If $\mathrm{pr}_1 : \mathbb{R}^m \to \mathbb{R}^n$ denotes the projection then $\mathcal{A} := \{(\tilde{\varphi} := \mathrm{pr}_1 \circ \varphi, U \cap N) \mid \varphi$ is an adapted coordinate system$\}$ is a $\mathcal{C}^\infty$-atlas for N. In addition, $j : N \hookrightarrow M$ is an embedding.*

Proof. Uniqueness is clear from 3.3.15. Let (φ_1, U_1), (φ_2, U_2) be adapted coordinate systems with $(U_1 \cap N) \cap (U_2 \cap N) \neq \emptyset$. We have to show that $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are $\mathcal{C}^\infty$-compatible. We first note that since the φ_i are homeomorphisms, so are the $\tilde{\varphi}_i$ as maps from $U_i \cap N$ with the trace topology onto $\mathrm{pr}_1(\varphi_i(U_i) \cap (\mathbb{R}^n \times \{0\}))$.

Let $\theta : \mathbb{R}^n \hookrightarrow \mathbb{R}^m$, $\theta(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$. Then $\tilde{\varphi}_i^{-1} = \varphi_i^{-1} \circ \theta$. It follows that $\tilde{\varphi}_1 \circ \tilde{\varphi}_2^{-1}$ is defined on $\tilde{\varphi}_2(U_1 \cap U_2 \cap N)$ $(= \mathrm{pr}_1(\varphi_2(U_1 \cap U_2) \cap (\mathbb{R}^n \times \{0\}))$, hence open in $\mathbb{R}^n$), and

$$\tilde{\varphi}_1 \circ \tilde{\varphi}_2^{-1} = (\mathrm{pr}_1 \circ \varphi_1) \circ (\mathrm{pr}_1 \circ \varphi_2)^{-1} = \mathrm{pr}_1 \circ \varphi_1 \circ \varphi_2^{-1} \circ \theta$$

is smooth. Consequently, $\mathcal{A}$ is an atlas for N and by 1.2.7 the natural manifold topology of N is precisely the trace topology of M on N. If (φ, U) is an adapted chart then $\varphi \circ j \circ \tilde{\varphi}^{-1} = \theta$, so j is an immersion. Since N carries the trace topology, $j : N \to (j(N), \mathcal{T}_M|_{j(N)})$ is a homeomorphism, so j is an embedding. □

3.3.20 Proposition. *Let M^m, N^n be manifolds, N compact and $i : N \to M$ an injective immersion. Then i is even an embedding and $i(N)$ is a submanifold of M that is diffeomorphic to N.*

Proof. We have to show that $i : (N, \mathcal{T}_N) \to (i(N), \mathcal{T}_M|_{i(N)})$ is a homeomorphism. We already know that this map is continuous and bijective. But also i^{-1} is continuous: Let $A \subseteq N$ be closed, hence compact. Then $(i^{-1})^{-1}(A) = i(A)$ is compact and therefore closed. The final claim follows from 3.3.17. $\square$

3.3.21 Corollary. *Let $f : N^n \to M^m$ be an immersion. Then every $p \in N$ has an open neighborhood U such that $f|_U : U \to M$ is an embedding. Thus the difference between an immersion and an embedding is of a global nature.*

Proof. By 3.3.3 there exist charts φ at p and ψ at $f(p)$ such that $\psi \circ f \circ \varphi^{-1} = (x^1, \dots, x^n) \mapsto (x^1, \dots, x^n, 0, \dots, 0)$. Thus there exists a compact neighborhood V of p such that $f|_V$ is injective. As in the proof of 3.3.20 it follows that $f|_V : V \to (f(V), \mathcal{T}_M|_{f(V)})$ is a homeomorphism. Let $U \subseteq V$ be an open neighborhood of p. Then $f|_U$ is an injective immersion and $f : U \to (f(U), \mathcal{T}_M|_{f(U)})$ is a homeomorphism, so $f : U \to M$ is an embedding. $\square$

3.3.22 Theorem. *Let M^m, N^n be manifolds and $f : N \to M$ smooth with* $\mathrm{rk}(f) \equiv k$ *on N ($k < n$). Let $q \in f(N)$. Then $f^{-1}(q)$ is a closed submanifold of N of dimension $n - k$.*

Proof. Since f is continuous, $f^{-1}(q)$ is closed in N. We show that $f^{-1}(q)$ possesses the submanifold property of dimension $n - k$. The claim then follows from 3.3.19. Let $p \in f^{-1}(q)$. Then by 3.3.3 there exist charts (φ, U) at p and (ψ, V) at $f(p) = q$ such that $\varphi(p) = 0$, $\psi(q) = 0$ and

$$f_{\psi\varphi}(x) = \psi \circ f \circ \varphi^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^k, 0, \dots, 0).$$

Here, $f_{\psi\varphi}$ is defined on $\varphi(U \cap f^{-1}(V)) =: \varphi(W)$. Then (φ, W) is a chart of N at p and

$$\begin{aligned}\varphi(f^{-1}(q) \cap W) = \varphi(f^{-1}(q)) \cap \varphi(W) &= \varphi(f^{-1}(\psi^{-1}(\psi(q)))) \cap \varphi(W) \\ &= f_{\psi\varphi}^{-1}(0) \cap \varphi(W) = (\{0\} \times \mathbb{R}^{n-k}) \cap \varphi(W).\end{aligned}$$

$\square$

3.3.23 Corollary. *Let $f : N^n \to M^m$ be smooth with $m < n$ and let $q \in N$. If* $\mathrm{rk}_p(f) = m$ *for all $p \in f^{-1}(q)$ then $f^{-1}(q)$ is a closed submanifold of N of dimension $n - m$.*

Proof. Let $p \in f^{-1}(q)$. Then f has maximal rank ($= m$) at p, hence by 3.3.4 even in an open neighborhood U of p in N. Therefore the rank of f equals m on an open neighborhood $\tilde{N}$ of $f^{-1}(q)$ in N. The claim now follows by applying 3.3.22 to $f : \tilde{N} \to M$. $\square$

3.3.24 Remark. For $N = \mathbb{R}^n$ and $M = \mathbb{R}^m$ this result reduces to the description of submanifolds as zero-sets of regular maps, cf. 1.1.8.

3.3.25 Proposition. *Under the assumptions of 3.3.22, let $L := f^{-1}(q)$ and let $p \in L$. Then* $T_pL = \ker(T_pf)$.

Proof. For any smooth curve c in L with $c(0) = p$, $f \circ c \equiv q$, so $0 = \frac{d}{dt}\big|_0 (f \circ c) = T_p f(c'(0))$. Hence (cf. 2.3.1) $T_p L \subseteq \ker(T_p M)$. Since $\dim(\ker T_p f) + \dim(\mathrm{im} T_p f) = \dim T_p N = n$, $\dim(\ker T_p f) = n - k = \dim T_p L$, and equality follows. $\square$

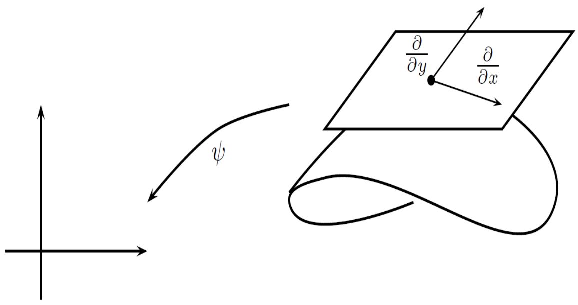
3.3.26 Example. Let $\pi : TM \to M^m$ be the canonical projection and let $p \in M$. Then π is smooth and $\mathrm{rk}(\pi) = m$ since with respect to a chart ψ of M we have $\psi \circ \pi \circ T\psi^{-1} = \mathrm{pr} : \mathbb{R}^{2m} \to \mathbb{R}^m$ (cf. 2.2.7). By 3.3.23 it follows that $\pi^{-1}(p) = T_p M$ is an m-dimensional submanifold of TM. Moreover, by 3.3.25, for $v_p \in T_p M$ we have $T_{v_p} T_p M = \ker(T_{v_p}\pi)$. By the proof of 3.3.22, the submanifold charts of $T_p M$ are given by $T\psi|_{T_p M} = T_p\psi$. As these are linear isomorphisms, the trace topology of TM on $T_p M$ is precisely the usual topology of $T_p M$ as a finite-dimensional vector space. Also, $T_p\psi$ is a diffeomorphism, so the manifold structure of $T_p M$ as well is its usual differentiable structure as a finite-dimensional vector space.

Chapter 4

Multilinearity and integration

4.1 Tensors

Heuristically, if we want to determine the area of a curved surface, or, more generally, the volume of some submanifold, we first have to approximate the surface ‘infinitesimally’ by its tangent space, then determine the area of these approximating spaces and then sum (resp. integrate) up the results.



Thus we first need a way of assigning volumes to parallelepipeds in vector spaces. A map ω that assigns a volume to a parallelepiped with edges u, v, w should possess the following properties:

(i) $\omega(\alpha u, v, w, \dots) = \omega(u, \alpha v, w, \dots) = \cdots = \alpha \cdot \omega(u, v, w, \dots)$

(ii) $\omega(u_1 + u_2, v, w, \dots) = \omega(u_1, v, w \dots) + \omega(u_2, v, w, \dots)$, and analogously for $v, w, \dots$

(iii) $\omega(u, u, w, \dots) = \omega(u, v, v, \dots) = \cdots = 0$

Since $0 = \omega(u+v, u+v, w, \dots) = \omega(u, v, w, \dots) + \omega(v, u, w, \dots)$, (iii) is equivalent to ω being antisymmetric (or skew-symmetric).

Due to (i),(ii) we have to consider multilinear mappings on vector spaces (in particular, on T_pM). The skew-symmetry (iii) will be taken into account in the following section. We therefore begin this section with a crash-course in multilinear algebra.

In what follows let $E_1, \dots, E_k, E, F$ be finite-dimensional vector spaces. Then by $L^k(E_1, \dots, E_k; F)$ we denote the space of multilinear maps from $E_1 \times \cdots \times E_k$ to

F. An important special case is $(k = 1)$: $L(E, \mathbb{R}) = E^*$, the dual space of E, i.e., the vector space of linear functionals on E. If $\mathcal{B}_E = \{e_1, \dots, e_n\}$ is a basis of E, then the functionals defined by

$$\alpha^j(e_i) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$(j = 1, \dots, n)$ form a basis of E^*, the *dual basis* of $\mathcal{B}_E$. For each $e \in E$ we have $e = \sum_{i=1}^n \alpha^i(e) e_i$ and for each $\alpha \in E^*$ we get $\alpha = \sum_{i=1}^n \alpha(e_i)\alpha^i$. The bidual space $E^{**} = (E^*)^*$ is canonically isomorphic to E: the map

$$\begin{aligned} i : E &\to E^{**} \\ i(e) &= \underbrace{\alpha}_{\in E^*} \mapsto \alpha(e) \end{aligned}$$

is a linear isomorphism.

4.1.1 Definition. *Let E be a vector space. Then*

$$T^r_s(E) := L^{r+s}(\underbrace{E^*, \dots, E^*}_{r}, \underbrace{E, \dots, E}_{s}; \mathbb{R})$$

is called the space of r-times contra- and s-times covariant tensors, or, for short, $\binom{r}{s}$-tensors. The elements of $T^r_s(E)$ are called tensors of type $\binom{r}{s}$.

For $t_1 \in T^{r_1}_{s_1}(E)$, $t_2 \in T^{r_2}_{s_2}(E)$, the tensor product $t_1 \otimes t_2 \in T^{r_1+r_2}_{s_1+s_2}(E)$ is defined by:

$$\begin{aligned} &t_1 \otimes t_2(\beta^1, \dots, \beta^{r_1}, \gamma^1, \dots, \gamma^{r_2}, f_1, \dots, f_{s_1}, g_1, \dots, g_{s_2}) \\ &:= t_1(\beta^1, \dots, \beta^{r_1}, f_1, \dots, f_{s_1}) \cdot t_2(\gamma^1, \dots, \gamma^{r_2}, g_1, \dots, g_{s_2}) \end{aligned}$$

$(\beta^j, \gamma^j \in E^*, f_j, g_j \in E)$.

Clearly, $\otimes$ is associative and bilinear.

4.1.2 Example.

(i) By definition, $T^0_1(E) = L(E, \mathbb{R}) = E^*$ and $T^1_0(E) = L(E^*, \mathbb{R}) = E^{**} = E$. Elements of E (vectors) therefore are $\binom{1}{0}$-tensors, elements of E^* (often called co-vectors) are $\binom{0}{1}$-tensors.

(*ii*) Let E be a vector space with scalar product $g(e, f) = \langle e, f\rangle$. Then g is a bilinear map $g : E \times E \to \mathbb{R}$, i.e., a $\binom{0}{2}$-tensor.

4.1.3 Proposition. *Let* $\dim(E) = n$. *Then* $\dim(T^r_s(E)) = n^{r+s}$. *If $\{e_1, \dots, e_n\}$ is a basis of E and $\{\alpha^1, \dots, \alpha^n\}$ is the corresponding dual basis, then*

$$\mathcal{B}^r_s := \{e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_s} \mid 1 \leq i_k, j_k \leq n\}$$

is a basis of $T^r_s(E)$.

Proof. $\mathcal{B}^r_s$ is linearly independent: let

$$\sum_{\substack{i_1,\dots,i_r \\ j_1,\dots,j_s}} \underbrace{t^{i_1\dots i_r}_{j_1\dots j_s}}_{\in\mathbb{R}} e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_s} = 0$$

Inserting $(\alpha^{k_1}, \dots, \alpha^{k_r}, e_{l_1}, \dots, e_{l_s})$, then since $\alpha^i(e_j) = e_j(\alpha^i) = \delta_{ij}$ it follows that all $t^{i_1\dots i_r}_{j_1\dots j_s}$ vanish.

$\mathcal{B}^r_s$ generates $T^r_s(E)$: each $t \in T^r_s(E)$ can be written as follows:

$$t = \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} t(\alpha^{i_1}, \dots, \alpha^{i_r}, e_{j_1}, \dots, e_{j_s}) e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_s}.$$

To see this, it suffices to show that both sides of this equation define the same multilinear map. Let $\beta^1 = \sum \lambda^1_{i_1} \alpha^{i_1}, \dots, \beta^r = \sum \lambda^r_{i_r} \alpha^{i_r} \in E^*$, and $x_1 = \sum \mu^{j_1}_1 e_{j_1}, \dots, x_s = \sum \mu^{j_s}_s e_{j_s} \in E$. Then

$$t(\beta^1, \dots, \beta^r, x_1, \dots, x_s) = \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} \lambda^1_{i_1} \dots \lambda^r_{i_r} \mu^{j_1}_1 \dots \mu^{j_s}_s t(\alpha^{i_1}, \dots, \alpha^{i_r}, e_{j_1}, \dots, e_{j_s})$$

$$= \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} t(\alpha^{i_1}, \dots, \alpha^{i_r}, e_{j_1}, \dots, e_{j_s}) e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_s}(\beta^1, \dots, x_s)$$

□

Every linear map $\varphi : E \to F$ possesses an adjoint map $\varphi^* \in L(F^*, E^*)$: for $\beta \in F^*$, $e \in E$ one sets $\varphi^*(\beta)(e) := \beta(\varphi(e))$. If A is the matrix of φ with respect to bases of E resp. F, then A^t is the matrix of φ^* with respect to the corresponding dual bases of F^* resp. E^*.

More generally, we now want to assign to any $\varphi \in L(E, F)$ a linear map $\varphi^r_s \in L(T^r_s(E), T^r_s(F))$. If φ is a linear isomorphism we may combine such a map from φ and φ^*:

4.1.4 Definition. *Let $\varphi \in L(E, F)$ be bijective. Then $T^r_s(\varphi) \equiv \varphi^r_s \in L(T^r_s E, T^r_s F)$ is defined as*

$$(\varphi^r_s(t))(\beta^1, \dots, \beta^r, f_1, \dots, f_s) := t(\varphi^*(\beta^1), \dots, \varphi^*(\beta^r), \varphi^{-1}(f_1), \dots, \varphi^{-1}(f_s))$$

for $t \in T^r_s(E)$, $\beta^1, \dots, \beta^r \in F^*$, $f_1, \dots, f_s \in F$.

4.1.5 Example. $\varphi^1_0 : E = T^1_0(E) \to T^1_0(F) = F$, $\varphi^1_0(e)(\beta) = e(\varphi^*(\beta)) = \varphi(e)(\beta)$. Thus we may identify φ^1_0 with φ.

$\varphi^0_1 : E^* = T^0_1(E) \to T^0_1(F) = F^*$, $\varphi^0_1(\alpha)(f) = \alpha(\varphi^{-1}(f)) = (\varphi^{-1})^*(\alpha)(f)$, so we may identify φ^0_1 with $(\varphi^{-1})^*$.

It follows that $T^r_s \varphi = \varphi^r_s$ is a simultaneous extension of φ and $(\varphi^{-1})^*$ to general tensor spaces.

4.1.6 Proposition. *Let $\varphi : E \to F$, $\psi : F \to G$ be linear isomorphisms. Then:*

(i) $(\psi \circ \varphi)^r_s = \psi^r_s \circ \varphi^r_s$

(ii) $(\mathrm{id}_E)^r_s = \mathrm{id}_{T^r_s(E)}$

(iii) *$\varphi^r_s : T^r_s E \to T^r_s F$ is a linear isomorphism, and $(\varphi^r_s)^{-1} = (\varphi^{-1})^r_s$.*

(iv) *If $t_1 \in T^{r_1}_{s_1}(E)$, $t_2 \in T^{r_2}_{s_2}(E)$, then $\varphi^{r_1+r_2}_{s_1+s_2}(t_1 \otimes t_2) = \varphi^{r_1}_{s_1}(t_1) \otimes \varphi^{r_2}_{s_2}(t_2)$.*

Proof. (i) We first note that for $\gamma \in G^*$, $e \in E$ we have

$$(\varphi^* \circ \psi^*)(\gamma)(e) = (\varphi^*(\psi^*(\gamma)))(e) = \psi^*(\gamma)(\varphi(e)) = \gamma(\psi(\varphi(e))) = (\psi \circ \varphi)^*(\gamma)(e).$$

Now let $\gamma^1, \dots, \gamma^r \in G^*, g_1, \dots, g_s \in G$ and $t \in T^r_s(E)$. Then

$$
\begin{aligned}
&(\psi^r_s(\varphi^r_s(t)))(\gamma^1, \dots, \gamma^r, g_1, \dots, g_s) \\
&\qquad = (\varphi^r_s(t))(\psi^*\gamma^1, \dots, \psi^*\gamma^r, \psi^{-1}(g_1), \dots, \psi^{-1}(g_s)) \\
&\qquad = t(\underbrace{\varphi^*(\psi^*\gamma^1)}_{(\psi\circ\varphi)^*\gamma^1}, \dots, \varphi^*(\psi^*\gamma^r), \underbrace{\varphi^{-1}(\psi^{-1})(g_1)}_{(\psi\circ\varphi)^{-1}(g_1)}, \dots, \varphi^{-1}(\psi^{-1}(g_s))) \\
&\qquad = ((\psi\circ\varphi)^r_s(t))(\gamma^1, \dots, \gamma^r, g_1, \dots, g_s).
\end{aligned}
$$

(ii) Since $\mathrm{id}_E^{-1} = \mathrm{id}_E$ and $\mathrm{id}_E^* = \mathrm{id}_{E^*}$ this is immediate from the definitions.
(iii) follows from (i) and (ii).
(iv)

$$
\begin{aligned}
&\varphi^{r_1+r_2}_{s_1+s_2}(t_1\otimes t_2)(\beta^1, \dots, \beta^{r_1+r_2}, f_1, \dots, f_{s_1+s_2}) \\
&\qquad = (t_1\otimes t_2)(\varphi^*\beta^1, \dots, \varphi^*\beta^{r_1+r_2}, \varphi^{-1}(f_1), \dots, \varphi^{-1}(f_{s_1+s_2})) \\
&\qquad = t_1(\varphi^*\beta^1, \dots, \varphi^*\beta^{r_1}, \varphi^{-1}(f_1), \dots, \varphi^{-1}(f_{s_1}))\cdot \\
&\qquad\quad\ t_2(\varphi^*\beta^{r_1+1}, \dots, \varphi^*\beta^{r_1+r_2}, \varphi^{-1}(f_{s_1+1}), \dots, \varphi^{-1}(f_{s_1+s_2})) \\
&\qquad = (\varphi^{r_1}_{s_1}t_1)\otimes(\varphi^{r_2}_{s_2}t_2)(\beta^1, \dots, \beta^{r_1+r_2}, f_1, \dots, f_{s_1+s_2}).
\end{aligned}
$$

□

To simplify notations, in what follows we will employ Einstein's *summation convention*: for every index which appears both as an upper and as a lower index, summation is carried out over its entire set of values. Thus, instead of

$$\sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}} t^{i_1,\dots,i_r}_{j_1,\dots,j_s} e_{i_1}\otimes\cdots\otimes e_{i_r}\otimes\alpha^{j_1}\otimes\cdots\otimes\alpha^{j_s}$$

we simply write $t^{i_1,\dots,i_r}_{j_1,\dots,j_s} e_{i_1}\otimes\cdots\otimes e_{i_r}\otimes\alpha^{j_1}\otimes\cdots\otimes\alpha^{j_s}$. Deviations from this convention will be mentioned explicitly.

Our next aim is to extend the above constructions of multilinear algebra to tangent vectors, i.e., to elements of certain vector bundles. To carry out this transfer we first consider the case of local vector bundles.

4.1.7 Definition. *Let $\varphi : U\times F \to U'\times F'$, $\varphi(u,f) = (\varphi_1(u), \varphi_2(u)f)$ be a local vector bundle isomorphism (cf.* 2.2.6 *(i)). Then define $\varphi^r_s : U\times T^r_sF \to U'\times T^r_sF'$ by*

$$\varphi^r_s(u,t) = (\varphi_1(u), (\varphi_2(u))^r_s(t)) \qquad (t\in T^r_sF)$$

Note that $\varphi_2(u)$ is an isomorphism for each u, so $(\varphi_2(u))^r_s$ is well-defined.

4.1.8 Lemma. *Under the assumptions of 4.1.7, $\varphi^r_s : U\times T^r_sF \to U'\times T^r_sF'$ is a local vector bundle isomorphism.*

Proof. By 4.1.6 (iii), every $(\varphi_2(u))^r_s$ is a linear isomorphism. Hence φ^r_s is bijective and it remains to show that $(u,t)\mapsto\varphi^r_s(u,t)$ is smooth (it then follows that also $(\varphi^r_s)^{-1} = (\varphi^{-1})^r_s$ is smooth). Clearly, φ_1 is smooth.

Concerning φ_2 we first note that on the space $L(F,F')$ of linear maps (i.e., matrices) the map $\varphi\mapsto\varphi^*$ ($= A\mapsto A^t$) is linear, hence smooth. Moreover, the space of invertible matrices $\mathrm{GL}(F,F')$ is open in $L(F,F')$ (since $\mathrm{GL}(F,F') = \{A\in L(F,F') \mid \det(A)\neq 0\}$) and $\varphi\mapsto\varphi^{-1}$ (corresponding to $A\mapsto A^{-1}$)) is smooth on $\mathrm{GL}(F,F')$ by the inversion formula for matrices. Thus the maps $u\mapsto\varphi_2(u)^*$ and $u\mapsto\varphi_2(u)^{-1}$

are smooth. Moreover, the maps $i_k,\ i'_k : (\beta^1, \dots \beta^r, f_1, \dots, f_s) \mapsto \beta^k$ resp. $\mapsto f_k$ are linear, hence smooth as well. Summing up,

$$\begin{array}{rcl}
(u,t) & \mapsto & (\varphi_2(u))^r_s(t) = \\
(u,t) & \mapsto & (t, \varphi_2(u)^*, \dots, \varphi_2(u)^*, \varphi_2(u)^{-1}, \dots, \varphi_2(u)^{-1}) \\
& \mapsto & t \circ (\varphi_2(u)^* \circ i_1, \dots, \varphi_2(u)^* \circ i_r, \varphi_2(u)^{-1} \circ i'_1, \dots, \varphi_2(u)^{-1} \circ i'_s)
\end{array}$$

is smooth since also the last of the above maps is multilinear, hence $\mathcal{C}^\infty$. □

After these preparations we may now assign to any vector bundle E the corresponding $\binom{r}{s}$ tensor bundle, which has precisely the $(E_b)^r_s$ as fibers:

4.1.9 Definition. *Let (E, B, π) be a vector bundle, with $E_b = \pi^{-1}(b)$ the fiber over b. Then let*

$$T^r_s(E) := \bigsqcup_{b \in B} T^r_s(E_b) = \bigcup_{b \in B} \{b\} \times (E_b)^r_s$$

be the $\binom{r}{s}$-tensor bundle over E. Let $\pi^r_s : T^r_s(E) \to B,\ \pi^r_s(e) = b$ for $e \in T^r_s(E_b)$ denote the canonical projection. For $A \subseteq B$ let $T^r_s(E)|_A := \bigsqcup_{b\in A} T^r_s(E_b)$.

We wish to turn $T^r_s(E)$ itself into a vector bundle with basis B. To this end we will produce vector bundle charts for $T^r_s(E)$ from those of E, according to the following pattern:

4.1.10 Definition. *Let $E,\ E'$ be vector bundles and $f : E \to E'$. f is called a vector bundle homomorphism, if for each $e \in E$ there exists a vector bundle chart (Ψ, W) around e and a vector bundle chart (Ψ', W') around $f(e)$, such that $f(W) \subseteq W'$ and $f_{\Psi'\Psi} := \Psi' \circ f \circ \Psi^{-1}$ is a local vector bundle homomorphism (cf. 2.2.6 (i)). If f in addition is a diffeomorphism and $f|_{E_b} : E_b \to E'_{f(b)}$ is a linear isomorphism for all $b \in B$ then f is called a vector bundle isomorphism. In this case we define $f^r_s : T^r_s E \to T^r_s E'$ by*

$$f^r_s|_{T^r_s(E_b)} := (f|_{E_b})^r_s \ \forall b \in B$$

It is straightforward to check that a smooth map $f : E \to E'$ is a vector bundle homomorphism if and only if f is fiber-linear, i.e., if and only if $f|_{E_b} : E_b \to E'_{f(b)}$ is linear for each $b \in B$.

4.1.11 Examples.

(i) Let $M,\ N$ be manifolds and $f : M \to N$ smooth. Then $Tf : TM \to TN$ is a vector bundle homomorphism. In fact, by 2.2.5 we have:

$$\begin{array}{rcl}
T\psi \circ Tf \circ T\varphi^{-1}(x,w) & = & T(\psi \circ f \circ \varphi^{-1})(x,w) \\
& = & (\psi \circ f \circ \varphi^{-1}(x), D(\psi \circ f \circ \varphi^{-1})(x)w).
\end{array}$$

If f is a diffeomorphism then $Tf : TM \to TN$ is a vector bundle isomorphism.

(ii) Let E be a vector bundle, and (Ψ, W) a vector bundle chart of E. Then $\Psi : W \to U \times \mathbb{R}^n$ is a vector bundle isomorphism. This holds, in particular, for $E = TM$ and $\Psi = T\psi$, where ψ is any chart of M.

4.1.12 Theorem. *Let (E, B, π) be a vector bundle with vector bundle atlas $\mathcal{A} = \{(\Psi_\alpha, W_\alpha) \mid \alpha \in A\}$. Then $(T^r_s E, B, \pi^r_s)$ is a vector bundle with vector bundle atlas $\mathcal{A}^r_s = \{((\Psi_\alpha)^r_s, (T^r_s E)|_{W_\alpha \cap B}) \mid \alpha \in A\}$. $(T^r_s E, B, \pi^r_s)$ is called the tensor bundle of type $\binom{r}{s}$ over E.*

Proof. Clearly the $(T^r_s E)|_{W_\alpha \cap B}$ form a covering of $T^r_s E$. Let Ψ_α, Ψ_β be vector bundle charts from $\mathcal{A}$ with $W_{\alpha\beta} := W_\alpha \cap W_\beta \neq \emptyset$. Since $E|_{W_{\alpha\beta}\cap B} \cong \bigcup_{b \in W_{\alpha\beta}\cap B}\{b\}\times E_b$, it follows that Ψ_α is of the form $\Psi_\alpha(b,e) = (\psi_{\alpha 1}(b), \psi_{\alpha 2}(b)\cdot e)$, with $b \in B$, $e \in E_b$, and $\psi_{\alpha 2}(b)$ linear for each b. Therefore, $(\Psi_\alpha)^r_s$ is defined as $(b,t) \mapsto (\psi_{\alpha 1}(b), (\psi_{\alpha 2}(b))^r_s t)$ $(t \in T^r_s(E_b))$.

Then

$$\begin{aligned}
\Psi_\beta \circ \Psi_\alpha^{-1}(x,w) &= \Psi_\beta(\underbrace{\psi_{\alpha 1}^{-1}(x)}_{=:b}, \psi_{\alpha 2}(b)^{-1}w) \\
&= (\psi_{\beta 1}(\psi_{\alpha 1}^{-1}(x)), \psi_{\beta 2}(b)\psi_{\alpha 2}(b)^{-1}w) \\
&=: (\psi_{\beta\alpha 1}(x), \psi_{\beta\alpha 2}(x)\cdot w).
\end{aligned}$$

Hence by 4.1.6 (i) and 4.1.7,

$$\begin{aligned}
(\Psi_\beta)^r_s \circ ((\Psi_\alpha)^r_s)^{-1}(x,t') &= (\Psi_\beta)^r_s(\psi_{\alpha 1}^{-1}(x), (\psi_{\alpha 2}(b)^{-1})^r_s(t')) \\
&= (\psi_{\beta 1}(\psi_{\alpha 1}^{-1}(x)), (\psi_{\beta 2}(b)\psi_{\alpha 2}(b)^{-1})^r_s(t')) \\
&= (\psi_{\beta\alpha 1}(x), (\psi_{\beta\alpha 2}(x))^r_s(t')) = (\Psi_\beta \circ \Psi_\alpha^{-1})^r_s(x,t'),
\end{aligned}$$

which, by 4.1.8, is a local vector bundle isomorphism. Thus $T^r_s(E)$ is a vector bundle. As in the proof of 2.2.4 (for TM) it follows that $T^r_s(E)$ is Hausdorff and second countable. □

For us the most important special case of this construction is $E = TM$:

4.1.13 Definition. *Let M be a manifold. Then $T^r_s(M) := T^r_s(TM)$ is called the bundle of r-times contra- and s-times covariant tensors on M (resp. of tensors of type $\binom{r}{s}$). $T^*M := T^0_1(M)$ is called the cotangent bundle of M.*

By 4.1.5 we have $T^1_0(M) = TM$: in fact, $\pi^{-1}(p) = T_pM$ $\forall p$ and $T^1_0(T_pM) = T_pM$. For each chart ψ of M, $(T\psi)^1_0 = T\psi$.

If $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ is an atlas of M, then by 4.1.12,

$$\mathcal{A}^r_s = \{((T\psi_\alpha)^r_s, (T^r_s M)|_{V_\alpha}) \mid \alpha \in A\}$$

is a vector bundle atlas of $T^r_s M$.

4.1.14 Definition. *Smooth sections of $T^r_s M$ (i.e., smooth maps $t : M \to T^r_s M$ with $\pi^r_s \circ t = \mathrm{id}_M$) are called $\binom{r}{s}$-tensors (resp. $\binom{r}{s}$-tensor fields) on M. The space $\Gamma(M, T^r_s M)$ of $\binom{r}{s}$-tensor fields is denoted by $\mathcal{T}^r_s(M)$. In particular, $\mathcal{T}^1_0(M) = \mathfrak{X}(M)$. We also write $\Omega^1(M)$ instead of $\mathcal{T}^0_1(M)$. The elements of $\Omega^1(M)$ are called differential forms of order* 1 *(1-forms, covector fields).*

If $t \in \mathcal{T}^r_s(M)$ and $f \in \mathcal{C}^\infty(M)$, then $ft : p \mapsto f(p)t(p) \in (T_pM)^r_s$ is a tensor field on M. Then $\mathcal{T}^r_s(M)$ with the pointwise operations $+$, $f\cdot$ is a $\mathcal{C}^\infty(M)$-module.

As in the case of $\mathfrak{X}(M) = \mathcal{T}^1_0(M)$ we also want to derive local representations of general tensor fields in charts. We first consider the special case $\Omega^1(M) = \mathcal{T}^0_1(M) = \Gamma(M, T^0_1 M)$. As a set,

$$T^0_1 M = \bigsqcup_{p \in M} (T_pM)^* = \bigcup_{p\in M} \{p\} \times (T_pM)^*.$$

The vector bundle charts of $T^0_1 M = T^*M$ are of the form $(T\psi)^0_1 : T^0_1 M\big|_V \to \psi(V) \times (\mathbb{R}^n)^0_1 = \psi(V) \times (\mathbb{R}^n)^*$ for any chart (ψ, V) of M. As in the case of $TM = T^1_0 M$

we want to use the vector bundle charts to define a basis of $(T_pM)^*$. Recall that for T_pM in this way we derived the basis $\{\frac{\partial}{\partial x^i}\big|_p \mid 1 \le i \le n\}$, where $\frac{\partial}{\partial x^i}\big|_p = (T_p\psi)^{-1}(e_i)$, i.e., $\frac{\partial}{\partial x^i} = p \mapsto (T\psi)^{-1}(\psi(p), e_i)$.

In the case of $T^0_1 M$ let $\{\alpha^j \mid 1 \le j \le n\}$ be the dual basis of $\{e_i \mid 1 \le i \le n\}$ in $(\mathbb{R}^n)^*$. Then for any $p \in V$ the family

$$dx^i\,|_p := [(T\psi)^0_1]^{-1}(\psi(p), \alpha^i) \qquad (1 \le i \le n)$$

is a basis of $(T_pM)^*$. We have

$$\begin{aligned} dx^i\big|_p &= [(T\psi)^0_1]^{-1}(\psi(p), \alpha^i) = \\ &= (p, [(T_p\psi)^0_1]^{-1}(\alpha^i)) \stackrel{4.1.5}{=} (p, (((T_p\psi)^{-1})^*)^{-1}(\alpha^i)) \\ &= (p, (T_p\psi)^*(\alpha^i)). \end{aligned} \tag{4.1.1}$$

Since $dx^j\big|_p \in (T_pM)^*$ and $\frac{\partial}{\partial x^i}\big|_p \in T_pM$, we can apply $dx^j\big|_p$ to $\frac{\partial}{\partial x^i}\big|_p$:

$$\begin{aligned} dx^j\big|_p\Big(\frac{\partial}{\partial x^i}\Big|_p\Big) &= (T_p\psi)^*(\alpha^j)((T_p\psi)^{-1}(e_i)) = \\ &= \alpha^j(T_p\psi((T_p\psi)^{-1}(e_i))) = \\ &= \alpha^j(e_i) = \delta_{ij} \end{aligned}$$

It follows that $\{dx^j\big|_p \mid 1 \le j \le n\}$ is precisely the dual basis of $\{\frac{\partial}{\partial x^i}\big|_p \mid 1 \le i \le n\}$ in $(T_pM)^*$.

Another way of looking at dx^i results from the following definition:

4.1.15 Definition. *Let $f \in \mathcal{C}^\infty(M)$. Then $df : M \to T^*M$, $p \mapsto T_pf$ is called the exterior derivative of f.*

4.1.16 Remark. (i) $df \in \mathcal{T}^0_1(M)$. In fact, for any $p \in M$, $T_pf \in L(T_pM, \mathbb{R}) = (T_pM)^*$. Moreover, df is smooth since for any chart ψ around p we have (setting $\psi(p) = x$):

$$\begin{aligned} (T\psi)^0_1 \circ df \circ \psi^{-1}(x) &= (x, ((T_p\psi)^{-1})^* \circ T_pf) = (x, T_pf \circ (T_p\psi)^{-1}) \\ &= (x, T_x(f \circ \psi^{-1})) = (x, D(f \circ \psi^{-1})(x)) \end{aligned}$$

$$\begin{array}{ccc} T^0_1 M & \xrightarrow{(T\psi)^0_1} & \psi(V) \times (\mathbb{R}^n)^* \\ {\scriptstyle df}\uparrow & & \uparrow{\scriptstyle \mathrm{id}\times D(f\circ\psi^{-1})} \\ M \supseteq V & \xrightarrow{\psi} & \psi(V) \end{array}$$

(ii) If $f \in \mathcal{C}^\infty(M)$ and $X \in \mathfrak{X}(M)$, then for all $p \in M$, $X_p \in T_pM$ and $df\big|_p \in T_pM^*$, so $df(X) := p \mapsto df\big|_p(X_p) : M \to \mathbb{R}$ is well-defined. We have:

$$df\big|_p(X_p) = T_pf(X_p) = X(f)\big|_p.$$

Thus $df(X) = X(f)$. In particular, $df(X) \in \mathcal{C}^\infty(M)$.

(iii) Let (ψ, V) be a chart, $\psi = (x^1, \dots, x^n)$. Then $d(x^i)$ in the sense of 4.1.15 is precisely the above dx^i. Indeed, by (ii) we have

$$d(x^j)\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial}{\partial x^i}(x^j) = \delta_{ij},$$

i.e., $\{d(x^j)\,|_p|\ 1 \le j \le n\}$ is precisely the dual basis of $\{\frac{\partial}{\partial x^i}\big|_p \mid 1 \le i \le n\}$ for all $p \in V$. Since $df_p \in T_pM^*$ it follows that

$$df = \sum_{i=1}^{n} df\Big(\frac{\partial}{\partial x^i}\Big)dx^i = \sum_{i=^1}^{n} \frac{\partial f}{\partial x^i}dx^i \tag{4.1.2}$$

(iv) For $f, g \in \mathcal{C}^\infty(M)$ we have $d(fg) = (df)g + f(dg)$. Indeed, for any $X \in \mathfrak{X}(M)$ we have

$$d(fg)(X) = X(fg) = X(f)g + fX(g) = ((df)g + f(dg))(X).$$

If (ψ, V) is a chart of M, $\psi = (x^1, \dots, x^n)$, then for all $p \in M$ the tuple $\{\frac{\partial}{\partial x^i}\big|_p \mid 1 \le i \le n\}$ is a basis of T_pM and $\{dx^j\big|_p \mid 1 \le j \le n\}$ is the corresponding dual basis of T_pM^*. Thus, by 4.1.3, for any $p \in M$ the tuple:

$$\{\frac{\partial}{\partial x^{i_1}}\bigg|_p \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}}\bigg|_p \otimes dx^{j_1}\big|_p \otimes \dots \otimes dx^{j_s}\big|_p \mid 1 \le i_k, j_k \le n\}$$

is a basis of $(T_pM)^r_s$. Hence if t is a section of T^r_sM then there are uniquely determined functions $t^{i_1\dots i_r}_{j_1\dots j_s}$ on V such that

$$t|_V = t^{i_1\dots i_r}_{j_1\dots j_s}\frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s} \tag{4.1.3}$$

(cf. the special case of vector fields in 2.2.9: $X|_V = X^i\frac{\partial}{\partial x^i}$.) As for vector fields we also have a characterization of smoothness for tensor fields in terms of local coordinates:

4.1.17 Proposition. *Let t be a section of the bundle $T^r_s(M)$. TFAE:*

(i) t is smooth, i.e., $t \in \mathcal{T}^r_s(M)$.

(ii) In every chart representation (4.1.3) *all coefficient functions $t^{i_1\dots i_r}_{j_1\dots j_s}$ are smooth.*

Proof. Let (ψ, V) be a chart of M. Then $(T\psi)^r_s$ is a vector bundle chart of T^r_sM. By definition, t is smooth if and only if the push-forward $\psi_*t := (T\psi)^r_s \circ t \circ \psi^{-1} : \psi(V) \to \psi(V) \times (\mathbb{R}^n)^r_s$ is smooth for every chart ψ.

$$\begin{array}{ccc} T^r_sM & \xrightarrow{(T\psi)^r_s} & \psi(V) \times (\mathbb{R}^n)^r_s \\ {\scriptstyle t}\big\uparrow & & \big\uparrow{\scriptstyle \psi_* t} \\ M \supseteq V & \xrightarrow{\psi} & \psi(V) \end{array}$$

For $x \in \psi(V)$ we have (setting $p := \psi^{-1}(x)$):

$$\begin{aligned}
(T\psi)^r_s \circ t \circ \psi^{-1}(x) &\overset{(4.1.3)}{=} (T\psi)^r_s\Big(t^{i_1\dots i_r}_{j_1\dots j_s}(p)\,\frac{\partial}{\partial x^{i_1}}\bigg|_p \otimes \dots \otimes dx^{j_s}\big|_p\Big) \\
&= (x, (T_p\psi)^r_s(t^{i_1\dots i_r}_{j_1\dots j_s}(p)\,\frac{\partial}{\partial x^{i_1}}\bigg|_p \otimes \dots \otimes dx^{j_s}\big|_p)) \\
&\overset{4.1.6(iv)}{=} (x, t^{i_1\dots i_r}_{j_1\dots j_s}(p)\,\underbrace{(T_p\psi)^1_0}_{=T_p\psi}(\frac{\partial}{\partial x^{i_1}}\bigg|_p) \otimes \dots \otimes \underbrace{(T_p\psi)^0_1}_{((T_p\psi)^*)^{-1}}(dx^{j_s}\big|_p)) \\
&= (x, t^{i_1\dots i_r}_{j_1\dots j_s}(\psi^{-1}(x))\,\underbrace{e_{i_1} \otimes \dots \otimes e_{i_r} \otimes \alpha^{j_1} \otimes \dots \otimes \alpha^{j_s}}_{\in(\mathbb{R}^n)^r_s})
\end{aligned}$$

This map is smooth if and only if all $t^{i_1\dots i_r}_{j_1\dots j_s}\circ\psi^{-1}$ are smooth, i.e., if and only if all $t^{i_1\dots i_r}_{j_1\dots j_s}$ are smooth on V. $\square$

If $t\in\mathcal{T}^r_s(M)$ and $\alpha^1,\dots,\alpha^r\in\Omega^1(M), X_1,\dots,X_s\in\mathfrak{X}(M)$, then

$$p\mapsto t(p)(\alpha^1(p),\dots,\alpha^r(p),X_1(p),\dots,X_s(p))$$

is a well-defined function $M\to\mathbb{R}$ which we denote by $t(\alpha^1,\dots,\alpha^r,X_1,\dots,X_s)$. For $f\in\mathcal{C}^\infty(M)$ we have

$$t(f\alpha^1,\alpha^2,\dots)=t(\alpha^1,f\alpha^2,\dots)=\cdots=t(\alpha^1,\dots,fX_s)=ft(\alpha^1,\dots,X_s).$$

Thus $(\alpha^1,\dots,\alpha^r,X_1,\dots,X_s)\mapsto t(\alpha^1,\dots,X_s)$ is $\mathcal{C}^\infty(M)$-multilinear.

4.1.18 Proposition. *Let t be a section of the bundle $T^r_s(M)$. TFAE:*

(i) t is smooth (i.e., $t\in\mathcal{T}^r_s(M)$).

(ii) $\forall\alpha^1,\dots,\alpha^r\in\Omega^1(M),\forall X_1,\dots,X_s\in\mathfrak{X}(M)$, the map $t(\alpha^1,\dots,\alpha^r,X_1,\dots,X_s)$ is in $\mathcal{C}^\infty(M)$.

Proof. Let (ψ,V) be a chart in M, $\psi=(x^1,\dots,x^n)$.
(i)$\Rightarrow$(ii): Let $X_k=X^{a_k}_k\frac{\partial}{\partial x^{a_k}}$ $(1\le k\le s)$, $\alpha^m=\alpha^m_{b_m}dx^{b_m}$ $(1\le m\le r)$ be the local representations with respect to ψ. By 4.1.17, all coefficient functions $X^{a_j}_j$, $\alpha^i_{b_i}$, $t^{b_1\dots b_r}_{a_1\dots a_s}$ are smooth on V. Hence so is

$$\begin{aligned}t(\alpha^1,\dots,X_s) &= \alpha^1_{b_1}\dots\alpha^r_{b_r}X^{a_1}_1\dots X^{a_s}_s t(dx^{b_1},\dots,dx^{b_r},\frac{\partial}{\partial x^{a_1}},\dots,\frac{\partial}{\partial x^{a_s}})\\ &\overset{(4.1.3)}{=} \alpha^1_{b_1}\dots\alpha^r_{b_r}X^{a_1}_1\dots X^{a_s}_s t^{b_1\dots b_r}_{a_1\dots a_s}.\end{aligned}$$

(ii)$\Rightarrow$ (i): By 4.1.17 we have to show that $t^{i_1\dots i_r}_{j_1\dots j_s}$ is smooth on V for all $i_1,\dots,i_r$, $j_1,\dots,j_s$. As in the proof of 2.2.10, (ii)$\Rightarrow$(iii), we extend $dx^{i_1},\dots,\frac{\partial}{\partial x^{j_s}}$ to elements of $\Omega^1(M)$ resp. $\mathfrak{X}(M)$. Then $t^{i_1\dots i_r}_{j_1\dots j_s}=t(dx^{i_1},\dots,\frac{\partial}{\partial x^{j_s}})$ is smooth by (ii). $\square$

The above observations lead to the following algebraic characterization of smooth tensor fields. Let

$$L^{r+s}_{\mathcal{C}^\infty(M)}(\underbrace{\Omega^1(M)\times\cdots\times\Omega^1(M)}_{r}\times\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{s},\mathcal{C}^\infty(M))$$

be the space of $\mathcal{C}^\infty(M)$-multilinear maps from $\Omega^1(M)\times\cdots\times\Omega^1(M)\times\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)$ to $\mathcal{C}^\infty(M)$. Then we have:

4.1.19 Theorem. *The map*

$$\begin{aligned}A:\mathcal{T}^r_s(M) &\to L^{r+s}_{\mathcal{C}^\infty(M)}(\Omega^1(M)\times\cdots\times\mathfrak{X}(M),\mathcal{C}^\infty(M))\\ t &\mapsto [(\alpha^1,\dots,\alpha^r,X_1,\dots,X_s)\mapsto t(\alpha^1,\dots,X_s)]\end{aligned}$$

is a $\mathcal{C}^\infty(M)$-linear isomorphism.

Proof. By 4.1.18, for all $t\in\mathcal{T}^r_s(M)$, $A(t)\in L^{r+s}_{\mathcal{C}^\infty(M)}(\Omega^1(M)\times\cdots\times\mathfrak{X}(M),\mathcal{C}^\infty(M))$ and clearly A is $\mathcal{C}^\infty(M)$-linear.

A is injective: If $A(t)=0$ then for all $p\in M$ and all $\alpha^1,\dots,X_s$ we have $t(p)(\alpha^1(p),\dots,X_s(p))=0$. All elements of T_pM resp. of $(T_pM)^*$ can be written in this way (i.e., are of the form $X_i(p)$ resp. $\alpha^j(p)$ for certain smooth fields X_i, α^j: this is seen by extending any given constant (co-)vector to a smooth field using a partition of unity, cf. the proof of 2.2.10, (ii)$\Rightarrow$(iii)). Hence it follows that $t(p) = 0\ \forall p$, i.e., $t = 0$.

A is surjective: Let $\Phi \in L^{r+s}_{\mathcal{C}^\infty(M)}(\Omega^1(M) \times \cdots \times \mathfrak{X}(M), \mathcal{C}^\infty(M))$. We have to show that there exists some $t \in \mathcal{T}^r_s(M)$ with $A(t) = \Phi$. To this end we first demonstrate that $\Phi(\alpha^1, \ldots, X_s)\big|_p$ depends only on $\alpha^1(p), \ldots, X_s(p)$. It suffices to show that $\alpha^1(p) = 0$ implies $\Phi(\alpha^1, \ldots, X_s)\big|_p = 0$ (and analogously for $\alpha^2, \ldots, X_s$). This we do in two steps

1) If V is an open neighborhood of p and $\alpha^1\big|_V = 0$, then $\Phi(\alpha^1, \ldots, X_s)\big|_p = 0$ (i.e., Φ depends only on the local behavior of α^1). Choose an open neighborhood U of p such that $\overline{U} \subseteq V$. By 1.3.14 there exists a partition of unity (χ_1, χ_2) subordinate to $\{V, M \setminus \overline{U}\}$. Then $\alpha^1 = \chi_2 \cdot \alpha^1$, and therefore

$$\Phi(\alpha^1, \ldots, X_s)\big|_p = \Phi(\chi_2\alpha^1, \alpha^2, \ldots, X_s)\big|_p = \underbrace{\chi_2(p)}_{=0}\, \Phi(\alpha^1, \alpha^2, \ldots, X_s)\big|_p = 0$$

2) Now let $\alpha^1(p) = 0$, let V be a chart neighborhood of p, and write $\alpha^1\big|_V = \alpha^1_j dx^j$. Then by 1),

$$\begin{aligned} \Phi(\alpha^1, \ldots, X_s)\big|_p &= \Phi(\alpha^1_j dx^j, \alpha^2, \ldots, X_s)\big|_p \\ &= \alpha^1_j(p)\ \Phi(dx^j, \alpha^2, \ldots, X_s)\big|_p = 0 \end{aligned}$$

Therefore, for each $p \in M$ we may define some $t(p) \in (T_pM)^r_s$ by

$$t(p)(\alpha^1(p), \ldots, X_s(p)) := \Phi(\alpha^1, \ldots, X_s)\big|_p.$$

(Recall that all elements of $T_pM^* \times \cdots \times T_pM$ can be written in this way, as was demonstrated above). Thus t is a section of T^r_sM. By construction, for all $\alpha^i \in \Omega^1(M)$ and all $X_j \in \mathfrak{X}(M)$ we have $t(\alpha^1, \ldots, X_s) = \Phi(\alpha^1, \ldots, X_s) \in \mathcal{C}^\infty(M)$, so $t \in \mathcal{T}^r_s(M)$ by 4.1.18. Obviously, $A(t) = \Phi$, so A is onto.

□

4.1.20 Example. (Kronecker delta) Let $\delta : \Omega^1(M) \times \mathfrak{X}(M) \to \mathcal{C}^\infty(M)$, $\delta(\alpha, X) := \alpha(X)$. This clearly defines a $\mathcal{C}^\infty(M)$-bilinear operation, hence by 4.1.19 is an element of $\mathcal{T}^1_1(M)$, called the Kronecker delta. It has the interesing property that its coordinate expression is the same in any local chart, namely

$$\delta = \frac{\partial}{\partial x^i} \otimes dx^i.$$

Indeed, given $X = X^j \frac{\partial}{\partial x^j}$ and $\alpha = \alpha_k dx^k$,

$$\delta(\alpha, X) = \alpha_i X^i = \Big(\frac{\partial}{\partial x^i} \otimes dx^i\Big)(\alpha, X).$$

All standard operations of multilinear algebra can be transferred fiber-wise to tensor fields. We have already encountered the following:

- $f \in \mathcal{C}^\infty(M),\ t \in \mathcal{T}^r_s(M) \Rightarrow\ f \cdot t := p \mapsto f(p) \cdot t(p) \in \mathcal{T}^r_s(M)$

- $t \in \mathcal{T}^r_s(M),\ \alpha^1, \dots, \alpha^r \in \Omega^1(M),\ X_1, \dots, X_s \in \mathfrak{X}(M) \Rightarrow\ t(\alpha^1, \dots, X_s) \in \mathcal{C}^\infty(M)$

Moreover, for $t_1 \in \mathcal{T}^{r_1}_{s_1}(M),\ t_2 \in \mathcal{T}^{r_2}_{s_2}(M)$ the tensor product $t_1 \otimes t_2 \in \mathcal{T}^{r_1+r_2}_{s_1+s_2}(M)$ is defined as

$$t_1 \otimes t_2 : p \mapsto t_1(p) \otimes t_2(p)$$

$t_1 \otimes t_2$ is smooth by 4.1.17 or also by 4.1.18.

4.2 Tensor calculus

In this section we are going to extend the Lie derivative to arbitrary tensor fields, thereby laying the ground for a powerful calculus with many applications. We already encountered the Lie derivative of functions and vector fields and the exterior derivative on smooth functions. Recall from 2.3.11 and 2.3.13 that, for any $f \in \mathcal{C}^\infty(M)$ and any $X, Y \in \mathfrak{X}(M)$ we have

$$L_X(f) = X(f) = df(X), \qquad L_X Y = [X, Y].$$

Moreover, by 2.2.15 the space of all derivations on $\mathcal{C}^\infty(M)$ coincides with $\mathfrak{X}(M)$. Thus any derivation on $\mathcal{C}^\infty(M)$ is of the form L_X for a unique $X \in \mathfrak{X}(M)$.

4.2.1 Proposition. *Let M, N be manifolds and let $X \in \mathfrak{X}(M)$.*

(i) $L_X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ and $L_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ are natural with respect to push-forward under any diffeomorphism $\varphi : M \to N$, i.e.: $L_{\varphi_* X}(\varphi_* f) = \varphi_*(L_X f)$ for all $f \in \mathcal{C}^\infty(M)$, and $L_{\varphi_* X}(\varphi_* Y) = \varphi_*(L_X Y)$.

$$\begin{tikzcd} \mathcal{C}^\infty(M) \arrow[r, "\varphi_*"] \arrow[d, "L_X"] & \mathcal{C}^\infty(N) \arrow[d, "L_{\varphi_* X}"] \\ \mathcal{C}^\infty(M) \arrow[r, "\varphi_*"] & \mathcal{C}^\infty(N) \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{X}(M) \arrow[r, "\varphi_*"] \arrow[d, "L_X"] & \mathfrak{X}(N) \arrow[d, "L_{\varphi_* X}"] \\ \mathfrak{X}(M) \arrow[r, "\varphi_*"] & \mathfrak{X}(N) \end{tikzcd}$$

(ii) Both operations are natural with respect to restrictions: Let $U \subseteq M$ be open, then $(L_X f)|_U = L_{X|_U}(f|_U)$ and $(L_X Y)|_U = L_{X|_U}(Y|_U)$.

$$\begin{tikzcd} \mathcal{C}^\infty(M) \arrow[r, "|_U"] \arrow[d, "L_X"] & \mathcal{C}^\infty(U) \arrow[d, "L_{X|_U}"] \\ \mathcal{C}^\infty(M) \arrow[r, "|_U"] & \mathcal{C}^\infty(U) \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{X}(M) \arrow[r, "|_U"] \arrow[d, "L_X"] & \mathfrak{X}(N) \arrow[d, "L_{X|_U}"] \\ \mathfrak{X}(M) \arrow[r, "|_U"] & \mathfrak{X}(N) \end{tikzcd}$$

Proof. (i) For $p \in M$ and $q := \varphi(p)$ we have

$$\begin{aligned} L_{\varphi_* X}(\varphi_* f)(q) &= d(f \circ \varphi^{-1})(\varphi_* X(q)) = T_q(f \circ \varphi^{-1}) \circ T_{\varphi^{-1}(q)}\varphi \circ X \circ \varphi^{-1}(q) \\ &= T_q f \circ X \circ \varphi^{-1}(q) = (Tf \circ X)(\varphi^{-1}(q)) = (\varphi_*(L_X f))(q). \end{aligned}$$

Furthermore, using the remark following 2.3.16, we have

$$L_{\varphi_* X}(\varphi_* Y) = [\varphi_* X, \varphi_* Y] = \varphi_*[X, Y] = \varphi_*(L_X Y).$$

(ii) This is immediate from the local representations (see 2.2.10 and 2.2.17) $\square$

We next want to extend the Lie derivative L_X to an operation on the full tensor algebra $\mathcal{T}(M) := \bigoplus_{r,s} \mathcal{T}^r_s(M)$. The type of operation we aim for is specified in the following definition:

4.2.2 Definition. *A* differential operator *on $\mathcal{T}(M)$ is a collection of maps $D^r_s(U) : \mathcal{T}^r_s(U) \to \mathcal{T}^r_s(U)$ ($r, s \geq 0$, $\mathcal{T}^0_0(U) := \mathcal{C}^\infty(U)$) for each open set $U \subseteq M$, such that (writing D for short)*

(DO1) D is a tensor derivation*: it is $\mathbb{R}$-linear and $D(t_1 \otimes t_2) = Dt_1 \otimes t_2 + t_1 \otimes Dt_2$ for any tensors t_1, t_2.*

(DO2) D is local (natural with respect to restrictions): For $U \subseteq V \subseteq M$ open and $t \in \mathcal{T}^r_s(V)$:

$$(Dt)|_U = D(t|_U) \in \mathcal{T}^r_s(U).$$

(DO3) $D\delta = 0$, where δ is the Kronecker delta from 4.1.20.

The key to extending L_X to $\mathcal{T}(M)$ is the following theorem:

4.2.3 Theorem. *(Willmore) Suppose that for any $U \subseteq M$ open we are given maps $E_U : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)$ and $F_U : \mathfrak{X}(U) \to \mathfrak{X}(U)$ that are $\mathbb{R}$-linear tensor derivations and natural with respect to restrictions, i.e.:*

(i) $E_U(f \otimes g) = (E_U f) \otimes g + f \otimes E_U g$ for all $f, g \in \mathcal{C}^\infty(U)$.

(ii) For all $f \in \mathcal{C}^\infty(M)$: $E_U(f|_U) = (E_M f)|_U$.

(iii) $F_U(f \otimes X) = (E_U f) \otimes X + f \otimes F_U X$.

(iv) For $X \in \mathfrak{X}(M)$, $F_U(X|_U) = (F_M X)|_U$.

Then there exists a unique differential operator D on $\mathcal{T}(M)$ such that $D|_{\mathcal{C}^\infty(U)} = E_U$ and $D|_{\mathfrak{X}(U)} = F_U$ for each $U \subseteq M$ open.

Proof. *Uniqueness:* Supposing that such a D exists, by (DO2) and (ii), (iv) it suffices to show uniqueness of each $D(U)$, where U is the domain of a chart φ with coordinates $x^1, \ldots, x^n$. Then any $t \in \mathcal{T}(U)$ can be written in the form $t = t^{i_1 \ldots i_r}_{j_1 \ldots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}$, where $t^{i_1 \ldots i_r}_{j_1 \ldots j_s} \in \mathcal{C}^\infty(U)$. Applying D to t and using (DO1) we obtain a sum of terms that can be expressed via E_U and F_U, except for terms of the form $D(dx^j)$. But for these we get, using (DO3) and 4.1.20:

$$0 = D\delta = D\Big(dx^i \otimes \frac{\partial}{\partial x^i}\Big) = D(dx^i) \otimes \frac{\partial}{\partial x^i} + dx^i \otimes D\Big(\frac{\partial}{\partial x^i}\Big). \tag{4.2.1}$$

Here, $D\Big(\frac{\partial}{\partial x^i}\Big) = F_U\Big(\frac{\partial}{\partial x^i}\Big)$, so inserting $(\frac{\partial}{\partial x^j}, dx^k)$ shows that $D(dx^i)$ is also completely determined.

Existence: Conversely, we define D on any t as above by the expressions we derived in the uniqueness proof using E_U and F_U. Then due to (i) and (iii), this D is well-defined and satisfies (DO1). Also, (ii) and (iv) imply (DO2). Finally, (DO3) holds since we defined $D(dx^i)$ via (4.2.1). □

Using this we can now extend L_X to all of $\mathcal{T}(M)$:

4.2.4 Corollary. *For any $X \in \mathfrak{X}(M)$ there exists a unique differential operator L_X on $\mathcal{T}(M)$ that coincided with L_X on each $\mathcal{C}^\infty(U)$ and $\mathfrak{X}(U)$.*

Proof. We have to verify the conditions from 4.2.3 for $E_U = L_X : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(U)$ and $F_U = L_X : \mathfrak{X}(U) \to \mathfrak{X}(U)$: (i) is (2.2.5), (ii) and (iv) follow from 4.2.1, and (iii) is a consequence of 2.2.17 (iv). □

4.2.5 Remark. Condition (DO3) is equivalent to the requirement that D commute with contractions, i.e., with

(DO4) For all $t \in \mathcal{T}^r_s(M)$, $\alpha_1, \dots, \alpha_r \in \Omega^1(M)$, $X_1, \dots, X_s \in \mathfrak{X}(M)$:

$$\begin{aligned} D(t(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s)) = (Dt)(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s) \\ + \sum_{j=1}^{r} t(\alpha^1, \dots, D\alpha^j, \dots \alpha^r, X_1, \dots, X_s) \\ + \sum_{k=1}^{s} t(\alpha^1, \dots \alpha^r, X_1, \dots, DX_k, \dots, X_s). \end{aligned}$$

Assuming (DO1), (DO2), the equivalence of (DO3) with (DO4) can be seen as follows: Write $t(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s)$ in local coordinates. Then noting that by (4.2.1) we have $D(\alpha(X)) = D\alpha(X) + \alpha(DX)$, a straightforward calculation using (DO1)–(DO3) gives (DO4). Conversely, applying (DO4) to the identity $\delta(\alpha, X) = \alpha(X)$ gives

$$(D\delta)(\alpha, X) + \delta(D\alpha, X) + \delta(\alpha, DX) = D(\delta(\alpha, X)) = D(\alpha(X)) = D\alpha(X) + \alpha(DX),$$

which implies $(D\delta)(\alpha, X) = 0$ for all α, X and thereby (DO3).

4.3 Differential Forms

In this section we wish to study alternating multilinear forms, first in the vector space setting and later on vector bundles.

4.3.1 Definition. *Let E be a finite dimensional vector space and $\Lambda^k E^* := L^k_{\text{alt}}(E, \mathbb{R})$ the space of all multilinear alternating maps from $E^k = E \times \cdots \times E$ to $\mathbb{R}$.*

4.3.2 Remark.

(i) $t \in L^k(E, \mathbb{R})$ is called alternating if

$$t(f_1, \dots, f_i, \dots, f_j, \dots, f_k) = -t(f_1 \dots, f_j, \dots, f_i, \dots, f_k) \ (1 \le i < j \le k).$$

Let $S_k := \{\varphi : \{1, \dots, k\} \to \{1, \dots, k\} \mid \varphi \text{ bijective }\}$ be the permutation group of order k. Then for $\sigma \in S_k$ and $t \in \Lambda^k E^*$ we have

$$t(f_{\sigma(1)}, \dots, f_{\sigma(k)}) = \operatorname{sgn}(\sigma) t(f_1, \dots, f_k)$$

For $\sigma, \tau \in S_k$, $\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}(\sigma) \cdot \operatorname{sgn}(\tau)$. Since S_k is a group, for all $\tau_0 \in S_k$ the map $\tau \mapsto \tau \circ \tau_0 : S_k \to S_k$ is a bijection.

(ii) We set $\Lambda^0 E^* = \mathbb{R}$. Moreover, $\Lambda^1 E^* = L^1_{\text{alt}}(E, \mathbb{R}) = L(E, \mathbb{R}) = E^*$.

(iii) $\Lambda^k E^*$ is a subspace of $T^0_k(E)$, the space of all multilinear maps $E \times \cdots \times E \to \mathbb{R}$.

4.3.3 Definition. *The map* $\operatorname{Alt} : T^0_k(E) \to T^0_k(E)$,

$$\operatorname{Alt}(t)(f_1, \dots, f_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) t(f_{\sigma(1)}, \dots, f_{\sigma(k)})$$

is called alternator.

4.3.4 Lemma. Alt *is a linear projection of* $T^0_k(E)$ *onto* $\Lambda^k E^*$, *i.e.,*

(i) Alt *is linear,* $\mathrm{Alt}(T^0_k(E)) \subseteq \Lambda^k E^*$.

(ii) $\mathrm{Alt}|_{\Lambda^k E^*} = \mathrm{id}_{\Lambda^k E^*}$.

(iii) $\mathrm{Alt} \circ \mathrm{Alt} = \mathrm{Alt}$.

(iv) $\mathrm{Alt}(T^0_k(E)) = \Lambda^k E^*$.

Proof. (i) Clearly, Alt is linear. Let $t \in T^0_k(E)$, $\tau \in S_k$. Then

$$\begin{aligned}\mathrm{Alt}(t)(f_{\tau(1)}, \ldots, f_{\tau(k)}) &= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) t(f_{\tau\sigma(1)}, \ldots, f_{\tau\sigma(k)}) \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\tau)\mathrm{sgn}(\tau \circ \sigma) t(f_{\tau\sigma(1)}, \ldots, f_{\tau\sigma(k)}) \\ &= \mathrm{sgn}(\tau)\mathrm{Alt}(t)(f_1, \ldots, f_k).\end{aligned}$$

(ii) If $t \in \Lambda^k E^*$, then

$$\mathrm{Alt}(t)(f_1, \ldots, f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) t(f_{\sigma(1)}, \ldots, f_{\sigma(k)}) = t(f_1, \ldots, f_k).$$

(iii) and (iv) follow from (i) and (ii). □

4.3.5 Definition. *Let* $\alpha \in T^0_k(E)$, $\beta \in T^0_l(E)$. *The exterior product (or wedge product) of* α *and* β *is defined as*

$$\alpha \wedge \beta := \frac{(k+l)!}{k!l!} \mathrm{Alt}(\alpha \otimes \beta)$$

For $\alpha \in T^0_0 E = \Lambda^0 E^* = \mathbb{R}$ *we set* $\alpha \wedge \beta = \beta \wedge \alpha = \alpha \cdot \beta$.

4.3.6 Example. Let $\alpha, \beta \in \Lambda^1 E^* = T^0_1(E) = E^*$. Then

$$\begin{aligned}(\alpha \wedge \beta)(f_1, f_2) &= \frac{2!}{1!1!} \frac{1}{2!} \sum_{\sigma \in S_2} \mathrm{sgn}(\sigma)(\alpha \otimes \beta)(f_{\sigma(1)}, f_{\sigma(2)}) \\ &= (\alpha \otimes \beta)(f_1, f_2) - (\alpha \otimes \beta)(f_2, f_1) = (\alpha \otimes \beta - \beta \otimes \alpha)(f_1, f_2).\end{aligned}$$

4.3.7 Proposition. *Let* $\alpha \in T^0_k(E)$, $\beta \in T^0_l(E)$, *and* $\gamma \in T^0_m(E)$. *Then:*

(i) $\alpha \wedge \beta = \mathrm{Alt}(\alpha) \wedge \beta = \alpha \wedge \mathrm{Alt}(\beta)$.

(ii) $\wedge$ *is bilinear.*

(iii) $\alpha \wedge \beta = (-1)^{k \cdot l} \beta \wedge \alpha$.

(iv) $\wedge$ *is associative:* $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

Proof. (i) For $\tau \in S_k$ and $\alpha \in T^0_k(E)$ let $(\tau\alpha)(f_1, \ldots, f_k) := \alpha(f_{\tau(1)}, \ldots, f_{\tau(k)})$. Then

$$\begin{aligned}\mathrm{Alt}(\tau\alpha)(f_1, \ldots, f_k) &= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \alpha(f_{\sigma\tau(1)}, \ldots, f_{\sigma\tau(k)}) \\ &= \mathrm{sgn}(\tau) \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma\tau) \alpha(f_{\sigma\tau(1)}, \ldots, f_{\sigma\tau(k)}) \\ &= \mathrm{sgn}(\tau)\mathrm{Alt}(\alpha)(f_1, \ldots, f_k).\end{aligned}$$

Therefore,

$$\mathrm{Alt}(\tau\alpha) = \mathrm{sgn}(\tau)\cdot \mathrm{Alt}(\alpha). \tag{4.3.1}$$

Using this, we obtain

$$\mathrm{Alt}(\mathrm{Alt}(\alpha)\otimes\beta)(f_1,\dots,f_{k+l}) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}} \mathrm{sgn}(\sigma)(\mathrm{Alt}(\alpha)\otimes\beta)(f_{\sigma(1)},\dots,f_{\sigma(k+l)})$$

$$= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}} \mathrm{sgn}(\sigma)\mathrm{Alt}(\alpha)(f_{\sigma(1)},\dots,f_{\sigma(k)})\beta(f_{\sigma(k+1)},\dots,f_{\sigma(k+l)})$$

$$= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}} \mathrm{sgn}(\sigma)\frac{1}{k!}\sum_{\tau\in S_k}\mathrm{sgn}(\tau)\underbrace{\alpha(f_{\sigma(\tau(1))},\dots,f_{\sigma(\tau(k))})}_{=(\tau\alpha)(f_{\sigma(1)},\dots,f_{\sigma(k)})}\beta(f_{\sigma(k+1)},\dots,f_{\sigma(k+l)})$$

$$= \frac{1}{k!}\sum_{\tau\in S_k}\mathrm{sgn}(\tau)\mathrm{Alt}((\tau\alpha)\otimes\beta)(f_1,\dots,f_{k+l}) = (*)$$

We define $\tau' \in S_{k+l}$ by

$$\tau'(1,\dots,k+l) := (\tau(1),\dots,\tau(k),k+1,\dots,k+l).$$

Then $\mathrm{sgn}(\tau') = \mathrm{sgn}(\tau)$ and $(\tau\alpha)\otimes\beta = \tau'(\alpha\otimes\beta)$. Therefore,

$$\begin{aligned} (*) &= \frac{1}{k!}\sum_{\tau\in S_k}\mathrm{sgn}(\tau')\mathrm{Alt}(\tau'(\alpha\otimes\beta))(f_1,\dots,f_{k+l}) \\ &\overset{(4.3.1)}{=} \mathrm{Alt}(\alpha\otimes\beta)(f_1,\dots,f_{k+l}), \end{aligned}$$

so $\mathrm{Alt}(\mathrm{Alt}(\alpha)\otimes\beta) = \mathrm{Alt}(\alpha\otimes\beta)$, and thereby $\mathrm{Alt}(\alpha)\wedge\beta = \alpha\wedge\beta$. The second equation in (i) follows in the same way.

(ii) is clear since $\otimes$ is bilinear and Alt is linear.

(iii) Let $\sigma_0 \in S_{k+l}$ be given by $\sigma_0(1,\dots,k+l) := (k+1,\dots,k+l,1,\dots,k)$. Then $\mathrm{sgn}(\sigma_0) = (-1)^{k\cdot l}$ and $\alpha\otimes\beta(f_1,\dots,f_{k+l}) = \beta\otimes\alpha(f_{\sigma_0(1)},\dots,f_{\sigma_0(k+l)})$. By (4.3.1) this entails

$$\alpha\wedge\beta = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\alpha\otimes\beta) = \frac{(k+l)!}{k!l!}\mathrm{Alt}(\sigma_0(\beta\otimes\alpha)) = (-1)^{kl}\beta\wedge\alpha.$$

(iv)

$$\begin{aligned} \alpha\wedge(\beta\wedge\gamma) &= \frac{(l+m)!}{l!m!}\alpha\wedge\mathrm{Alt}(\beta\otimes\gamma) \overset{(i)}{=} \frac{(l+m)!}{l!m!}\alpha\wedge(\beta\otimes\gamma) \\ &= \frac{(l+m)!}{l!m!}\frac{(k+l+m)!}{k!(l+m)!}\mathrm{Alt}(\underbrace{\alpha\otimes(\beta\otimes\gamma)}_{(\alpha\otimes\beta)\otimes\gamma}) \\ &= \frac{(k+l+m)!}{k!l!m!}\frac{(k+l)!m!}{(k+l+m)!}(\alpha\otimes\beta)\wedge\gamma \\ &\overset{(i)}{=} \underbrace{\frac{(k+l)!}{k!l!}\mathrm{Alt}(\alpha\otimes\beta)}_{=\alpha\wedge\beta}\wedge\gamma = (\alpha\wedge\beta)\wedge\gamma. \end{aligned}$$

□

4.3.8 Definition. $\Lambda E^* := \bigoplus_{k=0}^{\infty}\Lambda^k E^*$ *with the operations* $+$, $\lambda\cdot$ *and* $\wedge$ *is called the* exterior algebra *(or* Grassmann algebra*) of* E.

As we shall demonstrate in a moment, $\Lambda^k E^* = \{0\}$ for $k > n$, so in fact

$$\Lambda E^* = \bigoplus_{k=0}^{n} \Lambda^k E^*.$$

To prove this we need the following auxilliary result:

4.3.9 Lemma. *Let $\alpha^1, \dots, \alpha^k \in \Lambda^1 E^* = E^*$ and $f_1, \dots, f_k \in E$. Then*

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(f_1, \dots, f_k) = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \alpha^1(f_{\sigma(1)}) \cdot \cdots \cdot \alpha^k(f_{\sigma(k)})$$

Proof. We have to show that

$$\alpha^1 \wedge \cdots \wedge \alpha^k = k! \cdot \operatorname{Alt}(\alpha^1 \otimes \cdots \otimes \alpha^k)$$

This we do by induction, the case $k = 1$ being obvious. For $k - 1 \to k$ we calculate:

$$\begin{aligned}
\alpha^1 \wedge \cdots \wedge \alpha^k &\overset{4.3.7(iv)}{=} \alpha^1 \wedge (\alpha^2 \wedge \cdots \wedge \alpha^k) = \\
&\overset{\text{ind.hyp.}}{=} (k-1)! \alpha^1 \wedge \operatorname{Alt}(\alpha^2 \otimes \cdots \otimes \alpha^k) = \\
&\overset{4.3.7(i)}{=} (k-1)! \alpha^1 \wedge (\alpha^2 \otimes \cdots \otimes \alpha^k) = \\
&= (k-1)! \frac{(k-1+1)!}{(k-1)!1!} \operatorname{Alt}(\alpha^1 \otimes \cdots \otimes \alpha^k)
\end{aligned}$$

□

4.3.10 Proposition. *Let $n = \dim(E)$. Then $\dim(\Lambda^k E^*) = \binom{n}{k}$ for $0 \le k \le n$. For $k > n$, $\Lambda^k E^* = \{0\}$. Therefore, $\dim(\Lambda E^*) = \sum_{k=0}^n \binom{n}{k} = 2^n$. If $\{e_1, \dots, e_n\}$ is a basis of E and $\{\alpha^1, \dots, \alpha^n\}$ is the corresponding dual basis, then $\mathcal{B} := \{\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} \mid 1 \le i_1 < \cdots < i_k \le n\}$ is a basis of $\Lambda^k E^*$.*

Proof. $\mathcal{B}$ spans $\Lambda^k E^*$: Let $t \in \Lambda^k E^* \subseteq T^0_k(E)$. By 4.1.3, t is of the form

$$t = t(e_{i_1}, \dots, e_{i_k}) \alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}.$$

By 4.3.4 (ii) and 4.3.9,

$$t = \operatorname{Alt}(t) = t(e_{i_1}, \dots, e_{i_k}) \operatorname{Alt}(\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}) = \frac{1}{k!} t(e_{i_1}, \dots, e_{i_k}) \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}.$$

Since t is antisymmetric, all terms in this sum where two indices coincide vanish. In particular, $t = 0$ for $k > n$ (so $\Lambda^k E^* = \{0\}$ $\forall k > n$). If all i_j are distinct, then for any $\sigma \in S_k$ we have

$$t(e_{i_1}, \dots, e_{i_k}) \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} = \operatorname{sgn}(\sigma)^2 t(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(k)}}) \alpha^{i_{\sigma(1)}} \wedge \cdots \wedge \alpha^{i_{\sigma(k)}}$$

There are $k!$ such terms, so:

$$t = \sum_{1 \le i_1 < \cdots < i_k \le n} t(e_{i_1}, \dots, e_{i_k}) \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$$

$\mathcal{B}$ is linearly independent: let

$$\sum_{1 \le i_1 < \cdots < i_k \le n} t_{i_1 \dots i_k} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} = 0.$$

We have to show that all $t_{i_1\dots i_k}$ vanish. Let $1 \le i'_1 < \dots < i'_k \le n$ be some fixed k-tuple and choose $j'_{k+1} < \dots < j'_n$ such that $\{i'_1,\dots,i'_k\} \cup \{j'_{k+1},\dots,j'_n\} = \{1,\dots,n\}$. Then by 4.3.7,

$$\begin{aligned} 0 &= \sum_{1\le i_1<\dots<i_k\le n} t_{i_1\dots i_k}\alpha^{i_1}\wedge\dots\wedge\alpha^{i_k}\wedge\alpha^{j'_{k+1}}\wedge\dots\wedge\alpha^{j'_n} = \\ &= \pm t_{i'_1\dots i'_k}\alpha^1\wedge\dots\wedge\alpha^n. \end{aligned}$$

Since $\alpha^1\wedge\dots\wedge\alpha^n \neq 0$ (by 4.3.9, $\alpha^1\wedge\dots\wedge\alpha^n(e_1,\dots,e_n) = 1$), it follows that $t_{i'_1\dots i'_k} = 0$. □

4.3.11 Definition. *Let* $\dim(E) = n$, $\omega \in \Lambda^n E^*$, $\omega \neq 0$. *Then* ω *is called a volume element on* E. *Two volume elements* ω_1, ω_2 *are called equivalent if* $\omega_1 = c\cdot\omega_2$ *for some* $c > 0$ *(recall that* $\dim(\Lambda^n E^*) = 1$*). An equivalence class of volume elements on* E *is called an orientation of* E.

4.3.12 Proposition. *Let* $\dim(E) = n$, *and* $\varphi \in L(E,E)$. *Then there is a unique number* $\det\varphi \in \mathbb{R}$, *the determinant of* φ, *such that for the pullback-map*

$$\begin{aligned} \varphi^* : \Lambda^n E^* &\to \Lambda^n E^* \\ (\varphi^*\omega)(f_1,\dots,f_n) &:= \omega(\varphi(f_1),\dots,\varphi(f_n)) \end{aligned}$$

we have $\varphi^*\omega = \det\varphi\cdot\omega$, *for all* $\omega \in \Lambda^n E^*$.

Proof. Obviously, the map φ^* is linear: $\Lambda^n E^* \to \Lambda^n E^*$. By 4.3.10, $\dim(\Lambda^n E^*) = 1$. Thus with respect to any basis $\{\omega_0\}$ of $\Lambda^n E^*$, φ^* is given by a 1×1-matrix $c \in \mathbb{R}$. Hence for any $\omega = a\cdot\omega_0$ we have $\varphi^*\omega = c\cdot a\cdot\omega_0$, and we can set $\det\varphi := c$. □

4.3.13 Remark. The determinant in the sense of 4.3.12 is precisely the homonymous number from linear algebra: let $\mathcal{B} := \{e_1,\dots,e_n\}$ be a basis of E, $\{\alpha^1,\dots,\alpha^n\}$ the corresponding dual basis, and set $\omega := \alpha^1\wedge\dots\wedge\alpha^n$. Then

$$\begin{aligned} \det\varphi &= \det\varphi\,\omega(e_1,\dots,e_n) = \varphi^*\omega(e_1,\dots,e_n) = \omega(\varphi(e_1),\dots,\varphi(e_n)) \\ &\overset{4.3.9}{=} \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\alpha^1(\varphi(e_{\sigma(1)}))\cdot\dots\cdot\alpha^n(\varphi(e_{\sigma(n)})) \\ &= \sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\varphi_{1\sigma(1)}\dots\varphi_{n\sigma(n)} \end{aligned}$$

where $(\varphi_{ij})_{i,j}$ is the matrix representation of φ with respect to $\mathcal{B}$. □

4.3.14 Definition. *Let* $\varphi \in L(E,F)$, $\alpha \in T^0_k(F)$. *The* pullback *of* α *under* φ *is defined as* $\varphi^* : T^0_k(F) \to T^0_k(E)$,

$$\varphi^*(\alpha)(e_1,\dots,e_k) := \alpha(\varphi(e_1),\dots,\varphi(e_k)) \qquad (e_1,\dots,e_k \in E).$$

If φ *is bijective, then the* push-forward $\varphi_* : T^0_k(E) \to T^0_k(F)$ *is defined as* $\varphi_* := (\varphi^{-1})^*$. *Thus, for* $\alpha \in T^0_k(E)$,

$$\varphi_*(\alpha)(f_1,\dots,f_k) = \alpha(\varphi^{-1}(f_1),\dots,\varphi^{-1}(f_k)) \qquad (f_1,\dots,f_k \in F)$$

Then $\varphi_* = \varphi^0_k$ *in the sense of 4.1.4.*

4.3.15 Proposition. *Let $\varphi \in L(E,F)$, $\psi \in L(F,G)$. Then:*

(i) $\varphi^* : T_k^0(F) \to T_k^0(E)$ *is linear and* $\varphi^*(\Lambda^k F^*) \subseteq \Lambda^k E^*$.

(ii) $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

(iii) *If* $\varphi = \mathrm{id}_E$, *then* $\varphi^* = \mathrm{id}_{T_k^0(E)}$.

(iv) *If φ is bijective, then so is φ^* and* $(\varphi^*)^{-1} = (\varphi^{-1})^*$.

(v) *If φ is bijective, then so is φ_* and $(\varphi_*)^{-1} = \varphi^*$. If ψ is bijective, then* $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

(vi) *If* $\alpha \in \Lambda^k F^*$, $\beta \in \Lambda^l F^*$, *then* $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$.

Proof. (i) and (iii) are clear.

(ii) $(\psi \circ \varphi)^*\alpha(e_1, \dots, e_k) = \alpha(\psi(\varphi(e_1)), \dots, \psi(\varphi(e_k))) = (\psi^*\alpha)(\varphi(e_1), \dots, \varphi(e_k)) = \varphi^*(\psi^*\alpha)(e_1, \dots, e_k)$.

(iv) follows from (ii) and (iii).

(v) $(\varphi_*)^{-1} = ((\varphi^{-1})^*)^{-1} \stackrel{(iv)}{=} ((\varphi^*)^{-1})^{-1} = \varphi^*$.

(vi) $\varphi^*(\alpha \wedge \beta)(e_1, \dots, e_{k+l}) = (\alpha \wedge \beta)(\varphi(e_1), \dots, \varphi(e_{k+l})) \stackrel{4.3.3, 4.3.5}{=\!=\!=} ((\varphi^*\alpha) \wedge (\varphi^*\beta))\,(e_1, \dots, e_{k+l})$. □

We are now going to transfer the above constructions to the vector bundle setting, starting with the case of local vector bundles.

4.3.16 Definition. *Let $\varphi : U \times F \to U' \times F'$ be a local vector bundle isomorphism, $\varphi(u,f) = (\varphi_1(u), \varphi_2(u) \cdot f)$. Then let $\varphi_* : U \times \Lambda^k F^* \to U' \times \Lambda^k F'^*$,*

$$(u, \omega) \mapsto (\varphi_1(u), \varphi_2(u)_*(\omega)).$$

4.3.17 Remark. Since $\varphi_* = \varphi_k^0$, by 4.1.8 (and 4.3.15) φ_* is a local vector bundle isomorphism.

4.3.18 Definition. *Let (E, B, π) be a vector bundle, $E_b = \pi^{-1}(b)$ the fiber over $b \in B$. Then set*

$$\Lambda^k E^* := \bigsqcup_{b \in B} \Lambda^k E_b^* = \bigcup_{b \in B} \{b\} \times \Lambda^k E_b^*$$

and $\pi_k^0 : \Lambda^k E^* \to B$, $\pi_k^0(e) = b$ for $e \in \Lambda^k E_b^*$. For $A \subseteq B$ set $\Lambda^k E^*\big|_A = \bigsqcup_{b \in A} \Lambda^k E_b^*$.

4.3.19 Theorem. *Let (E, B, π) be a vector bundle with atlas $\mathcal{A} = \{(\Psi_\alpha, W_\alpha) \mid \alpha \in A\}$. Then $(\Lambda^k E^*, B, \pi_k^0)$ is a vector bundle with atlas $\tilde{\mathcal{A}} := \{((\Psi_\alpha)_*, \Lambda^k E^*\big|_{W_\alpha \cap B}) \mid \alpha \in A\}$, where $(\Psi_\alpha)_* = (\Psi_\alpha)_k^0$ (cf. 4.1.10), i.e., $(\Psi_\alpha)_*|_{\Lambda^k E_b^*} = (\Psi_\alpha|_{E_b})_*$.*

Proof. Clearly the $\Lambda^k E^*\big|_{W_\alpha \cap B}$ cover $\Lambda^k E^*$. By 4.3.15 (v), the $(\Psi_\alpha)_*|_{\Lambda^k E_b^*}$ are linear isomorphisms with image $\{\psi_{\alpha 1}(b)\} \times \Lambda^k(\mathbb{R}^n)^*$. The vector bundle chart transitions are local vector bundle isomorphisms according to 4.1.12 and 4.3.15. In fact, $(\Psi_\alpha)_* = (\Psi_\alpha)_k^0$. That $\Lambda^k E^*$ is Hausdorff and second countable follows again as in 2.2.4. □

Again our main interest is in the case $E = TM$:

4.3.20 Definition. *Let M be a manifold. Then $\Lambda^k T^*M := \Lambda^k(TM)^*$ is called the vector bundle of exterior k-forms on TM. The space of smooth sections of $\Lambda^k T^*M$ is denoted by $\Omega^k(M)$. Its elements are called* differential forms of order k *or (exterior) k-forms on M.*

Note that $\Omega^0(M) = \mathcal{C}^\infty(M)$ and $\Omega^1(M)$ is the space of 1-forms (cf. 4.1.14).

4.3.21 Remark.

(i) Due to 4.3.2 (iii), every fiber of $\Lambda^k T^*M$, i.e., every $\Lambda^k T_p^*M$ is precisely the subspace of $(T_pM)^0_k$ consisting of the alternating $\binom{0}{k}$-tensors. Thus, sections of $\Lambda^k T^*M$ are certain $\binom{0}{k}$-tensor fields, namely those which in every $p \in M$ are alternating multilinear maps.

(ii) Let (ψ, V) be a chart of M, $\psi = (x^1, \ldots, x^n)$. By 4.3.10, for every $p \in V$ the tuples $\{ dx^{i_1}\big|_p \wedge \cdots \wedge dx^{i_k}\big|_p \mid 1 \le i_1 < \cdots < i_k \le n\}$ form a basis of $\Lambda^k T_pM^*$. Hence every section ω of $\Lambda^k T^*M$ can locally be written as

$$\omega|_V = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{4.3.2}$$

with $\omega_{i_1 \ldots i_k} = \omega(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}})$. Since the vector bundle charts of $\Lambda^k T^*M$ are of the form $(T\psi)^0_k$, ω is smooth if and only if for every chart (ψ, V) the map $\psi_*\omega = (T\psi)^0_k \circ \omega \circ \psi^{-1} = (T\psi)_* \circ \omega \circ \psi^{-1}$ is smooth. As in the proof of 4.1.17 (only using 4.3.15 (vi) instead of 4.1.6 (iv)) it follows that

$$\psi_*\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \ldots i_k} \circ \psi^{-1} \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}.$$

Hence ω is smooth if and only if for every chart all local components $\omega_{i_1 \ldots i_k}$ are smooth.

(iii) By (i) and 4.1.18, a section ω of $\Lambda^k T^*M$ is smooth if and only if for all vector fields $X_1, \ldots, X_k \in \mathfrak{X}(M)$, $\omega(X_1, \ldots, X_k) \in \mathcal{C}^\infty(M)$.

(iv) By (i) and 4.1.19, $\Omega^k(M)$ is precisely the space of all $\mathcal{C}^\infty(M)$-multilinear and alternating maps $(\mathfrak{X}(M))^k \to \mathcal{C}^\infty(M)$.

(v) Apart from the operations $+$, $f\cdot$ and $\otimes$ studied so far, for differential forms also the exterior product is available: let $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$. Then set $\alpha \wedge \beta := p \mapsto \alpha(p) \wedge \beta(p) \in \Lambda^{k+l} T_pM^*$. It follows that $\alpha \wedge \beta \in \Omega^{k+l}(M)$ (smoothness follows from (ii) or (iii)).

$\Omega(M) := \bigoplus_{k=0}^n \Omega^k(M)$ with these operations is called the algebra of differential forms on M.

In 4.1.15, 4.1.16 we introduced the exterior derivative df of a smooth function f. We now wish to extend this operation from $\Omega^0(M)$ to general $\Omega^k(M)$.

4.3.22 Theorem. *Let M be a manifold. For every open $U \subseteq M$ there exists a uniquely determined family of maps $d^k(U) : \Omega^k(U) \to \Omega^{k+1}(U)$, denoted simply by d, such that:*

(i) d is $\mathbb{R}$-linear and for $\alpha \in \Omega^k(U)$, $\beta \in \Omega^l(U)$ we have:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta.$$

(ii) *For $f \in \Omega^0(U) = \mathcal{C}^\infty(U)$, df is the exterior derivative from 4.1.15.*

(iii) $d \circ d = 0$.

(iv) *If U, V are open, $U \subseteq V \subseteq M$ and $\alpha \in \Omega^k(V)$, then $d(\alpha|_U) = (d\alpha)|_U$, i.e.,*

$$\begin{array}{ccc} \Omega^k(V) & \xrightarrow{\ |_U\ } & \Omega^k(U) \\ d\big\downarrow & & \big\downarrow d \\ \Omega^{k+1}(V) & \xrightarrow{\ |_U\ } & \Omega^{k+1}(U) \end{array}$$

d is called exterior derivative.

Proof. *Uniqueness:* By (iv) it suffices to show that d is uniquely determined on any chart (ψ, U). Thus let $\omega \in \Omega^k(U)$. By 4.3.21 (ii),

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Hence due to (i), (ii), (iii) we necessarily have:

$$\begin{aligned} d\omega &= \sum_{1 \le i_1 < \cdots < i_k \le n} d(\omega_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ &= \sum_{1 \le i_1 < \cdots < i_k \le n} d\omega_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad (*) \\ &\quad + \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \ldots i_k} \underbrace{d(dx^{i_1})}_{=0} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \\ &\quad + 0 + \cdots + 0, \end{aligned}$$

and uniqueness follows.

Existence: For any chart domain we define d by $(*)$ above. We first show that this d has the claimed properties (i)–(iv):

(i): Linearity being obvious, it suffices to calculate $d(\alpha \wedge \beta)$ for $\alpha = f_0 df_1 \wedge \cdots \wedge df_k$, $\beta = g_0 dg_1 \wedge \cdots \wedge dg_l$. We first note that By 4.1.16 (iv), $d(f_0 g_0) = g_0 df_0 + f_0 dg_0$. Thus

$$\begin{aligned} d(\alpha \wedge \beta) &= d(f_0 g_0 df_1 \wedge \cdots \wedge df_k \wedge dg_1 \wedge \cdots \wedge dg_l) \\ &\overset{(*)}{=} d(f_0 g_0) \wedge df_1 \wedge \cdots \wedge df_k \wedge dg_1 \wedge \cdots \wedge dg_l \\ &= g_0 df_0 \wedge df_1 \wedge \cdots \wedge dg_l + f_0 dg_0 \wedge df_1 \wedge \cdots \wedge \cdots \wedge dg_l \\ &= d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \end{aligned}$$

(ii) and(iv) are obvious.

(iii): It suffices to show that $d(df) = 0$ for all $f \in \mathcal{C}^\infty(U)$. By (4.1.2), $df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i$. Hence,

$$d(df) = \sum_{i=1}^n d\Big(\frac{\partial f}{\partial x^i}\Big) \wedge dx^i = \sum_{i,j} \underbrace{\frac{\partial}{\partial x^j}\Big(\frac{\partial f}{\partial x^i}\Big)}_{\text{symm. in } i,j} \underbrace{dx^j \wedge dx^i}_{\text{antisymm}} = 0.$$

It remains to show that the above gives a well-defined global object on M. To this end, let $\tilde\psi = (y^1, \ldots, y^n)$ be another chart, w.l.o.g. with the same domain U. Define $\tilde d$ by $(*)$ (with $x \leftrightarrow y$). By the proof of uniqueness, it follows that

$$\tilde d\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \underbrace{\tilde d\omega_{i_1 \ldots i_k}}_{\overset{(ii)}{=} d\omega_{i_1 \ldots i_k}} \wedge \underbrace{\tilde d x^{i_1}}_{=dx^{i_1}} \wedge \cdots \wedge \underbrace{\tilde d x^{i_k}}_{=dx^{i_k}} = d\omega.$$

Thus d looks the same in any chart, hence is globally well-defined. □

4.3.23 Example.

(i) Let $\omega = P(x,y)dx + Q(x,y)dy$ be a 1-form on $\mathbb{R}^2$. Then

$$\begin{aligned}
d\omega &= dP \wedge dx + dQ \wedge dy = \\
&= (\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy) \wedge dx + (\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy) \wedge dy = \\
&= (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy.
\end{aligned}$$

(ii) Let $\omega = P(x,y,z)dy \wedge dz + Q(x,y,z)dz \wedge dx + R(x,y,z)dx \wedge dy$. Then

$$d\omega = (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dx \wedge dy \wedge dz.$$

4.3.24 Definition. *Let M, N be manifolds, and $F : M \to N$ smooth. For $\omega \in \mathcal{T}_k^0(N)$, the pullback of ω under F is defined as $F^*\omega(p) := (T_pF)^*(\omega(F(p)))$ (cf. 4.3.14). For $X_1, \dots, X_k \in T_pM$ we therefore have*

$$\begin{aligned}
F^*\omega(p)(X_1, \dots X_k) &= \omega(F(p))(T_pF(X_1), \dots, T_pF(X_k)) \\
&\overset{4.3.14}{=} (T_pF)^*(\omega|_{F(p)}).
\end{aligned} \tag{4.3.3}$$

In particular, $F^*f = f \circ F$ for $f \in \mathcal{C}^\infty(N) = \Omega^0(N)$.

4.3.25 Lemma. *Let $F : M \to N$, $G : N \to P$ be smooth. Then*

(i) $F^* : \mathcal{T}_k^0(N) \to \mathcal{T}_k^0(M)$, $F^* : \Omega^k(N) \to \Omega^k(M)$.

(ii) $(G \circ F)^* = F^* \circ G^*$.

(iii) $\mathrm{id}_M^* = \mathrm{id}_{\Omega^k(M)}$ *(resp.* $= \mathrm{id}_{\mathcal{T}_k^0(M)}$*).*

(iv) If F is a diffeomorphism, then F^* is a linear isomorphism and $(F^*)^{-1} = (F^{-1})^*$.

Proof. (i) By 4.3.15 (i), $(T_pF)^*(\omega(F(p))) \in T_k^0(T_pM)$ resp. $\in \Lambda^k(T_pM)^*$. Thus we only have to show that $F^*\omega$ is smooth. To this end, let (φ, U), (ψ, V) be charts of M resp. N with $F(U) \subseteq V$. Then both $F_{\psi\varphi} = \psi \circ F \circ \varphi^{-1}$ and $\psi_*\omega = (T\psi)_* \circ \omega \circ \psi^{-1}$ are smooth (see 4.1.17, 4.3.21 (ii)).

By 4.3.15 (ii) we get (setting $p = \varphi^{-1}(x)$):

$$\begin{aligned}
(DF_{\psi\varphi}(x))^* &= (T_xF_{\psi\varphi})^* \\
&= (T_{F(p)}\psi \circ T_pF \circ (T_p\varphi)^{-1})^* \\
&= \underbrace{((T_p\varphi)^{-1})^*}_{\overset{4.3.15(v)}{=}(T_p\varphi)_*} \circ (T_pF)^* \circ (T_{F(p)}\psi)^* \qquad (*)
\end{aligned}$$

Hence, by 4.1.17, 4.3.21 (ii), the local representation $\varphi_*(F^*\omega)(x)$ of $F^*\omega$ with respect to φ is given by

$$\begin{aligned}
& (T\varphi)_* \circ F^*\omega \circ \varphi^{-1}(x) \\
= \;& (T_p\varphi)_* \circ (T_pF)^*(\omega \circ F \circ \varphi^{-1}(x)) \\
= \;& (T_p\varphi)_* \circ (T_pF)^* \circ (T_{F(p)}\psi)^* \big((T_{F(p)}\psi)_* \circ \omega \circ \psi^{-1} \circ \psi \circ F \circ \varphi^{-1}(x)\big) \\
\overset{(*)}{=} \;& \underbrace{(DF_{\psi\varphi}(x))^*}_{\mathcal{C}^\infty}(\underbrace{(\psi_*\omega)}_{\mathcal{C}^\infty} \circ \underbrace{F_{\psi\varphi}}_{\mathcal{C}^\infty}(x))
\end{aligned}$$

which is smooth by the chain rule.

(ii)

$$\begin{aligned}
(G\circ F)^*(\omega)(p) \quad &= \quad (T_p(G\circ F))^*(\omega(G\circ F(p))) = \\
&= \quad (T_{F(p)}G \circ T_pF)^*(\omega(G\circ F(p))) = \\
&\overset{4.3.15(ii)}{=} \quad (T_pF)^* \circ (T_{F(p)}G)^*(\omega(G(F(p)))) = \\
&= \quad (T_pF)^*(G^*\omega(F(p))) = F^*(G^*\omega)(p)
\end{aligned}$$

(iii) Obvious.

(iv) Follows from (ii) and (iii). $\square$

4.3.26 Theorem. *Let $F : M \to N$ be smooth. Then:*

(i) $F^* : \Omega(N) \to \Omega(M)$ is an algebra homomorphism, i.e., it is linear and $F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$.

(ii) For all $\omega \in \Omega(N)$, $F^*(d\omega) = d(F^*\omega)$.

Proof. (i) To begin with, let $\alpha = f \in \Omega^0(N) = \mathcal{C}^\infty(N)$. Then

$$\begin{aligned}
F^*(f\wedge\beta)(p) &= F^*(f\cdot\beta)(p) = \\
&= (T_pF)^*(f(F(p))\beta(F(p))) = \\
&= \underbrace{f(F(p))}_{F^*f(p)}\,\underbrace{(T_pF)^*(\beta(F(p)))}_{F^*\beta(p)} \\
&= (F^*f \wedge F^*\beta)(p).
\end{aligned}$$

In the general case we have

$$\begin{aligned}
F^*(\alpha\wedge\beta)(p) \quad &= \quad (T_pF)^*(\alpha(F(p)) \wedge \beta(F(p))) = \\
&\overset{4.3.15(vi)}{=} \quad (T_pF)^*(\alpha(F(p))) \wedge (T_pF)^*(\beta(F(p))) = \\
&= \quad ((F^*\alpha)\wedge(F^*\beta))(p).
\end{aligned}$$

(ii) By definition of F^* and 4.3.22 (iv) it suffices to show that every $p \in M$ has a neighborhood U with $d(F^*\omega|_U) = (F^*d\omega)|_U$ for all $\omega \in \Omega(N)$. Let (ψ, V) be a chart around $F(p)$, and U a neighborhood of p with $F(U) \subseteq V$. Then for $\omega \in \Omega^k(V)$ we have

$$\begin{aligned}
\omega \quad &= \quad \sum_{1\le i_1<\cdots<i_k\le n} \omega_{i_1\ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\
d\omega \quad &= \quad \sum_{1\le i_1<\cdots<i_k\le n} d\omega_{i_1\ldots i_k} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}
\end{aligned}$$

By (i),

$$F^*\omega|_U = \sum F^*\omega_{i_1\dots i_k} F^*(dx^{i_i}) \wedge \dots \wedge F^*(dx^{i_k}) \qquad (*)$$

In general, for $f \in \mathcal{C}^\infty(N)$, $F^*(df) = d(F^*f)$. In fact, if $X \in T_pM$, then

$$\begin{aligned} F^*(df)(p)(X) &= df(F(p))(T_pF(X)) = T_{F(p)}f(T_pF(X)) \\ &= T_p(f \circ F)(X) = d(\underbrace{f \circ F}_{=F^*f})(p)(X). \end{aligned}$$

Thus, from $(*)$ we conclude that

$$\begin{aligned} d(F^*\omega|_U) &= d(\sum F^*\omega_{i_1\dots i_k} d(F^*x^{i_1}) \wedge \dots \wedge d(F^*x^{i_k})) \\ &= \sum d(F^*\omega_{i_1\dots i_k}) \wedge d(F^*x^{i_1}) \wedge \dots \wedge d(F^*x^{i_k}) \\ &= \sum F^*(d\omega_{i_1\dots i_k}) \wedge F^*(dx^{i_1}) \wedge \dots \wedge F^*(dx^{i_k}) \\ &= F^*(\sum d\omega_{i_1\dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}) \\ &= (F^*d\omega)|_U\,. \end{aligned}$$

□

4.3.27 Proposition. *Let M be a manifold, $p \in M$, (φ, V), (ψ, V) charts around p, $\varphi = (x^1, \dots, x^n)$, $\psi = (y^1, \dots, y^n)$. Then:*

(i) $dx^i\big|_p = \sum_{k=1}^n D_k(\varphi^i \circ \psi^{-1})(\psi(p))\, dy^k\big|_p = \sum_{k=1}^n \dfrac{\partial x^i}{\partial y^k}\Big|_p\, dy^k\big|_p$

(ii) $dx^1 \wedge \dots \wedge dx^n\big|_p = \det D(\varphi \circ \psi^{-1})(\psi(p))\, dy^1 \wedge \dots \wedge dy^n\big|_p$

(iii) If $\omega \in \Omega^n(M)$, $\varphi_*\omega = \omega_\varphi \alpha^1 \wedge \dots \wedge \alpha^n$, $\psi_*\omega = \omega_\psi \alpha^1 \wedge \dots \wedge \alpha^n$ ($\alpha^1, \dots, \alpha^n$ the standard basis of $(\mathbb{R}^n)^*$), then:

$$\omega_\psi(y) = \omega_\varphi(\varphi \circ \psi^{-1}(y)) \cdot \det D(\varphi \circ \psi^{-1})(y) \qquad \forall y \in \psi(V)$$

Proof. (i) Since $\{dx^i\big|_p \mid 1 \le i \le n\}$ is the dual basis of $\{\frac{\partial}{\partial x^j}\big|_p \mid 1 \le j \le n\}$ it suffices to show that

$$\left(\sum_{k=1}^n \frac{\partial x^i}{\partial y^k}\Big|_p dy^k\big|_p\right)(\frac{\partial}{\partial x^j}\Big|_p) = \delta_{ij}.$$

In fact,

$$\sum_{k=1}^n \frac{\partial x^i}{\partial y^k}\Big|_p \underbrace{dy^k\big|_p(\frac{\partial}{\partial x^j}\Big|_p)}_{=\frac{\partial y^k}{\partial x^j}\big|_p} = \sum_{k=1}^n \underbrace{D_k(\varphi^i \circ \psi^{-1})(\psi(p))}_{[D(\varphi\circ\psi^{-1})]_{ik}} \cdot \underbrace{D_j(\psi^k \circ \varphi^{-1})(\varphi(p))}_{[D(\psi\circ\varphi^{-1})]_{kj}} = \delta_{ij}.$$

(ii) By (i) we obtain (recall the summation convention!):

$$\begin{aligned}
dx^1 \wedge \cdots \wedge dx^n\big|_p &= \Big(\frac{\partial x^1}{\partial y^{\sigma_1}}\Big|_p dy^{\sigma_1}\big|_p\Big) \wedge \cdots \wedge \Big(\frac{\partial x^n}{\partial y^{\sigma_n}}\Big|_p dy^{\sigma_n}\big|_p\Big) = \\
&= \frac{\partial x^1}{\partial y^{\sigma_1}}\Big|_p \cdots \frac{\partial x^n}{\partial y^{\sigma_n}}\Big|_p \underbrace{dy^{\sigma_1} \wedge \cdots \wedge dy^{\sigma_n}\big|_p}_{=\begin{cases} \operatorname{sgn}(\sigma)\, dy^1\wedge\cdots\wedge dy^n|_p & \sigma\in S_n \\ 0 & \text{else}\end{cases}} \\
&= \underbrace{\left(\sum_{\sigma\in S_n} \frac{\partial x^1}{\partial y^{\sigma_1}}\Big|_p \cdot \ldots \frac{\partial x^n}{\partial y^{\sigma_n}}\Big|_p \cdot \operatorname{sgn}(\sigma)\right)}_{=\det(D(\varphi\circ\psi^{-1})(\psi(p)))} \cdot dy^1 \wedge \cdots \wedge dy^n\big|_p
\end{aligned}$$

(iii) Let $\omega = f\,dx^1 \wedge \cdots \wedge dx^n = g\,dy^1 \wedge \cdots \wedge dy^n$. Then by 4.3.21 (ii), $\omega_\varphi = f \circ \varphi^{-1}$, $\omega_\psi = g \circ \psi^{-1}$. Thus (ii) gives

$$\begin{aligned}
f(p)\; dx^1 \wedge \cdots \wedge dx^n\big|_p &= f(p) \det D(\varphi\circ\psi^{-1})(\psi(p))\; dy^1 \wedge \cdots \wedge dy^n\big|_p = \\
&= g(p)\; dy^1 \wedge \cdots \wedge dy^n\big|_p .
\end{aligned}$$

Hence,

$$\begin{aligned}
\omega_\psi(y) &= g(\psi^{-1}(y)) = f(\psi^{-1}(y)) \det D(\varphi\circ\psi^{-1})(y) \\
&= \omega_\varphi(\varphi\circ\psi^{-1}(y)) \det D(\varphi\circ\psi^{-1})(y)
\end{aligned}$$

$\square$

4.3.28 Remark. A direct proof of 4.3.27 (iii) can be based on 4.3.12: Let $\psi_*\omega = \omega_\psi \alpha^1 \wedge \cdots \wedge \alpha^n$, $\varphi_*\omega = \omega_\varphi \alpha^1 \wedge \cdots \wedge \alpha^n$. Then

$$\begin{aligned}
\omega_\psi(y)\alpha^1 \wedge \cdots \wedge \alpha^n &= (\psi^{-1})^* \circ \varphi^*(\omega_\varphi \alpha^1 \wedge \cdots \wedge \alpha^n)(y) \\
&= (T_y(\varphi\circ\psi^{-1}))^*(\omega_\varphi(\varphi\circ\psi^{-1}(y))\alpha^1 \wedge \cdots \wedge \alpha^n) \\
&\overset{4.3.12}{=} \det(D(\varphi\circ\psi^{-1}))(y)\omega_\varphi(\varphi\circ\psi^{-1}(y))\alpha^1 \wedge \cdots \wedge \alpha^n,
\end{aligned}$$

so $\omega_\psi(y) = \omega_\varphi(\varphi\circ\psi^{-1}(y)) \cdot \det D(\varphi\circ\psi^{-1})(y)$.

In 4.3.24 we defined the pullback of any element of $\mathcal{T}^0_k(N)$ under any smooth map $F : M \to N$. If we additionally suppose that F is a diffeomorphism, then we can similarly define the pullback of an arbitrary tensor field $t \in \mathcal{T}^r_s(N)$ under F, by setting (for $X_i \in T_pM$, $\alpha_j \in T^*_pM$)

$$\begin{aligned}
&(F^*t)(p)(\alpha^1, \ldots, \alpha^r, X_1, \ldots, X_s) := \\
&\qquad t(F(p))((T_pF)^0_1(\alpha^1), \ldots, (T_pF)^0_1(\alpha^r), T_pF(X_1), \ldots, T_pF(X_s)).
\end{aligned} \tag{4.3.4}$$

Here, according to 4.1.5, $(T_pF)^0_1 = ((T_pF)^{-1})^*$ is the inverse transpose of T_pF (which exists since F is a diffeomorphism). Explicitly, $(T_pF)^0_1(\alpha) = \alpha \circ T_pF^{-1}$. Also, for $\alpha \in \Omega^1(N)$, $(F^*\alpha)_p = \alpha_{F(p)} \circ T_pF$. As usual, we define push-forward via the inverse map: $F_* := (F^{-1})^*$. Note that, using the notation from 4.1.4, we have

$$(F^*t)_p = (T_{F(p)}F^{-1})^r_s(t(F(p))). \tag{4.3.5}$$

The following result collects some basic properties of the pullback operation.

4.3.29 Proposition. *Let $F : M \to N$ be a diffeomorphism.*

(i) For $t_1, t_2 \in \mathcal{T}(N)$ we have $F^*(t_1 \otimes t_2) = (F^*t_1) \otimes (F^*t_2)$.

(ii) Let $t \in \mathcal{T}^r_s(N)$, $X_i \in \mathfrak{X}(N)$, $\alpha^j \in \Omega^1(N)$, then

$$F^*(t(\alpha^1, \dots, \alpha^r, X_1, \dots, X_s)) = (F^*t)(F^*\alpha^1, \dots, F^*\alpha^r, F^*X_1, \dots, F^*X_s).$$

(iii) $F^*(\delta^N) = \delta^M$ (with δ^M, δ^N the Kronecker delta tensors on M, N).

(iv) For $\alpha \in \Omega^1(N)$ and $X \in \mathfrak{X}(N)$, $F^*(\alpha(X)) = (F^*\alpha)(F^*X)$.

(v) If also $G : N \to P$ is a diffeomorphism and $t \in \mathcal{T}(P)$, then $(G \circ F)^*t = F^*(G^*t)$.

Proof. (i) follows directly from the definitions.

(ii) To keep notations in reasonable bounds, we show this for $t \in \mathcal{T}^1_1$ (the general case being completely analogous):

$$\begin{aligned}(F^*t)|_p((F^*\alpha)_p,(F^*X)_p) &\overset{(4.3.4)}{=} (F^*t)|_p(\alpha_{F(p)} \circ T_pF, (TF^{-1} \circ X \circ F)_p)\\ &\overset{(4.3.4)}{=} t|_{F(p)}((T_pF)^0_1)(\alpha_{F(p)} \circ T_pF), T_pF(TF^{-1} \circ X \circ F)_p)\\ &= t|_{F(p)}(\alpha_{F(p)}, X_{F(p)}) = (t(\alpha, X))_{F(p)}.\end{aligned}$$

(iii) Let $\alpha \in T^*_pM$, $X \in T_pM$. Then

$$\begin{aligned}(F^*\delta^N)_p(\alpha, X) &= \delta^N_{F(p)}(\alpha \circ T_pF^{-1}, T_pF(X)) = \alpha(T_pF \circ T_pF^{-1}X)\\ &= \alpha(X) = \delta^M(\alpha, X).\end{aligned}$$

(iv)

$$\begin{aligned}F^*(\alpha(X))_p = F^*(\delta^N(\alpha, X))_p &\overset{(ii)}{=} (F^*\delta^N)(F^*\alpha, F^*X)\\ &\overset{(iii)}{=} \delta^M(F^*\alpha, F^*X) = F^*\alpha(F^*X).\end{aligned}$$

(v) is immediate from (4.3.5) and 4.1.6 (i). □

As an application, we prove the naturality of the Lie derivative on the full tensor algebra with respect to push-forward under diffeomorphisms:

4.3.30 Corollary. *Let M, N be manifolds, $F : M \to N$ a diffeomorphism, and $X \in \mathfrak{X}(M)$. Then L_X is natural with respect to push-forward under F, i.e.: $L_{F_*X}(F_*t) = F_*(L_Xt)$ for all $t \in \mathcal{T}^r_s(M)$ $(r, s \geq 0)$.*

$$\begin{array}{ccc}\mathcal{T}^r_s(M) & \xrightarrow{F_*} & \mathcal{T}^r_s(N)\\ \downarrow{\scriptstyle L_X} & & \downarrow{\scriptstyle L_{F_*X}}\\ \mathcal{T}^r_s(M)) & \xrightarrow{F_*} & \mathcal{T}^r_s(N)\end{array}$$

Proof. Writing t locally as $t = t^{i_1\dots i_r}_{j_1\dots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$, we have $F_*t = F_*t^{i_1\dots i_r}_{j_1\dots j_s} F_*\Big(\frac{\partial}{\partial x^{i_1}}\Big) \otimes \dots \otimes F_*\Big(\frac{\partial}{\partial x^{i_r}}\Big) \otimes F_*(dx^{j_1}) \otimes \dots \otimes F_*(dx^{j_s})$. Using the product rule (DO1) from 4.2.2, together with 4.3.29 (i), we can therefore reduce the claim to the special cases $t = t^{i_1\dots i_r}_{j_1\dots j_s} \in \mathcal{C}^\infty(U)$, $t = \frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$, both of which follow from 4.2.1, and $t = dx^j$ a one-form. It therefore remains to settle the case of a one-form α, for which we use (DO4) from 4.2.5: We have $L_X\alpha(Y) = L_X(\alpha(Y)) - \alpha(L_XY)$. Applying F_* to both sides of this equation and using 4.3.29 (ii) and the result for smooth functions and vector fields, we obtain:

$$F_*(L_X\alpha)(F_*Y) = L_{F_*X}((F_*\alpha)(F_*Y)) - (F_*\alpha)(L_{F_*X}F_*Y) \stackrel{\text{(DO4)}}{=} (L_{F_*X}F_*\alpha)(F_*Y).$$

Since F_*Y can be any vector field here, the claim follows for α as well. $\square$

Recall from 2.3.10 and 2.3.12 the definition of L_X on smooth functions and vector fields via differentiation of the pullback under the flow of X. We next want to show that such a description remains valid also for the extension of the Lie derivative to the full tensor algebra. Thus let $p \in M$ and let (V, a, Fl^X) be a flow box at p (cf. 2.3.5). Then for each $\lambda \in I_a = (-a, a)$ the flow map $\mathrm{Fl}^X_\lambda : V \to V_\lambda := \mathrm{Fl}^X_\lambda(V)$ is a diffeomorphism. For $t \in \mathcal{T}^r_s(M)$ we now set $t_\lambda := (\mathrm{Fl}^X_\lambda)^*(t|_{V_\lambda}) \in \mathcal{T}^r_s(V)$. Now set

$$\begin{aligned} t_\sharp(p) : I_a &\to T^r_s(T_pM) \\ \lambda &\mapsto t_\lambda(p). \end{aligned}$$

Using this definition we have:

4.3.31 Theorem. *$t_\sharp(p)$ is a smooth curve in (the vector space) $T^r_s(T_pM)$ and*

$$L_Xt(p) = t_\sharp(p)'(0) = \frac{d}{d\lambda}\Big|_{\lambda=0} (\mathrm{Fl}^X_\lambda)^*(t)_p$$

Proof. We first show smoothness of the curve $t_\sharp(p)$: Writing t locally as $t = t^{i_1\dots i_r}_{j_1\dots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$, by 4.3.29 (i) and (v) we have

$$\begin{aligned} (\mathrm{Fl}^X_\lambda)^*t = (\mathrm{Fl}^X_\lambda)^* t^{i_1\dots i_r}_{j_1\dots j_s} (\mathrm{Fl}^X_\lambda)^*\Big(\frac{\partial}{\partial x^{i_1}}\Big)\otimes& \\ \otimes \dots \otimes (\mathrm{Fl}^X_\lambda)^*\Big(\frac{\partial}{\partial x^{i_r}}\Big) \otimes (\mathrm{Fl}^X_\lambda)^*&(dx^{j_1}) \otimes \dots \otimes (\mathrm{Fl}^X_\lambda)^*(dx^{j_s}) \end{aligned}$$

Here, $(\mathrm{Fl}^X_\lambda)^* t^{i_1\dots i_r}_{j_1\dots j_s} = t^{i_1\dots i_r}_{j_1\dots j_s} \circ \mathrm{Fl}^X_\lambda$ depends smoothly on (λ, p), and so does each $(\mathrm{Fl}^X_\lambda)^*(dx^j)$ (by 4.3.26 (ii) and (4.1.2)). For $(\mathrm{Fl}^X_\lambda)^*\Big(\frac{\partial}{\partial x^i}\Big)$ we have, writing $X = \sum_i X^i \frac{\partial}{\partial x^i}$:

$$(\mathrm{Fl}^X_\lambda)^*X = \sum_i X^i \circ \mathrm{Fl}^X_\lambda \cdot (\mathrm{Fl}^X_\lambda)^*\Big(\frac{\partial}{\partial x^i}\Big).$$

Furthermore, $(\mathrm{Fl}^X_\lambda)^*\big(\frac{\partial}{\partial x^i}\big|_p\big) = T(\mathrm{Fl}^X_\lambda)^{-1}\big(\frac{\partial}{\partial x^i}\big|_{\mathrm{Fl}^X_\lambda(p)}\big)$, which again is smooth in (λ, p) by (2.1.5) (with $\varphi = \psi$). Altogehter, $(\lambda, p) \mapsto ((\mathrm{Fl}^X_\lambda)^*t)_p$ is smooth as a map into T^r_sM by 4.1.17. For fixed p it takes values in $T^r_s(T_pM)$, hence by the analogue of 3.3.26 for T^r_sM, it is smooth also as a map into the vector space $T^r_s(T_pM)$. Since $\mathrm{Fl}^X_0 = \mathrm{id}_M$, $t_\sharp(p)(0) = t_p$.

We now define an operator $E_X : \mathcal{T}(M) \to \mathcal{T}(M)$ by $E_Xt(p) := t_\sharp(p)'|_{\lambda=0}$. Then E_X is $\mathbb{R}$-linear, and it is a tensor derivation, as can immediately be read off from the local representation given above: inserting basis vectors we see that the λ-derivative distributes over the tensor product by the usual Leibnitz rule. Moreover, E_X is natural with respect to restrictions because it is defined locally. Furthermore, using 4.3.29 (iii) we have

$$E_X\delta = \frac{d}{d\lambda}\Big|_{\lambda=0} (\mathrm{Fl}^X_\lambda)^*(\delta) = \frac{d}{d\lambda}\Big|_{\lambda=0} \delta = 0.$$

By Willmore's theorem 4.2.3 we conclude that E_X is a differential operator on $\mathcal{T}(M)$. Moreover, by 2.3.10 and 2.3.12, E_X coincides with L_X on both $\mathcal{C}^\infty(M)$ and $\mathfrak{X}(M)$. Applying 4.2.3 once more we conclude that $E_X = L_X$. $\square$

4.3.32 Corollary. *Let $t \in \mathcal{T}(M)$ and $X \in \mathfrak{X}(M)$. Then*

$$\frac{d}{d\lambda}(\mathrm{Fl}^X_\lambda)^* t = (\mathrm{Fl}^X_\lambda)^* L_X t = L_X((\mathrm{Fl}^X_\lambda)^* t).$$

Proof. We have

$$\frac{d}{d\lambda}(\mathrm{Fl}^X_\lambda)^* t = \frac{d}{d\mu}\bigg|_0 (\mathrm{Fl}^X_{\lambda+\mu})^* t = \frac{d}{d\mu}\bigg|_0 (\mathrm{Fl}^X_\mu)^* (\mathrm{Fl}^X_\lambda)^* t = L_X((\mathrm{Fl}^X_\lambda)^* t).$$

On the other hand, by 4.3.30, $L_X((\mathrm{Fl}^X_\lambda)^* t) = (\mathrm{Fl}^X_\lambda)^*(L_{(\mathrm{Fl}^X_{-\lambda})^* X} t)$, and since $[X, X] = 0$, 2.3.18 shows that $(\mathrm{Fl}^X_{-\lambda})^* X = X$, which implies the second equality. $\square$

4.3.33 Theorem. *Let $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$. Then $L_X\omega \in \Omega^k(M)$ and $dL_X\omega = L_X d\omega$:*

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{L_X} & \Omega^k(M) \\ \Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle d} \\ \Omega^{k+1}(M) & \xrightarrow{L_X} & \Omega^{k+1}(M) \end{array}$$

Proof. By 4.3.6 we have $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$ for any one-forms α, β. Using this and the fact that L_X is a tensor derivation, induction therefore gives

$$L_X(\alpha^1 \wedge \cdots \wedge \alpha^k) = L_X\alpha^1 \wedge \cdots \wedge \alpha^k + \cdots + \alpha^1 \wedge \cdots \wedge L_X\alpha^k \tag{4.3.6}$$

for any $\alpha^1, \ldots, \alpha^k \in \Omega^1(M)$. Locally, any $\omega \in \Omega^k(M)$ is a linear combination of such terms, hence $L_X\omega \in \Omega^k(M)$.

Moreover, using 4.3.31 and the linearity of d, we have

$$dL_X\omega = d\Big(\frac{d}{d\lambda}\bigg|_0 ((\mathrm{Fl}^X_\lambda)^*\omega)\Big) = \frac{d}{d\lambda}\bigg|_0 (d(\mathrm{Fl}^X_\lambda)^*\omega) \overset{4.3.26}{=} \frac{d}{d\lambda}\bigg|_0 (\mathrm{Fl}^X_\lambda)^* d\omega = L_X d\omega.$$

$\square$

The final operation of tensor calculus we are going to study is the *inner product* operator:

4.3.34 Definition. *Let $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^{k+1}(M)$ and define $i_X\omega \in \mathcal{T}^0_k(M)$ by*

$$i_X\omega(X_1, \ldots, X_k) := \omega(X, X_1, \ldots, X_k).$$

If $\omega \in \Omega^0(M) \equiv \mathcal{C}^\infty(M)$ we set $i_X\omega := 0$. $i_X\omega$ is called the inner product *of X and ω.*

4.3.35 Theorem. *For $k = 1, \ldots, n$, $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$. Moreover, let $\alpha \in \Omega^k(m)$, $\beta \in \Omega^l(M)$ and $f \in \mathcal{C}^\infty(M)$, then:*

(i) i_X is a $\wedge$-antiderivation, i.e., it is $\mathbb{R}$-linear and

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_X\beta).$$

(ii) $i_X \circ i_X = 0$.

(iii) $i_{fX}\alpha = f i_X \alpha$.

(iv) $i_X df = L_X f$.

(v) $L_X\alpha = i_X d\alpha + d i_X \alpha$ *(Cartan's magic formula).*

(vi) $L_{fX}\alpha = f L_X \alpha + df \wedge i_X \alpha$.

Proof. That $i_X\alpha \in \Omega^{k-1}(m)$ is clear by 4.1.19.

(i) $\mathbb{R}$-linearity of i_X is clear. Since what we claim is a tensor identity, we may argue with individual tangent vectors $v \equiv v_1, v_2, \ldots, v_{k+l} \in T_pM$. We note that it suffices to take α and β of the form $\alpha = \alpha^1 \wedge \cdots \wedge \alpha^k$, $\beta = \beta^1 \wedge \cdots \wedge \beta^l$ where $\alpha^i, \beta^i \in \Omega^1(M)$ since general α, β are locally given as sums of such forms. We first show that

$$i_v(\alpha^1 \wedge \cdots \wedge \alpha^k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v) \alpha^1 \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k, \tag{4.3.7}$$

where the hat indicates that the term is omitted. This means that, for all v_i

$$(\alpha^1 \wedge \cdots \wedge \alpha^k)(v_1, \ldots, v_k) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v_1) \alpha^1 \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k (v_2, \ldots, v_k). \tag{4.3.8}$$

To see this, note first that by 4.3.13 the left hand side is the determinant of the matrix $A = (a_{ij})$ with $a_{ij} = \alpha^i(v_j)$. For the same reason, $\alpha^1 \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k(v_2, \ldots, v_k)$ is the determinant of the sub-matrix of A that is obtained by deleting the i-th row and first column. Thus (4.3.8) is simply the expansion of $\det A$ along its first column.

Using (4.3.7) we obtain

$$\begin{aligned}
i_v(\alpha \wedge \beta) &= i_v(\alpha^1 \wedge \ldots \alpha^k \wedge \beta^1 \wedge \cdots \wedge \beta^l) \\
&= \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k \wedge \beta^1 \wedge \cdots \wedge \beta^l \\
&\qquad + \sum_{i=k+1}^{k+l} (-1)^{i-1} \beta^i(v) \alpha^1 \wedge \cdots \wedge \alpha^k \wedge \beta^1 \wedge \cdots \wedge \widehat{\beta^i} \cdots \wedge \beta^l \\
&= (i_v\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_v \beta).
\end{aligned}$$

(ii) is clear by antisymmetry.

(iii) follows from point-wise multilinearity.

(iv) $i_X df = df(X) = X(f) = L_X f$.

(v) We prove this by induction. For $k = 0$, this is just (iv), so suppose that (v) holds for $\alpha \in \Omega^k(M)$. Any $(k+1)$-form can be written locally as a sum of terms of the form $df \wedge \alpha$ for α a k-form, so these are the only forms we need to consider. By (4.3.6),

$$L_X(df \wedge \alpha) = df \wedge L_X\alpha + L_X df \wedge \alpha,$$

and for the right hand side of the claimed identity we employ (i) and 4.3.22 (i) to obtain

$$\begin{aligned}
i_X d(df \wedge \alpha) + d i_X(df \wedge \alpha) &= -i_X(df \wedge d\alpha) + d(i_X df \wedge \alpha - df \wedge i_X\alpha) \\
&= -i_X df \wedge d\alpha + df \wedge i_X d\alpha + d i_X df \wedge \alpha + i_X df \wedge d\alpha + df \wedge d i_X \alpha \\
&\overset{(iv)}{=} df \wedge (i_X d\alpha + d i_X \alpha) + dL_X f \wedge \alpha \overset{\text{Ind.}}{=} df \wedge L_X\alpha + dL_X f \wedge \alpha \\
&\overset{4.3.33}{=} df \wedge L_X \alpha + L_X df \wedge \alpha
\end{aligned}$$

(vi) Using the previous results and 4.3.22 we have

$$\begin{aligned} L_{fX}\alpha &= i_{fX}d\alpha + di_{fX}\alpha = fi_X d\alpha + d(fi_X\alpha) \\ &= fi_X d\alpha + df \wedge i_X\alpha + f di_X\alpha = fL_X\alpha + df \wedge i_X\alpha. \end{aligned}$$

□

Concerning the naturality of the inner product with respect to diffeomorphisms we have:

4.3.36 Proposition. *Let $F : M \to N$ be a diffeomorphism, $\omega \in \Omega^k(N)$ and $X \in \mathfrak{X}(N)$. Then $i_{F^*X}(F^*\omega) = F^*(i_X\omega)$.*

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{F^*} & \Omega^k(M) \\ \big\downarrow{\scriptstyle i_X} & & \big\downarrow{\scriptstyle i_{F^*X}} \\ \Omega^{k-1}(N) & \xrightarrow{F^*} & \Omega^{k-1}(M) \end{array}$$

Consequently, for $Y \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$, $i_{F_*Y}(F_*\omega) = F_*(i_Y\omega)$.

Proof. Since $F_* = (F^{-1})^*$, it suffices to prove the first claim. Let $v_1, \dots, v_{k-1} \in T_pM$ and set $q := F(p)$, then

$$\begin{aligned} &i_{F^*X}(F^*\omega)(p)(v_1, \dots, v_{k-1}) = F^*\omega|_p(F^*X(p), v_1, \dots, v_{k-1}) \\ &\quad = F^*\omega|_p(T_qF^{-1} \circ X(q), v_1, \dots, v_{k-1}) \overset{4.3.25}{=} \omega|_q(X_q, T_pF(v_1), \dots, T_pF(v_{k-1})) \\ &\quad = i_X\omega|_q(T_pF(v_1), \dots, T_pF(v_{k-1})) = (F^*i_X\omega)|_p(v_1, \dots, v_{k-1}). \end{aligned}$$

□

The fact that L_X is a tensor derivation and the relation between d and L_X are given in the following result:

4.3.37 Theorem. *Let $\alpha \in \Omega^1(M)$, $X, X_i, Y \in \mathfrak{X}(M)$, and $\omega \in \Omega^k(M)$. Then:*

(i) $(L_X\omega)(X_1, \dots, X_k) = L_X(\omega(X_1, \dots, X_k)) - \sum_{i=1}^k \omega(X_1, \dots, L_XX_i, \dots, X_k)$.

(ii)

$$\begin{aligned} d\omega(X_0, X_1, \dots, X_k) &= \sum_{i=0}^k (-1)^i L_{X_i}(\omega(X_0, \dots, \widehat{X_i}, \dots, X_k)) \\ &+ \sum_{0\le i<j\le k} (-1)^{i+j}\omega(L_{X_i}(X_j), X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_k). \end{aligned}$$

Proof. (i) is precisely (DO4) from 4.2.5 for the differential operator L_X.

(ii) We use induction. For $k = 0$ (i.e., $\omega \in \Omega^0(M) = \mathcal{C}^\infty(M)$), the claim reduces to $d\omega(X_0) = L_{X_0}\omega$, which is true by 2.3.11. So suppose the statement is true for $k-1$ and let $\omega \in \Omega^k(M)$. Then by 4.3.35 (v) we have

$$\begin{aligned} &d\omega(X, X_1, \dots, X_k) = (i_Xd\omega)(X_1, \dots, X_k) \\ &\quad = (L_X\omega)(X_1, \dots, X_k) - (d(i_X\omega))(X_1, \dots, X_k) \\ &\quad \overset{(i)}{=} L_X(\omega(X_1, \dots, X_k)) - \sum_{i=1}^k \omega(X_1, \dots, L_XX_i, \dots, X_k) - (d(i_X\omega))(X_1, \dots, X_k) \end{aligned}$$

Applying the induction assumption to $i_X\omega \in \Omega^{k-1}(M)$ and using skew-symmetry we obtain

$$(d(i_X\omega))(X_1,\ldots,X_k) = \sum_{i=1}^{k}(-1)^{i-1}L_{X_i}(\omega(X,X_1,\ldots,\widehat{X_i},\ldots,X_k))$$
$$- \sum_{1\le i<j\le k}(-1)^{i+j}\omega(L_{X_i}(X_j),X,X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_k).$$

Inserting this into the above yields the claim. □

4.3.38 Definition. *A differential form $\omega \in \Omega^k(M)$ is called* closed *if $d\omega = 0$ and* exact *if there exists some $\alpha \in \Omega^{k-1}(M)$ such that $\omega = d\alpha$.*

Before we can analyze the relation between these notions we need an auxilliary result on the exterior derivative in $\mathbb{R}^n$:

4.3.39 Lemma. *Let $U \subseteq \mathbb{R}^n$ open and let $\omega \in \Omega^k(U)$. Then for any $x \in U$ and $v_0,\ldots,v_k \in \mathbb{R}^n$ we have*

$$d\omega(x)(v_0,\ldots,v_k) = \sum_{i=0}^{k}(-1)^i D\omega(x)\cdot v_i(v_0,\ldots,\widehat{v_i},\ldots,v_k)$$

Proof. Note that, since $\omega : U \to L^k(\mathbb{R}^n,\mathbb{R})$, we have $D\omega : U \to L(\mathbb{R}^n, L^k(\mathbb{R}^n,\mathbb{R}))$, so $D\omega(x)\cdot v_i \in L^k(\mathbb{R}^n,\mathbb{R})$. Also, immediately from the definition we see that the d defined here is a map $\Omega^k(U) \to \Omega^{k+1}(U)$. Hence we are left with verifying (i)–(iv) from 4.3.22. Of these, $\mathbb{R}$-linearity, as well as (ii) and (iv) are clear, and since $\wedge$ is bilinear, we have $D(\alpha\wedge\beta) = \alpha\wedge D\beta + D\alpha\wedge\beta$, which implies (i) (by inserting and changing the summation index in the second sum). Finally, that $d\circ d = 0$ follows exactly as in the proof of (iii) in 4.3.22. □

We now can prove the following fundamental result:

4.3.40 Theorem.

(i) Every exact form is closed.

(ii) (Poincaré Lemma) Locally, also the converse of (i) is true: If ω is closed then for each $p \in M$ there is a neighborhood U of p such that $\omega|_U \in \Omega^k(U)$ is exact.

Proof. (i) is clear from $d\circ d = 0$.

(ii) Using a local chart and the naturality properties of d (4.3.26 (ii) and 4.3.22 (iv)) it follows that it suffices to consider the case where $\omega \in \Omega^k(U)$ and U is an open ball around 0 in $\mathbb{R}^n$. Our strategy is to construct an $\mathbb{R}$-linear map $H : \Omega^k(U) \to \Omega^{k-1}(U)$ such that $d\circ H + H\circ d = \mathrm{id}_{\Omega^k(U)}$. This will give the result because $d\omega = 0$ then implies that $d(H\omega) = \omega$.

Let $v_1,\ldots,v_k \in \mathbb{R}^n$ and set

$$H\omega(x)(v_1,\ldots,v_{k-1}) := \int_0^1 t^{k-1}\omega(tx)(x,v_1,\ldots,v_{k-1})\,dt,$$

which is well-defined since $tx \in U$ for all $t \in [0,1]$ and all $x \in U$, and obviously lies in $\Omega^{k-1}(U)$. Using 4.3.39, we calculate:

$$
\begin{aligned}
d(H\omega)(x)(v_1,\dots,v_k) &= \sum_{i=1}^{k}(-1)^{i+1}D(H\omega)(x)\cdot v_i(v_1,\dots,\widehat{v_i},\dots,v_k)\\
&= \sum_{i=1}^{k}(-1)^{i+1}\int_0^1 t^{k-1}\omega(tx)(v_i,v_1,\dots,\widehat{v_i},\dots,v_k)\,dt\\
&\quad+\sum_{i=1}^{k}(-1)^{i+1}\int_0^1 t^{k}D\omega(tx)\cdot v_i(x,v_1,\dots,\widehat{v_i},\dots,v_k)\,dt
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
H(d\omega)(x)(v_1,\dots,v_k) &= \int_0^1 t^k d\omega(tx)(x,v_1,\dots,v_k)\,dt\\
&\overset{4.3.39}{=} \int_0^1 t^k D\omega(tx)\cdot x(v_1,\dots,v_k)\,dt\\
&\quad+\sum_{i=1}^{k}(-1)^{i}\int_0^1 t^{k}D\omega(tx)\cdot v_i(x,v_1,\dots,\widehat{v_i},\dots,v_k)\,dt
\end{aligned}
$$

Altogether, we arrive at

$$
\begin{aligned}
&[d(H\omega)(x)+H(d\omega)(x)](v_1,\dots,v_k)\\
&\quad= \int_0^1 kt^{k-1}\omega(tx)(v_1,\dots,v_k)\,dt+\int_0^1 t^kD\omega(tx)\cdot x(v_1,\dots,v_k)\,dt\\
&\quad= \int_0^1 \frac{d}{dt}[t^k\omega(tx)(v_1,\dots,v_k)]\,dt=\omega(x)(v_1,\dots,v_k).
\end{aligned}
$$

□

The Poincaré lemma is the starting point for de Rham cohomology. We do not have time to go into any details, and instead only define (for later use) the most basic notions.

4.3.41 Definition. *For any $k\in\mathbb{N}_0$, consider the following vector spaces:*

$$
\begin{aligned}
\mathcal{Z}^k(M) &:= \ker(d:\Omega^k(M)\to\Omega^{k+1}(M))=\{\text{closed } k\text{–forms on} M\}\\
\mathcal{B}^k(M) &:= \mathrm{Im}(d:\Omega^{k-1}(M)\to\Omega^{k}(M))=\{\text{exact } k\text{–forms on} M\}
\end{aligned}
$$

It is understood that $\Omega^k(M):=\{0\}$ for $k<0$ or $k>n$. The k-th de Rham cohomology group *is the quotient vector space*

$$
H^k_{dR}(M):=\mathcal{Z}^k(M)/\mathcal{B}^k(M).
$$

The Poincaré Lemma 4.3.40 states that any point has a neighborhood U for which $H^1_{dR}(U)=0$. If $H^1_{dR}(M)=0$ then for any closed 1-form $\omega\in\Omega^1(M)$ there exists some $f\in\mathcal{C}^\infty(M)$ with $\omega=df$. For more information on de Rham cohomology we refer to [5].

4.4 Integration, Stokes' Theorem

Our aim in this section is to develop a theory of integrating differential forms on manifolds. Based on this we will prove Stokes' theorem, which provides a far-reaching generalization of the classical integration theorems of analysis (Gauss, Stokes, Green). As a fundamental tool we will need the transformation rule for integrals:

4.4.1 Theorem. *Let $U, V \subseteq \mathbb{R}^n$ be open, $\Phi : U \to V$ a diffeomorphism, $f \in \mathcal{C}(V)$,* supp$f$ *compact. Then:*

$$\int_U f(\Phi(x))|\det D\Phi(x)|d^n x = \int_V f(y)d^n y \tag{4.4.1}$$

Our strategy for defining $\int_M \omega$ for $\omega \in \Omega^n_c(V)$, (Ω^n_c denoting the space of compactly supported n-forms, V a chart neighborhood) will be to set

$$\int_M \omega := \int_{\varphi(V)} \omega_\varphi(x)d^n x.$$

To make this a well-defined expression it should be independent of the chosen chart. The transformation behavior of ω_φ according to 4.3.27 (iii), however, differs from (4.4.1) (no absolute value of $\det D(\varphi \circ \psi^{-1})$). We therefore consider manifolds with distinguished atlasses:

4.4.2 Definition. *A manifold M is called* orientable *if it possesses an* oriented atlas $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ *such that* $\det D(\psi_\beta \circ \psi_\alpha^{-1})(x) > 0$ $\forall x \in \psi_\alpha(V_\alpha \cap V_\beta)$ $\forall \alpha$ $\forall \beta$*. As in the case of smooth manifolds, also for oriented manifolds one can define corresponding $\mathcal{C}^\infty$-structures (allowing only oriented atlasses). Charts in an oriented atlas are called* positively oriented*. A manifold M together with an oriented atlas is called* oriented*.*

4.4.3 Remark.

(i) Not every manifold is orientable. The most famous example of a non-orientable manifold is the Möbius strip.

(ii) One can show that the following are equivalent:

- M is orientable.
- $\exists \omega \in \Omega^n(M)$ with $\omega(p) \neq 0$ $\forall p \in M$. Such an ω is called *volume form* on M (cf. 4.3.11).
- The $\mathcal{C}^\infty(M)$-module $\Omega^n(M)$ is one-dimensional (every volume form provides a basis).

In the special case $M = \mathbb{R}^n$ we proceed as follows: For $\omega = a(x^1, \dots, x^n)\alpha^1 \wedge \dots \wedge \alpha^n$ with compact support $K \subseteq U$, U open in $\mathbb{R}^n$, let $\int_U \omega := \int_K a(x)d^n x$. To extend this definition to general manifolds we first consider the case $\omega \in \Omega^n_c(M)$ such that $\mathrm{supp}(\omega) \subseteq U$, where (φ, U) is a chart of M. Then put

$$\int_{(\varphi)} \omega := \int \varphi_*(\omega|_U) = \int_{\varphi(U)} \omega_\varphi(x)d^n x$$

4.4.4 Lemma. *Let M be an oriented manifold, $\omega \in \Omega^n_c(M)$, (φ, U), (ψ, V) positively oriented charts and* $\mathrm{supp}(\omega) \subseteq U \cap V$*. Then $\int_{(\varphi)} \omega = \int_{(\psi)} \omega$. Thus we may simply write $\int \omega$ for this common value.*

Proof. Let $\varphi_*\omega = \omega_\varphi \alpha^1 \wedge \cdots \wedge \alpha^n$, $\psi_*\omega = \omega_\psi \alpha^1 \wedge \cdots \wedge \alpha^n$. Then

$$\begin{aligned}
\int_{(\psi)} \omega &= \int_{\psi(V)} \omega_\psi(y) d^n y = \int_{\psi(U\cap V)} \omega_\psi(y) d^n y = \\
&\overset{4.3.17(iii)}{=} \int_{\psi(U\cap V)} \omega_\varphi(\varphi\circ\psi^{-1}(y)) \underbrace{\det D(\varphi\circ\psi^{-1})(y)}_{=|\det D(\varphi\circ\psi^{-1})(y)|} d^n y = \\
&= \int_{\varphi(U\cap V)} \omega_\varphi(x) d^n x = \int_{\varphi(U)} \omega_\varphi(x) d^n x = \int_{(\varphi)} \omega.
\end{aligned}$$

□

4.4.5 Definition. *Let M be an oriented manifold and $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ an oriented atlas. Let $\{\chi_\alpha \mid \alpha \in A\}$ be a partition of unity subordinate to $\{V_\alpha \mid \alpha \in A\}$. Let $\omega \in \Omega^n_c(M)$ and $\omega_\alpha := \chi_\alpha \cdot \omega$ (hence $\operatorname{supp}(\omega_\alpha)$ is compact and contained in V_α). Then let*

$$\int_M \omega := \sum_{\alpha\in A} \int \omega_\alpha.$$

4.4.6 Proposition.

(*i*) *The sum in 4.4.5 contains only finitely many non-vanishing terms.*

(*ii*) *Definition 4.4.5 is independent of the chosen oriented atlas (in the given oriented $\mathcal{C}^\infty$-structure) and partition of unity.*

Proof. (i) Since $\{\operatorname{supp}\chi_\alpha \mid \alpha \in A\}$ is locally finite, only finitely many $\operatorname{supp}\chi_\alpha$ intersect the compact set $\operatorname{supp}(\omega)$ (every $p \in \operatorname{supp}(\omega)$ has a neighborhood intersecting only finitely many $\operatorname{supp}\chi_\alpha$, finitely many such neighborhoods cover $\operatorname{supp}(\omega)$).

(ii) Let $\mathcal{A}' = \{(\varphi_\beta, U_\beta) \mid \beta \in B\}$ be another oriented atlas in the same oriented $\mathcal{C}^\infty$-structure, $\{\mu_\beta \mid \beta \in B\}$ a partition of unity subordinate to $\{U_\beta \mid \beta \in B\}$. Then

$$\sum_{\alpha\in A} \int \omega_\alpha \overset{\sum_\beta \mu_\beta = 1}{=} \sum_{\alpha\in A} \int \sum_{\beta\in B} \mu_\beta \chi_\alpha \omega = \sum_{\alpha,\beta} \int \mu_\beta \chi_\alpha \omega = \cdots = \sum_{\beta\in B} \int \mu_\beta \omega.$$

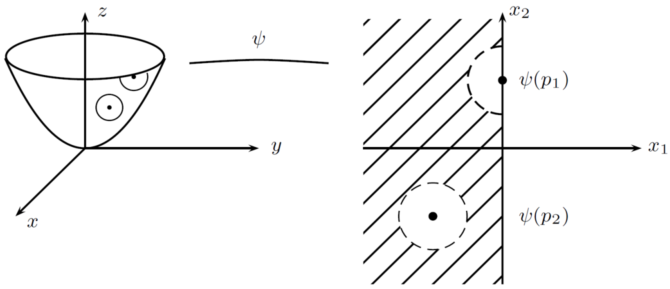
□

In the integral theorems of vector analysis, typical domains of integration are n-dimensional domains with boundary, where the boundary itself forms an $(n-1)$-dimensional domain of integration. Such domains are currently not covered by our notion of manifold:

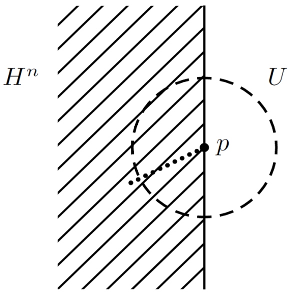
4.4.7 Example. Let $M = \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 + y^2, z \leq z_0, z_0 > 0\}$.

M is not a manifold since points like p_1 (see the figure below) do not have open neighborhoods in M that are homeomorphic to $\mathbb{R}^2$. On the other hand it is quite obvious that M has charts which are homeomorphic to relatively open subsets of a suitable half-space. Points like p_2 form the boundary (but not in the topological sense!) of M, which itself is a 1-dimensional manifold (without boundary).

We now want to make precise these observations in the following definition.



4.4.8 Definition. *Let the half-space $H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^1 \le 0\}$ be equipped with the trace topology of $\mathbb{R}^n$ (i.e., $V \subseteq H^n$ is open $\Leftrightarrow \exists U \subseteq \mathbb{R}^n$ open s.t. $U \cap H^n = V$). Let $V \subseteq H^n$ be open. Then $f : V \to \mathbb{R}^m$ is called smooth on V if there exists an open subset $U \supseteq V$ of $\mathbb{R}^n$ and a smooth extension $\tilde{f}$ of f to U. For any $p \in V$ we then set $Df(p) := D\tilde{f}(p)$.*



We have to check that $Df(p)$ is independent of $\tilde{f}$: This is clear if $V \subseteq (H^n)^\circ$. Thus let $p = (0, x^2, \ldots, x^n)$ and $\tilde{f}$, $\tilde{\tilde{f}}$ be two extensions of f to an open neighborhood U of p in $\mathbb{R}^n$. Set $g := \tilde{f} - \tilde{\tilde{f}}$. We have to show that $Dg(p) = 0$. To this end, pick a sequence of points $p_m \in (H^n)^\circ$ with $p_m \to p$. Then $Dg(p_m) = 0$ for all m, so also $Dg(p) = \lim_{m\to\infty} Dg(p_m) = 0$.

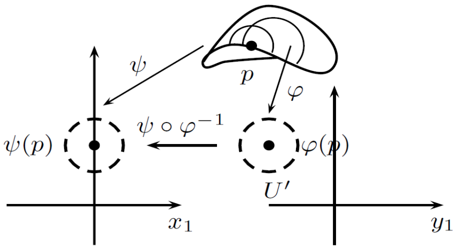
4.4.9 Definition. *A* manifold with boundary *is a set M together with an atlas $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ of bijective maps $\psi_\alpha : V_\alpha \to \psi_\alpha(V_\alpha) \subseteq H^n$ (relatively) open, such that $\bigcup_{\alpha\in A} V_\alpha = M$ and for all α, β with $V_\alpha \cap V_\beta \neq \emptyset$ we have $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(V_\alpha \cap V_\beta) \to \psi_\beta(V_\alpha \cap V_\beta)$ is smooth in the sense of 4.4.8. As in the case of manifolds without boundary we require M with its natural topology (induced by the charts) to be Hausdorff and second countable.*

4.4.10 Lemma. *Let M be a manifold with boundary. A point $p \in M$ is called* boundary point *of M if there exists a chart $(\psi = (x^1, \ldots, x^n), V)$ with $x^1(p) = 0$. If p is a boundary point, denoted by $p \in \partial M$ then for any chart $(\varphi = (y^1, \ldots, y^n), U)$ around p we have $y^1(p) = 0$.*

Proof. Suppose to the contrary that there would exist a chart $\varphi = (y^1, \ldots, y^n)$ with $y^1(p) < 0$.

Choose a neighborhood U' of $\varphi(p)$ which is open in $\mathbb{R}^n$ and contained in $\varphi(U \cap V) \subseteq H^n$. Since $\det(D(\psi \circ \varphi^{-1}))(\varphi(p)) \neq 0$, by 1.1.1, $\psi \circ \varphi^{-1}$ is a diffeomorphism onto a neighborhood of $\psi \circ \varphi^{-1}(\varphi(p)) = \psi(p)$ which is open in $\mathbb{R}^n$. This neighborhood must therefore be contained in H^n, contradicting $\psi^1(p) = x^1(p) = 0$. □

All constructions we already know for manifolds without boundary like tangent space, tensors, differential forms, orientability, etc. work out completely analogously for manifolds with boundary. The next result shows that ∂M itself is a manifold (without boundary).



4.4.11 Proposition. *Let M be an n-dimensional manifold with boundary. Then ∂M is an $(n-1)$-dimensional manifold (without boundary). If M is oriented then the orientation of M induces an orientation of ∂M.*

Proof. Let $\mathcal{A} = \{(\psi_\alpha, V_\alpha) \mid \alpha \in A\}$ be an atlas of M. Set $A' := \{\alpha \in A \mid V_\alpha \cap \partial M \neq \emptyset\}$, $\mathcal{A}' := \{(\psi_\alpha|_{V_\alpha \cap \partial M}, V_\alpha \cap \partial M) \mid \alpha \in A'\}$. We show that $\mathcal{A}'$ is an atlas for ∂M. Set $\tilde{V}_\alpha := V_\alpha \cap \partial M$, $\tilde{\psi}_\alpha := \psi_\alpha|_{\tilde{V}_\alpha}$. Then $\tilde{\psi}_\alpha : \tilde{V}_\alpha \to \tilde{\psi}_\alpha(\tilde{V}_\alpha)$ is bijective and by 4.4.10 it follows that $\tilde{\psi}_\alpha(\tilde{V}_\alpha) = \psi_\alpha(V_\alpha) \cap \{x^1 = 0\}$. Clearly, $\bigcup_{\alpha \in A'} \tilde{V}_\alpha = \partial M$. Now let α, $\beta \in A'$ such that $\tilde{V}_\alpha \cap \tilde{V}_\beta \neq \emptyset$. Since $\psi_\alpha(V_\alpha \cap V_\beta) \subseteq H^n$ is open, $\tilde{\psi}_\alpha(\tilde{V}_\alpha \cap \tilde{V}_\beta) = \psi_\alpha(V_\alpha \cap V_\beta) \cap \{x^1 = 0\}$ is open in $\{x^1 = 0\} \cong \mathbb{R}^{n-1}$. Moreover, $\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1}$ is smooth on $\psi_\alpha(\tilde{V}_\alpha \cap \tilde{V}_\beta)$ as a restriction of the smooth map $\psi_\beta \circ \psi_\alpha^{-1}$.

Suppose now that $\mathcal{A}$, in addition, is oriented, i.e., that $\det D(\psi_\beta \circ \psi_\alpha^{-1}) > 0$ for all α, β with $V_\alpha \cap V_\beta \neq \emptyset$. Let $\psi_\alpha = (x_\alpha^1, \dots, x_\alpha^n)$, $\psi_\beta = (x_\beta^1, \dots, x_\beta^n)$. Then for every $(0, x_\alpha^2, \dots, x_\alpha^n) \in \tilde{\psi}_\alpha(\tilde{V}_\alpha \cap \tilde{V}_\beta)$, $\psi_\beta \circ \psi_\alpha^{-1}(0, \underbrace{x_\alpha^2, \dots, x_\alpha^n}_{=: x'_\alpha}) = (0, \tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1}(x'_\alpha))$.

Therefore,

$$D(\psi_\beta \circ \psi_\alpha^{-1})(0, x'_\alpha) = \left.\begin{pmatrix} \frac{\partial(\psi_\beta^1 \circ \psi_\alpha^{-1})}{\partial x^1} & 0 \ \dots \ 0 \\ * & \\ \vdots & D(\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1}) \\ * & \end{pmatrix}\right|_{(0, x'_\alpha)}$$

$$\Rightarrow \ \det D(\psi_\beta \circ \psi_\alpha^{-1})(0, x'_\alpha) = \left.\frac{\partial(\psi_\beta^1 \circ \psi_\alpha^{-1})}{\partial x^1}\right|_{(0, x'_\alpha)} \det D(\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1})(0, x'_\alpha) \quad (*)$$

Now $\psi_\beta^1 \circ \psi_\alpha^{-1}(0, x'_\alpha) = 0$ and $\psi_\beta^1 \circ \psi_\alpha^{-1}(x^1, x'_\alpha) < 0$ for $x^1 < 0$ (since $\psi_\beta \circ \psi_\alpha^{-1} : H^n \to H^n$). Therefore, $\frac{\partial(\psi_\beta^1 \circ \psi_\alpha^{-1})}{\partial x^1} \geq 0$ and $\neq 0$ (by $(*)$), hence > 0. Again by $(*)$ it follows that $\det D(\tilde{\psi}_\beta \circ \tilde{\psi}_\alpha^{-1}) > 0$, so $\mathcal{A}'$ is oriented. $\square$

As the final ingredient for Stokes' theorem we consider the restriction of differential forms defined on M to ∂M: Let $i : \partial M \hookrightarrow M$ be the natural inclusion. We first note that i is smooth since for any chart $\psi = (x^1, \dots, x^n)$ of M we have:

$$\begin{tikzcd} \partial M \arrow[r, "i"] \arrow[d, "\tilde{\psi}"'] & M \arrow[d, "\psi"] \\ \tilde{\psi}(\tilde{V}) \arrow[r, "j"] & \psi(V) \end{tikzcd}$$

where $j : (x^2, \dots, x^n) \mapsto (0, x^2, \dots, x^n)$. This is obviously smooth.

The restriction of any $\omega \in \Omega^k(M)$ is defined as $i^*\omega \in \Omega^k(\partial M)$. As in (4.3.2), the local representation of ω with respect to ψ can be written as

$$\omega = \sum_{1\le i_1<\dots<i_k\le n} \omega_{i_1\dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

Then $\psi_*\omega$ is given by

$$\sum_{1\le i_1<\dots<i_k\le n} \omega_{i_1\dots i_k} \circ \psi^{-1} \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k} =: \sum_{1\le i_1<\dots<i_k\le n} \omega^{\psi}_{i_1\dots i_k} \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k}.$$

The local representation of $i^*\omega$ with respect to $\tilde\psi$ therefore is

$$\begin{aligned}\tilde\psi_*(i^*\omega) &= (\tilde\psi^{-1})^*(i^*\omega) = (i\circ\tilde\psi^{-1})^*\omega = (\psi^{-1}\circ j)^*\omega = j^*((\psi^{-1})^*\omega) = \\ &= j^*(\psi_*\omega) \overset{4.3.26(i)}{=} \sum_{1\le i_1<\dots<i_k\le n} \omega^{\psi}_{i_1\dots i_k} \circ j\ j^*(\alpha^{i_1}) \wedge \dots \wedge j^*(\alpha^{i_k}).\end{aligned}$$

Observing now that

$$j^*(\alpha^k)(v)\big|_x \overset{4.3.24}{=} \alpha^k(\underbrace{Dj(x)}_{=j \text{ by linearity}}(v)) = \alpha^k(j(v)) = \begin{cases} 0 & k=1 \\ v_k = \alpha^k(v) & k\neq 1 \end{cases}$$

we finally arrive at

$$\tilde\psi_*(i^*\omega) = \sum_{\underset{\uparrow}{1<i_1<\dots<i_k\le n}} \omega^{\psi}_{i_1\dots i_k} \circ j\ \alpha^{i_1} \wedge \dots \wedge \alpha^{i_k} \tag{4.4.2}$$

4.4.12 Theorem. *(Stokes' theorem) Let M be an oriented manifold with boundary, $\omega \in \Omega^{n-1}_c(M)$, and $i : \partial M \hookrightarrow M$. Then:*

$$\int_{\partial M} i^*\omega = \int_M d\omega$$

Proof. Denote by K the compact support of ω. We consider the following two cases:

1.) There exists a chart $(\psi = (x^1,\dots,x^n), V)$ with $K \subseteq V$. Since $\omega \in \Omega^{n-1}(M)$, the local representation of ω with respect to ψ reads

$$\omega = \sum_{k=1}^n \omega_k dx^1 \wedge \dots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \dots \wedge dx^n \tag{4.4.3}$$

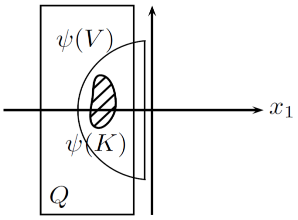
where $\omega_j \in \mathcal{C}^\infty(V)$ for all j. Hence

$$d\omega = \Big(\sum_{k=1}^n (-1)^{k-1} \frac{\partial \omega_k}{\partial x^k}\Big) dx^1 \wedge \dots \wedge dx^n \tag{4.4.4}$$

with $\frac{\partial\omega_k}{\partial x^k} = D_k(\omega_k \circ \psi^{-1})(\psi(.))$. We now distinguish the following sub-cases:

1a) $V \cap \partial M = \emptyset$. Then $i^*\omega = 0$ (cf., e.g., (4.4.2)), hence $\int_{\partial M} i^*\omega = 0$ and we have to show that also

$$\int_M d\omega \overset{4.4.4}{=} \int_{\psi(V)} \psi_*(d\omega) \overset{(4.4.4),4.3.21(ii)}{=} \int_{\psi(V)} \sum_{k=1}^n (-1)^{k-1} \frac{\partial \omega^{\psi}_k}{\partial x^k} dx^1 \dots dx^n = 0.$$

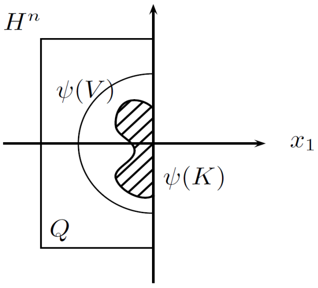


We now choose a parallelepiped $Q \subsetneq H^n$ of the form $Q = \{(x^1, \ldots, x^n \mid a^k \le x^k \le b^k \ (1 \le k \le n)\}$ such that $\psi(K)$ lies in the interior of Q. Then if we extend the (compactly supported) ω_k^ψ by 0 to all of H^n, we obtain (applying the fundamental theorem of calculus):

$$
\begin{aligned}
\int_{\psi(V)} \sum_{k=1}^{n} (-1)^{k-1} \frac{\partial \omega_k^\psi}{\partial x^k} dx^1 \ldots dx^n &= \sum_{k=1}^{n} (-1)^{k-1} \int_Q \frac{\partial \omega_k^\psi}{\partial x^k} dx^1 \ldots dx^n \\
= \;& \sum_{k=1}^{n} (-1)^{k-1} \int \big(\underbrace{\omega_k^\psi(x^1, \ldots, x^{k-1}, b^k, x^{k+1}, \ldots, x^n)}_{=0} \\
& - \underbrace{\omega_k^\psi(x^1, \ldots, x^{k-1}, a^k, x^{k+1}, \ldots, x^n)}_{=0}\big) dx^1 \ldots dx^{k-1} dx^{k+1} \ldots dx^n \\
= \;& 0
\end{aligned}
$$

1b) Now suppose that $V \cap \partial M \neq \emptyset$. Then

$$
\begin{aligned}
\int_{\partial M} i^*\omega \;&\overset{4.4.4}{=}\; \int_{\tilde\psi(V \cap \partial M)} \tilde\psi_*(i^*\omega) \\
&\overset{(4.4.3),(4.4.2)}{=}\; \underbrace{\int_{\tilde\psi(V \cap \partial M)} \omega_1^\psi(0, x^2, \ldots, x^n) dx^2 \ldots dx^n}_{=\int_{\psi(K)\cap\{x^1=0\}}} \qquad (4.4.5)
\end{aligned}
$$



Again we extend the ω_k^ψ by 0 to all of H^n and choose a parallelepiped $Q \subseteq H^n$, this time of the form $Q = [a^1, 0] \times [a^2, b^2] \times \cdots \times [a^n, b^n]$ such that $\psi(K) \subseteq Q^\circ \cup \{x^1 = 0\}$. Then as in the previous case we obtain:

$$
\begin{aligned}
\int_M d\omega &= \sum_{k=1}^{n}(-1)^{k-1}\int_Q \frac{\partial\omega_k^\psi}{\partial x^k}dx^1\dots dx^n\\
&= \int_{[a^2,b^2]\times\cdots\times[a^n,b^n]\ (=Q\cap\{x^1=0\})}\big(\omega_1^\psi(0,x^2,\dots,x^n)-\underbrace{\omega_1^\psi(a^1,x^2,\dots,x^n)}_{=0}\big)dx^2\dots dx^n\\
&\quad+\sum_{k=2}^{n}(-1)^{k-1}\int\big(\underbrace{\omega_k^\psi(x^1,\dots,b^k,\dots,x^n)}_{=0}\\
&\quad-\underbrace{\omega_k^\psi(x^1,\dots,a^k,\dots,x^n)}_{=0}\big)dx^1\dots dx^{k-1}dx^{k+1}\dots dx^n\\
&= \int_{\psi(K)\cap\{x^1=0\}}\omega_1^\psi(0,x^2,\dots,x^n)dx^2\dots dx^n\\
&\overset{(4.4.5)}{=} \int_{\partial M} i^*\omega
\end{aligned}
$$

2.) The general case: Let $\{(\psi_\alpha,V_\alpha)\mid\alpha\in A\}$ be an oriented atlas, $\{\chi_\alpha\mid\alpha\in A\}$ a subordinate partition of unity. Then the $\omega_\alpha:=\chi_\alpha\cdot\omega$ satisfy the assumptions of case 1.). Also, $\sum_\alpha d\chi_\alpha = d(\sum_\alpha\chi_\alpha)=d(1)=0$. Thus $\omega=\sum_\alpha\omega_\alpha$ and

$$\sum_\alpha d\omega_\alpha=\sum_\alpha d(\chi_\alpha\cdot\omega)=\sum_\alpha(d\chi_\alpha)\omega+\sum_\alpha\chi_\alpha d\omega=d\omega.$$

From this we finally obtain

$$\int_M d\omega=\sum_\alpha\int_M d\omega_\alpha\overset{1.)}{=}\sum_\alpha\int_{\partial M}i^*\omega_\alpha=\int_{\partial M}i^*\Big(\sum_\alpha\omega_\alpha\Big)=\int_{\partial M}i^*\omega.$$

□

4.4.13 Examples.

(i) Applying 4.4.12 to the ω from 4.3.23 (i), we obtain Green's theorem in the plane:

$$\int_{\partial M}Pdx+Qdy=\int_M\Big(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\Big)dxdy$$

(ii) From 4.3.23 (ii) and 4.4.12 we derive Gauss' divergence theorem (in $\mathbb{R}^3$):

$$\int_M\Big(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\Big)dxdydz=\int_{\partial M}Pdy\wedge dz+Qdz\wedge dx+Rdx\wedge dy$$

Chapter 5

Symplectic Manifolds

As an application of the machinery of global analysis developed so far, in this final chapter we glimpse at a branch of differential geometry that has diverse applications, spanning from classical mechanics to differential topology and Fourier integral operators, to mention only a few. We closely follow [5, Ch. 22].

5.1 Symplectic linear algebra

In this section we develop the algebraic underpinnings of symplectic geometry, which basically amounts to the study of certain 2-forms. Throughout, V will be a finite dimensional vector space and we freely use the notations introduced in Chapter 4.

5.1.1 Definition. *A 2-form $\omega \in \Lambda^2 V^*$ is called* nondegenerate *if the associated linear map $\hat{\omega} : V \to V^*$, $\hat{\omega}(v) := \omega(v, .)$ is invertible. A nondegenerate 2-form is called a* symplectic tensor*. A vector space equipped with a symplectic tensor is called a* symplectic vector space*.*

A 2-form ω is nondegenerate if and only if $v \mapsto \omega(v, .)$, $V \to V^*$ is injective, i.e., if and only if $\omega(v, w) = 0$ for all w implies that $v = 0$.

5.1.2 Example. Let $\dim(V) = 2n$ and choose a basis $(v_1, w_1, \ldots, v_n, w_n)$ of V. Let $(\alpha^1, \beta^1, \ldots, \alpha^n, \beta^n)$ be the corresponding dual basis of V^* and consider the 2-form

$$\omega := \sum_{i=1}^{n} \alpha^i \wedge \beta^i. \tag{5.1.1}$$

Then ω is a symplectic tensor on V: Indeed, we have

$$\omega(v_i, v_j) = \omega(w_i, w_j) = 0, \qquad \omega(v_i, w_j) = -\omega(w_j, v_i) = \delta_{ij}. \tag{5.1.2}$$

Thus if $v = a^i v_i + b^i w_i \in V$ and if $\omega(v, w) = 0$ for all w then $0 = \omega(v, w_i) = a^i$ and $0 = \omega(v, v_i) = -b^i$ for all i, so $v = 0$.

5.1.3 Definition. *Let (V, ω) be a symplectic vector space and let $S \subseteq V$ be a linear subspace. Then the* symplectic complement *of S is the subspace*

$$S^\perp := \{v \in V \mid \omega(v, w) = 0 \ \forall w \in S\}$$

5.1.4 Lemma. *Let* (V, ω) *be a symplectic vector space. Then for any linear subspace* S *we have* $\dim S + \dim S^\perp = \dim V$.

Proof. Define $F : V \to S^*$, $F(v) := w \mapsto \omega(v, w)$. Let $\alpha \in S^*$ and let $\tilde{\alpha} \in V^*$ be any linear extension of α to all of V. Since $\hat{\omega} : V \to V^*$ is an isomorphism, there exists some $v \in V$ such that $\hat{\omega}(v) = \tilde{\alpha}$, hence $F(v) = \alpha$, showing that F is surjective. Since $S^\perp = \ker F$, the dimension of $S^\perp$ is $\dim V - \dim S^* = \dim V - \dim S$. □

Note that, contrary to the case of scalar products, $S \cap S^\perp \neq \{0\}$ in general. Indeed, if $\dim S = 1$, then by skew-symmetry of ω, $S \subseteq S^\perp$.

The following result, whose proof may be viewed as a symplectic analogue of the Gram-Schmidt algorithm, demonstrates that *any* symplectic tensor has a basis representation of the form (5.1.1).

5.1.5 Proposition. *Let* ω *be a symplectic tensor on an* m*-dimensional vector space. Then* m *is even,* $m = 2n$ *and there exists a basis for* V *with respect to which* ω *is of the form* (5.1.1).

Proof. As seen in 5.1.2, we have to construct a basis $(v_1, w_1, \dots, v_n, w_n)$ of V satisfying (5.1.2), which we do by induction over m. For $m = 0$, there is nothing to do, so suppose that $m \geq 1$ and we already have the result for all dimensions less than m. Take $v_1 \neq 0$ in V, then since ω is nondegenerate there exists some $w_1 \in V$ with $\omega(v_1, w_1) \neq 0$. Scaling w_1 if necessary, we may assume that $\omega(v_1, w_1) = 1$. Since ω is skew-symmetric, v_1 is not proportional to w_1, hence $\{v_1, w_1\}$ is linearly independent, implying that $\dim V \geq 2$. Now set $S := \mathrm{span}(v_1, w_1)$. Then as in 5.1.2 it follows that if $v = \alpha v_1 + \beta w_1 \in S \cap S^\perp$, then $\alpha = \beta = 0$, so $S \cap S^\perp = \{0\}$. Moreover, for any $v \in V$ we have

$$v - \omega(v, w_1)v_1 + \omega(v, v_1)w_1 \in S^\perp,$$

so $V = S + S^\perp$. It follows that $S^\perp$ is symplectic itself: if $v \in S^\perp$ and $\omega(v, w) = 0$ for all $w \in S^\perp$ then in fact $\omega(v, w) = 0$ for all $w \in V$, hence $v = 0$. Therefore we can apply the induction hypothesis to $S^\perp$, obtaining that it must be even-dimensional and have a basis $(v_2, w_2, \dots, v_n, w_n)$ that satisfies (5.1.2) for $2 \leq i, j \leq n$. Hence $(v_1, w_1, \dots, v_n, w_n)$ is the required basis for V. □

A basis of a symplectic vector space satisfying (5.1.2) is called a *symplectic basis*.

5.2 Symplectic structures on manifolds

We now want to implement symplectic constructions on a smooth manifold M. We call a smooth 2-form $\omega \in \Omega^2(M)$ *nondegenerate* if ω_p is nondegenerate in $\Lambda^2 T_p^* M$ for each $p \in M$. A *symplectic form* on M is a closed nondegenerate 2-form. A manifold M together with a symplectic form ω is called a *symplectic manifold*, or a manifold with a *symplectic structure*.

By 5.1.5, a symplectic manifold is necessarily even-dimensional. A diffeomorphism $F : (M_1, \omega_1) \to (M_2, \omega_2)$ between symplectic manifolds is called a symplectomorphism if $F^*\omega_2 = \omega_1$. Symplectic geometry may be described as the study of properties of symplectic manifolds that are invariant under symplectomorphisms.

5.2.1 Example. Denote the standard coordinates on $M = \mathbb{R}^{2n}$ by $(x^1, \dots, x^n, y^1, \dots, y^n)$. Then the 2-form

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i \tag{5.2.1}$$

is symplectic: Clearly, $d\omega = 0$, and in any point it equals the standard symplectic form from 5.1.2. It is therefore called the *standard symplectic form* on $\mathbb{R}^{2n}$.

An (immersed or regular) submanifold $S \subseteq M$ of a symplectic manifold is called a (immersed or regular) symplectic submanifold if $(T_pS, (\omega|_p)_{T_pS \times T_pS})$ is symplectic for each $p \in S$.

The paradigmatic example of a symplectic manifold is the cotangent bundle T^*M of any manifold M. Its symplectic structure is built up as follows. We begin by defining a natural 1-form τ on T^*M, the so-called *tautological 1-form*. To this end, we write any element φ of T^*M as (q, φ), i.e., $\varphi \in T^*_qM$, so for the bundle projection $\pi : T^*M \to M$ we have $\pi(q, \varphi) = q$. At q we have a linear pullback map under $T_{(q,\varphi)}\pi : T_{(q,\varphi)}(T^*M) \to T_qM$:

$$\begin{aligned}(T_{(q,\varphi)}\pi)^* : T^*_qM &\to T^*_{(q,\varphi)}(T^*M)\\ \alpha &\mapsto \alpha \circ T_{(q,\varphi)}\pi.\end{aligned}$$

Using this map, we define a 1-form on the manifold T^*M (*not* M!) by

$$\tau_{(q,\varphi)} := (T_{(q,\varphi)}\pi)^*\varphi. \tag{5.2.2}$$

Untangling the definitions, this means that, for any $v \in T_{(q,\varphi)}(T^*M)$ we have

$$\tau_{(q,\varphi)}(v) = \varphi(T_{(q,\varphi)}\pi(v)). \tag{5.2.3}$$

5.2.2 Theorem. *The tautological 1-form τ is a smooth 1-form on T^*M, i.e., $\tau \in \Omega^1(T^*M)$, and $\omega := -d\tau \in \Omega^2(T^*M)$ is a symplectic form on the cotangent bundle T^*M.*

Proof. Let $\psi : U \to \mathbb{R}^n$ be a chart with coordinates $(x^1, \dots, x^n)$, let $(e_1, \dots, e_n)$ be the standard basis of $\mathbb{R}^n$ and $(\alpha^1, \dots, \alpha^n)$ the corresponding dual basis on $(\mathbb{R}^n)^*$. A standard chart for T^*M then is $T^0_1\psi : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$. Any element (q, φ) of T^*U can be written as $\sum_{i=1}^n \xi_i dx^i|_q$ for some $q \in U$. Then writing $\psi(q) = (x^1, \dots, x^n)$, by (4.1.1) we have

$$T^0_1\psi\Big(\sum_{i=1}^n \xi_i dx^i|_q\Big) = (x^1, \dots, x^n, \xi_1, \dots, \xi_n).$$

as the local expression of (q, φ) in terms of the chart $T^0_1\psi$. In these coordinates on T^*U, any vector $v \in T_{(q,\varphi)}(T^*M)$ can be written as

$$v = \sum_{i=1}^n v_i \left.\frac{\partial}{\partial x^i}\right|_{(q,\varphi)} + \sum_{i=1}^n w_i \left.\frac{\partial}{\partial \xi^i}\right|_{(q,\varphi)} \tag{5.2.4}$$

and since the bundle projection π locally takes the form $\pi(x, \xi) = x$ it follows that

$$T_{(q,\varphi)}\pi(v) = \sum_{i=1}^n v_i \left.\frac{\partial}{\partial x^i}\right|_q \tag{5.2.5}$$

It is important to keep in mind the different roles that the coordinate functions x^i are playing in (5.2.4), where they are the first n coordinates in T^*M (i.e., the first n components of $T^0_1\psi$, hence live on T^*U) as compared to (5.2.5), where they are the coordinates in M (i.e., the components of ψ, hence live on U). This is a slight abuse of notation, but is standard usage in the field, so we also comply.

Consequently,

$$\begin{aligned}\tau_{(q,\varphi)}(v) = \varphi(T_{(q,\varphi)}\pi(v)) &= \Big(\sum_{j=1}^n \xi^i dx^i|_q\Big)\Big(\sum_{i=1}^n v_i \left.\frac{\partial}{\partial x^i}\right|_q\Big) \\ &= \sum_{i=1}^n v_i \xi_i = \sum_{i=1}^n \xi_i dx^i|_{(q,\varphi)}(v)\end{aligned} \tag{5.2.6}$$

In other words, in terms of the local coordinates $(x^1, \dots, x^n, \xi_1, \dots, \xi_n)$ of T^*M, $\tau = \xi_i dx^i$, hence in particular it is smooth.

From this, we get that $\omega := -d\tau \in \Omega^2(T^*M)$, and clearly ω is closed (see 4.3.40 (i)). From (5.2.6) we obtain the following local expression:

$$\omega = \sum_{i=1}^n dx^i \wedge d\xi_i,$$

which is precisely the standard symplectic form (5.1.1) in these coordinates, so ω is symplectic. □

5.3 The Darboux theorem

Our aim in this section is to prove an analogue of 5.1.5 for symplectic forms on manifolds. We will see that given a symplectic manifold (M, ω), around any point there exist local coordinates with respect to which ω is of the form (5.2.1). This is a distinguishing feature of symplectic geometry, and makes it very different from, e.g., Riemannian geometry, where such a distinguished local form for the metric is in general unattainable (curvature being an obstruction against it).

Before we can prove the result we require some technical preparations that are basically analogues of 4.3.31 for time-dependent vector fields (and covariant tensors).

5.3.1 Proposition. *Let $X : I \times M \to TM$ be a smooth time-dependent vector field and let $\Psi : W \to M$ be its time-dependent flow (as given in 3.2.1). Then for any covariant tensor field $\omega \in \mathcal{T}^0_k(M)$ and any $(t_1, t_0, p) \in W$ we have:*

$$\left.\frac{d}{dt}\right|_{t=t_1} (\Psi^*_{t,t_0}\omega)_p = \Psi^*_{t_1,t_0}(L_{X_{t_1}}\omega). \tag{5.3.1}$$

Proof. We first treat the case $t_1 = t_0$. Then $\Psi_{t_0,t_0} = \mathrm{id}_M$ and (5.3.1) reduces to

$$\left.\frac{d}{dt}\right|_{t=t_0} (\Psi^*_{t,t_0}\omega)_p = L_{X_{t_0}}\omega. \tag{5.3.2}$$

To show this we proceed similarly to Section 4.2 and first consider the scalar case $\omega = f \in \mathcal{C}^\infty(M)$. Then

$$\left.\frac{d}{dt}\right|_{t=t_0} (\Psi^*_{t,t_0}f)_p = \left.\frac{\partial}{\partial t}\right|_{t=t_0} f(\Psi(t,t_0,p)) = X(t_0, \Psi(t_0,t_0,p))f = (L_{X_{t_0}}f)(p)$$

since $t \mapsto \Psi(t, t_0, p)$ is an integral curve of X. As the next special case we consider $\omega = df$ for some $f \in \mathcal{C}^\infty(M)$. Note now that in any local coordinate system, the function $\Psi^*_{t,t_0}f(x) = f(\Psi(t,t_0,x))$ is a smooth function of the variables

$(t, x^1, \dots, x^n)$, so differentiation with respect to t commutes with differentiation with respect to any of the x^i. It follows that the exterior derivative d commutes with $\frac{\partial}{\partial t}$, which together with 4.3.26 (ii) gives

$$\begin{aligned}\frac{d}{dt}\Big|_{t=t_0}(\Psi^*_{t,t_0}df)_p &= \frac{\partial}{\partial t}\Big|_{t=t_0} d(\Psi^*_{t,t_0}f)_p = d\Big(\frac{\partial}{\partial t}\Big|_{t=t_0}\Psi^*_{t,t_0}f\Big)_p \\ &= d(L_{X_{t_0}}f)_p \overset{4.3.33}{=} (L_{X_{t_0}}df)_p\end{aligned}$$

Thus we have shown (5.3.2) for smooth functions and 1-forms. To extend it to arbitrary covariant tensors, let $\omega = \alpha \otimes \beta$ and suppose the result is true for α and β. Then since (by 4.2.4) the Lie derivative is a differential operator,

$$(L_{X_{t_0}}(\alpha \otimes \beta))_p = (L_{X_{t_0}}\alpha) \otimes \beta_p + \alpha_p \otimes (L_{X_{t_0}}\beta)_p$$

On the other hand, by the product rule in local coordinates, for the left-hand side of (5.3.2) we get

$$\frac{d}{dt}\Big|_{t=t_0}(\Psi^*_{t,t_0}(\alpha\otimes\beta))_p = \Big(\frac{d}{dt}\Big|_{t=t_0}(\Psi^*_{t,t_0}\alpha)_p\Big) \otimes \beta_p + \alpha_p \otimes \Big(\frac{d}{dt}\Big|_{t=t_0}(\Psi^*_{t,t_0}\beta)_p\Big).$$

Combining these results shows the claim for $\alpha \otimes \beta$, and since locally any covariant tensor is of the form $\omega = \omega_{i_1\dots i_k}dx^{i_1} \otimes \dots \otimes dx^{i_k}$, (5.3.2) follows.

Turning now to the case of arbitrary t_1, recall from 3.2.1 that $\Psi_{t,t_0} = \Psi_{t,t_1} \circ \Psi_{t_1,t_0}$ whenever the right hand side exists. Moreover, since Ψ_{t_1,t_0} does not depend on t and the pullback of tensor fields is fiber-linear by (4.3.3), pullback under Ψ_{t_1,t_0} commutes with the t-derivative, so we obtain

$$\begin{aligned}\frac{d}{dt}\Big|_{t=t_1}\Psi^*_{t,t_0}\omega &= \frac{d}{dt}\Big|_{t=t_1}\Psi^*_{t_1,t_0}\circ\Psi^*_{t,t_1}\omega = \Psi^*_{t_1,t_0}\Big(\frac{d}{dt}\Big|_{t=t_1}\Psi^*_{t,t_1}\omega\Big) \\ &\overset{(5.3.2)}{=} \Psi^*_{t_1,t_0}(L_{X_{t_1}}\omega).\end{aligned}$$

□

Generalizing the notion of a time-dependent vector field, we say that a smooth map $\omega : I \times M \to T^0_k M$ (where $I \subseteq M$ is an open interval) is a *time-dependent tensor field* if $\omega(t,p) \in T^0_k(T_pM)$ for each $(t,p) \in I \times M$. In other words, for each fixed t, $\omega_t : p \mapsto \omega(t,p) \in \mathcal{T}^0_k(M)$. As a final preparatory result we need to know how to differentiate time-dependent tensor fields with respect to time-dependent vector fields:

5.3.2 Proposition. *Let $X : I \times M \to TM$ be a time-dependent vector field with time-dependent flow $\Psi : W \to M$ and $\omega : I \times M \to T^0_k M$ is a time-dependent tensor field. Then for any (t_1, t_0, p) we have:*

$$\frac{d}{dt}\Big|_{t=t_1}(\Psi^*_{t,t_0}\omega_t)_p = \Big(\Psi^*_{t_1,t_0}\Big(L_{X_{t_1}}\omega_{t_1} + \frac{d}{dt}\Big|_{t=t_1}\omega_t\Big)\Big)_p. \tag{5.3.3}$$

Proof. For small $\varepsilon > 0$ we have a smooth map $F : (t_1-\varepsilon, t_1+\varepsilon)\times(t_1-\varepsilon,t_1+\varepsilon) \to T^0_k(T_pM)$,

$$F(u,v) := (\Psi^*_{u,t_0}\omega_v)_p \overset{(4.3.3)}{=} (T_p\Psi_{u,t_0})^*(\omega_v|_{\Psi_{u,t_0}(p)}).$$

This defines a smooth map into the finite-dimensional vector space $T^0_k(T_pM)$, so we may apply the chain rule, in conjunction with 5.3.1:

$$\begin{aligned}\frac{d}{dt}\Big|_{t=t_1}F(t,t) &= \frac{\partial F}{\partial u}(t_1,t_1) + \frac{\partial F}{\partial v}(t_1,t_1) \\ &= (\Psi^*_{t_1,t_0}(L_{X_{t_1}}\omega_{t_1}))_p + \frac{\partial}{\partial v}\Big|_{v=t_1}(T_p\Psi_{t_1,t_0})^*(\omega_v|_{\Psi_{t_1,t_0}(p)}).\end{aligned}$$

As in the previous proof, since Ψ_{t_1,t_0} is independent of v, we may switch the v-derivative in the last term to the inside, finishing the proof. $\square$

We are now prepared to give a modern proof of the Darboux theorem, due to A. Weinstein:

5.3.3 Theorem. *(Darboux) Let (M,ω) be a $2n$-dimensional symplectic manifold. Then for any $p \in M$ there exist local coordinates $(x^1,\dots,x^n,y^1,\dots,y^n)$ around p in terms of which ω has the local representation*

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i. \tag{5.3.4}$$

Such coordinates are called Darboux coordinates *(or also* symplectic *or* canonical *coordinates).*

Proof. For the course of this proof, let ω_0 denote the given symplectic form on M and fix any $p_0 \in M$. We are looking for a coordinate chart (U_0,φ) around p_0 such that $\varphi^*\omega_1 = \omega_0$, where $\omega_1 = \sum_{i=1}^n dx^i \wedge dy^i$ is the standard symplectic form on $\mathbb{R}^{2n}$. Since this is a local problem, we may without loss of generality assume that $M = U \subseteq \mathbb{R}^{2n}$ is an open ball. Due to 5.1.5 we can apply a linear coordinate transformation so as to arrange that $\omega_0|_{p_0} = \omega_1|_{p_0}$.

Set $\eta := \omega_1 - \omega_0$. Then η is closed, so by the Poincaré Lemma 4.3.40 there exists a smooth 1-form on U such that $d\alpha = -\eta$. Moreover, subtracting a constant-coefficient (hence closed) 1-form if necessary, we may assume that $\alpha_{p_0} = 0$. For $t \in \mathbb{R}$ we consider the following closed 2-form ω_t on U:

$$\omega_t := \omega_0 + t\eta = (1-t)\omega_0 + t\omega_1.$$

Let $I \supseteq [0,1]$ be a bounded open interval. For each t, $\omega_t|_{p_0} = \omega_0|_{p_0}$ is nondegenerate. Considering the bilinear form ω as a matrix, this means that $\det\omega \neq 0$ on the compact set $\bar{I} \times \{p_0\}$ and by continuity of det it follows that there exists an open neighborhood $U_1 \subseteq U$ of p_0 such that ω_t is nondegenerate on U_1 for all $t \in \bar{I}$. This means that for each $t \in \bar{I}$ and each $p \in U_1$ the map $\hat{\omega}_t : T_pU_1 \to T_p^*U_1$, $\hat{\omega}_t(X) := i_X\omega_t = \omega_t(X, .\,)$ is a linear isomorphism. Therefore we may define a time-dependent vector field $X : I \times U_1 \to TU_1$ by setting $X_t := \hat{\omega}_t^{-1}\alpha$, i.e., $i_{X_t}\omega_t = \alpha$. Calculating $\hat{\omega}_t^{-1}$ amounts to inverting the matrix corresponding to $\hat{\omega}_t$, hence X_t is smooth.

Since $\alpha_{p_0} = 0$ we also have $X_t|_{p_0} = 0$ for all $t \in I$. Let $\Psi : W \to U_1$ denote the time-dependent flow of X (see 3.2.1). Then $\Psi(t,0,p_0) = p_0$ for all $t \in I$, whereby $I \times \{0\} \times \{p_0\} \subseteq W$. Since W is open in $I \times I \times U_1$ and $[0,1] \times \{0\} \times \{p_0\}$ is compact, there exists a neighborhood U_0 of p_0 such that $[0,1] \times \{0\} \times U_0 \subseteq W$. We may therefore apply 5.3.2, to obtain (via Cartan's magic formula 4.3.35 (v)) for any $t_1 \in [0,1]$:

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_1} (\Psi_{t,0}^*\omega_t) &= \Psi_{t_1,0}^*\Big(L_{X_{t_1}}\omega_{t_1} + \left.\frac{d}{dt}\right|_{t=t_1}\omega_t\Big) \\
&= \Psi_{t_1,0}^*(i_{X_{t_1}}\underbrace{d\omega_{t_1}}_{=0} + d(i_{X_{t_1}}\omega_{t_1}) + \eta) \\
&= \Psi_{t_1,0}^*(d\alpha + \eta) = 0.
\end{aligned}$$

Consequently, $\Psi_{t,0}^*\omega_t = \Psi_{0,0}^*\omega_0 = \omega_0$ for all $t \in [0,1]$, so in particular $\Psi_{1,0}^*\omega_1 = \omega_0$. By 3.2.1 (iii), $\Psi_{1,0}$ is a diffeomorphism onto its image, hence it can serve as a local chart. Finally, $\Psi_{1,0}(p_0) = p_0$, i.e., this chart is indeed centered at p_0. $\square$

5.4 Hamiltonian vector fields

Non-degeneracy of the symplectic form provides a means to implement a duality between the exterior derivative of a function and a corresponding vector field, which turns out to be very useful in applications. Let (M, ω) be a symplectic manifold and let $f \in \mathcal{C}^\infty(M)$. The *Hamiltonian vector field* $X_f \in \mathfrak{X}(M)$ is defined by

$$X_f := \hat{\omega}^{-1}(df),$$

where $\hat{\omega} : TM \to T^*M$ is the vector bundle isomorphism induced by ω (cf. the proof of the Darboux theorem to see that X_f is indeed smooth). Equivalently,

$$i_{X_f}\omega = df, \tag{5.4.1}$$

i.e., for any $Y \in \mathfrak{X}(M)$ we have $\omega(X_f, Y) = df(Y) = Y(f)$.

5.4.1 Example. Let us calculate X_f explicitly for ω in Darboux coordinates (in particular: for the standard symplectic form on $\mathbb{R}^{2n}$). We make the ansatz

$$X_f = \sum_{i=1}^n \Big(a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i} \Big)$$

for smooth functions a^i, b^i to be determined. By 4.3.35 we have

$$i_{X_f}\omega = i_{\sum_{i=1}^n (a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i})} \Big(\sum_{j=1}^n dx^j \wedge dy^j \Big) = \sum_{j=1}^n (a^i dy^i - b^i dx^i),$$

while

$$df = \sum_{j=1}^n \Big(\frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^i} dy^i \Big).$$

By (5.4.1) we conclude that $a^i = \frac{\partial f}{\partial y^i}$ and $b^i = -\frac{\partial f}{\partial x^i}$, i.e.,

$$X_f = \sum_{i=1}^n \Big(\frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial}{\partial y^i} \Big). \tag{5.4.2}$$

Some important properties of Hamiltonian vector fields are given in the following result:

5.4.2 Proposition. *Let (M, ω) be a symplectic manifold and let $f \in \mathcal{C}^\infty(M)$. Then*

- *(i) f is constant along each integral curve of X_f.*
- *(ii) At each regular point p of f ($df|_p \neq 0$), X_f is tangent to the level set of f.*

Proof. By skew-symmetry of ω we have

$$X_f(f) = df(X_f) = i_{X_f}\omega(X_f) = \omega(X_f, X_f) = 0.$$

This gives (i). Moreover, if $L = f^{-1}(q)$ is a level set of f containing p, then by 3.3.25 we have $T_pL = \ker(df|_p)$, so again by the above equation $X_f(p) \in T_pL$. $\square$

We call $X \in \mathfrak{X}(M)$ a *symplectic vector field* if ω is invariant under the flow of X, i.e., if $(\mathrm{Fl}^X_t)^*\omega = \omega$ for all t. By 4.3.31, X is symplectic if and only if $L_X\omega = 0$. X is called (globally) *Hamiltonian* if there exists some $f \in \mathcal{C}^\infty(M)$ with $X = X_f$. It is called *locally Hamiltonian* if every point has a neighborhood on which X is Hamiltonian.

5.4.3 Proposition. *Let (M,ω) be a symplectic manifold and let $X \in \mathfrak{X}(M)$. TFAE:*

(i) X is symplectic.

(ii) X is locally Hamiltonian.

Moreover, TFAE:

(iii) Every locally Hamiltonian vector field is globally Hamiltonian.

(iv) $H^1_{dR}(M) = 0$.

Proof. Cartan's magic formula 4.3.35 (v) gives

$$L_X\omega = d(i_X\omega) + i_X(d\omega) = d(i_X\omega) \tag{5.4.3}$$

as ω is closed. Hence X is symplectic if and only if $i_X\omega$ is closed.

(i)⇒(ii) By the Poincaré lemma 4.3.40, any point p has a neighborhood U on wich $i_X\omega$ is exact, i.e., $i_X\omega = df \overset{(5.4.1)}{=} i_{X_f}\omega$ on U for some $f \in \mathcal{C}^\infty(U)$. Since ω is nondegenerate, this implies $X = X_f$ on U.

(ii)⇒(i): Let $X = X_f$ on some open set U. Then $i_X\omega = i_{X_f}\omega = df$, which is clearly closed.

(iii)⇒(iv) Let $\eta \in \Omega^1(M)$ be closed and define $X \in \mathfrak{X}(M)$ by $X := \hat{\omega}^{-1}(\eta)$. Then by (5.4.3) we get $L_X\omega = di_X\omega = d\eta = 0$, so X is symplectic and thereby locally Hamiltonian. (iii) then says that $X = X_f$ for some global $f \in \mathcal{C}^\infty(M)$. Thus

$$\eta = \omega(X_f, \,.\,) = i_{X_f}\omega \overset{(5.4.1)}{=} df,$$

showing that η is exact.

(iv)⇒(iii): By (ii)⇒(i), if X is locally Hamiltonian then it is symplectic, hence by (5.4.3) we get that $i_X\omega$ is closed. Thus by assumption there is some $f \in \mathcal{C}^\infty(M)$ with $i_X\omega = df$. This means that $X = X_f$, so X is globally Hamiltonian. □

A symplectic manifold (M,ω), together with a function $H \in \mathcal{C}^\infty(M)$ is called a *Hamiltonian system*. The function H is called the *Hamiltonian* of this system. The terminology comes from classical mechanics, where the Hamiltonian is used to describe the total energy of a system. The flow of the corresponding Hamiltonian vector field X_H is called its *Hamiltonian flow*. The integral curves of X_H are called the *trajectories* (or orbits) of the system. In Darboux coordinates, a trajectory has to satisfy (writing a dot for the t-derivative) $(\dot{x}(t), \dot{y}(t)) = X_H(x(t), y(t))$, which by (5.4.2) translates into:

$$\begin{aligned} \dot{x}^i(t) &= \frac{\partial H}{\partial y^i}(x(t), y(t)) \\ \dot{y}^i(t) &= -\frac{\partial H}{\partial x^i}(x(t), y(t)) \end{aligned} \tag{5.4.4}$$

This ODE system is called *Hamilton's equations.*

5.4.4 Example. To give some connection to classical mechanics, where these constructions originate, let us look at the n-body problem. Consider n point particles with masses m_k located at points $q_k(t) \in \mathbb{R}^3$ at time t, with $q_k(t) =$

$(q_k^1(t), q_k^2(t), q_k^3(t))$. The evolution of the entire system is then encoded in the following curve in $\mathbb{R}^{3n}$:

$$q(t) = (q_1^1(t), q_1^2(t), q_1^3(t), \ldots, q_n^1(t), q_n^2(t), q_n^3(t)).$$

We do not allow collisions, so we only look at curves in the open set $Q := \mathbb{R}^{3n} \setminus \{q \in \mathbb{R}^{3n} \mid q_k = q_l \text{ for some } k \neq l\}$. Our assumption is that the particles move under the influence of forces that depend exclusively on the position of all particles. We write the force acting on the k-th particle as $F_k(q) = (F_k^1(q), F_k^2(q), F_k^3(q))$. Then by Newton's second law, the coordinates of our particles satisfy the following $3n \times 3n$ system of ODEs:

$$\begin{aligned}
m_k \ddot{q}_k^1(t) &= F_k^1(q(t)) \\
m_k \ddot{q}_k^2(t) &= F_k^2(q(t)) \\
m_k \ddot{q}_k^3(t) &= F_k^3(q(t)) \qquad (k = 1, \ldots, n).
\end{aligned}$$

To rewrite this system more concisely, write $q(t) = (q^1(t), \ldots, q^{3n}(t))$, $F(q) = (F_1(q), \ldots, F_{3n}(q))$, and $M = (M_{ij}) := \operatorname{diag}(m_1, m_1, m_1, \ldots, m_n, m_n, m_n)$. Then we obtain for the equations of motion:

$$M_{ij} \ddot{q}^j(t) = F_i(q(t)) \qquad (1 \leq i \leq 3n). \tag{5.4.5}$$

We may view the F_i as the components of a smooth 1-form F on Q, and we also make the assumption that F is *conservative*, i.e., it comes from a potential: there exists some $V \in \mathcal{C}^\infty(Q)$ with $F = -dV$. Since masses are positive, the matrix M is positive definite, hence it induces a scalar product $\langle\,.\,,\,.\,\rangle$ on $\mathbb{R}^{3n}$ via $(v, w) \mapsto v^t \cdot M \cdot w$. This scalar product then induces the standard isomorphism between any $T_qQ \cong \mathbb{R}^{3n}$ and T_q^*Q, namely $v \mapsto \langle v,\,.\,\rangle$, providing a vector bundle isomorphism $\widehat{M} : TQ \to T^*Q$ (note that both bundles are actually trivial). Denote the coordinates on TQ by (q^i, v^i) and the ones on T^*Q by (q^i, p_i). Then we can write $v^t \cdot M \cdot w = M_{ij} v^i w^j$, and for $\widehat{M}$ we get

$$(q^i, p_i) = \widehat{M}(q^i, v^i) = (q^i, M_{ij} v^j).$$

If $\dot{q}(t)$ is the velocity vector of the system, then the corresponding 1-form $p(t) = \widehat{M}(\dot{q}(t))$ in coordinates reads

$$p_i(t) = M_{ij} \dot{q}^j(t). \tag{5.4.6}$$

From the physics point of view, $p(t)$ contains the momenta of the particles. Combining (5.4.5) and (5.4.6) we see that

$$M_{ij} \ddot{q}^j = F_i(q(t)) = -\frac{\partial V}{\partial q^i}(q(t)),$$

so a curve $q(t)$ in Q satisfies the Newtonian equations of motion (a second order system of ODEs) if and only if the corresponding curve $\gamma(t) := (q(t), p(t))$ in T^*Q is a solution to the first order system of ODEs

$$\begin{aligned}
\dot{q}^i(t) &= M^{ij} p_j(t) \\
\dot{p}_i(t) &= -\frac{\partial V}{\partial q^i}(q(t)),
\end{aligned} \tag{5.4.7}$$

where M^{ij} are the coefficients of the inverse matrix of M. Now define the *total energy* $H \in \mathcal{C}^\infty(T^*Q)$ of the system by

$$H(p, q) := V(q) + K(p),$$

where V is the potential energy from above and K is the total kinetic energy, $K(p) = \frac{1}{2}M^{ij}p_i p_j$. In terms of the Darboux coordinates (q^i, p_i) on T^*Q, (5.4.7) is precisely the system of Hamilton's equations (5.4.4) for this Hamiltonian. From 5.4.2 we know that H is constant along the trajectories (i.e., the solutions to (5.4.7)), which means that the total energy is conserved along the solutions of the equations of motion.

5.5 Poisson brackets

5.5.1 Definition. *Let (M, ω) be a symplectic manifold and let $f, g \in \mathcal{C}^\infty(M)$. Then the* Poisson bracket *$\{f, g\} \in \mathcal{C}^\infty(M)$ of f and g is given by*

$$\{f, g\} := \omega(X_f, X_g) \overset{(5.4.1)}{=} df(X_g) = X_g(f). \tag{5.5.1}$$

The last equality (together with 2.3.11) provides a geometric interpretation of the Poisson bracket: $\{f, g\}$ measures the rate of change of f along the flow of the Hamiltonian vector field corresponding to g. Using (5.4.2) we obtain the following local expression for the Poisson bracket in Darboux coordinates:

$$\{f, g\} = \sum_{i=1}^{n} \Big(\frac{\partial g}{\partial y^i} \frac{\partial f}{\partial x^i} - \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial y^i} \Big) \tag{5.5.2}$$

The fundamental algebraic properties of $\{\,.\,,\,.\,\}$ are collected in the following result:

5.5.2 Proposition. *Let (M, ω) be a symplectic manifold and let $f, g, h \in \mathcal{C}^\infty(M)$. Then*

(i) $\{\,.\,,\,.\,\}$ *is $\mathbb{R}$-bilinear.*

(ii) $\{f, g\} = -\{g, f\}$ *(anti-symmetry).*

(iii) $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ *(Jacobi-identity).*

(iv) $X_{\{f,g\}} = -[X_f, X_g]$.

Proof. (i) and (ii) follow directly from $\{f, g\} = \omega(X_f, X_g)$ and the fact that $\omega \in \Omega^2(M)$, because $f \mapsto X_f = \hat{\omega}^{-1}(df)$ is linear in f.

(iv) Since ω is nondegenerate, it suffices to show that

$$\omega(X_{\{f,g\}}, Y) + \omega([X_f, X_g], Y) = 0 \tag{5.5.3}$$

for each $Y \in \mathfrak{X}(M)$. Due to (5.4.1) and (5.5.2) we have

$$\omega(X_{\{f,g\}}, Y) = d(\{f, g\})(Y) = Y\{f, g\} = Y X_g f. \tag{5.5.4}$$

Furthermore, X_g is symplectic by 5.4.3, so

$$\begin{aligned} 0 &= (L_{X_g}\omega)(X_f, Y) \\ &\overset{4.3.37}{=} X_g(\omega(X_f, Y)) - \omega([X_g, X_f], Y) - \omega(X_f, [X_g, Y]). \end{aligned} \tag{5.5.5}$$

Here,

$$\begin{aligned} X_g(\omega(X_f, Y)) &= X_g(df(Y)) = X_g Y f \\ \omega(X_f, [X_g, Y]) &= df([X_g, Y]) = [X_g, Y](f) = X_g Y f - Y X_g f \\ &\overset{(5.5.4)}{=} X_g Y f - \omega(X_{\{f,g\}}, Y). \end{aligned}$$

Inserting this into (5.5.5) gives (5.5.3).

(iii)

$$\begin{aligned}\{f,\{g,h\}\} &\overset{(5.5.1)}{=} X_{\{g,h\}}f \overset{(iv)}{=} -[X_g,X_h]f = -X_gX_hf + X_hX_gf\\ &\overset{(5.5.1)}{=} -X_g\{f,h\} + X_h\{f,g\} = -\{\{f,h\},g\} + \{\{f,g\},h\}\\ &\overset{(ii)}{=} -\{g,\{h,f\}\} - \{h,\{f,g\}\}.\end{aligned}$$

□

Points (i)–(iii) of the previous result state that $(\mathcal{C}^\infty(M),\{\,.\,,\,.\,\})$ is a Lie algebra (cf. 2.2.18) on any symplectic manifold.

5.5.3 Definition. *Let (M,ω,H) be a Hamiltonian system. A function $f\in\mathcal{C}^\infty(M)$ that is constant along every integral curve of X_H is called a* conserved quantity *of the system. A vector field $X\in\mathfrak{X}(M)$ is called an* infinitesimal symmetry *of the system if both ω and H are invariant under the flow of X.*

5.5.4 Proposition. *Let (M,ω,H) be a Hamiltonian system.*

(i) $f\in\mathcal{C}^\infty(M)$ is a conserved quantity if and only if $\{f,H\}=0$.

(ii) $X\in\mathfrak{X}(M)$ is an infinitesimal symmetry if and only if it is symplectic and $X(H)=0$.

(iii) Let X be an infinitesimal symmetry and γ a trajectory of the system. Then for any $s\in\mathbb{R}$, $\mathrm{Fl}^X_s\circ\gamma$ is a trajectory as well (on its domain).

Proof. (i) By 4.3.31, f is a conserved quantity if and only if

$$0 = X_H(f) \overset{(5.5.1)}{=} \{f,H\}.$$

(ii) By definition (and 4.3.32), X is an infinitesimal symmetry if and only if $L_X\omega=0$ and $X(H)=0$. Here, the first condition says that X is symplectic.

(iii) Setting $c(t):=\mathrm{Fl}^X_s(\gamma(t))$, we have to show that $\dot c(t)=X_H(c(t))$ on the domain of c. Now

$$\begin{aligned}\dot c(t) &= (T_{\gamma(t)}\mathrm{Fl}^X_s)(\dot\gamma(t)) = (T_{\gamma(t)}\mathrm{Fl}^X_s)(X_H(\gamma(t)))\\ &= (T_{\gamma(t)}\mathrm{Fl}^X_s)(X_H((\mathrm{Fl}^X_s)^{-1}(c(t)))) = ((\mathrm{Fl}^X_s)^*X_H)(c(t)).\end{aligned}$$

Hence if we can show that $(\mathrm{Fl}^X_s)^*X_H = X_H$, the result will follow. By 2.3.18 this amounts to showing that $[X,X_H]=0$. It suffices to establish this on any open set, so since X is symplectic by (ii) we may assume (using 5.4.3) that $X=X_f$ for some smooth function f. Then

$$\{f,H\} \overset{(5.5.1)}{=} -X_f(H) \overset{(ii)}{=} 0,$$

so indeed

$$[X,X_H] = [X_f,X_H] \overset{5.5.2}{=} -X_{\{f,H\}} = 0.$$

□

We conclude this chapter with a central theorem of mathematical physics, which establishes a deep connection between conserved quantities and infinitesimal symmetries.

5.5.5 Theorem. *(Noether's theorem) Let (M, ω, H) be a Hamiltonian system.*

(i) If f is a conserved quantity, then X_f is an infinitesimal symmetry.

(ii) Conversely, if $H^1_{dR}(M) = 0$, then each infinitesimal symmetry X is the Hamiltonian vector field $X = X_f$ of a conserved quantity $f \in \mathcal{C}^\infty(M)$. This f is unique up to addition of a function that is constant on each connected component of M.

Proof. (i) By 5.5.4 (i), $\{f, H\} = 0$, so $X_f H = \{H, f\} = 0$. Moreover, 5.4.3 shows that X_f is symplectic, so 5.5.4 (ii) gives the claim.

(ii) By definition, X is symplectic, hence by 5.4.3 it is globally Hamiltonian, say $X = X_f$. Also,

$$\{H, f\} = X_f H = X(H) = 0$$

since X is an infinitesimal symmetry, and so f is a conserved quantity by 5.5.4 (i). To show uniqueness, suppose that also $X_g = X$ for some $g \in \mathcal{C}^\infty(M)$. Then

$$d(g - f) \overset{(5.4.1)}{=} i_{(X_g - X_f)}\omega = i_0\omega = 0,$$

so $g - f$ has to be constant on each connected component of M. $\square$

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Index