Proseminar zu Lie-Gruppen

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13. Look up the definition of a vector bundle in [DG1], 2.5.5. Verify that for any manifold M, $TM = \bigcup_{p\in M}\{p\} \times T_pM$ is a vector bundle in this sense. *Hint:* The vector bundle charts are given as $T\varphi : TU \to U \times \mathbb{R}^n$, where (φ, U) is a chart of M. Verify that the changes of chart $T\varphi \circ (T\psi)^{-1}$ are local vector bundle homomorphisms. Show that M is the base of this vector bundle and that the fibers are precisely the T_pM.

14. Let M be a smooth manifold, $X \in \mathfrak{X}(M)$, V a finite-dimensional vector space and $f \in \mathcal{C}^\infty(M, V)$. For a given basis $\{v_1, \dots, v_n\}$ let $f = \sum_i f_i v_i$ with $f \in \mathcal{C}^\infty(M, \mathbb{R})$. Define $X(f) := \sum_i X(f_i)v_i$. Show that

    (a) $X(f)$ is independent of the chosen basis.

    (b) If $\alpha \in \mathcal{C}^\infty(M, \mathbb{R})$ then $X(\alpha f) = \alpha X(f) + X(\alpha)f$.

15. The exterior derivative of differential k-forms $\varphi \in \Omega^k(M)$ satisfies:

    (i) For $X_0, \dots, X_k \in \mathfrak{X}(M)$,

    $$\begin{aligned} d\varphi(X_0, \dots, X_k) &= \sum_i (-1)^i X_i(\varphi(X_0, \dots, \hat{X}_i, \dots, X_k)) \\ &\quad + \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) \end{aligned}$$

    (ii) For any $f \in \mathcal{C}^\infty(M, N)$ and any $\varphi \in \Omega^k(N)$, $d(f^*\varphi) = f^*(d\varphi)$.

    (iii) $d \circ d = 0$.

    Check that, knowing these properties (do you?), the same results can be achieved for vector-valued differential forms $\varphi \in \Omega^k(M, V)$ by defining $d\varphi = \sum_i d\varphi_i v_i$, where for any basis $\{v_1, \dots, v_n\}$ of V, $\varphi = \sum_i \varphi_i v_i$.

16. Let f, g be smooth maps from some manifold M into a Lie group G. Then

    $$\delta^r(f \cdot g)(p) = \delta^r f(p) + \mathrm{Ad}(f(p))(\delta^r g(p)).$$