1.1 Submanifolds

In [3, Sec. 2.1] we introduced submanifolds of $\mathbb{R}^n$: $M \subseteq \mathbb{R}^n$ is called a submanifold of dimension k if for every $p \in M$ there exists an open neighborhood W of p in $\mathbb{R}^n$, an open subset U of $\mathbb{R}^k$ and an immersion $\varphi : U \to \mathbb{R}^n$ such that $\varphi : U \to \varphi(U)$ is a homeomorphism and $\varphi(U) = W \cap M$. Then φ is called a local parametrisation of M. By [3, 2.2.8], any such M is an abstract manifold whose natural manifold topology is precisely the trace topology of $\mathbb{R}^n$ on M.

We now want to introduce appropriate notions of submanifolds for abstract manifolds in general. To this end we first need a few results on maps between manifolds.

1.1.1 Definition. *Let M, N be manifolds and let $f : M \to N$ be smooth. The rank $\mathrm{rk}_p(f)$ of f at $p \in M$ is the rank of the linear map $T_pf : T_pM \to T_{f(p)}N$.*

If $\varphi = (x^1, \ldots, x^m)$ is a chart of M at p and $(y^1, \ldots, y^n)$ a chart of N at $f(p)$ then the matrix of $T_pf : T_pM \to T_{f(p)}N$ with respect to the bases $(\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^m}\big|_p)$ of T_pM and $(\frac{\partial}{\partial y^1}\Big|_{f(p)}, \ldots, \frac{\partial}{\partial y^n}\Big|_{f(p)})$ is the Jacobi matrix of of $\psi \circ f \circ \varphi^{-1}$ at $\varphi(p)$ (see [3, 2.4]). Thus $\mathrm{rk}_p(f) = \mathrm{rk}_{\varphi(p)}(\psi \circ f \circ \varphi^{-1})$.

1.1.2 Definition. *Let $f : M \to N$ be smooth. f is called* immersion *(*submersion*) if T_pf is injective (surjective) for every $p \in M$.*

If $\dim(M) = m$ and $\dim(N) = n$ (which henceforth we will indicate by writing M^m and N^n, respectively) then f is an immersion (resp. submersion) if and only if $\mathrm{rk}_p(f) = m$ (resp. $= n$) for all $p \in M$. The following result shows that maps of constant rank locally always are of a particularly simple form.

1.1.3 Theorem. *(Rank Theorem) Let M^m, N^n be manifolds and let $f : M \to N$ be smooth. Let $p \in M$ and suppose that $\mathrm{rk}_p(f) = k$ in a neighborhood of p. Then there exist charts (φ, U) of M at p and (ψ, V) of N at $f(p)$ such that $\varphi(p) = 0 \in \mathbb{R}^m$, $\psi(f(p)) = 0 \in \mathbb{R}^n$ and*

$$\psi \circ f \circ \varphi^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0).$$

Proof. By the above, the rank of f is independent of the chosen charts, so without loss of generality we may assume that $f : W \to W'$, where W is open in $\mathbb{R}^m$ and W' is open in $\mathbb{R}^n$, $p = 0$, $f(p) = 0$ and $\mathrm{rk}(f) \equiv k$ on W. Since $\mathrm{rk}(Df(0)) = k$ there exists an invertible $k \times k$ submatrix of $Df(0)$ and without loss we may assume that this matrix is given by $(\frac{\partial f^i}{\partial x^j})_{i,j=1}^k$. Now consider the smooth map $\varphi : W \to \mathbb{R}^m$,

$$\varphi(x^1, \ldots, x^m) = (f^1(x^1, \ldots, x^m), \ldots, f^k(x^1, \ldots, x^m), x^{k+1}, \ldots, x^m).$$

Then $\varphi(0) = 0$ and

$$D\varphi(0) = \begin{pmatrix} \left(\frac{\partial f^i}{\partial x^j}\right)_{i,j=1}^k & * \\ 0 & I_{m-k} \end{pmatrix}$$

is invertible. By the inverse function theorem φ thereby is a diffeomorphism from some open neighborhood $W_1 \subseteq W$ of 0 onto some open neighborhood U_1 of 0 in $\mathbb{R}^m$. Then on U_1 we have

$$f \circ \varphi^{-1}(x) = f \circ \varphi^{-1}(x^1, \ldots, x^k, x^{k+1}, \ldots, x^m) = (x^1, \ldots, x^k, \bar{f}^{k+1}(x), \ldots, \bar{f}^n(x))$$

for suitable smooth functions $\bar f^{k+1},\dots,\bar f^n$. Consequently,

$$D(f\circ\varphi^{-1})(0)=\begin{pmatrix} I_k & 0\\ * & \left(\frac{\partial \bar f^r}{\partial x^s}\right)_{\substack{r=k+1,\dots,n\\ s=k+1,\dots,m}}\end{pmatrix}.$$

Since $D(f\circ\varphi^{-1}) = Df\circ D\varphi^{-1}$ and $D\varphi^{-1}$ is bijective it follows that $\mathrm{rk}(D(f\circ\varphi^{-1})) = \mathrm{rk}(Df)\equiv k$ on U_1. Then necessarily $\frac{\partial \bar f^r}{\partial x^s}=0$ for $r=k+1,\dots,n$ and $s=k+1,\dots,m$, i.e., $\bar f^{k+1},\dots,\bar f^n$ depend only on $x^1,\dots,x^k$. Now set

$$T(y^1,\dots,y^k,y^{k+1},\dots,y^m):= \\ (y^1,\dots,y^k,y^{k+1}+\bar f^{k+1}(y^1,\dots,y^k),\dots,y^n+\bar f^n(y^1,\dots,y^k)).$$

Then $T(0)=0$ and

$$DT(y)=\begin{pmatrix} I_k & 0\\ * & I_{n-k}\end{pmatrix},$$

so T is a diffeomorphism from some open neighborhood $\tilde V$ of 0 in $\mathbb{R}^n$ onto some open $0\in V\subseteq W'$. Choose $\tilde U\subseteq U_1$ open such that $f\circ\varphi^{-1}(\tilde U)\subseteq V$ and let $U:=\varphi^{-1}(\tilde U)$. Let $\psi:=T^{-1}$, then

$$\tilde U\xrightarrow{\varphi^{-1}} U\xrightarrow{f} V\xrightarrow{\psi}\tilde V$$

and

$$\psi\circ f\circ\varphi^{-1}(x^1,\dots,x^k,x^{k+1},\dots,x^m)= \\ \psi(x^1,\dots,x^k,\bar f^{k+1}(x^1,\dots,x^k),\dots,\bar f^n(x^1,\dots,x^k))=(x^1,\dots,x^k,0,\dots,0)$$

on $\tilde U$. □

1.1.4 Lemma. *Let $f: M^m\to N^n$ be smooth, let $p\in M$ and suppose that $\mathrm{rk}_p(f)=k$. Then there exists a neighborhood U of p in M such that $\mathrm{rk}_q(f)\ge k$ for all $q\in U$. In particular, if $k=\min(m,n)$ then $\mathrm{rk}_q(f)=k$ for all $q\in U$.*

Proof. Picking charts φ around p and ψ around $f(p)$, $\mathrm{rk}_p(f)=k$ if and only if there exists a $k\times k$-submatrix of $(D(\psi\circ f\circ\varphi^{-1}))$ with nonzero determinant. By continuity, the same is then true on an entire neighborhood of p. This means that the rank cannot drop locally. If $k=\min(m,n)$ then it also cannot increase. □

1.1.5 Theorem. (Inverse function theorem) *Let $f: M^m\to N^n$ be smooth, let $p\in M$ and suppose that $T_pf: T_pM\to T_{f(p)}N$ is bijective. Then there exist open neighborhoods U of p in M and V of $f(p)$ in N such that $f:U\to V$ is a diffeomorphism.*

Proof. For charts φ of M at p, and ψ at $f(p)$ in N the map $D(\psi\circ f\circ\varphi^{-1})(\varphi(p))= T_{f(p)}\psi\circ T_pf\circ T_{\varphi(p)}\varphi^{-1}$ is invertible. Hence by the classical inverse function theory, $\psi\circ f\circ\varphi^{-1}$ is a diffeomorphism around $\varphi(p)$ and the claim follows. □

1.1.6 Proposition. (Local characterization of immersions) *Let $f: M^m\to N^n$ be smooth and let $p\in M$. TFAE:*

(i) T_pf is injective.

(ii) $\mathrm{rk}_p(f) = m$.

(iii) If $\psi = (\psi^1, \ldots, \psi^n)$ *is a chart at* $f(p)$ *in* N *then there exist* $1 \le i_1 < \cdots < i_m \le n$ *such that* $(\psi^{i_1}, \ldots, \psi^{i_m})$ *is a chart at* p *in* M.

Proof. Clearly, (i)$\Leftrightarrow$ (ii).
(ii)$\Rightarrow$(iii): Let φ be a chart at p in M. Then $\mathrm{rk}(D(\psi \circ f \circ \varphi^{-1})(\varphi(p))) = m$, hence there exist $1 \le i_1 < \cdots < i_m \le n$ with $\det D((\psi^{i^1}, \ldots, \psi^{i^m}) \circ f \circ \varphi^{-1})(\varphi(p)) \neq 0$. By 1.1.5, then, $(\psi^{i^1}, \ldots, \psi^{i^m})$ is a diffeomorphism locally around p, hence a chart.
(iii)$\Rightarrow$(ii): The linear map $D((\psi^{i^1}, \ldots, \psi^{i^m}) \circ f \circ \varphi^{-1})(\varphi(p))$ is bijective, so $\mathrm{rk}(D(\psi \circ f \circ \varphi^{-1})(\varphi(p))) = m$. $\square$

1.1.7 Proposition. (Local characterization of submersions) *Let* $f : M^m \to N^n$ *be smooth and let* $p \in M$. *TFAE:*

(i) $T_p f$ *is surjective.*

(ii) $\mathrm{rk}_p(f) = n$.

(iii) If $\psi = (\psi^1, \ldots, \psi^n)$ *is any chart at* $f(p)$ *in* N *then there exists a chart* φ *of* M *at* p *such that* $(\psi^1 \circ f, \ldots \psi^n \circ f, \varphi^{n+1}, \ldots, \varphi^m)$ *is a chart at* p *in* M.

Proof. Again, (i)$\Leftrightarrow$(ii) is obvious.
(ii)$\Rightarrow$(iii): Let $\tilde{\varphi}$ and ψ be charts at p and $f(p)$, respectively. Since $\mathrm{rk}(D(\psi \circ f \circ \varphi^{-1})(\varphi(p))) = n$, the Jacobi matrix $D(\psi \circ f \circ \varphi^{-1})(\varphi(p))$ possesses n linearly independent columns. By permuting the coordinates of $\tilde{\varphi}$ we obtain a chart φ such that the first n columns of $D(\psi \circ f \circ \varphi^{-1})(\varphi(p))$ are linearly independent. Now set $\chi := (\psi^1 \circ f, \ldots, \psi^n \circ f, \varphi^{n+1}, \ldots, \varphi^m)$. Then

$$D(\chi \circ \varphi^{-1})(\varphi(p)) = \begin{pmatrix} \left(\frac{\partial \psi^i \circ f \circ \varphi^{-1}}{\partial x^j}(\varphi(p))\right)_{i,j=1}^{n} & * \\ 0 & I_{m-n} \end{pmatrix} \tag{1.1.1}$$

Hence, by 1.1.5, $\chi \circ \varphi^{-1}$ is a diffeomorphism around $\varphi(p)$, and so χ is a chart at p.
(iii)$\Rightarrow$(ii): Since $\mathrm{rk}(D(\chi \circ \varphi^{-1})(\varphi(p))) = m$, (1.1.1) implies that $\mathrm{rk}(D(\psi \circ f \circ \varphi^{-1})(\varphi(p))) = m$. $\square$

1.1.8 Proposition. *Let* M^m, N^n, R^r *be manifolds,* $f : M \to N$ *continuous and* $g : N \to R$ *an immersion. If* $g \circ f$ *is smooth then so is* f.

Proof. Given $p \in M$, by 1.1.3 we may choose charts (φ, U) around $f(p)$ in N, and (ψ, V) around $g(f(p))$ in R such that

$$g_{\psi\varphi} := \psi \circ g \circ \varphi^{-1}(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0). \tag{1.1.2}$$

Let and (χ, W) be a chart in M around p and set $f_{\varphi\chi} := \varphi \circ f \circ \chi^{-1}$.

$$\begin{array}{ccccc} M & \xrightarrow{f} & N & \xrightarrow{g} & R \\ \chi\downarrow & & \downarrow\varphi & & \downarrow\psi \\ \mathbb{R}^m & \xrightarrow{f_{\varphi\chi}} & \mathbb{R}^n & \xrightarrow{g_{\psi\varphi}} & \mathbb{R}^r \end{array}$$

Then $\psi\circ(g\circ f)\circ\chi^{-1}$ is defined on $\chi((g\circ f)^{-1}(V)\cap W)$, $f_{\varphi\chi}$ is defined on $\chi(f^{-1}(U)\cap W)$, and $g_{\psi\varphi}$ is defined on $\varphi(g^{-1}(V)\cap U)$. It follows that $g_{\psi\varphi}\circ f_{\varphi\chi}$ is defined on

$$\begin{aligned}\chi(f^{-1}(U)\cap W)\cap f_{\varphi\chi}^{-1}(\varphi(g^{-1}(V)\cap U)) &= \chi(f^{-1}(U)\cap W)\cap\chi(f^{-1}(g^{-1}(V)\cap U))\\ &= \chi(f^{-1}(g^{-1}(V))\cap f^{-1}(U)\cap W)\end{aligned}$$

Since f is continuous, this shows that $g_{\psi\varphi}\circ f_{\varphi\chi}$ is a restriction of $\psi\circ(g\circ f)\circ\chi^{-1}$ to an open set, hence is smooth. By (1.1.2), $(g_{\psi\varphi}\circ f_{\varphi\chi})^i = f^i_{\varphi\chi}$ for $1\le i\le n$, hence $f_{\varphi\chi}$ is smooth. Thus, finally, f is smooth. □

1.1.9 Proposition. *Let M^m, N^n, R^r be manifolds, $f : M\to N$ a surjective submersion and $g : N\to R$ arbitrary. If $g\circ f$ is smooth then so is g.*

Proof. Using the same notations as in the proof of 1.1.8, by 1.1.3 we may choose the charts (χ, W) around p and (φ, U) around $f(p)$ in such a way that $f_{\varphi\chi} = \varphi\circ f\circ\chi^{-1} = (x^1,\dots,x^m)\mapsto(x^1,\dots,x^n)$. As in the proof of 1.1.8, $g_{\psi\varphi}\circ f_{\varphi\chi}$ is a restriction of $\psi\circ(g\circ f)\circ\chi^{-1}$ to an open set, hence is smooth. Thus $(x^1,\dots,x^m)\mapsto g_{\psi\varphi}(x^1,\dots,x^n)$ and thereby $g_{\psi\varphi}$ itself is smooth, which implies smoothness of g. □

After these preparations we are now ready to introduce the notion of submanifold of an abstract manifold.

1.1.10 Definition. *Let M^m and N^n be manifolds with $N\subseteq M$ and denote by $j : N\hookrightarrow M$ the inclusion map. N is called an* immersive submanifold *of M if j is an immersion. N is called a* submanifold *(or sometimes a* regular submanifold*), if it is an immersive submanifold and in addition N is a topological subspace of M, i.e., if the natural manifold topology of N is the trace topology of the natural manifold topology on M.*

This definition is a natural generalization of the notion of submanifold of $\mathbb{R}^n$, cf. [3, 2.1.5]. The figure-eight manifold from [3, 2.1.5] (with atlas $\{N, j^{-1}\}$) is an example of an immersive submanifold that is not a regular submanifold.

1.1.11 Remark. If N is a submanifold of M then for each $p\in N$, the map $T_pj : T_pN\to T_pM$ is injective. Hence $T_pj(T_pN)$ is a subspace of T_pM that is isomorphic to T_pN. We will therefore henceforth identify $T_pj(T_pN)$ with T_pN and notationally suppress the map T_pj, i.e., we will consider T_pN directly as a subspace of T_pM.

1.1.12 Theorem. *Let N^n be an immersive submanifold of M^m. TFAE:*

(i) N is a submanifold of M (i.e., N carries the trace topology of M).

(ii) Around any $p\in N$ there exists an adapted coordinate system*, i.e., for every $p\in N$ there exists a chart (φ, U) around p in M such that $\varphi(p)=0$, $\varphi(U\cap N)=\varphi(U)\cap(\mathbb{R}^n\times\{0\})$ (with $0\in\mathbb{R}^{m-n}$) and such that $\varphi|_{U\cap N}$ is a chart of N around p.*

(iii) Every $p\in N$ possesses a neighborhood basis $\mathcal{U}$ in M such that $U\cap N$ is connected in N for every $U\in\mathcal{U}$.

Proof. (i)⇒(ii): Let $p\in N$. By assumption, $j : N\hookrightarrow M$ is an immersion. Thus by 1.1.3 there exist charts (ψ, V) around p in N and $(\varphi,\tilde U)$ around $j(p)=p$ in M, with $\varphi(p)=0$, such that

$$\varphi\circ j\circ\psi^{-1} = (x^1,\dots,x^n)\mapsto(x^1,\dots,x^n,0,\dots,0).$$

The domain of $\varphi \circ j \circ \psi^{-1}$ is $\psi(V \cap j^{-1}(\tilde{U}))$. Since j is continuous, $j^{-1}(\tilde{U})$ is open in N. Shrinking V to $V \cap j^{-1}(\tilde{U})$ if necessary, we can assume w.l.o.g. that $V \subseteq j^{-1}(\tilde{U})(= \tilde{U} \cap N)$. The domain of definition of $\varphi \circ j \circ \psi^{-1}$ then is $\psi(V)$. By (i) there exists some open subset W of M such that $V = W \cap N$ and without loss we may assume that $W = \tilde{U}$ (otherwise replace both $\tilde{U}$ and W by $\tilde{U} \cap W$). Then $V = \tilde{U} \cap N$.

Denote by $\mathrm{pr}_1 : \mathbb{R}^m \to \mathbb{R}^n$ the projection map. We have

$$\varphi(V) = \varphi(j(V)) = \varphi \circ j \circ \psi^{-1}(\psi(V)) = \psi(V) \times \{0\},$$

so $\mathrm{pr}_1(\varphi(V)) = \psi(V)$, which is open in $\mathbb{R}^n$. Hence the set

$$U := \varphi^{-1}((\mathrm{pr}_1(\varphi(V)) \times \mathbb{R}^{m-n}) \cap \varphi(\tilde{U}))$$

is open in M and contains p. It follows that (φ, U) is a chart of M around p and we claim that $\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\})$.

To see '$\subseteq$', note that obviously $\varphi(U \cap N) \subseteq \varphi(U)$ and $U \cap N \subseteq \tilde{U} \cap N = V$, so $\varphi(U \cap N) \subseteq \varphi(V) \subseteq \mathbb{R}^n \times \{0\}$. Conversely,

$$\varphi(U) \cap (\mathbb{R}^n \times \{0\}) = (\mathrm{pr}_1(\varphi(V)) \times \{0\}) \cap \varphi(\tilde{U}) = (\psi(V) \times \{0\}) \cap \varphi(\tilde{U})$$

Now let $\varphi(u) \in \varphi(U) \cap (\mathbb{R}^n \times \{0\})$. Then for some $v \in V$ we have

$$\varphi(u) = (\psi(v), 0) = \varphi \circ j \circ \psi^{-1}(\psi(v)) = \varphi(j(v)) = \varphi(v),$$

so $u = v \in V \subseteq N$ and thereby $\varphi(u) \in \varphi(U \cap N)$.

Finally, $\varphi|_{U \cap N}$ is a chart of N around p since $U \cap N = j^{-1}(U)$ is an open neighborhood of p in N and

$$\begin{aligned}\varphi|_{U\cap N} \circ \psi^{-1} = \varphi|_{U \cap N} \circ j \circ \psi^{-1} &= \varphi \circ j \circ \psi^{-1}|_{U \cap N} \\ &= (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, 0, \ldots, 0).\end{aligned}$$

Identifying $\mathbb{R}^n \times \{0\}$ with $\mathbb{R}^n$, this latter map is the identity on $\mathbb{R}^n$, so $\varphi|_{U \cap N} = \psi|_{U \cap N}$, hence it is a chart.

(ii)$\Rightarrow$(iii): Let (φ, U) be a chart as in (ii). Pick $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(0) \subseteq \varphi(U)$ and let $U_\varepsilon := \varphi^{-1}(B_\varepsilon(0))$ for $\varepsilon < \varepsilon_0$. Then $\mathcal{U} := \{U_\varepsilon \mid \varepsilon < \varepsilon_0\}$ is a neighborhood basis of p in M and

$$\varphi(U_\varepsilon \cap N) = \varphi(U_\varepsilon \cap U \cap N) = B_\varepsilon(0) \cap \varphi(U \cap N) = B_\varepsilon(0) \cap (\mathbb{R}^n \times \{0\})$$

is connected in $\mathbb{R}^n$. Thus $\mathcal{U}$ serves the desired purpose.

(iii)$\Rightarrow$(i): Denote by $\mathcal{T}_M$ and $\mathcal{T}_N$ the topologies on M and N, respectively. Since $j : N \hookrightarrow M$ is continuous, for every $W \in \mathcal{T}_M$ we get $j^{-1}(W) = W \cap N \in \mathcal{T}_N$, so $\mathcal{T}_M|_N \leq \mathcal{T}_N$. Conversely we will show that any $\mathcal{T}_N$-neighborhood of any $p \in N$ is also a $\mathcal{T}_M|_N$-neighborhood of p. To this end let $p \in N$ and let U be a neighborhood of p in N such that is homeomorphic to a ball in $\mathbb{R}^n$ (e.g. the inverse image of such a ball under a chart). Then ∂U is compact in N, so also $j(\partial U) = \partial U$ is compact in M (since j is continuous). Since $p \in U^\circ$, $p \notin \partial U$ and so by (iii) there exists some $V \in \mathcal{U}$ with $V \cap \partial U = \emptyset$. If we can show that $V \cap N \subseteq U$ then we are done since $V \cap N$ is a neighborhood of p in $\mathcal{T}_M|_N$. Assume, therefore, that $V \cap N \not\subseteq U$. This means that $(V \cap N) \cap (N \setminus U) \neq \emptyset$. Thus $V \cap N$ is connected and $(p \in)(V \cap N) \cap U \neq \emptyset$ as well as $(V \cap N) \cap (N \setminus U) \neq \emptyset$. But this implies $(V \cap N) \cap \partial U \neq \emptyset$ and thereby $V \cap \partial U \neq \emptyset$, a contradiction. $\square$

1.1.13 Remark. (i) For $M = \mathbb{R}^m$, condition (ii) from 1.1.12 is precisely (T) from [3, 2.1.8] (local trivialization). Therefore, submanifolds of $\mathbb{R}^m$ in the sense of [3] are exactly submanifolds of $\mathbb{R}^m$ in the sense of 1.1.10.

(ii) Consider the subset N of $\mathbb{R}^2$ that consists of the interval $[-1, 1]$ on the y-axis, plus the graph of $\sin(1/x)$ between $x = 0$ and $x = 1$. Then N is an immersive submanifold of $\mathbb{R}^2$ that is not a submanifold due to 1.1.12 (iii): in fact, any ball around $(0, 0)$ of radius less than 1 intersects N in a non-connected set.

1.1.14 Proposition. *Let N be a submanifold of M and let $f : P \to M$ be smooth and such that $f(P) \subseteq N$. Then also $f : P \to N$ is smooth.*

Proof. Since N carries the trace topology of M and $f : P \to M$ is continuous, also $f : P \to N$ is continuous. Also, $j : N \hookrightarrow M$ is an immersion and by assumption $j \circ f$ is smooth. The claim therefore follows from 1.1.8. □

1.1.15 Corollary. *Let M be a manifold and let N be a subset of M. Then N can be endowed with the structure of a submanifold of M in at most one way.*

Proof. By definition, N has to carry the trace topology of M. Suppose that there are two differentiable structures that make N a submanifold of M and denote N with these structures by N_1, N_2. Since $j : N_i \to M$ is smooth for $i = 1, 2$, 1.1.14 shows that both $\mathrm{id} : N_1 \to N_2$ and $\mathrm{id} : N_2 \to N_1$ are smooth. Hence $\mathrm{id} : N_1 \to N_2$ is a diffeomorphism and so the differentiable structures on N coincide. □

1.1.16 Definition. *Let M, N be manifolds. A smooth map $i : N \to M$ is called an* embedding *if i is an injective immersion and if i is a homeomorphism from N onto $(i(N), \mathcal{T}_M|_{i(N)})$.*

1.1.17 Remark. (i) If $i : N \to M$ is an embedding then $i(N)$ can be turned into a submanifold of M by declaring i to be a diffeomorphism. The charts of $i(N)$ then are the $\psi \circ i^{-1}$, where ψ is any chart of N. This manifold $i(N)$ then is a submanifold of M: Let $j : i(N) \hookrightarrow M$ be the inclusion map. Then $i = j \circ i$ is an immersion and i is a diffeomorphism by definition, so j is an immersion. Also, $i(M)$ carries the trace topology by assumption. By 1.1.15 this manifold structure on $i(N)$ is the only one possible.

Next we want to check how to tell whether a given subset N of M can be made into a submanifold of M. We first generalized the condition from 1.1.12 (ii):

1.1.18 Definition. *Let M^m be a manifold and let N be a subset of M. We say that N possesses the submanifold-property of dimension n if for every $p \in N$ there exists a chart (φ, U) of p in M such that $\varphi(p) = 0$ and $\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\})$. (φ, U) then is called an adapted coordinate system.*

1.1.19 Theorem. *Let M^m be a manifold and let N be a subset of M possessing the submanifold-property of dimension n. Then N can be equipped in a unique way with a differentiable structure such that it becomes an n-dimensional submanifold of M. If $\mathrm{pr}_1 : \mathbb{R}^m \to \mathbb{R}^n$ denotes the projection then $\mathcal{A} := \{(\tilde{\varphi} := \mathrm{pr}_1 \circ \varphi, U \cap N) \mid \varphi$ is an adapted coordinate system$\}$ is a $\mathcal{C}^\infty$-atlas for N. In addition, $j : N \hookrightarrow M$ is an embedding.*

Proof. Uniqueness is clear from 1.1.15. Let (φ_1, U_1), (φ_2, U_2) be adapted coordinate systems with $(U_1 \cap N) \cap (U_2 \cap N) \neq \emptyset$. We have to show that $\tilde\varphi_1$ and $\tilde\varphi_2$ are $\mathcal{C}^\infty$-compatible. We first note that since the φ_i are homeomorphisms, so are the $\tilde\varphi_i$ as maps from $U_i \cap N$ with the trace topology onto $\mathrm{pr}_1(\varphi_i(U_i) \cap (\mathbb{R}^n \times \{0\}))$.

Let $\theta : \mathbb{R}^n \hookrightarrow \mathbb{R}^m$, $\theta(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0)$. Then $\tilde\varphi_i^{-1} = \varphi_i^{-1} \circ \theta$. It follows that $\tilde\varphi_1 \circ \tilde\varphi_2^{-1}$ is defined on $\tilde\varphi_2(U_1 \cap U_2 \cap N)$ $(= \mathrm{pr}_1(\varphi_2(U_1 \cap U_2)) \cap (\mathbb{R}^n \times \{0\})$, hence open in $\mathbb{R}^n$), and

$$\tilde\varphi_1 \circ \tilde\varphi_2^{-1} = (\mathrm{pr}_1 \circ \varphi_1) \circ (\mathrm{pr}_1 \circ \varphi_2)^{-1} = \mathrm{pr}_1 \circ \varphi_1 \circ \varphi_2^{-1} \circ \theta$$

is smooth. Consequently, $\mathcal{A}$ is an atlas for N and by [3, 2.2.7] the natural manifold topology of N is precisely the trace topology of M on N. If (φ, U) is an adapted chart then $\varphi \circ j \circ \tilde\varphi^{-1} = \theta$, so j is an immersion. Since N carries the trace topology, $j : N \to (j(N), \mathcal{T}_M|_{j(N)})$ is a homeomorphism, so j is an embedding. □

1.1.20 Proposition. *Let M^m, N^n be manifolds, N compact and $i : N \to M$ an injective immersion. Then i is even an embedding and $i(N)$ is a submanifold of M that is diffeomorphic to N.*

Proof. We have to show that $i : (N, \mathcal{T}_M|_{i(N)})$ is a homeomorphism. We already know that this map is continuous and bijective. But also i^{-1} is continuous: Let $A \subseteq N$ be closed, hence compact. Then $(i^{-1})^{-1}(A) = i(A)$ is compact and therefore closed. The final claim follows from 1.1.17 (i). □

1.1.21 Corollary. *Let $f : N^n \to M^m$ be an immersion. Then every $p \in N$ has an open neighborhood U such that $f|_U : U \to M$ is an embedding. Thus the difference between an immersion and an embedding is of a global nature.*

Proof. By 1.1.3 there exist charts φ at p and ψ at $f(p)$ such that $\psi \circ f \circ \varphi^{-1} = (x^1, \dots, x^n) \mapsto (x^1, \dots, x^n, 0, \dots, 0)$. Thus there exists a compact neighborhood V of p such that $f|_V$ is injective. As in the proof of 1.1.20 it follows that $f|_V : V \to (f(V), \mathcal{T}_M|_{f(V)})$ is a homeomorphism. Let $U \subseteq V$ be an open neighborhood of p. Then $f|_U$ is an injective immersion and $f : U \to (f(U), \mathcal{T}_M|_{f(U)})$ is a homeomorphism, so $f : U \to M$ is an embedding. □

1.1.22 Theorem. *Let M^m, N^n be manifolds and $f : N \to M$ smooth with* $\mathrm{rk}(f) \equiv k$ *on N $(k < n)$. Let $q \in f(N)$. Then $f^{-1}(q)$ is a closed submanifold of N of dimension $n - k$.*

Proof. Since f is continuous, $f^{-1}(q)$ is closed in N. We show that $f^{-1}(q)$ possesses the submanifold property of dimension $n - k$. The claim then follows from 1.1.19. Let $p \in f^{-1}(q)$. Then by 1.1.3 there exist charts (φ, U) at p and (ψ, V) at $f(p) = q$ such that $\varphi(p) = 0$, $\psi(q) = 0$ and

$$f_{\psi\varphi}(x) = \psi \circ f \circ \varphi^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^k, 0, \dots, 0).$$

Here, $f_{\psi\varphi}$ is defined on $\varphi(U \cap f^{-1}(V)) =: \varphi(W)$. Then (φ, W) is a chart of N at p and

$$\begin{aligned}\varphi(f^{-1}(q) \cap W) = \varphi(f^{-1}(q)) \cap \varphi(W) &= \varphi(f^{-1}(\psi^{-1}(\psi(q)))) \cap \varphi(W) \\ &= f_{\psi\varphi}^{-1}(0) \cap \varphi(W) = (\{0\} \times \mathbb{R}^{n-k}) \cap \varphi(W).\end{aligned}$$

□

1.1.23 Corollary. *Let $f : N^n \to M^m$ be smooth with $m < n$and let $q \in N$. If $\mathrm{rk}_p(f) = m$ for all $p \in f^{-1}(q)$ then $f^{-1}(q)$ is a closed submanifold of N of dimension $n - m$.*

Proof. Let $p \in f^{-1}(q)$. Then f has maximal rank ($= m$) at p, hence by 1.1.4 even in an open neighborhood U of p in N. Therefore the rank of f equals m on an open neighborhood $\tilde{N}$ of $f^{-1}(q)$ in N. The claim now follows by applying 1.1.22 to $f : \tilde{N} \to M$. □

1.1.24 Remark. For $N = \mathbb{R}^n$ and $M = \mathbb{R}^m$ this result reduces to the description of submanifolds as zero-sets of regular maps, cf. [3, 2.1.8].

1.1.25 Proposition. *Under the assumptions of 1.1.22, let $L := f^{-1}(q)$ and let $p \in L$. Then $T_pL = \ker(T_pf)$.*

Proof. For any smooth curve c in L with $c(0) = p$, $f \circ c \equiv q$, so $0 = \frac{d}{dt}\big|_0 (f \circ c) = T_pf(c'(0))$. Hence $T_pL \subseteq \ker(T_pM)$. Since $\dim(\ker T_pf) + \dim(\mathrm{im} T_pf) = \dim T_pN = n$, $\dim(\ker T_pf) = n - k = \dim T_pL$, and equality follows. □

1.1.26 Example. Let $\pi : TM \to M^m$ be the canonical projection and let $p \in M$. Then π is smooth and $\mathrm{rk}(\pi) = m$ since with respect to a chart ψ of M we have $\psi \circ \pi \circ T\psi^{-1} = \mathrm{pr} : \mathbb{R}^{2m} \to \mathbb{R}^m$ (cf. [3, 2.5.6]). By 1.1.23 it follows that $\pi^{-1}(p) = T_pM$ is an m-dimensional submanifold of TM. Moreover, by 1.1.25, for $v_p \in T_pM$ we have $T_{v_p}T_pM = \ker(T_{v_p}\pi)$. By the proof of 1.1.22, the submanifold charts of T_pM are given by $T\psi|_{T_pM} = T_p\psi$. As these are linear isomorphisms, the trace topology of TM on T_pM is precisely the usual topology of T_pM as a finite-dimensional vector space. Also, $T_p\psi$ is a diffeomorphism, so the manifold structure of T_pM as well is its usual differentiable structure as a finite-dimensional vector space.

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