Proseminar zu Lie-Gruppen

Michael Kunzinger

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30. With X_1,...,X_k as in problem 29, let $f^1, \dots, f^{n-k}$ be solutions of

$$X^1(f) = 0, \dots, X^k(f) = 0 \qquad (*)$$

on an open subset V of U with $\mathrm{rk}(f) = n - k$. Again let Ω be the distribution spanned by $X_1, \dots, X_k$. Show that:

    (a) Locally around any $x_0 \in V$ there exist functions $y^1, \dots, y^k$ such that $\varphi := (y^1, \dots, y^k, f^1, \dots, f^{n-k})$ is a chart around x_0.
    (b) This chart is flat for Ω.
    (c) The general solution of $(*)$ is of the form $g = h(f^1, \dots, f^{n-k})$ for an arbitrary smooth function h locally around x_0. (Hint: g solves $(*)$ if and only if $T_x g$ vanishes on $\Omega(x) = \mathrm{span}(\partial_{y^1}, \dots, \partial_{y^k})$).

31. Let Ω be a distribution on M. TFAE:

    (i) Ω is involutive.
    (ii) If a smooth 1-form ω annihilates Ω (i.e., if $\omega(X) = 0$ for every local vector field $X \in \Omega$), then so does $d\omega$ (i.e., $d\omega(X, Y) = 0$ for all $X, Y \in \Omega$).

    (Hint: use (10.2) from the lecture course.)

32. Let $H := \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^3$. Then H is a 2-dimensional submanifold of $\mathbb{R}^3$ that is a Lie subgroup of $(\mathbb{R}^3, +)$. Show that the charts $\varphi_y : (x, y, 0) \mapsto x$ ($y \in \mathbb{R}$) form an atlas on H for a new differentiable structure in which H is a 1-dimensional Lie subgroup of $(\mathbb{R}^3, +)$.