Proseminar zu Lie-Gruppen

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WS2017/18

21. Let ρ be the following equivalence relation on $\mathbb{R}^2$: $x \sim_\rho y :\Leftrightarrow |x| = |y|$. Show that $\mathbb{R}^2/\rho$ cannot be endowed with a quotient manifold structure of $\mathbb{R}^2$. Nevertheless, if one replaces $\mathbb{R}^2$ by $\mathbb{R}^2 \setminus \{0\}$, then such a structure exists.

22. The vector field $X := e^{x_2} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$ defines a one-dimensional distribution Ω on $\mathbb{R}^2$. Find a chart around $(0,0)$ that is flat for Ω.

23. Let $X_1, \ldots, X_k$ be smooth vector fields on a manifold M such that $[X_i, X_j] = 0$ for all $i,\ j$. Then the distribution Ω spanned by $X_1, \ldots, X_k$ is integrable by the Frobenius theorem. Show that for any $p \in M$ and $\varepsilon > 0$ sufficiently small the set
$$M' := \{\mathrm{Fl}^{X_1}_{t^1} \circ \ldots \circ \mathrm{Fl}^{X_k}_{t^k}(p) \mid (t^1, \ldots, t^k) \in (-\varepsilon, \varepsilon)^k\}$$
can be equipped with a manifold structure that makes it an integral manifold of Ω containing p. (*Hint:* Examine the proof of the Frobenius theorem.)

24. Let $M := \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z > 0\}$, $X := x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y}$, $Y := y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$.
    (i) Show that X, Y span an involutive distribution Ω on M.
    (ii) Determine the integral manifolds of Ω.

25. Let $c : I \to M$ be a maximal integral curve of a vector field $X \in \mathfrak{X}(M)$ and suppose that $c'(t) \neq 0$ for all $t \in I$. Equip $C := c(I)$ with its natural manifold structure as described in 17.18. Show that C is diffeomorphic either to $\mathbb{R}$ or to S^1.