Proseminar zu Lie-Gruppen

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26. Let $i : (0, 2\pi) \to \mathbb{R}^2$, $i(s) = (\sin(2s), \sin(s))$. Then $M' := i((0, 2\pi))$ is an immersive submanifold of $\mathbb{R}^2$ (the figure-eight manifold) with global chart $\varphi : M' \to (0, 2\pi)$, $(\sin(2s), \sin(s)) \mapsto s$.

    Let $f : S^1 \to \mathbb{R}^2$, $f(x_1, x_2) = (2x_1x_2, x_2)$. Show that although f is smooth and $f(S^1) = M'$, the induced map from S^1 to M' is not smooth (in fact, not even continuous).

27. Let M' be a connected integral manifold of an integrable distribution Ω on M. Prove that, if M' is a closed subset of M, then M' is a leaf of Ω.

28. Let U be an open neighborhood of 0 in $\mathbb{R}^2$ and let F, $G : U \times \mathbb{R} \to \mathbb{R}$ be smooth functions. Consider the following system of partial differential equations:

$$\frac{\partial z}{\partial x} = F(x, y, z(x, y))$$
$$\frac{\partial z}{\partial y} = G(x, y, z(x, y))$$

    (a) Show that the following integrability condition is a necessary requirement for the existence of a solution z:

$$F_y + GF_z = G_x + FG_z$$

    (b) Show that (the graphs of) solutions z of this system are integral manifolds of the distribution Ω spanned by X and Y, where

$$X = \partial_x + F(x, y, z)\partial_z, \quad Y = \partial_y + G(x, y, z)\partial_z$$

    (c) What does the integrability condition from (a) mean in terms of X and Y?

29. Let $X_1, \ldots, X_k$ be smooth linearly independent vector fields on an open subset U of $\mathbb{R}^n$. The following are equivalent:

    (i) For every $x_0 \in U$ there exist a neighborhood V of x_0 and $n - k$ smooth solutions $f^1, \ldots, f^{n-k}$ of the following system of PDEs on V:

$$X_1(f) = 0, \ldots, X_k(f) = 0$$

    such that $T_x f^1, \ldots T_x f^{n-k}$ are linearly independent for each $x \in V$.

    (ii) The distribution Ω spanned by $X_1, \ldots, X_k$ is involutive.

    (Hints: for (i)⇒(ii): the map $f := (f^1, \ldots, f^{n-k})$ is a submersion. Show that the level surfaces of f are integral manifolds of Ω. For (ii)⇒(i): consider the component functions of a flat chart for Ω.)