Proseminar Lie Groups

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WS2017/18

1. By a matrix group we mean a closed subgroup of the general linear group $GL(n,\mathbb{R})$. Verify that the following sets are matrix groups in this sense:
   (a) $SL(n,\mathbb{R}) := \{A \in GL(n,\mathbb{R}) \mid \det(A) = 1\}$.
   (b) The space $UT(n,\mathbb{R})$ of regular upper triangular matrices.
   (c) The affine group $\mathrm{Aff}(n,\mathbb{R}) := \left\{ \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix} \mid A \in GL(n,\mathbb{R}),\ t \in \mathbb{R}^n \right\}$ (as a subgroup of $GL(n+1,\mathbb{R})$).
   (d) The orthogonal group: $O(n) := \{A \in GL(n,\mathbb{R}) \mid AA^t = I\}$.
   (e) The Lorentz group $\{A \in GL(n,\mathbb{R}) \mid A^t D A = D\}$, where $D = \mathrm{diag}(-1, 1, \ldots, 1)$.
2. Prepare a short presentation about equivalent descriptions of submanifolds of $\mathbb{R}^n$ and of their tangent spaces.
3. Consider the determinant function as a map $\det : GL(n,\mathbb{R}) \to \mathbb{R}$ and show that for $A \in GL(n,\mathbb{R})$ and $B \in M(n,\mathbb{R})$ we have:
   $$D\det(A)(B) = \det(A)\mathrm{tr}(A^{-1}B)$$
   (where tr denotes the trace). *Hint:* Calculate $\frac{d}{dt}\big|_0 \det(A + tB)$.
4. Conclude from the above that $SL(n,\mathbb{R})$ is a submanifold of $GL(n,\mathbb{R})$ and calculate its tangent space at I.