# Lie Groups

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# Preface

These lecture notes accompany a three hour per week introduction to the theory of Lie groups, held at the Faculty of Mathematics at Vienna University in winter term 2023. The prerequisites are a solid knowledge of analysis on manifolds, as provided, e.g., by my course [5]. The presentation is based mainly on [1, 9, 2, 11]. Some proofs of auxiliary results have been outsourced to appendices so as to not distract too much from the main line of the text.

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Michael Kunzinger, winter term 2023

# Contents

# 1 Definitions and first examples

**1.1 Definition.** *A Lie group is a group $G$ that simultaneously is a differentiable manifold and such that the group multiplication*

$$\mu : G \times G \to G$$
$$\mu(g, h) = g \cdot h$$

*is smooth.*

**1.2 Remark.** By a manifold we mean here a set $M$ equipped with a $\mathcal{C}^\infty$-structure. However, we make no further assumptions on the topology of $M$ (like Hausdorff ($T_2$) or second countable). Recall from [5, Sec. 1.2] that any manifold automatically inherits a topology, called the natural manifold topology, via its charts (i.e., by declaring the charts to be local homeomorphisms). By 'smooth' we will always mean $\mathcal{C}^\infty$.

**1.3 Examples.** (i) $(\mathbb{R}^n, +)$ with its natural manifold structure is a Lie group.

(ii) The matrix group $\mathrm{GL}(n, \mathbb{R})$, with $\mu$ the standard multiplication of matrices is a Lie group, cf. [5, 1.1.9 (iii)]. (Seemingly) more generally, let $V$ be a finite-dimensional real vector space. Then analogously,

$$\mathrm{GL}(V) := \{f : V \to V \mid f \text{ linear and invertible}\}$$

is a Lie group. Indeed, after the choice of some basis in $V$ we have $\mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{R})$. Analogous claims hold for $\mathrm{GL}(n, \mathbb{C})$. $\mathrm{GL}(V)$ is the paradigmatic Lie group.

(iii) $\mathbb{R} \setminus \{0\}$ and $\mathbb{C} \setminus \{0\}$ are Lie groups with respect to multiplication.

(iv) Given Lie groups $G$ and $H$, also $G \times H$ is a Lie group, with multiplication

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

It is called the product of the Lie groups $G$ and $H$.

(v) $U(1) := \{z \in \mathbb{C} \mid |z| = 1\}$ and hence, by (iv), also the $n$-dimensional *torus group* $\mathbb{T}^n := U(1)^n$ are Lie groups.

(vi) Lie groups that are subgroups of some $\mathrm{GL}(V)$ are called *linear Lie groups.* As we shall see, all the standard matrix groups $(\mathrm{O}(n), \mathrm{U}(n), \mathrm{SO}(n, \mathbb{R}), \mathrm{SO}(n, \mathbb{C}), \dots)$ are linear Lie groups.

**1.4 Definition.** *Let $G$ be a Lie group and let $g \in G$. The* left translation $L_g$ *is defined as*

$$L_g : G \to G$$
$$L_g(h) := gh.$$

*Analogously, the* right translation $R_g$ *is*

$$R_g : G \to G$$
$$R_g(h) := hg.$$

We have $L_g(h) = \mu(g, h)$, so $L_g = \mu(g, \,.\,)$. Also, $L_g^{-1} = L_{g^{-1}}$. Consequently, $L_g$ is a diffeomorphism (and analogously for $R_g$). In addition, $L_g \circ L_h = L_{gh}$, $R_g \circ R_h = R_{hg}$, and $L_e = R_e = \mathrm{id}_G$.

**1.5 Lemma.** *Let $G$ be a Lie group.*

*(i) Let $g,\ h \in G$, $v \in T_gG$, and $w \in T_hG$. Then*

$$T_{(g,h)}\mu(v,w) = T_hL_g(w) + T_gR_h(v).$$

*(ii) Denote by $\nu : G \to G$, $\nu(g) := g^{-1}$ the inversion map on $G$. Then $\nu$ is smooth and for any $g \in G$ we have*

$$T_g\nu = -T_eR_{g^{-1}} \circ T_gL_{g^{-1}} = -T_eL_{g^{-1}} \circ T_gR_{g^{-1}}.$$

*In particular, $T_e\nu = -\mathrm{id}_{T_eG}$.*

**Proof.** (i) Note first that

$$T_{(g,h)}\mu(v,w) = T_{(g,h)}\mu(v,0) + T_{(g,h)}\mu(0,w).$$

Now let $c : (-\varepsilon,\varepsilon) \to G$ be a smooth curve with $c(0) = g$ and $c'(0) = v$. Then

$$T_{(g,h)}\mu(v,0) = \frac{d}{dt}\Big|_0 \mu(c(t),h) = \frac{d}{dt}\Big|_0 R_h(c(t)) = T_gR_h(v).$$

Analogously, $T_{(g,h)}\mu(0,w) = T_hL_g(w)$.

(ii) Let $\Phi : G \times G \to G \times G$, $\Phi(g,h) := (g,gh)$, i.e., $\Phi = \mathrm{pr}_1 \times \mu$, so $\Phi \in \mathcal{C}^\infty$. For $v,w \in T_eG$ we have by (i):

$$T_{(e,e)}\Phi(v,w) = (T_{(e,e)}\mathrm{pr}_1(v,w), T_{(e,e)}\mu(v,w)) = (v, T_eR_e(v) + T_eL_e(w)) = (v, v+w).$$

Therefore, $T_{(e,e)}\Phi : T_eG \times T_eG \to T_eG \times T_eG$ is invertible. By the inverse function theorem, locally around $(e,e)$ the map $\Phi$ possesses a smooth inverse $\tilde{\Phi}$. Now $\tilde{\Phi} = \Phi^{-1} = (g,h) \mapsto (g, g^{-1}h)$, so that $\tilde{\Phi}(g,e) = (g,\nu(g))$. This shows that $\nu$ is smooth in a neighborhood of $e$. Now for any $g \in G$ we have $\nu = R_{g^{-1}} \circ \nu \circ L_{g^{-1}}$, implying that $\nu$ is smooth in a neighborhood of $g$. Since $g$ was arbitrary, $\nu : G \to G$ is $\mathcal{C}^\infty$.

Next, for any $g \in G$ we have $e = \mu(g,\nu(g))$. From this, by differentiation we obtain for any $v \in T_gG$:

$$0 = T_{(g,g^{-1})}\mu(v, T_g\nu(v)) \underset{(i)}{=} T_{g^{-1}}L_g(T_g\nu(v)) + T_gR_{g^{-1}}(v).$$

Due to $(T_{g^{-1}}L_g)^{-1} = T_eL_{g^{-1}}$ (note that $L_g^{-1} = L_{g^{-1}}$) we get

$$T_g\nu(v) = -T_eL_{g^{-1}}(T_gR_{g^{-1}}(v)).$$

The second formula follows analogously by differentiating $e = \mu(\nu(g),g)$. □

# 2 Some topological properties of Lie groups

The fact that any Lie group is a manifold for which $\mu : G\times G \to G$ and $\nu : G \to G$ are (smooth, therefore) continuous automatically implies several improved properties of the topology of $G$. To derive these we first consider the general case of topological groups:

**2.1 Definition.** *A topological group is a group that is also a topological space such that the group operations*

$$\mu : G \times G \to G,\ \ \mu(g,h) = gh$$
$$\nu : G \to G,\ \ \nu(g) = g^{-1}$$

*are continuous.*

Thus any Lie group is in particular a topological group. In any topological group all $L_g$ and $R_g$ are homeomorphisms.

**2.2 Remark.** Let $G$ be a topological group and let $U$ be open in $G$. Then for any $g \in G$ also the coset $L_g U \equiv gU$ is open (and analogously for $R_g U \equiv Ug$). If $U, V \subseteq G$ are open, then so is $U \cdot V = \mu(U, V)$ because $U \cdot V = \bigcup_{g \in U} g \cdot V$.

If $U$ is a neighborhood of $e$ then by the continuity of $\mu$ there exists a neighborhood $W$ of $e$ with $W^2 := W \cdot W \subseteq U$. A subgroup $H$ of $G$ that is an open (closed) subset of $G$ is called open (closed) subgroup of $G$. Any open subgroup of $G$ is automatically closed as well because $G = \bigcup_{g \in G} gH$, so $G \setminus H = \bigcup_{g \in G \setminus H} gH$ is open.

**2.3 Remark.** Let $G$ be a topological group and let $U$ be open in $G$. Then also $\nu(U) \equiv U^{-1} = \{g^{-1} \mid g \in U\}$ is open. Let $U$ and $W$ be neighborhoods of $e$ with $W^2 \subseteq U$. Then $V := W \cap W^{-1}$ is a neighborhood of $e$ with $V = V^{-1}$ and $V^{-1} \cdot V \subseteq U$.

Any subgroup of a topological group is itself a topological group (when endowed with the trace topology).

**2.4 Proposition.** *Let $G$ be a topological group and let $G_e$ denote the connected component of $e$ in $G$. Then*

*(i) $G_e$ is a normal subgroup and any connected component of $G$ is a coset of $G_e$.*

*(ii) If $G$ is a Lie group, then $G_e$ is open and closed.*

**Proof.** (i) As $G_e$ is connected, so are $G_e \times G_e$ and $\mu(G_e \times G_e)$. Thus $\mu(G_e \times G_e)$ is connected and contains $e$, implying $\mu(G_e \times G_e) \subseteq G_e$. Also $G_e^{-1} = \nu(G_e)$ is connected and contains $e$, so $G_e^{-1} \subseteq G_e$. This shows that $G_e$ is a subgroup. If $g \in G$, then $gG_eg^{-1}$ is connected and contains $e$, hence is also contained in $G_e$, showing that $G_e$ is normal. Finally, let $g \in G$ and denote by $G_g$ the connected component of $g$. Then $gG_e$ is a connected superset of $g$, hence $gG_e \subseteq G_g$. Analogously, $g^{-1}G_g \subseteq G_e$, so $G_g = gG_e$.

(ii) Being a manifold, $G$ is locally connected, hence any connected component of $G$ is open. $G_e$ is then also closed by Remark 2.2. □

**2.5 Proposition.** *Let $G$ be a connected topological group and let $U$ be any neighborhood of $e$. Then $G$ is generated by $U$, i.e., any element of $G$ is a product of certain elements of $U$.*

**Proof.** Let $H := \langle U \rangle$ be the subgroup of $G$ generated by $U$. If $h \in H$, then $hU$ is a neighborhood of $h$ contained in $H$. Hence $H$ is open. By Remark 2.2 it is also closed. As $G$ is connected, $G = H$. □

**2.6 Proposition.** *Any topological group $G$ that is $T_1$ is automatically also $T_2$.*

**Proof.** First, let $g \neq e$. Since $G$ is $T_1$, there exists a neighborhood $U$ of $e$ such that $g \notin U$. By Remark 2.3, there exists some neighborhood $V$ of $e$ such that $V^{-1}V \subseteq U$. Then $Vg$ and $V$ are disjoint neighborhoods of $g$ and $e$, respectively: indeed, if $h \in Vg \cap V$, then there is some $k \in V$ with $kg = h$, so $g = k^{-1}h \in V^{-1}V \subseteq U$, a contradiction.

Finally, if $g \neq g'$, by the above there exist disjoint neighborhoods $W$, $W'$ of $e$ and $g^{-1}g'$. Then $gW$ and $g'W'$ separate $g$ and $g'$. □

Since the natural topology of a manifold is always $T_1$ (cf. [5, 1.3.1]), we obtain:

**2.7 Corollary.** *Any Lie group is a Hausdorff space.*

**2.8 Proposition.** *Any connected Lie group is second countable.*

**Proof.** It suffices to show that any such $G$ possesses a countable atlas (cf. [5, 1.3.7]). Let $U$ be a chart neighborhood around $e$ with $U = U^{-1}$. Since $U$ is homeomorphic to some open subset of $\mathbb{R}^n$ there exists a countable dense subset $S$ of $U$. Let $H := \langle S \rangle$ be the subgroup of $G$ generated by $S$. Since it consists of finite products of elements of $S$, $H$ is itself countable. Also, since any $hU$ is a chart domain, it suffices to show that $\{hU \mid h \in H\}$ is a covering of $G$.

To show this, let $g \in G$. As $G$ is connected, by 2.2 it is generated by $U$, so there exist $p \in \mathbb{N}$ and $g_1, \dots, g_p \in U$ with $g = g_1 \cdots g_p$. Since $\mu$ is continuous, for any $i \in \{1, \dots, p\}$ there exists some open neighborhood $U_i \subseteq U$ of $g_i$ such that $U_1 \cdots U_p \subseteq gU$. Since $S$ is dense in $U$ and $U_i \subseteq U$, there exist $s_i \in U_i \cap S$ $(1 \le i \le p)$. Finally, let $h := s_1 \cdots s_p$. Then $h \in H$ and $h \in gU$. Thus $g \in hU^{-1} = hU$. □

**2.9 Corollary.** *Any Lie group is paracompact.*

**Proof.** By Proposition 2.4, $G_e$ is an open subgroup of $G$ and hence is itself a Lie group. By Proposition 2.8 $G_e$ is second countable, and the same is thereby true for any connected component of $G$ by Proposition 2.4. Thus the claim follows from [5, 1.3.15]. (Alternatively, [5, 1.3.14] implies that any connected component admits a partition of unity, hence so does $G$ itself, and thereby is paracompact). □

# 3 Left-invariant vector fields

Since left translations $L_g$ are diffeomorphisms, their tangent maps at $e$ are linear isomorphisms between $\mathfrak{g} := T_eG$ and $T_gG$. For $v \in \mathfrak{g}$ and $g \in G$ we set

$$L^v(g) := T_eL_g(v) \in T_gG.$$

If, on the other hand, $X \in \mathfrak{X}(G)$ is a smooth vector field on $G$, then we may form the pullback of $X$ under $L_g$: $L_g^*X := (TL_g)^{-1} \circ X \circ L_g \in \mathfrak{X}(G)$.

$$\begin{tikzcd} TG \arrow[r, "TL_g"] & TG \\ G \arrow[u, "L_g^*X"] \arrow[r, "L_g"] & G \arrow[u, "X"'] \end{tikzcd}$$

**3.1 Definition.** *Let $G$ be a Lie group and let $X \in \mathfrak{X}(G)$. Then $X$ is called* left-invariant *if $L_g^*X = X$ for each $g \in G$. The space of all left-invariant vector fields is denoted by $\mathfrak{X}_L(G)$.*

**3.2 Proposition.** *Let $G$ be a Lie group.*

*(i) The map*

$$\begin{aligned} L : G \times \mathfrak{g} &\to TG \\ (g, v) &\mapsto L^v(g) \end{aligned}$$

*is a diffeomorphism.*

*(ii) For any $v \in \mathfrak{g}$, $L^v : G \to TG$ is a smooth left-invariant vector field on $G$, $L^v \in \mathfrak{X}_L(G)$. The map $v \mapsto L^v$, $\mathfrak{g} \to \mathfrak{X}_L(G)$ is a linear isomorphism with inverse $X \mapsto X(e)$. Hence $\mathfrak{g} = T_eG$ can be identified with $\mathfrak{X}_L(G)$.*

**Proof.** (i) Consider the map

$$\begin{aligned} \varphi : G \times \mathfrak{g} &\to TG \times TG \\ (g, v) &\mapsto (0_g, v), \end{aligned}$$

with $0_g \in T_gG \subseteq TG$, $v \in T_eG \subseteq TG$. Then $\varphi$ is smooth and by Lemma 1.5 (i) we have

$$T\mu \circ \varphi(g, v) = T\mu(0_g, v) = T_{(g,e)}\mu(0_g, v) = T_eL_g(v) + \underbrace{T_gR_e(0_g)}_{=0} = T_eL_g(v) = L^v(g).$$

In particular, $L^v$ is smooth. Next, let

$$\begin{aligned} \psi : TG &\to TG \times TG \\ v_g &\mapsto (0_{g^{-1}}, v) = (0_{\nu(g)}, v). \end{aligned}$$

Since $\nu$ is smooth, so is $\psi$. Moreover, Lemma 1.5 (i) implies

$$T\mu \circ \psi(v_g) = T_{(g^{-1},g)}\mu(0_{g^{-1}}, v) = (T_gL_{g^{-1}})(v) + 0 \in T_eG = \mathfrak{g}.$$

It follows that the map $\Psi : TG \to G \times \mathfrak{g}$, $v_g \mapsto (g, T_gL_{g^{-1}}(v_g))$ is smooth. Now

$$\Psi \circ L(g, v) = \Psi(L^v(g)) = (g, T_gL_{g^{-1}}(L^v(g))) = (g, T_gL_{g^{-1}}(T_eL_g(v))) = (g, v),$$

and

$$L \circ \Psi(v_g) = L(g, T_gL_{g^{-1}}(v)) = L^{T_gL_{g^{-1}}(v)}(g) = T_eL_g \cdot T_gL_{g^{-1}}(v) = v,$$

so $\Psi = L^{-1}$ and $L$ is a diffeomorphism.
(ii) We have $L^v(g) = T_eL_g(v) \in T_gG$ and $g \mapsto L^v(g)$ is smooth by (i). Thus $L^v \in \mathfrak{X}(G)$. Also, $L^v$ is left-invariant: Let $h \in G$, then

$$\begin{aligned} ((L_g)^* L^v)(h) &= T_{gh}L_{g^{-1}} \circ L^v(gh) = T_{gh}L_{g^{-1}} \circ T_e(\underbrace{L_{gh}}_{=L_g \circ L_h})(v) \\ &= T_{gh}L_{g^{-1}} \circ T_hL_g \circ T_eL_h(v) = T_eL_h(v) = L^v(h). \end{aligned}$$

Hence $L^v \in \mathfrak{X}_L(G)$. Moreover, for $v \in \mathfrak{g}$ we get

$$L^v(e) = T_eL_e(v) = \mathrm{id}_{T_eG}(v) = v,$$

and for $X \in \mathfrak{X}_L(G)$, setting $v := X(e)$ we obtain

$$X(g) = ((L_{g^{-1}})^* X)(g) = T_eL_g(X(g^{-1}g)) = T_eL_g(v) = L^v(g).$$

This means that $X = L^v = L^{X(e)}$. □

**3.3 Remark.** The diffeomorphism $L : G \times \mathfrak{g} \to TG$ is called *left trivialization* of $TG$. By Proposition 3.2 (ii), $L$ is a vector bundle isomorphism (since it is fiber-linear). This means that the tangent bundle of any Lie group is trivializable. If $(v_1, \ldots, v_n)$ is a basis of $\mathfrak{g}$ then the evaluations of the corresponding left-invariant vector fields $L^{v_1}, \ldots, L^{v_n}$ at any $g$ form a basis of $T_gG$. In particular it follows that manifolds that do not admit nowhere-vanishing vector fields (e.g., $S^n$ for $n$ even) cannot be made into Lie groups.

# 4 The Lie algebra of a Lie group

**4.1 Definition.** *A vector space* $\mathfrak{a}$ *is called a* Lie algebra *if it is endowed with a bilinear operation* $(v,w) \mapsto [v,w]$ *that is antisymmetric and satisfies the Jacobi identity:*

$$[v,w] = -[w,v]$$
$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$

*for all* $u,v,w \in \mathfrak{a}$.

Given any Lie group $G$, our first aim is to equip the vector space $\mathfrak{g} = T_eG$ with a natural Lie algebra structure via Proposition 3.2. To do this, we require the following concept from differential geometry:

**4.2 Definition.** *Let* $f : M \to N$ *be a smooth map between two manifolds. Then vector fields* $X \in \mathfrak{X}(M)$ *and* $Y \in \mathfrak{X}(N)$ *are called* $f$*-related, denoted by* $X \sim_f Y$ *if* $T_pf(X_p) = Y_{f(p)}$ *for each* $p \in M$.

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ {\scriptstyle X}\uparrow & & \uparrow{\scriptstyle Y} \\ M & \xrightarrow{f} & N \end{array}$$

**4.3 Lemma.** *Smooth vector fields* $X \in \mathfrak{X}(M)$ *and* $Y \in \mathfrak{X}(N)$ *are* $f$*-related if and only if for each* $g \in \mathcal{C}^\infty(N)$ *we have*

$$X(g \circ f) = Y(g) \circ f.$$

**Proof.**

$$\begin{aligned} X(g \circ f) = Y(g) \circ f \ \forall g &\Leftrightarrow X_p(g \circ f) = Y_{f(p)}(g) \ \forall g \ \forall p \\ &\underset{(*)}{\Leftrightarrow} T_pf(X_p)(g) = Y_{f(p)}(g) \ \forall g \ \forall p \Leftrightarrow X \sim_f Y, \end{aligned}$$

where $(*)$ holds due to [5, (2.1.4)]. □

**4.4 Lemma.** *Let* $X_1, X_2 \in \mathfrak{X}(M)$, $Y_1, Y_2 \in \mathfrak{X}(N)$, $f \in \mathcal{C}^\infty(M,N)$ *and suppose that* $X_1 \sim_f Y_1$ *and* $X_2 \sim_f Y_2$. *Then also* $[X_1, X_2] \sim_f [Y_1, Y_2]$.

**Proof.** Using Lemma 4.3 we calculate:

$$\begin{aligned} [X_1,X_2](g \circ f) &= X_1(X_2(g \circ f)) - X_2(X_1(g \circ f)) = X_1(Y_2(g) \circ f) - X_2(Y_1(g) \circ f) \\ &= Y_1(Y_2(g)) \circ f - Y_2(Y_1(g)) \circ f = [Y_1, Y_2](g) \circ f. \end{aligned}$$

□

In particular, if $f : M \to N$ is a diffeomorphism, $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$, then $X \sim_f Y$ if and only if $X = f^*Y$. Lemma 4.4 then implies that $f^*[Y_1, Y_2] = [f^*Y_1, f^*Y_2]$.

Returning now to the Lie group setting, let $G$ be a Lie group, $X, Y \in \mathfrak{X}_L(G)$ and $g \in G$. Then

$$L_g^*[X,Y] = [L_g^*X, L_g^*Y] = [X,Y],$$

so also $[X,Y] \in \mathfrak{X}_L(G)$. Thus if $v, w \in \mathfrak{g}$, then also $[L^v, L^w] \in \mathfrak{X}_L(G)$. We may therefore define an operation on $\mathfrak{g}$ by setting

$$[v,w] := [L^v, L^w](e) \qquad (v, w \in \mathfrak{g}).$$

By Proposition 3.2 (ii) we then have $[L^v, L^w] = L^{[v,w]}$, since both sides lie in $\mathfrak{X}_L(G)$ and attain the same value at $e$.

**4.5 Definition.** *Let $G$ be a Lie group. The* Lie algebra *of $G$ is the vector space $\mathfrak{g} = T_eG$, endowed with the operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $[v, w] := [L^v, L^w](e)$.*

Bilinearity, anti-symmetry and the Jacobi identity for $[\ ,\ ]$ follow from the corresponding properties of the Lie bracket of vector fields (cf. [5, 2.2.17]). Hence $(\mathfrak{g}, [\ ,\ ])$ is indeed a Lie algebra in the sense of Definition 4.1.

**4.6 Examples.** (i) Let $V$ be a finite-dimensional vector space. Then $(V, +)$ is a Lie group. If $v \in T_0V \cong V$ and $g \in V$, we have $L^v(g) = T_0L_g(v) = v$, because $L_g = h \mapsto h + g$. Hence $L = (g, v) \mapsto (g, v)$ and the left trivialization is the usual identification $TV \cong V \times V$. The left-invariant vector fields are therefore precisely the constant ones and in particular, the Lie bracket vanishes: $[\ ,\ ] = 0$ on $T_0V$, so the Lie algebra is commutative. We shall see later that this is a general property of commutative Lie groups.

(ii) Given Lie groups $G$, $H$, by Example 1.3 (iv) also $G \times H$ is a Lie group and $T_{(e,e)}(G \times H) = T_eG \times T_eH = \mathfrak{g} \oplus \mathfrak{h}$. Also, $\mathfrak{X}_L(G \times H) \cong \mathfrak{X}_L(G) \times \mathfrak{X}_L(H)$, because for $(v, w) \in T_{(g,h)}(G \times G)$ we have

$$\begin{aligned} L^{(v,w)}(g, h) &= T_{(e,e)}L_{(g,h)}(v, w) = T_{(e,e)}(L_g \times L_h)(v, w) = (T_eL_g \times T_eL_h)(v, w) \\ &= (L^v(g), L^w(h)) = (L^v \times L^w)(g, h). \end{aligned}$$

Moreover,

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2], [Y_1, Y_2]),$$

so the Lie bracket on $\mathfrak{g} \oplus \mathfrak{h}$ is formed componentwise. We call $\mathfrak{g} \oplus \mathfrak{h}$ the *direct sum* of $\mathfrak{g}$ and $\mathfrak{h}$.

(iii) The single most important example of a Lie group is $G = \mathrm{GL}(n, \mathbb{R})$. It is an open subset of the vector space $M_n(\mathbb{R})$, so $\mathfrak{g} = T_IG = M_n(\mathbb{R})$. Vector fields on $G$ can be identified with smooth maps $G \to M_n(\mathbb{R})$ (but note that this is *not* the left trivialization).

Let $A, B, C \in M_n(\mathbb{R})$. Then

$$L_A(B + \lambda C) = A \cdot (B + \lambda C) = AB + \lambda AC = L_AB + \lambda L_AC,$$

so left translation is a linear map. It follows that the left-invariant vector field $L^C$ corresponding to some $C \in T_IG = M_n(\mathbb{R})$ is given by

$$L^C(A) = T_IL_A(C) = L_A(C) = A \cdot C \qquad (A \in G).$$

In other words, $L^C$ is the right-multiplication by $C$. We have shown this for $G = \mathrm{GL}(n, \mathbb{R})$, but since $G$ is dense in $M_n(\mathbb{R})$[1], we may extend $L^C$ continuously to all of $M_n(\mathbb{R})$. For the Lie bracket of vector fields on open subsets of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ we have (cf. [6, 3.1.6 (ii)]):

$$[X, Y](x) = DY(x)(X(x)) - DX(x)(Y(x)).$$

Also $L^C$ is linear by the above, so

$$D(L^{C'})(I)(C) = L^{C'}(C) = CC' \qquad (C, C' \in M_n(\mathbb{R})).$$

Thus, finally,

$$[C, C'] = [L^C, L^{C'}](I) = CC' - C'C.$$

This means that the Lie bracket on $M_n(\mathbb{R})$ is exactly the commutator of matrices.

[1] Let $A \in M_n(\mathbb{R})$. Since the polynomial $p : t \mapsto \det(A - tI)$ has at most $n$ zeros, we may find $t$ arbitrarily small with $p(t) \neq 0$, so $A - tI \in \mathrm{GL}(n, \mathbb{R})$.

## 5 Lie group homomorphisms

**5.1 Definition.** *Let $G$ and $H$ be Lie groups. A smooth map $\varphi : G \to H$ that is a group homomorphism is called a Lie group homomorphism. If $\varphi$ is (smoothly) invertible then also $\varphi^{-1}$ is a Lie group homomorphism, and $\varphi$ is called a Lie group isomorphism. A Lie group isomorphism $\varphi : G \to G$ is called Lie group automorphism.*

Thus for each Lie group homomorphism $\varphi$ we have $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$, $\varphi(e) = e$, and $\varphi(g^{-1}) = \varphi(g)^{-1}$.

**5.2 Examples.** (i) The inclusion $\mathrm{SO}(n, \mathbb{R}) \hookrightarrow \mathrm{GL}(n, \mathbb{R})$ is a Lie group homomorphism.

(ii) The conjugation map $\mathrm{conj}_g : G \to G$, $h \mapsto ghg^{-1}$ is a Lie group automorphism. We have

$$\mathrm{conj}_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g.$$

(iii) $t \mapsto e^{it}$ is a Lie group homomorphism from $\mathbb{R}$ to $S^1$.

**5.3 Lemma.** *Let $G$ be a connected Lie group, $H$ a Lie group and $\varphi_1, \varphi_2 : G \to H$ Lie group homomorphisms that coincide on some neighborhood of $e \in G$. Then $\varphi_1 = \varphi_2$.*

**Proof.** This is immediate from Proposition 2.5. □

**5.4 Definition.** *Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. A linear map $\alpha : \mathfrak{g} \to \mathfrak{h}$ with $\alpha([v_1, v_2]) = [\alpha(v_1), \alpha(v_2)]$ for all $v_1, v_2 \in \mathfrak{g}$ is called a* Lie algebra homomorphism.

**5.5 Proposition.** *Let $G$ and $H$ be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively.*

*(i) If $\varphi : G \to H$ is a Lie group homomorphism, then $\varphi' := T_e\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

*(ii) If $G$ is commutative, then the Lie bracket on $\mathfrak{g}$ vanishes identically.*

**Proof.** (i) For all $g, h \in G$ we have $\varphi(gh) = \varphi(g)\varphi(h)$, so $\varphi \circ L_g = L_{\varphi(g)} \circ \varphi$. Differentiating this identity at $e$ we obtain $T_g\varphi \circ T_e L_g = T_e L_{\varphi(g)} \circ \varphi'$, and inserting $v \in \mathfrak{g} = T_e G$ leads to

$$T_g\varphi(L^v(g)) = L^{\varphi'(v)}(\varphi(g)).$$

This means that $L^v \in \mathfrak{X}(G)$ and $L^{\varphi'(v)} \in \mathfrak{X}(H)$ are $\varphi$-related for any $v \in \mathfrak{g}$. Hence by Lemma 4.4 we obtain for any $v, w \in \mathfrak{g}$: $T\varphi \circ [L^v, L^w] = [L^{\varphi'(v)}, L^{\varphi'(w)}] \circ \varphi$. Inserting $e$ we arrive at $\varphi'([v, w]) = [\varphi'(v), \varphi'(w)]$.

(ii) If $G$ is commutative, then $\nu : g \mapsto g^{-1}$ is a Lie group homomorphism because

$$\nu(gh) = (gh)^{-1} = h^{-1}g^{-1} = g^{-1}h^{-1} = \nu(g)\nu(h).$$

Hence by (i), $\nu' : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism. Moreover, since $\nu' = -\mathrm{id}$ by Lemma 1.5 (ii), we get

$$-[v, w] = \nu'([v, w]) = [\nu'(v), \nu'(w)] = [-v, -w] = [v, w],$$

so $[v, w] = 0$. □

**5.6 Example.** Consider the subgroup $\mathrm{SL}(n,\mathbb{R}) := \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\} \subseteq \mathrm{GL}(n,\mathbb{R})$. Since det is regular on $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{R})$ is an $(n^2-1)$-dimensional submanifold of $\mathrm{GL}(n,\mathbb{R})$ (cf. [5, 1.1.9 (iv)]). Thus the Lie algebra $\mathfrak{sl}(n,\mathbb{R})$ of $\mathrm{SL}(n,\mathbb{R})$ is given by $\ker(D(\det)(I))$ ([5, 2.1.1 (iii)]). One can show that $\mathfrak{sl}(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$. The inclusion $i : \mathrm{SL}(n,\mathbb{R}) \hookrightarrow \mathrm{GL}(n,\mathbb{R})$ is a Lie group homomorphism with derivative $i' = \mathrm{incl} : \mathfrak{sl}(n,\mathbb{R}) \hookrightarrow M_n(\mathbb{R})$. By Proposition 5.5 (i), for $v, w \in \mathfrak{sl}(n,\mathbb{R})$ we have

$$[v,w]_{\mathfrak{sl}} = i'([v,w]_{\mathfrak{sl}}) = [i'(v), i'(w)]_{M_n} = [v,w]_{M_n},$$

so also in $\mathfrak{sl}(n,\mathbb{R})$ the Lie bracket is given by the commutator of matrices (cf. Example 4.6 (iii)).

# 6 Right-invariant vector fields

Analogously to Section 3 we may as well use right translations to trivialize $TG$. In this way we obtain the *right trivialization*

$$\begin{aligned} R : G \times \mathfrak{g} &\to TG \\ (g,v) &\mapsto R^v(g). \end{aligned}$$

Here, for any $v \in \mathfrak{g}$ the corresponding right-invariant vector field $R^v$ is defined by $R^v(g) = T_e R_g(v)$, and a vector field is called *right-invariant*, $X \in \mathfrak{X}_R(G)$, if $R_g^* X = X$ for all $g \in G$. That $R^v$ is indeed right-invariant follows since for each $h \in G$ we have

$$\begin{aligned} (R_g^* R^v)(h) &= T_{hg} R_{g^{-1}} \circ R^v(hg) = T_{hg} R_{g^{-1}} \circ T_e R_{hg}(v) \\ &= T_{hg} R_{g^{-1}} \circ T_h R_g \circ T_e R_h(v) = T_e R_h(v) = R^v(h). \end{aligned}$$

As in Proposition 3.2 it follows that $R$ is a diffeomorphism with inverse

$$v_g \mapsto (g, T_g R_{g^{-1}}(v)).$$

Due to $R_g^*[X,Y] = [R_g^* X, R_g^* Y]$, $\mathfrak{X}_R(G)$ is a Lie-subalgebra of $\mathfrak{X}(G)$ and as in Proposition 3.2 (ii) it follows that $X \mapsto X(e)$ and $v \mapsto R^v$ are mutually inverse isomorphisms between $\mathfrak{X}_R(G)$ and $\mathfrak{g}$.

**6.1 Proposition.** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and inversion $\nu : g \mapsto g^{-1}$.*

*(i)* $R^v = \nu^*(L^{-v})$ $\forall v \in \mathfrak{g}$. *Hence $\nu^* : \mathfrak{X}_L(G) \to \mathfrak{X}_R(G)$ is a linear isomorphism.*

*(ii)* $\forall v, w \in \mathfrak{g}$: $[R^v, R^w] = R^{-[v,w]}$.

*(iii)* $\forall v, w \in \mathfrak{g}$: $[L^v, R^w] = 0$.

**Proof.** (i) Since $(gh)^{-1} = h^{-1}g^{-1}$, we have $\nu \circ R_h = L_{h^{-1}} \circ \nu$. Thus for $X \in \mathfrak{X}_L(G)$ we get:

$$R_h^* \nu^* X = (\nu \circ R_h)^* X = (L_{h^{-1}} \circ \nu)^* X = \nu^* (L_{h^{-1}})^* X = \nu^* X,$$

showing that $\nu^* X$ is right-invariant. Moreover, $\nu^* X(e) = (T_e \nu)^{-1}(X_e) = -X_e$, i.e., $R^{-X_e} = \nu^* X = \nu^*(L^{X_e})$.

(ii) By (i),

$$[R^v, R^w] = [\nu^* L^{-v}, \nu^* L^{-w}] = \nu^* [L^{-v}, L^{-w}] = \nu^* L^{[-v,-w]} = R^{-[v,w]}.$$

(iii) $(0, L^v) : G \times G \to T(G \times G)$, $(g,h) \mapsto (0_g, L^v(h))$ is a vector field on $G \times G$ that is $\mu$-related to $L^v$: indeed by Lemma 1.5 we have

$$T_{(g,h)}\mu(0_g, L^v(h)) = T_h L_g(L^v(h)) = T_h L_g(T_e L_h(v)) = T_e L_{gh}(v) = L^v(\mu(g,h)).$$

Analogously, we conclude that $(R^w, 0)$ is $\mu$-related to $R^w$. By Lemma 4.4 we conclude:

$$0 = [(0, L^v), (R^w, 0)] \sim_\mu [L^v, R^w],$$

i.e., $0 = [L^v, R^w] \circ \mu$, and since $\mu$ is surjective, this gives $[L^v, R^w] = 0$. □

# 7 One-parameter subgroups

In this section we study the flow maps of left- resp. right-invariant vector fields. This will provide the foundation for the definition of the exponential map in the next section. We first recall some basics of ODE theory on differentiable manifolds from [5].

Let $M$ be a Hausdorff manifold and let $X \in \mathfrak{X}(M)$. A curve $c \in \mathcal{C}^\infty(I, M)$ is called an *integral curve* of $X$ if $c'(t) = X(c(t))$ for all $t \in I$. Here,

$$c'(t) = T_t c(1) = T_t c\Big(\frac{d}{dt}\Big|_1\Big).$$

Given $p \in M$, there exists a unique integral curve $c_p : I_p \to M$ of $X$ with $c(0) = p$ and such that $I_p = (t_-^p, t_+^p)$ is maximal. The set

$$\mathcal{D}(X) := \{(t,p) \in \mathbb{R} \times M \mid t \in I_p\}$$

is an open neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$. The *flow* of $X$,

$$\begin{aligned} \mathrm{Fl}^X : \mathcal{D}(X) &\to M \\ (t,p) &\mapsto c_p(t) \end{aligned}$$

is $\mathcal{C}^\infty$ and $\mathcal{D}(X)$ is the maximal domain of $\mathrm{Fl}^X$. We also write $\mathrm{Fl}^X(t,p) = \mathrm{Fl}^X_t(p)$. We have the flow property

$$\mathrm{Fl}^X_{s+t}(p) = \mathrm{Fl}^X_s(\mathrm{Fl}^X_t(p)),$$

whenever the right-hand side of this equation is defined ([5, 2.3.3 (iii)]). $X$ is called *complete* if $\mathcal{D}(X) = \mathbb{R} \times M$, i.e., if $I_p = \mathbb{R}$ for each $p \in M$. Any vector field with compact support (in particular, any vector field on a compact manifold) is complete ([5, 2.3.7]).

**7.1 Lemma.** *Let $X \in \mathfrak{X}(M)$ and suppose that there exists some $\varepsilon > 0$ such that $[-\varepsilon, \varepsilon] \subseteq I_p$ for each $p \in M$. Then $X$ is complete.*

**Proof.** Let $p \in M$. Then $t \mapsto \mathrm{Fl}^X_{t-\varepsilon}(\mathrm{Fl}^X_\varepsilon(p))$ is an integral curve of $X$ through $p$ defined on $[0, 2\varepsilon]$ (because $[-\varepsilon, \varepsilon] \subseteq I_{\mathrm{Fl}^X_\varepsilon(p)}$ by assumption). By uniqueness this integral curve coincides with $c_p$ on $[0, 2\varepsilon]$, so $[0, 2\varepsilon] \subseteq I_p$. By induction it follows that $[0, \infty) \subseteq I_p$. Analogously, $(-\infty, 0] \subseteq I_p$. □

**7.2 Lemma.** *Let $M, N$ be $T_2$-manifolds, $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$, $f \in \mathcal{C}^\infty(M, N)$ and $X \sim_f Y$. Then $f \circ \mathrm{Fl}^X_t = \mathrm{Fl}^Y_t \circ f$ for all $t$ such that the left-hand side of this equality is defined.*

**Proof.** Let $c : I \to M$ be an integral curve of $X$, so $c'(t) = X(c(t))$ for all $t \in I$. Then

$$(f \circ c)'(t) = T_{c(t)} f(c'(t)) = T_{c(t)} f(X(c(t))) = Y_{f(c(t))},$$

showing that $f \circ c$ is an integral curve of $Y$. Thus, for all $t \in I$, $f \circ \mathrm{Fl}_t^X = \mathrm{Fl}_t^Y \circ f$. □

Returning now to the Lie group setting, we have:

**7.3 Lemma.** *Let $X \in \mathfrak{X}_L(G)$ (resp. $X \in \mathfrak{X}_R(G)$), then*

*(i)* $\mathrm{Fl}_t^X(g) = g\mathrm{Fl}_t^X(e)$ *(resp.* $\mathrm{Fl}_t^X(g) = \mathrm{Fl}_t^X(e)g$*) for all* $g \in G$.

*(ii) $X$ is complete.*

**Proof.** (i) $X \in \mathfrak{X}_L(G)$ means that $L_g^* X = X$ for all $g$, i.e., $TL_g \circ X = X \circ L_g$. In other words, $X$ is $L_g$-related to itself: $X \sim_{L_g} X$. By Lemma 7.2 it follows that

$$g\mathrm{Fl}_t^X(e) = L_g \circ \mathrm{Fl}_t^X(e) = \mathrm{Fl}_t^X(L_g(e)) = \mathrm{Fl}_t^X(g).$$

The argument for $X \in \mathfrak{X}_R(G)$ is analogous.
(ii) By (i), $I_e \subseteq I_g$ for each $g \in G$. Thus the claim follows from Lemma 7.1. □

**7.4 Definition.** *A one-parameter subgroup of a Lie group $G$ is a Lie group homomorphism $\alpha : (\mathbb{R}, +) \to G$, i.e., a smooth curve $\alpha : \mathbb{R} \to G$ such that $\alpha(s+t) = \alpha(s)\alpha(t)$ for all $s, t \in \mathbb{R}$. In particular, $\alpha(0) = e$.*

**7.5 Lemma.** *Let $\alpha : \mathbb{R} \to G$ be a smooth curve with $\alpha(0) = e$. TFAE:*

*(i) $\alpha$ is a one-parameter subgroup with $\alpha'(0) = v$ ($\in T_eG = \mathfrak{g}$).*

*(ii)* $\alpha(t) = \mathrm{Fl}_t^{L^v}(e)$ $\forall t \in \mathbb{R}$.

*(iii)* $\alpha(t) = \mathrm{Fl}_t^{R^v}(e)$ $\forall t \in \mathbb{R}$.

*(iv)* $g \cdot \alpha(t) = \mathrm{Fl}_t^{L^v}(g)$ $\forall g \in G$ $\forall t \in \mathbb{R}$.

*(v)* $\alpha(t) \cdot g = \mathrm{Fl}_t^{R^v}(g)$ $\forall g \in G$ $\forall t \in \mathbb{R}$.

**Proof.** (i)⇒(ii):

$$\begin{aligned}\alpha'(t) &= \frac{d}{ds}\Big|_{s=0} \alpha(t+s) = \frac{d}{ds}\Big|_{s=0} (\alpha(t)\alpha(s)) = \frac{d}{ds}\Big|_{s=0} L_{\alpha(t)}\alpha(s) \\ &= T_e L_{\alpha(t)}(\alpha'(0)) = T_e L_{\alpha(t)}(v) = L^v(\alpha(t)).\end{aligned}$$

Therefore, $\alpha$ is an integral curve of $L^v$. Since $\alpha(0) = e$ it follows that $\alpha(t) = \mathrm{Fl}_t^{L^v}(e)$.
(ii)⇒(iv): This follows from Lemma 7.3 (i).
(iv)⇒(ii): Set $g = e$.
(ii)⇒(i): The curve $\alpha(t) := \mathrm{Fl}_t^{L^v}(e)$ is smooth, with $\alpha(0) = e$ and $\alpha'(0) = L^v(e) = v$. Moreover,

$$\frac{d}{ds}\alpha(t)\alpha(s) = T_{\alpha(s)} L_{\alpha(t)} \frac{d}{ds} \mathrm{Fl}_s^{L^v}(e) = T_{\alpha(s)} L_{\alpha(t)} L^v(\alpha(s)) = L^v(\alpha(t)\alpha(s)),$$

because $L^v$ is left-invariant. Also, $\alpha(t)\alpha(0) = \alpha(t)$. By definition of $\mathrm{Fl}^{L^v}$ we therefore obtain

$$\alpha(t)\alpha(s) = \mathrm{Fl}_s^{L^v}(\alpha(t)) = \mathrm{Fl}_s^{L^v}(\mathrm{Fl}_t^{L^v}(e)) = \mathrm{Fl}_{t+s}^{L^v}(e) = \alpha(t+s).$$

Analogously, one shows (i)⇒(iii)⇒(v)⇒(iii)⇒(i). □

# 8 The exponential map

The fundamental translation mechanism between the Lie algebra of a Lie group and the Lie group itself is provided by the exponential map, which we now proceed to introduce. According to Lemma 7.3, any $X \in \mathfrak{X}_L(G)$ is complete. Thus the following definition makes sense:

**8.1 Definition.** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. The* exponential map *is given by*

$$\begin{aligned}\exp : \mathfrak{g} &\to G \\ v &\mapsto \mathrm{Fl}_1^{L^v}(e).\end{aligned}$$

**8.2 Theorem.**

*(i)* $\exp$ *is smooth,* $\exp(0) = e$*, and* $T_0 \exp = \mathrm{id}_{\mathfrak{g}}$*. Consequently,* $\exp$ *is a diffeomorphism from some neighborhood of* $0 \in \mathfrak{g}$ *onto some neighborhood of* $e \in G$*.*

*(ii)* $\forall v \in \mathfrak{g}\ \forall g \in G$*:* $\mathrm{Fl}_t^{L^v}(g) = g \cdot \exp(tv)$.

*(iii)* $\forall v \in \mathfrak{g}\ \forall g \in G$*:* $\mathrm{Fl}_t^{R^v}(g) = \exp(tv) \cdot g$.

**Proof.** By Proposition 3.2 the map $\mathfrak{g} \times G \to TG$, $(v, g) \mapsto L^v(g)$ is a diffeomorphism, hence in particular is smooth. Thus $(v, g) \mapsto (0_v, L^v(g))$ defines a smooth vector field $Z$ on $\mathfrak{g} \times G$. Since $\mathrm{Fl}_t^Z(v, e) = (v, \mathrm{Fl}_t^{L^v}(e))$, exp is smooth.
Let $c : I \to G$ be an integral curve of $L^v$ and let $a \in \mathbb{R}$. Then $t \mapsto c(at)$ is an integral curve of $a \cdot L^v = L^{av}$ $(L^{av}(g) = T_e L_g(av) = a T_e L_g(v))$. Therefore,

$$\mathrm{Fl}_t^{L^{av}}(g) = \mathrm{Fl}_{at}^{L^v}(g).$$

From this we conclude $\mathrm{Fl}_t^{L^v}(e) = \mathrm{Fl}_1^{L^{tv}}(e) = \exp(tv)$. By Lemma 7.5, $\mathrm{Fl}_t^{L^v}(e) = \mathrm{Fl}_t^{R^v}(e)$, establishing (ii) and (iii) for $g = e$. For general $g$, Lemma 7.5 (iv), (v) shows that

$$\begin{aligned}\mathrm{Fl}_t^{L^v}(g) &= g \cdot \mathrm{Fl}_t^{L^v}(e) = g \cdot \exp(tv) \\ \mathrm{Fl}_t^{R^v}(g) &= \mathrm{Fl}_t^{R^v}(e) \cdot g = \exp(tv) \cdot g.\end{aligned}$$

Since $\exp(0) = \mathrm{Fl}_t^0(e) = e$, $T_0 \exp : T_0\mathfrak{g} = \mathfrak{g} \to T_e G = \mathfrak{g}$, and

$$T_0 \exp(v) = \frac{d}{dt}\Big|_0 \exp(tv) = \frac{d}{dt}\Big|_0 \mathrm{Fl}_t^{L^v}(e) = L^v(e) = v,$$

i.e., $T_0 \exp = \mathrm{id}_{\mathfrak{g}}$. By the inverse function theorem it follows that exp is a diffeomorphism on some neighborhood of $0 \in \mathfrak{g}$, so also (i) is proved. □

**8.3 Corollary.**

*(i)* $t \mapsto \exp(tv)$ *is a one-parameter group, i.e.,*

$$\forall v \in \mathfrak{g}\ \forall s, t \in \mathbb{R} : \ \exp((s + t)v) = \exp(sv) \exp(tv).$$

*(ii) Any one-parameter group of* $G$ *is of the form* $t \mapsto \exp(tv)$ *for some* $v \in \mathfrak{g}$*.*

**Proof.** (i) By Theorem 8.2 (ii),

$$\exp(sv)\exp(tv) = \mathrm{Fl}_s^{L^v}(e)\cdot\exp(tv) = \mathrm{Fl}_t^{L^v}(\mathrm{Fl}_s^{L^v}(e)) = \mathrm{Fl}_{s+t}^{L^v}(e) = \exp((s+t)v).$$

(ii) Let $\alpha$ be a one-parameter group, then by Lemma 7.5 (ii) and Theorem 8.2 (i) we have

$$\alpha(t) = \mathrm{Fl}_t^{L^v}(e) = \exp(tv).$$

□

**8.4 Example.** The exponential map on $G = \mathrm{GL}(n,\mathbb{R})$.

Let $A \in \mathfrak{g} = M_n(\mathbb{R})$. By Theorem 8.2 (ii), $\exp(tA) = \mathrm{Fl}_t^{L^A}(I)$. Also, from Example 4.6 (iii) we know that $L^A(B) = B\cdot A$. To determine $\mathrm{Fl}_t^{L^A}(I)$ we therefore have to solve the initial value problem

$$\begin{aligned} c'(t) &= L^A(c(t)) = c(t)\cdot A \\ c(0) &= I. \end{aligned}$$

The unique solution is given by

$$c(t) = e^{tA} = \sum_{k=0}^{\infty}\frac{t^k}{k!}A^k,$$

so $\exp(tA) = e^{tA}$, the matrix exponential. Recall that this series converges with respect to the operator norm because $\|A^k\| \le \|A\|^k$. Indeed, it converges uniformly on compact sets, in all derivatives. If $AB \ne BA$ then $\exp(A+B) \ne \exp(A)\exp(B)$.

**8.5 Remark.** Canonical coordinates of first and second kind.

Let $V$ be an open neighborhood of $0 \in \mathfrak{g}$ such that $\exp : V \to U := \exp(V)$ is a diffeomorphism. Then $(U, (\exp|_V)^{-1})$ is a local chart for $G$ around $\exp(0) = e$. After choosing a basis in $\mathfrak{g}$, the corresponding coordinates are called *canonical coordinates of the first kind.* For arbitrary $g \in G$ then $(L_g(U), (L_g \circ \exp)^{-1})$ is a chart around $g$.

*Canonical coordinates of the second kind* arise as follows: Let $(v_1,\dots,v_n)$ be a basis of $\mathfrak{g}$ and consider the map

$$\begin{aligned} f : \mathbb{R}^n &\to G \\ (t_1,\dots,t_n) &\mapsto \exp(t_1v_1)\cdot\exp(t_2v_2)\cdots\exp(t_nv_n). \end{aligned}$$

Then $\frac{\partial f}{\partial t_i}(0) = v_i$ $(1 \le i \le n)$, so

$$\begin{aligned} T_0 f : \mathbb{R}^n &\to \mathfrak{g} \\ (a_1,\dots,a_n) &\mapsto \sum_{i=1}^n a_i v_i. \end{aligned}$$

Consequently, $T_0 f$ is bijective, so $f$ is a diffeomorphism of a neighborhood of $0 \in \mathbb{R}^n$ onto some neighborhood of $e$ in $G$. $f^{-1}$ therefore is a local chart, whose coordinates are called canonical coordinates of the second kind.

The following result is a first example of "automatic regularity", which is quite typical for Lie group theory.

**8.6 Theorem.** *Let $\varphi : G \to H$ be a continuous homomorphism of the Lie groups $G$ and $H$. Then $\varphi$ is smooth, hence is a Lie group homomorphism.*

**Proof.** We first show that any continuous one-parameter group $\alpha : \mathbb{R} \to G$ is smooth. By Theorem 8.2 (i) there exists some $r > 0$ such that exp is a diffeomorphism from some ball (with respect to any chosen norm on $\mathfrak{g}$) $B_{2r}(0) \subseteq \mathfrak{g}$ onto some open neighborhood of $e$ in $G$. Let $\beta : [-\varepsilon, \varepsilon] \to B_r(0)$, $\beta := (\exp|_{B_r(0)})^{-1} \circ \alpha$. For $|t| \leq \varepsilon/2$ we have:

$$\exp(\beta(2t)) = \alpha(2t) = \alpha(t)^2 = \exp(2\beta(t)),$$

so $\beta(2t) = 2\beta(t)$ and thereby $\beta(s/2) = \beta(s)/2$ for all $s \in [-\varepsilon, \varepsilon]$. Hence for all $k, n \in \mathbb{N}$:

$$\alpha\left(\frac{n\varepsilon}{2^k}\right) = \alpha\left(\frac{\varepsilon}{2^k}\right)^n = \exp\left(\beta\left(\frac{\varepsilon}{2^k}\right)\right)^n = \exp\left(\frac{n}{2^k}\beta(\varepsilon)\right).$$

Due to $\alpha(-t) = \alpha(t)^{-1}$ and $\exp(-v) = \exp(v)^{-1}$ we obtain:

$$\alpha(t) = \exp\left(t\frac{1}{\varepsilon}\beta(\varepsilon)\right) \qquad \forall t \in \left\{\frac{n\varepsilon}{2^k} \mid k \in \mathbb{N},\ n \in \mathbb{Z}\right\} \subseteq \mathbb{R}. \tag{8.1}$$

As this set is dense in $\mathbb{R}$ and both sides are continuous, (8.1) indeed holds for each $t \in \mathbb{R}$, showing that $\alpha$ is smooth.

Now let $\varphi : G \to H$ be any continuous Lie group homomorphism. We use canonical coordinates of the second kind (cf. Remark 8.5). Let $(v_1, \ldots, v_n)$ be a basis of $\mathfrak{g}$ and let $f(t_1, \ldots, t_n) := \exp(t_1 v_1) \cdot \exp(t_2 v_2) \cdots \exp(t_n v_n)$ be the inverse of the chart. Then

$$(\varphi \circ f)(t_1, \ldots, t_n) = \varphi(\exp(t_1 v_1)) \cdots \varphi(\exp(t_n v_n)).$$

Every factor in this product is a continuous one-parameter group, hence is smooth by the above. Thus $\varphi \circ f$ is smooth, implying that $\varphi$ is $\mathcal{C}^\infty$ near $e$. Since $\varphi$ is a homomorphism, for any $g \in G$ we have $\varphi = L_{\varphi(g)} \circ \varphi \circ L_{g^{-1}}$, showing that $\varphi$ is smooth near $g$. □

**8.7 Corollary.** *Let $\varphi : G \to H$ be a bijective continuous homomorphism between Lie groups. If $G$ is separable then $\varphi$ is a diffeomorphism.*

**Proof.** By Theorem 8.6 it suffices to show that $\varphi^{-1}$ is continuous. Let $V$ be an open neighborhood of $e$ in $G$. Since $(g, h) \mapsto gh^{-1}$ is continuous, there exists a compact neighborhood $K$ of $e$ in $G$ with $KK^{-1} \subseteq V$. By assumption, $G$ is separable, so there exists a countable dense subset $\{a_n \mid n \in \mathbb{N}\}$ in $G$. It follows that $G = \bigcup_{m \in \mathbb{N}} a_m \cdot K$: indeed, if $g \in G$ then $gK^{-1}$ is a neighborhood of $g$, hence contains some $a_m$. Moreover $H$, being locally compact, is a Baire space. Therefore, since

$$H = \bigcup_{m \in \mathbb{N}} \varphi(a_m)\varphi(K),$$

where each $\varphi(a_m)\varphi(K)$ is closed, there exists some $m \in \mathbb{N}$ with $(\varphi(a_m)\varphi(K))^\circ \neq \emptyset$, and thereby $\varphi(K)^\circ \neq \emptyset$. Pick any $\varphi(g) \in \varphi(K)^\circ$ (noting that $\varphi$ is surjective), then $e_H = \varphi(g)\varphi(g)^{-1}$ is an interior point of $\varphi(K) \cdot \varphi(K)^{-1} = \varphi(KK^{-1}) \subseteq \varphi(V)$, i.e., $\varphi(V)$ is a neighborhood of $e_H$. Hence $\varphi^{-1}$ is continuous at $e_H$. For general $h \in H$, we have $\varphi^{-1} = L_{\varphi^{-1}(h)} \circ \varphi^{-1} \circ L_{h^{-1}}$, showing that $\varphi^{-1}$ is continuous at $h$. □

We know from Proposition 2.4 that the connected component $G_e$ of $G$ is an open and normal subgroup. Thus $G_e$ itself is a Lie group. The quotient group $G/G_e$ is called the *component group*.

**8.8 Theorem.** *Let $G$, $H$ be Lie groups with Lie algebras $\mathfrak{g}$, $\mathfrak{h}$ and exponential maps $\exp^G$, $\exp^H$.*

*(i) Let $\varphi : G \to H$ be a Lie group homomorphism with derivative $\varphi' = T_e\varphi : \mathfrak{g} \to \mathfrak{h}$. Then $\varphi \circ \exp^G = \exp^H \circ \varphi'$:*

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\varphi'"] \arrow[d, "\exp^G"'] & \mathfrak{h} \arrow[d, "\exp^H"] \\
G \arrow[r, "\varphi"] & H
\end{tikzcd}$$

*(ii) $G_e$ is the subgroup of $G$ generated by $\exp(\mathfrak{g})$.*

*(iii) If $\varphi, \psi : G \to H$ are Lie group homomorphisms with $\varphi' = \psi'$, then $\varphi|_{G_e} = \psi|_{G_e}$.*

**Proof.** (i) Let $v \in \mathfrak{g}$. By the proof of Proposition 5.5, $L^v \sim_\varphi L^{\varphi'(v)}$. Together with Lemma 7.2 we get

$$\varphi(\exp^G(v)) = \varphi(\mathrm{Fl}_1^{L^v}(e)) = \mathrm{Fl}_1^{L^{\varphi'(v)}}(\varphi(e)) = \exp^H(\varphi'(v)).$$

(ii) Since $e \in \exp(\mathfrak{g})$, which is connected, $\exp(\mathfrak{g}) \subseteq G_e$. Let $\tilde{G} := \langle \exp(\mathfrak{g}) \rangle$. Then $\tilde{G} \subseteq G_e$ and $\tilde{G}$ is open: take $U$, $V$ open such that $\exp : U \to V$ is a diffeomorphism. Then $V \subseteq \tilde{G}$. Let $g \in \tilde{G}$, then $gV$ is an open neighborhood of $g$ in $\tilde{G}$. Since $G_e$ is a topological group, the claim follows from Proposition 2.5.

(iii) By (i) we have $\varphi|_{\exp^G(\mathfrak{g})} = \psi|_{\exp^G(\mathfrak{g})}$, so the claim follows from (ii). □

# 9 The adjoint representation

**9.1 Definition.** *A representation of a Lie group $G$ on a finite-dimensional vector space $V$ is a Lie group homomorphism $G \to \mathrm{GL}(V)$. A representation of a Lie algebra $\mathfrak{g}$ on $V$ is a Lie algebra homomorphism $\mathfrak{g} \to L(V, V)$.*

From Proposition 5.5 we know that if $\varphi : G \to \mathrm{GL}(V)$ is a representation of the Lie group $G$ on $V$ then $\varphi' : \mathfrak{g} \to L(V, V)$ is a representation of the corresponding Lie algebra $\mathfrak{g}$ on $V$.

Any Lie group $G$ possesses a natural representation on its Lie algebra $\mathfrak{g}$: Consider the conjugation map $\mathrm{conj}_g : G \to G$, $\mathrm{conj}_g(h) = ghg^{-1}$ (cf. Example 5.2 (ii)). Its derivative $\mathrm{Ad}(g) := \mathrm{conj}_g' : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism.

Since $\mathrm{conj}_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$,

$$\mathrm{Ad}(g) = T_{g^{-1}}L_g \circ T_eR_{g^{-1}} = T_gR_{g^{-1}} \circ T_eL_g. \tag{9.1}$$

Moreover, $\mathrm{conj}_{gh} = \mathrm{conj}_g \circ \mathrm{conj}_h$, so

$$\mathrm{Ad}(gh) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h), \tag{9.2}$$

and $\mathrm{conj}_{g^{-1}} = (\mathrm{conj}_g)^{-1}$ implies $\mathrm{Ad}(g^{-1}) = \mathrm{Ad}(g)^{-1}$. Altogether, this establishes that $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a group homomorphism. We now want to show that it is also smooth. For this it suffices to prove that

$$\begin{aligned} G \times \mathfrak{g} &\to \mathfrak{g} \\ (g, v) &\mapsto \mathrm{Ad}(g)(v) \end{aligned}$$

is smooth. Now let $\varphi : G \times \mathfrak{g} \to TG \times TG \times TG$, $\varphi(g, v) := (0_g, v, 0_{g^{-1}})$, which is $\mathcal{C}^\infty$. Then by Lemma 1.5,

$$T\mu \circ (\mathrm{id}_{TG} \times T\mu)(\varphi(g, v)) = T\mu(0_g, T_eR_{g^{-1}}(v)) = T_{g^{-1}}L_g(T_eR_{g^{-1}}(v)) = \mathrm{Ad}(g)(v),$$

giving the claim. Consequently, $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a Lie group homomorphism, called the *adjoint representation* of $G$.

The derivative $\mathrm{ad} := \mathrm{Ad}' = T_e\mathrm{Ad} : \mathfrak{g} \to L(\mathfrak{g},\mathfrak{g})$ is therefore a representation of the Lie algebra $\mathfrak{g}$ of $G$ (so $\mathrm{ad}([v,w]) = [\mathrm{ad}(v),\mathrm{ad}(w)]$ for all $v,w \in \mathfrak{g}$), called the *adjoint representation* of $\mathfrak{g}$.

**9.2 Proposition.** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Let $g \in G$ and $v,w \in \mathfrak{g}$. Then:*

*(i)* $L^v(g) = R^{\mathrm{Ad}(g)(v)}(g)$.

*(ii)* $\mathrm{ad}(v)(w) = [v,w]$.

*(iii)* $\exp(t\mathrm{Ad}(g)(v)) = g\exp(tv)g^{-1} = \mathrm{conj}_g(\exp(tv))$ $(t \in \mathbb{R})$.

*(iv)* $\mathrm{Ad}(\exp(v))(w) = e^{\mathrm{ad}(v)}(w) = w + [v,w] + \frac{1}{2}[v,[v,w]] + \frac{1}{3!}[v,[v,[v,w]]] + \dots$, *i.e.,*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{ad}} & L(\mathfrak{g},\mathfrak{g}) \\ {\scriptstyle \exp^G}\downarrow & & \downarrow{\scriptstyle \exp^{\mathrm{GL}}} \\ G & \xrightarrow{\mathrm{Ad}} & \mathrm{GL}(\mathfrak{g}) \end{array}$$

**Proof.** (i) Due to $L_g = R_g \circ \mathrm{conj}_g$ we have:

$$L^v(g) = T_eL_g(v) = T_eR_g \circ T_e\mathrm{conj}_g(v) = T_eR_g \circ \mathrm{Ad}(g)(v) = R^{\mathrm{Ad}(g)(v)}(g).$$

(ii) Pick some basis $(v_1,\dots,v_n)$ of $\mathfrak{g}$ and let $v,w \in \mathfrak{g}$. Since $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is smooth, we have $\mathrm{Ad}(g)w = \sum_{i=1}^n f^i(g)v_i$ for certain $f^i \in \mathcal{C}^\infty(G,\mathbb{R})$. Let $\mathrm{ev}_w : L(\mathfrak{g},\mathfrak{g}) \to \mathfrak{g}$ be the evaluation map $A \mapsto Aw$. This map is linear, so

$$\begin{aligned} T_e(\mathrm{Ad}(\,.\,)\cdot w)\cdot v &= T_e(\mathrm{ev}_w \circ \mathrm{Ad})\cdot v = T_I\mathrm{ev}_w(T_e\mathrm{Ad}(v)) = \mathrm{ev}_w(T_e\mathrm{Ad}(v)) \\ &= \mathrm{ev}_w(\mathrm{ad}(v)) = \mathrm{ad}(v)(w) \end{aligned}$$

Therefore,

$$\begin{aligned} \mathrm{ad}(v)(w) &= T_e(\mathrm{Ad}(\,.\,)\cdot w)\cdot v = T_e\Big(\sum_{i=1}^n f^i(\,.\,)v_i\Big)\cdot v = \sum_{i=1}^n T_ef^i(v)\cdot v_i \\ &= \sum_{i=1}^n v(f^i)\cdot v_i = \sum_{i=1}^n L^v(f^i)|_e \cdot v_i \end{aligned} \tag{9.3}$$

By (i),

$$L^w(g) = R^{\mathrm{Ad}(g)w}(g) = R^{\sum f^i(g)v_i}(g) = \sum_{i=1}^n f^i(g)R^{v_i}(g),$$

so

$$[L^v, L^w] = [L^v, \sum_{i=1}^n f^iR^{v_i}] = \sum_{i=1}^n f^i \underbrace{[L^v, R^{v_i}]}_{=0} + \sum_{i=1}^n L^v(f^i)R^{v_i}. \tag{9.4}$$

Combining (9.3) with (9.4), we finally arrive at

$$[v,w] = [L^v,L^w](e) = \sum_{i=1}^n L^v(f^i)(e)R^{v_i}(e) = \sum_{i=1}^n L^v(f^i)(e)v_i = \mathrm{ad}(v)(w).$$

(iii) Inserting $\varphi = \mathrm{conj}_g$ in Theorem 8.8 (i), we get $\varphi' = \mathrm{Ad}(g)$ and

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}(g)"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{conj}_g"] & G \end{tikzcd}$$

Observing $t\mathrm{Ad}(g)(v) = \mathrm{Ad}(g)(tv)$, the claim follows.

(iv) This time, set $\varphi := \mathrm{Ad}$ in 8.8 (i). Then $\varphi' = \mathrm{ad}$ and

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp^G"'] & L(\mathfrak{g}, \mathfrak{g}) \arrow[d, "\exp^{\mathrm{GL}}"] \\ G \arrow[r, "\mathrm{Ad}"] & \mathrm{GL}(\mathfrak{g}) \end{tikzcd}$$

To conclude the proof, recall that $\exp^{\mathrm{GL}}$ is the matrix exponential by Example 8.4. □

# 10 The Maurer–Cartan form

The Maurer–Cartan form is an important tool for describing smooth maps taking values in a Lie group. It is a $\mathfrak{g}$-valued one-form, so we first derive some basic properties of vector-valued differential forms.

Let $M$ be a smooth manifold and $V$ a finite-dimensional vector space. A $V$-valued $k$-form $\varphi$ on $M$ is a smooth section of the vector bundle $\Lambda^k T^*M \otimes V$. Thus $\varphi$ is a smooth map $M \to \Lambda^k T^*M \otimes V$ with $\pi \circ \varphi = \mathrm{id}$. For each $p \in M$ we have

$$\varphi(p) \in \Lambda^k T_p^*M \otimes V \cong L^k_{\mathrm{alt}}(T_pM, V),$$

i.e., $\varphi(p) : (T_pM)^k \to V$ is $k$-linear and alternating. Smoothness of $\varphi$ is equivalent to the fact that for any $X_1, \dots, X_k \in \mathfrak{X}(M)$, the map

$$\varphi(X_1, \dots, X_k) = p \mapsto \varphi(p)(X_1(p), \dots, X_k(p))$$

lies in $\mathcal{C}^\infty(M, V)$. We denote the space of $V$-valued $k$-forms by $\Omega^k(M, V)$. As a $\mathcal{C}^\infty(M)$-module,

$$\Omega^k(M, V) \cong L^k_{\mathrm{alt}}(\mathfrak{X}(M)^k, \mathcal{C}^\infty(M, V)).$$

Pullback under smooth maps $f : M \to N$ works analogously to the case $V = \mathbb{R}$: If $\varphi \in \Omega^k(N, V)$, then $f^*\varphi \in \Omega^k(M, V)$ is given by

$$f^*\varphi(p)(v_1, \dots, v_k) := \varphi(f(p))(T_pf(v_1), \dots, T_pf(v_k)) \quad (v_i \in T_pM). \tag{10.1}$$

For $X \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M, \mathbb{R})$, $X(f) \in \mathcal{C}^\infty(M)$ is defined by

$$X(f)|_p = X(p)(f) = T_pf(X_p).$$

To extend this definition to $f \in \mathcal{C}^\infty(M, V)$, pick any basis $(v_1, \dots, v_n)$ of $V$ and write $f = \sum_{i=1}^n f_i v_i$ with $f_i \in \mathcal{C}^\infty(M)$. Then let $X(f) := \sum_{i=1}^n X(f_i)v_i$. It is easily verified that this definition is independent of the chosen basis in $\mathfrak{g}$. In this way, for any $X \in \mathfrak{X}(M)$ and any $f \in \mathcal{C}^\infty(M, V)$ we obtain a well-defined $X(f) \in \mathcal{C}^\infty(M, V)$. The map $f \mapsto X(f)$ is $\mathbb{R}$-linear and satisfies the product rule $X(\alpha f) = \alpha X(f) + X(\alpha)f$ for $\alpha \in \mathcal{C}^\infty(M, \mathbb{R})$.

Based on these constructions we may now define a vector-valued exterior derivative $d : \Omega^k(M,V) \to \Omega^{k+1}(M,V)$, again in close analogy to the scalar case: For $X_0,\dots,X_k \in \mathfrak{X}(M)$ and $\varphi \in \Omega^k(M,V)$ we set

$$\begin{aligned} d\varphi(X_0,\dots,X_k) := &\sum_{i=0}^{k}(-1)^i X_i(\varphi(X_0,\dots,\hat{X}_i,\dots,X_k)) \\ &+\sum_{i<j}(-1)^{i+j}\varphi([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots X_k). \end{aligned} \tag{10.2}$$

Then $d\varphi \in \Omega^{k+1}(M,V)$, and for $V = \mathbb{R}$ (10.2) reduces to the usual exterior derivative (cf. [5, 4.3.37]). Alternatively, one may expand $\varphi$ in a basis of $V$ and apply the scalar exterior derivative component-wise. We have $d(f^*\varphi) = f^*(d\varphi)$ and $d^2 = d \circ d = 0$.

Now let $G$ be a Lie group. From Proposition 3.2 we know the left trivialization of $TG$:

$$\begin{aligned} L : G \times \mathfrak{g} &\to TG \\ (g,v) &\mapsto L^v(g) \end{aligned}$$

with inverse $\Psi : TG \to G \times \mathfrak{g}$, $\Psi(v_g) = (g, T_gL_{g^{-1}}(v))$. This may alternatively be formulated in terms of a $\mathfrak{g}$-valued one-form:

**10.1 Definition.** *The* left Maurer–Cartan form $\omega \in \Omega^1(G,\mathfrak{g})$ *is given by*

$$\omega(g)(v) := T_gL_{g^{-1}}(v) \qquad (g \in G, v \in T_gG).$$

By the proof of Proposition 3.2, $\Psi$ is smooth, so the same is true of $\omega$: If $X \in \mathfrak{X}(G)$, then $\omega(X) = g \mapsto \mathrm{pr}_2 \circ \Psi \circ X(g)$. Thus indeed $\omega \in \Omega^1(G,\mathfrak{g})$.

**10.2 Proposition.** *The Maurer–Cartan form $\omega \in \Omega^1(G,\mathfrak{g})$ satisfies:*

*(i)* $(L_g)^*\omega = \omega \ \forall g \in G$.

*(ii)* $(R_g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega \ \forall g \in G$.

*(iii)* $\omega(L^v) = v \ \forall v \in \mathfrak{g}$.

*(iv) For each $g \in G$, $\omega(g) : T_gG \to \mathfrak{g}$ is a linear isomorphism.*

*(v)* Maurer–Cartan equation*: For all $X, Y \in \mathfrak{X}(G)$:*

$$d\omega(X,Y) + [\omega(X),\omega(Y)] = 0.$$

**Proof.** (i) By definition, $\omega(h) = T_hL_{h^{-1}}$. Using this and (10.1) we get

$$\begin{aligned} ((L_g)^*\omega)(h) &= \omega(L_g(h)) \circ T_hL_g = T_{gh}L_{h^{-1}g^{-1}} \circ T_hL_g \\ &= T_hL_{h^{-1}} \circ \underbrace{T_{gh}L_{g^{-1}} \circ T_hL_g}_{=\mathrm{id}_{T_hG}} = T_hL_{h^{-1}} = \omega(h). \end{aligned}$$

(ii)

$$\begin{aligned} ((R_g)^*\omega)(h) &= \omega(hg) \circ T_hR_g \underset{(i)(g\leftrightarrow h)}{=} T_gL_{g^{-1}} \circ T_{hg}L_{h^{-1}} \circ T_hR_g \\ &= T_gL_{g^{-1}} \circ T_h(\underbrace{L_{h^{-1}} \circ R_g}_{=R_g \circ L_{h^{-1}}}) = T_gL_{g^{-1}} \circ T_eR_g \circ T_hL_{h^{-1}} \\ &\underset{(9.1)}{=} \mathrm{Ad}(g^{-1}) \circ T_hL_{h^{-1}} = \mathrm{Ad}(g^{-1}) \circ (\omega(h)). \end{aligned}$$

(iii)

$$\omega(L^v)(h) = \omega(h)(L^v(h)) = T_h L_{h^{-1}}(T_e L_h(v)) = T_e(L_{h^{-1}} \circ L_h)(v) = v.$$

(iv) $\omega(g) = T_g L_{g^{-1}}$ is an isomorphism because $L_{g^{-1}}$ is a diffeomorphism.

(v) Note first that the value of $d\omega(X,Y) + [\omega(X),\omega(Y)]$ at $g \in G$ depends only on $X(g)$ and $Y(g)$. Any $X(g)$ (resp. $Y(g)$) can be written in the form $L^v(g)$ for a suitable $v \in \mathfrak{g}$ (because $T_e L_g : T_e G \to T_g G$ is a linear isomorphism). So we may assume without loss of generality that $X = L^v$, $Y = L^w$ $(v, w \in \mathfrak{g})$. Then by (iii) both $\omega(L^v) = v$ and $\omega(L^w) = w$ are constant, and (10.2) gives

$$\begin{aligned} d\omega(L^v, L^w) &= \underbrace{L^v(\omega(L^w))}_{=0} - \underbrace{L^w(\omega(L^v))}_{=0} - \omega([L^v, L^w]) = -\omega(L^{[v,w]}) \\ &\underset{(iii)}{=} -[v,w] = -[\omega(L^v), \omega(L^w)]. \end{aligned}$$

□

# 11 Logarithmic derivatives

The analogue of the exterior derivative $d : \mathcal{C}^\infty(M,\mathbb{R}) \to \Omega^1(M)$ for maps $f \in \mathcal{C}^\infty(M,G)$ is the so-called (left-) *logarithmic derivative.*

**11.1 Definition.** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, $M$ a smooth manifold and $f \in \mathcal{C}^\infty(M,G)$. The* left-logarithmic derivative *(or* Darboux derivative*) $\delta^l f \in \Omega^1(M,\mathfrak{g})$ is given by $\delta^l f := f^*\omega$, where $\omega$ is the Maurer–Cartan form on $G$.*

According to (10.1), we have

$$\delta^l f(p)(v) = \omega(f(p))(T_p f \cdot v) = T_{f(p)} L_{f(p)^{-1}}(T_p f \cdot v) \in T_e G = \mathfrak{g}$$

for $p \in M$ and $v \in T_p M$. Analogously, we define the *right-logarithmic derivative* $\delta^r f \in \Omega^1(M,\mathfrak{g})$ by

$$\delta^r f(p)(v) = T_{f(p)} R_{f(p)^{-1}}(T_p f \cdot v) \in \mathfrak{g} \qquad (p \in M,\ v \in T_p M).$$

**11.2 Example.** Let $G = (\mathbb{R}^+, \cdot)$ and let $f \in \mathcal{C}^\infty(M,G) = \mathcal{C}^\infty(M,\mathbb{R}^+)$. Then $df \in \Omega^1(M)$ satisfies

$$df(p)(v) = T_p f(v) \in T_{f(p)}\mathbb{R}^+ = \mathbb{R}.$$

In this example, left translations are linear, so

$$\delta^l f(p)(v) = T_{f(p)} L_{f(p)^{-1}}(T_p f \cdot v) = L_{f(p)^{-1}}(df(p)v) = \frac{1}{f(p)} df(p)(v),$$

i.e., $\delta^l f = \frac{df}{f} = d\log(f)$, hence the name.

**11.3 Proposition.** *Let $f, g : M \to G$ smooth.*

*(i) For the pointwise product of $f$ and $g$ we have:*

$$\begin{aligned} \delta^l(f \cdot g)(p) &= \delta^l g(p) + \mathrm{Ad}(g(p)^{-1})(\delta^l f(p)) \\ \delta^r(f \cdot g)(p) &= \delta^r f(p) + \mathrm{Ad}(f(p))(\delta^r g(p)). \end{aligned}$$

*(ii) The pointwise inverse $\nu \circ f$ of $f$ satisfies:*

$$\delta^l(\nu \circ f)(p) = -\mathrm{Ad}(f(p))(\delta^l f(p))$$
$$\delta^r(\nu \circ f)(p) = -\mathrm{Ad}(f(p)^{-1})(\delta^r f(p)).$$

*(iii) If $h : N \to M$ is smooth and $\delta = \delta^l$ or $\delta = \delta^r$, then*

$$\delta(f \circ h)(p)(v) = \delta f(h(p))(T_p h(v)).$$

*(iv) $\delta^r f(p) = \mathrm{Ad}(f(p)) \cdot \delta^l f(p)$.*

**Proof.** (i) $f \cdot g = \mu(f, g)$, so Lemma 1.5 (i) gives

$$T_p(f \cdot g) = T_{(f(p),g(p))}\mu(T_p f, T_p g) = T_{g(p)} L_{f(p)} \circ T_p g + T_{f(p)} R_{g(p)} \circ T_p f.$$

Therefore,

$$\begin{aligned}\delta^l(f \cdot g)(p) &= T_{f(p)g(p)} L_{g(p)^{-1} f(p)^{-1}} \circ T_p(f \cdot g)\\ &= T_{f(p)g(p)} L_{g(p)^{-1} f(p)^{-1}} \circ T_{g(p)} L_{f(p)} \circ T_p g\\ &\qquad + T_{f(p)g(p)} L_{g(p)^{-1} f(p)^{-1}} \circ T_{f(p)} R_{g(p)} \circ T_p f =: (1) + (2).\end{aligned}$$

Here, $(1) = T_{g(p)} L_{g(p)^{-1}} \circ T_p g = \delta^l g(p)$, and

$$\begin{aligned}(2) &= T_{f(p)}(\underbrace{L_{g(p)^{-1} f(p)^{-1}} \circ R_{g(p)}}_{=R_{g(p)} \circ L_{g(p)^{-1} f(p)^{-1}}}) \circ T_p f = T_{g(p)^{-1}} R_{g(p)} \circ T_{f(p)} \underbrace{L_{g(p)^{-1} f(p)^{-1}}}_{=L_{g(p)^{-1}} \circ L_{f(p)^{-1}}} \circ T_p f\\ &= T_{g(p)^{-1}} R_{g(p)} \circ T_e L_{g(p)^{-1}} \circ T_{f(p)} L_{f(p)^{-1}} \circ T_p f\\ &\underset{(9.1)}{=} \mathrm{Ad}(g(p)^{-1}) \circ T_{f(p)} L_{f(p)^{-1}} \circ T_p f = \mathrm{Ad}(g(p)^{-1}) \circ \delta^l f(p).\end{aligned}$$

The claim for $\delta^r$ follows analogously (or by using (iv)).
(ii) This follows from (i) because $\delta^l(f \cdot (\nu \circ f)) = \delta^l(e) = 0$ resp. $\delta^r((\nu \circ f) \cdot f) = \delta^r(e) = 0$.
(iii)

$$\delta^l(f \circ h)(p)(v) = T_{f(h(p))} L_{f(h(p))^{-1}}(\underbrace{T_p(f \circ h)(v)}_{=T_{h(p)} f \circ T_p h(v)}) = (\delta^l f)(h(p))(T_p h(v)).$$

(iv)

$$\begin{aligned}\mathrm{Ad}(f(p))\delta^l f(p) &\underset{(9.1)}{=} T_{f(p)} R_{f(p)^{-1}} \circ T_e L_{f(p)} \delta^l f(p)\\ &= T_{f(p)} R_{f(p)^{-1}} \circ \underbrace{T_e L_{f(p)} \circ T_{f(p)} L_{f(p)^{-1}}}_{=\mathrm{id}}(T_p f \cdot v)\\ &= T_{f(p)} R_{f(p)^{-1}}(T_p f \cdot v) = \delta^r f(p)(v).\end{aligned}$$

□

**11.4 Lemma.** *For $\exp : \mathfrak{g} \to G$ and $\alpha := z \mapsto \frac{e^z - 1}{z} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} z^k$ we have, for each $v \in \mathfrak{g}$:*

$$\delta^r \exp(v) = T_{\exp v}(R_{\exp(-v)}) \circ T_v \exp = \alpha(\mathrm{ad}(v)) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} (\mathrm{ad} v)^k \in L(\mathfrak{g}, \mathfrak{g}).$$

**Proof.** Note that $L(\mathfrak{g},\mathfrak{g})$ is a Banach algebra, and $\alpha(\mathrm{ad}(v))$ converges absolutely (with respect to the operator norm) for each $v \in \mathfrak{g}$. To begin with, let $f : V \to G$ be smooth from a finite-dimensional vector space $V$ into $G$, and for $t \in \mathbb{R}$ let $m_t : V \to V$, $m_t(v) = t \cdot v$. By definition,

$$\delta^r f(tv) = T_{f(tv)} R_{f(tv)^{-1}} \circ T_{tv} f.$$

Noting that $T_v m_t = m_t$ for each $v$, we get from Proposition 11.3 (iii):

$$\delta^r(f(t\,.\,))(v) = \delta^r(f \circ m_t)(v) = \delta^r f(tv) \circ m_t.$$

Hence

$$\delta^r(f(t\,.\,))(v) = t\delta^r f(tv). \tag{11.1}$$

By definition, $\delta^r \exp(v)$ is a map from $T_v\mathfrak{g}$ to $\mathfrak{g}$, and $T_v\mathfrak{g} = \mathfrak{g}$ (being a finite-dimensional vector space). Let $M : \mathfrak{g} \to L(\mathfrak{g},\mathfrak{g})$, $M(v) := \delta^r \exp(v)$. Then

$$\begin{aligned}
(s+t)M((s+t)v) &= (s+t)\delta^r \exp((s+t)v) \underset{(11.1)}{=} \delta^r(\exp((s+t)\,.\,))(v) \\
&= \delta^r(\exp(s\,.\,)\exp(t\,.\,))(v) \underset{11.3(i)}{=} \delta^r(\exp(s\,.\,))(v) + \mathrm{Ad}(\exp(sv))\delta^r(\exp(t\,.\,))(v) \\
&\underset{(11.1)}{=} s\delta^r \exp(sv) + \mathrm{Ad}(\exp(sv))(t\delta^r \exp(tv)) \\
&= sM(sv) + \mathrm{Ad}(\exp(sv))tM(tv).
\end{aligned}$$

Setting $N : \mathbb{R} \to L(\mathfrak{g},\mathfrak{g})$, $N(t) := tM(tv)$, this implies

$$N(s+t) = N(s) + \mathrm{Ad}(\exp(sv))N(t).$$

Fixing $t$ here and applying $\frac{d}{ds}\big|_0$ gives

$$N'(t) = N'(0) + \mathrm{ad}(v)N(t).$$

Here

$$\begin{aligned}
N'(0) &= \frac{d}{dt}\Big|_0 tM(tv) = M(0) + 0 \cdot \frac{d}{dt}\Big|_0 M(tv) = M(0) \\
&= \delta^r \exp(0) = T_e R_e \circ \underbrace{T_0 \exp}_{=\mathrm{id}} = \mathrm{id}_{\mathfrak{g}}.
\end{aligned}$$

Therefore, $N$ satisfies the following initial value problem:

$$\begin{aligned}
N'(t) &= \mathrm{id}_{\mathfrak{g}} + \mathrm{ad}(v)N(t) \\
N(0) &= 0.
\end{aligned}$$

The unique solution to this ODE is $N(t) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(\mathrm{ad}v)^k t^{k+1}$: Indeed, $N(0) = 0$ and

$$N'(t) = \sum_{k=0}^{\infty} \frac{k+1}{(k+1)!}(\mathrm{ad}v)^k t^k = \mathrm{id}_{\mathfrak{g}} + \sum_{k=1}^{\infty} \frac{1}{k!}(\mathrm{ad}v)^k t^k = \mathrm{id}_{\mathfrak{g}} + \mathrm{ad}(v)N(t).$$

Altogether, we arrive at

$$\delta^r \exp(v) = M(v) = N(1) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}(\mathrm{ad}v)^k = \alpha(\mathrm{ad}(v)).$$

□

**11.5 Corollary.** *Let $v \in \mathfrak{g}$, then the following are equivalent:*

*(i)* $T_v \exp$ *is bijective (and thereby* $\exp$ *is a local diffeomorphism around* $v$*).*

*(ii) No eigenvalue of* $\mathrm{ad}(v) : \mathfrak{g} \to \mathfrak{g}$ *is of the form* $2k\pi i$ *for some* $k \in \mathbb{Z} \setminus \{0\}$*.*

**Proof.** Using the Jordan normal form (or functional calculus), it follows that for any $A \in L(\mathfrak{g}, \mathfrak{g})$ we have $\mathrm{EV}(\alpha(A)) = \alpha(\mathrm{EV}(A))$, where EV is the set of eigenvalues. Using this, we have:

$$\begin{aligned}
T_v \exp \text{ is bijective } &\Leftrightarrow \underbrace{T_{\exp(v)} R_{\exp(-v)}}_{\text{bij.}} \circ T_v \exp = \delta^r \exp(v) \text{ is bijective} \\
&\Leftrightarrow 0 \text{ is not an eigenvalue of } \delta^r \exp(v) \underset{11.4}{=} \alpha(\mathrm{ad}(v)) \\
&\Leftrightarrow 0 \notin \mathrm{EV}(\alpha(\mathrm{ad}(v))) = \alpha(\mathrm{EV}(\mathrm{ad}(v))) \\
&\Leftrightarrow \text{ no eigenvalue of } \mathrm{ad}(v) \text{ is a zero of } \alpha.
\end{aligned}$$

Since the zeros of $\alpha(z) = \frac{e^z - 1}{z}$ are precisely the $2k\pi i$ for $k \in \mathbb{Z} \setminus \{0\}$, the claim follows. □

# 12 The Campbell–Baker–Hausdorff formula

**12.1 Theorem.** *Let* $G$ *be a Lie group with Lie algebra* $\mathfrak{g}$*. For* $z$ *near* $1 \in \mathbb{C}$ *set*

$$\beta(z) := \frac{\log(z)}{z-1} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m+1} (z-1)^m.$$

*Then for* $v, w$ *near* $0 \in \mathfrak{g}$*:*

$$\exp(v) \cdot \exp(w) = \exp C(v, w),$$

*where*

$$\begin{aligned}
C(v, w) &= w + \int_0^1 \beta\big(e^{t\,\mathrm{ad}(v)} \cdot e^{\mathrm{ad}(w)}\big) \cdot v \, dt \\
&= v + w + \sum_{m=1}^{\infty} \frac{(-1)^m}{m+1} \int_0^1 \Big( \sum_{\substack{k,l \geq 0 \\ k+l \geq 1}} \frac{t^k}{k! l!} \mathrm{ad}(v)^k \mathrm{ad}(w)^l \Big)^m v \, dt \\
&\underset{9.2(ii)}{=} v + w + \frac{1}{2}[v, w] + \frac{1}{12}([v, [v, w]] - [w, [w, v]]) + \ldots
\end{aligned}$$

**Proof.** Set $C(v, w) := \exp^{-1}(\exp(v) \exp(w))$, which is well-defined for $v, w$ in a suitable neighborhood of $0 \in \mathfrak{g}$ and set $C(t) := C(tv, w)$ (for $t \in [0, 1]$). Then

$$\begin{aligned}
T_{\exp C(t)} R_{\exp(-C(t))} \underbrace{\frac{d}{dt} \exp C(t)}_{= T_t(\exp C(\,.\,)) \cdot 1} &\underset{11.1}{=} \delta^r (\exp \circ C)(t) \cdot 1 \\
&\underset{11.3}{=} \delta^r \exp(C(t)) \dot{C}(t) \underset{11.4}{=} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \mathrm{ad}(C(t))^k \dot{C}(t) \\
&= \alpha(\mathrm{ad}(C(t))) \cdot \dot{C}(t).
\end{aligned} \tag{12.1}$$

Since $\exp C(t) = \exp(tv) \exp w$, we have

$$\exp(-C(t)) = (\exp C(t))^{-1} = \exp(-w) \exp(-tv),$$

so

$$
\begin{aligned}
&T_{\exp C(t)}R_{\exp(-C(t))}\frac{d}{dt}\exp C(t)\\
&\qquad = T_{\exp C(t)}(R_{\exp(-w)\exp(-tv)})\frac{d}{dt}(R_{\exp w}\circ \exp(tv))\\
&\qquad = T_{\exp(tv)}R_{\exp(-tv)}\underbrace{T_{\exp C(t)}R_{\exp(-w)}\circ T_{\exp(tv)}R_{\exp(w)}}_{=\mathrm{id}}\circ\frac{d}{dt}\exp(tv)
\end{aligned}
$$

Here,

$$
\frac{d}{dt}\exp(tv) = \frac{d}{dt}\mathrm{Fl}_t^{R^v}(e) = R^v(\mathrm{Fl}_t^{R^v}(e)) = R^v(\exp(tv)),
$$

so

$$
\begin{aligned}
T_{\exp C(t)}R_{\exp(-C(t))}\frac{d}{dt}\exp C(t) &= T_{\exp(tv)}R_{\exp(-tv)}\circ R^v(\exp(tv))\\
&= \big((R_{\exp(tv)})^* R^v\big)(e) = R^v(e) = v
\end{aligned}
\tag{12.2}
$$

since $R^v$ is right-invariant. Combining (12.1) with (12.2) we obtain $v = \alpha(\mathrm{ad}(C(t)))\cdot \dot C(t)$. Next,

$$
\begin{aligned}
e^{\mathrm{ad}(C(t))} &\underset{9.2(iv)}{=} \mathrm{Ad}(\exp C(t)) = \mathrm{Ad}(\exp(tv)\exp(w)) \underset{(9.2)}{=} \mathrm{Ad}(\exp(tv))\mathrm{Ad}(\exp(w))\\
&= e^{\mathrm{ad}(tv)}\cdot e^{\mathrm{ad}(w)}.
\end{aligned}
$$

Hence for $v, w$ small we get

$$
\mathrm{ad}(C(t)) = \log(e^{\mathrm{ad}(tv)}\cdot e^{\mathrm{ad}(w)}),
$$

where $\log(z)$ is defined by the power series $\sum_{m=1}^\infty \frac{(-1)^{m+1}}{m}(z-1)^m$. It follows that

$$
v = \alpha(\mathrm{ad}(C(t)))\dot C(t) = \alpha(\log(e^{\mathrm{ad}(tv)}\cdot e^{\mathrm{ad}(w)}))\dot C(t).
$$

For $z$ near 1 we have $\beta(z) := \frac{\log(z)}{z-1} = \sum_{m=0}^\infty \frac{(-1)^m}{m+1}(z-1)^m$, so

$$
\alpha(\log(z))\cdot\beta(z) = \frac{e^{\log z}-1}{\log z}\cdot\frac{\log z}{z-1} = \frac{z-1}{z-1} = 1 \Rightarrow \alpha(\log z) = \beta(z)^{-1}.
$$

Therefore, $v = \beta\big(e^{\mathrm{ad}(tv)}\cdot e^{\mathrm{ad}(w)}\big)^{-1}\dot C(t)$, so $\dot C(t) = \beta\big(e^{\mathrm{ad}(tv)}\cdot e^{\mathrm{ad}(w)}\big)v$ and $C(0) = C(0,w) = w$. Thus, finally,

$$
\begin{aligned}
C(v,w) = C(1) = C(0) + \int_0^1 \dot C(t)\,dt &= w + \int_0^1 \beta\big(e^{t\mathrm{ad}(v)}\cdot e^{\mathrm{ad}(w)}\big)\cdot v\,dt\\
&= v + w + \sum_{m=1}^\infty \frac{(-1)^m}{m+1}\int_0^1\Big(\sum_{\substack{k,l\ge 0\\ k+l\ge 1}}\frac{t^k}{k!l!}\mathrm{ad}(v)^k\mathrm{ad}(w)^l\Big)^m v\,dt\\
&\underset{9.2(ii)}{=} v + w + \frac{1}{2}[v,w] + \frac{1}{12}([v,[v,w]] - [w,[w,v]]) + \dots
\end{aligned}
$$

□

**12.2 Remark.** This result has far-reaching consequences:

(i) On a neighborhood of $e$, the product in $G$ is completely determined by the Lie bracket in $\mathfrak{g}$.

(ii) Let $\tilde U$, $U$ be neighborhoods of $0 \in \mathfrak{g}$ resp. of $e$ such that $\exp : \tilde U \to U$ is a diffeomorphism and pick a neighborhood $V \subseteq U$ of $e$ such that $V\cdot V \subseteq U$. Then the product $\mu$ on $V \times V$ is given in terms of the chart $(U, \exp^{-1})$ by the map $(v, w) \mapsto C(v, w)$, hence is real-analytic: $\exp^{-1} \circ \mu \circ (\exp \times \exp) = C$. Setting $\tilde{V} := \exp^{-1}(V)$ we have

$$\begin{tikzcd} G \times G \supseteq V \times V \arrow[r, "\mu"] \arrow[d, "\exp^{-1} \times \exp^{-1}"'] & U \subseteq G \arrow[d, "\exp^{-1}"] \\ \mathfrak{g} \times \mathfrak{g} \supseteq \tilde{V} \times \tilde{V} \arrow[r, "C"] & \tilde{U} \subseteq \mathfrak{g} \end{tikzcd}$$

By left translation we can then construct a real analytic atlas for $G$: Let $g \in G$. Then $\varphi_g := \exp^{-1} \circ L_{g^{-1}} : L_g(V) \to \tilde{V} \subseteq \mathfrak{g}$ is a chart around $g$. Now pick an open neighborhood $W$ of $e$ with $W \cdot W^{-1} \subseteq V$. Then $(\varphi_g, L_g(W))_{g \in G}$ is an atlas for $G$ that is real-analytic: Let $L_g(W) \cap L_h(W) \neq \varnothing$. Then there exist $w_1, w_2 \in W$ with $gw_1 = hw_2$, so $g^{-1}h = w_1 w_2^{-1} \in V$ and thereby $g^{-1}h = \exp(\tilde{v})$ for some $\tilde{v} \in \tilde{V}$. Therefore, $\varphi_g \circ \varphi_h^{-1} : \varphi_h(L_g(W) \cap L_h(W)) \to \mathfrak{g}$, with

$$\begin{aligned} \varphi_g \circ \varphi_h^{-1} &= \exp^{-1} \circ L_{g^{-1}} \circ L_h \circ \exp = \exp^{-1} \circ L_{g^{-1}h} \circ \exp \\ &= \exp^{-1} \circ \mu(g^{-1}h, \,.\,) \circ \exp = \exp^{-1} \circ \mu \circ (\exp \times \exp)(\tilde{v}, \,.\,) = C(\tilde{v}, \,.\,), \end{aligned}$$

which is $C^\omega$. Note that all the constructions in this lecture course so far would have worked only assuming the Lie group to be a $C^2$-manifold. Hence any $C^2$ Lie group is automatically real analytic.

# 13 Submanifolds

To develop a suitable notion of Lie subgroups we require certain fundamental facts about (immersive) submanifolds, which we collect in this and the following section. We will build on [5, Sec. 3.3] for results that won't be proved here.

**13.1 Definition.** *Let $M$, $N$ be manifolds and let $f \in \mathcal{C}^\infty(M, N)$. $f$ is called an* immersion *(resp. a* submersion*) if for all $p \in M$ the tangent map $T_p f$ is injective (resp. surjective).*

**13.2 Remark.** (i) By the rank theorem ([5, 3.3.3]) immersions are locally of the form $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_m, 0, \dots, 0)$, and submersions are of the form $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$ (where $m = \dim M > n = \dim N$).

(ii) Let $f : M \to N$ be continuous and $g : N \to R$ an immersion. If $g \circ f$ is $\mathcal{C}^\infty$, then so is $f$ ([5, 3.3.8]).

(iii) Let $f : M \to N$ be a surjective submersion and let $g : N \to R$ be any map. If $g \circ f$ is $\mathcal{C}^\infty$, then so is $g$ ([5, 3.3.9]).

**13.3 Definition.** *Let $M, N$ be manifolds with $N \subseteq M$ and let $j : N \hookrightarrow M$ be the inclusion map. $N$ is called* immersive submanifold *of $M$ if $j$ is an immersion. $N$ is called submanifold (or* regular *submanifold) if in addition $N$ is a topological subspace of $M$, i.e., if the natural manifold topology of $N$ is the trace topology of the natural manifold topology on $M$.*

Note that in this terminology, the prize for dropping the adjective 'regular' in 'regular submanifold' (which saves a lot of repetitions) is that there are immersive submanifolds that are not submanifolds in this sense.

**13.4 Theorem.** *Let $N^n$ be an immersive submanifold of $M^m$. TFAE:*

*(i) $N$ is a submanifold of $M$.*

*(ii) Around any $p \in N$ there exists an* adapted coordinate system*, i.e., for any $p \in N$ there exists a chart $(\varphi, U)$ around $p$ in $M$ such that $\varphi(p) = 0$, $\varphi(U \cap N) = \varphi(U) \cap (\mathbb{R}^n \times \{0\}) \subseteq \mathbb{R}^m$ and $\varphi|_{U \cap N}$ is a chart of $N$ around $p$.*

**Proof.** See [5, 3.3.12]. □

**13.5 Remark.** (i) If $N$ is a submanifold of $M$, $f : P \to M$ is smooth, and $f(P) \subseteq N$, then also $f : P \to N$ is smooth ([5, 3.3.14]).

(ii) If $N$ is a subset of a manifold $M$, then $N$ can be turned into a submanifold of $M$ in at most one way ([5, 3.3.15]).

**13.6 Lemma.** *Let $P$ be an immersive submanifold of $M$, $N$ a submanifold of $M$ and $P \subseteq N$. Then $P$ is an immersive submanifold of $N$.*

**Proof.** We have the following inclusions:

$$\begin{tikzcd} & M \\ P \arrow[ur, hook, "j"] \arrow[r, hook, "j_P"'] & N \arrow[u, hook, "j_N"'] \end{tikzcd}$$

We need to show that $j_P$ is an immersion. We have $j = j_N \circ j_P$, and $j$ is an immersion by assumption. Hence Remark 13.5 (i) shows that $j_P$ is smooth. Moreover, for any $p \in P$,

$$\underbrace{T_p j}_{inj.} = \underbrace{T_p j_N}_{inj.} \circ T_p j_P,$$

so $T_p j_P$ is injective, giving the claim. □

**13.7 Definition.** *Let $M, N$ be manifolds. A map $i : N \to M$ is called an* embedding *if $i$ is an injective immersion and a homeomorphism from $N$ onto $(i(N), \mathcal{T}_M|_{i(N)})$ (i.e., $i(N)$ equipped with the trace topology from $M$).*

Locally, any immersion is an embedding: Let $f : N \to M$ be an immersion. Then each $p \in N$ has a neighborhood $U$ such that $f : U \to M$ is an embedding ([5, 3.3.21]). In this sense, the difference between immersions and embeddings is a global property, not a local one.

**13.8 Theorem.** *Let $M^m$ and $N^n$ be manifolds and $f : N \to M$ smooth with $\mathrm{rk}(f) \equiv k$ on $N$. Let $q \in f(N)$. Then $f^{-1}(q)$ is a closed submanifold of $N$ of dimension $n - k$.*

**Proof.** See [5, 3.3.22]. □

**13.9 Corollary.** *Let $f : N^n \to M^m$ be smooth, with $m < n$. If $\mathrm{rk}_p(f) = m$ for all $p \in f^{-1}(q)$ (where $q \in f(N)$), then $L := f^{-1}(q)$ is a closed submanifold of $N$ of dimension $n - m$. Moreover, for all $p \in L$, $T_p L = \ker T_p f$.*

**Proof.** See [5, 3.3.23, 3.3.24]. □

# 14 Topological properties of submanifolds

**14.1 Proposition.** *Let $M'$ be an immersive submanifold of a manifold $M$ with $\dim M' = \dim M$. Then $M'$ is an open submanifold of $M$.*

**Proof.** Let $p \in M'$ and $j : M' \hookrightarrow M$. Since $\dim M' = \dim M$, $T_p j : T_p M' \to T_p M$ is bijective. Thus there is an open neighborhood $U$ of $p$ in $M'$ such that $U = j(U)$ is open in $M$, showing that $M'$ is open in $M$. Therefore, we can turn $M'$ into an open submanifold $\tilde{M}$ of $M$ (by restriction of the charts of $M$ to $M'$). Then $\mathrm{id} = j : M' \to \tilde{M}$ is smooth by Remark 13.5 (i). Furthermore,

$$T_p\mathrm{id} = T_p j : T_p M' \to T_p \tilde{M} = T_p M$$

is bijective. Consequently, id is a local and thereby a global diffeomorphism, so $M' = \tilde{M}$. $\square$

**14.2 Proposition.** *Let $M'$ be a submanifold of $M$ with $\dim M' < \dim M$. Then $M'$ is not dense in $M$.*

**Proof.** Let $n = \dim M$, $l = \dim M'$. Given any $p \in M'$, using Remark 13.2 (i) pick charts $(\varphi, U)$, $(\psi, V)$ around $p$ in $M$ resp. $M'$ such that $V \subseteq U$ and $\varphi \circ j \circ \psi^{-1} = x \mapsto (x, 0)$ on $\psi(V)$. Since $M'$ is a submanifold there exists an open set $W$ in $M$ with $V = M' \cap W$. Then $\varphi_1 := \varphi|_{U \cap W}$ is a chart around $p$ in $M$ and $\varphi_1(V) \subseteq \mathbb{R}^l \times \{0\}$. Thus $\varphi_1^{-1}(\mathbb{R}^l \times \mathbb{R}^{n-l} \setminus \{0\})$ is open in $W$ (and thereby in $M$) and disjoint from $V$. Since $V = M' \cap W$, this open set is also disjoint from $M'$, so $M'$ cannot be dense in $M$. $\square$

Recall from Remark 1.2 that manifolds are always supposed to be equipped with their natural topology (which is induced by their charts). In particular this holds for immersive submanifolds, whose natural topology is not necessarily equal to the trace topology of the surrounding manifold.

**14.3 Example.** We equip $\mathbb{R}^2$ with a new $\mathcal{C}^\infty$-structure: For $a \in \mathbb{R}$ let $\varphi_a : \mathbb{R} \times \{a\} \to \mathbb{R}$, $(x, a) \mapsto x$. Then $(\mathbb{R} \times \{a\}, \varphi_a)_{a \in \mathbb{R}}$ forms a $\mathcal{C}^\infty$-atlas, because the chart domains do not intersect for different values of $a$. Call the resulting manifold $\mathbb{R}'$. $\mathbb{R}'$ is a one-dimensional immersive submanifold of $\mathbb{R}^2$: For $(x, a) \in \mathbb{R}'$, the map $\mathrm{id}_{\mathbb{R}^2} \circ j \circ \varphi_a^{-1} = x \mapsto (x, a)$ has rank 1.

Note that $\mathbb{R}'$ is *not* a submanifold of $\mathbb{R}^2$ because $\mathbb{R} \times \{a\}$ is open in $\mathbb{R}'$ (being a chart domain), but not open in the trace topology of $\mathbb{R}^2$ on $\mathbb{R}'$ (which just is the standard Euclidean topology). Since $\mathbb{R}'$ is the disjoint union of the open subsets $\mathbb{R} \times \{a\}$ $(a \in \mathbb{R})$, $\mathbb{R}'$ is not connected as a manifold, although it is connected as a subset of $\mathbb{R}^2$.

Nevertheless, we have:

**14.4 Lemma.** *Let $M'$ be a connected immersive submanifold of $M$. Then $M'$ is a connected subset of $M$.*

**Proof.** $j : M' \hookrightarrow M$ is smooth, hence continuous. Thus $j(M') = M'$ is a connected subset of $M$. $\square$

**14.5 Proposition.** *If $M$ is Hausdorff, then so is any immersive submanifold $M'$ of $M$.*

**Proof.** Let $p \neq q$ be points in $M'$ and choose disjoint open neighborhoods $U$ of $p$ and $V$ of $q$ in $M$. Then $j^{-1}(U) = U \cap M'$, $j^{-1}(V) = V \cap M'$ are disjoint, as well as open because $j$ is continuous. $\square$

**14.6 Remark.** If $M$ is second countable, then so is any submanifold because it carries the trace topology. For immersive submanifolds, however, this need not hold in general. Indeed, the manifold $\mathbb{R}'$ from Example 14.3 is the disjoint union of the uncountably many open sets $\mathbb{R} \times \{a\}$ $(a \in \mathbb{R})$, hence is not second countable.

Nevertheless, we have:

**14.7 Proposition.** *Let $M$ be a second countable manifold and $M'$ a connected immersive submanifold of $M$. Then $M'$ is itself second countable.*

**Proof.** See Appendix A. $\square$

# 15 Quotient manifolds

For our analysis of Lie group actions on manifolds later on we need some general results on quotients of manifolds, which in turn rely on a closer study of submersions.

**15.1 Definition.** *Let $f : M \to M'$ be a smooth map. A* local section *of $f$ around $p \in M$ is a smooth map $g : M' \supseteq V \to M$ ($V$ open) with $p \in g(V)$ and $f \circ g = \mathrm{id}_V$.*

**15.2 Proposition.** *Let $f : M \to M'$ be a smooth map. TFAE:*

*(i) $f$ is a submersion.*

*(ii) For each $p \in M$ there exists a local section of $f$ around $p$.*

**Proof.** (i)⇒(ii): By Remark 13.2 (i) there exist charts $(\varphi, U)$ around $p$ in $M$ and $(\psi, W)$ around $f(p)$ in $M'$ such that $\psi \circ f \circ \varphi^{-1} = (x, y) \mapsto x = \mathrm{pr}_1$. Without loss of generality let $U \subseteq f^{-1}(W)$, so that $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^l$ ($l = \dim M'$). Thus $f \circ \varphi^{-1} = \psi^{-1} \circ \mathrm{pr}_1|_{\varphi(U)}$.

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & W \arrow[d, "\psi"] \\ \varphi(U) \arrow[r, "\mathrm{pr}_1"] & \psi(W) \end{tikzcd}$$

Let $\varphi(p) = (x_0, y_0)$ and $h : \mathbb{R}^l \to \mathbb{R}^n := x \mapsto (x, y_0)$. Then

$$f \circ \varphi^{-1} \circ h \circ \psi = \psi^{-1} \circ \underbrace{\mathrm{pr}_1 \circ h}_{=\mathrm{id}} \circ \psi|_{(h\circ\psi)^{-1}(\varphi(U))} = \mathrm{id}|_{(h\circ\psi)^{-1}(\varphi(U))}.$$

Set $g := \varphi^{-1} \circ h \circ \psi$. Then $g$ is smooth, and defined on the open set $V := (h \circ \psi)^{-1}(\varphi(U))$. Also, $f \circ g = \mathrm{id}_V$ and $p \in g(V)$: Indeed, let $v := f(p)$. Then $v \in V$ because $h \circ \psi(v) = h \circ \psi \circ f(p) = h(\psi \circ f \circ \varphi^{-1}(\varphi(p))) = h(\mathrm{pr}_1(x_0, y_0)) = (x_0, y_0) = \varphi(p) \in \varphi(U)$. Finally, $g(v) = \varphi^{-1} \circ h \circ \psi \circ f(p) = \varphi^{-1}(x_0, y_0) = p$, so $p \in g(V)$.

(ii)⇒(i): Let $p \in M$, let $g$ be a local section of $f$ around $p$ and let $v \in V$ be such that $g(v) = p$. Then $f \circ g = \mathrm{id}_V$, so $T_p f \circ T_v g = \mathrm{id}_{T_v M'}$, implying that $T_p f$ is surjective. $\square$

**15.3 Corollary.** *Any submersion is an open mapping.*

**Proof.** Let $f : M \to M'$ be a submersion and let $U \subseteq M$ be open. Let $m' \in f(U)$ and $m \in U$ such that $f(m) = m'$. By Proposition 15.2 there exists a local section $g : V \to M$ of $f$ around $m$ and some $v \in V$ with $g(v) = m$. Then $v = (f \circ g)(v) = f(m) = m'$. Hence $g(m') = m$, so $g^{-1}(U)$ is an open neighborhood of $m'$ and $g^{-1}(U) \subseteq f(U)$ ($p \in g^{-1}(U) \Rightarrow g(p) \in U \Rightarrow p = f \circ g(p) \in f(U)$). Consequently, $f(U)$ is open. $\square$

**15.4 Definition.** *Let $f : M \to M'$. The fibers of $f$ are the sets $f^{-1}(m')$ for $m' \in f(M)$.*

**15.5 Proposition.**

*(i) Let $f : M^m \to N^n$ be a submersion. If $m = n$ then any fiber of $f$ is discrete. If $m > n$ then any fiber of $f$ is a submanifold of $M$ of dimension $m - n$.*

*(ii) Let $f : M^m \to N^n$ be smooth and let $m > n$. If $\mathrm{rk}_p(f) = n$ for each $p$ in a fiber $S$ of $f$, then $S$ is a submanifold of $M$ of dimension $m - n$. If $p \in S$, then $T_pS = \ker(T_pf)$.*

**Proof.** (i) If $m = n$ then $f$ is a local diffeomorphism. Thus any $p \in M$ has a neighborhood that does not contain any other points of the fiber $f^{-1}(f(p))$. Consequently, the fiber is discrete. If $m > n$, the claim follows from Corollary 13.9.

(ii) This also follows from Corollary 13.9. □

**15.6 Definition.** *Let $M$ be a manifold and let $\rho$ be an equivalence relation on $M$. Denote by $M' := M/\rho$ the quotient space and by $\pi : M \to M/\rho$ the quotient map. If $M'$ is endowed with a differentiable structure such that $\pi : M \to M'$ is a submersion, then $M'$ is called a* quotient manifold *of $M$.*

**15.7 Example.** (i) Let $M := \mathbb{R}^{n+1} \setminus \{0\}$ and define $(x, y) \in \rho$ if $x = \alpha y$ for some $\alpha \in \mathbb{R} \setminus \{0\}$. Then $M/\rho = \mathbb{P}^n(\mathbb{R})$ is called *projective space* and is a quotient manifold of $\mathbb{R}^{n+1} \setminus \{0\}$.

(ii) Let $f : M \to M'$ be a surjective submersion and define $m_1 \sim_\rho m_2 :\Leftrightarrow f(m_1) = f(m_2)$. Let

$$\begin{aligned} F : M/\rho &\to M' \\ \pi(m) &\mapsto f(m). \end{aligned}$$

Then $F$ is well-defined and bijective and we define a manifold structure on $M/\rho$ by declaring $F$ to be a diffeomorphism. Then $\pi : M \to M/\rho$ is a submersion because $F \circ \pi = f$ is one, and so $M/\rho$ is a quotient manifold of $M$.

**15.8 Remark.** If $M' := M/\rho$ admits a $\mathcal{C}^\infty$-structure as a quotient manifold of $M$ then this structure is unique. Indeed, let $M'_1$ be another such structure and let $i : M' \to M'_1$ be the identity map. Also, let $\pi, \pi_1$ be the corresponding quotient maps.

$$\begin{tikzcd} & M \arrow[dl, "\pi"'] \arrow[dr, "\pi_1"] & \\ M' \arrow[rr, "i"] & & M'_1 \end{tikzcd}$$

Then $\pi_1 = i \circ \pi$, so $i$ is smooth by Remark 13.2 (iii). Analogously, $i^{-1}$ is $\mathcal{C}^\infty$, so $i$ is a diffeomorphism, i.e., $M' = M'_1$.

**15.9 Proposition.** *The natural manifold topology of a quotient manifold is the quotient topology.*

**Proof.** The quotient map $\pi : M \to M'$ is $\mathcal{C}^\infty$, hence continuous and by Corollary 15.3 also open. Let $\tau$ be the manifold topology on $M'$ and $\tau_Q$ the quotient topology.

$$\begin{tikzcd} (M', \tau_Q) \arrow[r, "\mathrm{id}"] & (M', \tau) \\ M \arrow[u, "\pi"] \arrow[ur, "\mathrm{id}\circ\pi = \pi"'] & \end{tikzcd}$$

Since $\pi = \mathrm{id} \circ \pi$ is continuous, id is continuous by the universal property of $\tau_Q$, so $\tau_Q \geq \tau$. Conversely, let $U \in \tau_Q$, then $\pi^{-1}(U)$ is open in $M$, and since $\pi$ is open

$U = \pi(\pi^{-1}(U)) \in \tau$, so also $\tau \geq \tau_Q$. □

Quotient manifolds of $T_2$ manifolds need not be $T_2$ themselves in general. The situation is better concerning second countability:

**15.10 Proposition.** *If $M$ is second countable, then so is any quotient manifold $M'$ of $M$.*

**Proof.** Let $\{B_n \mid n \in \mathbb{N}\}$ be a basis for the topology of $M$ and let $U \subseteq M'$ be open. Then $\pi^{-1}(U) = \bigcup_{k\in\mathbb{N}} B_{n_k}$ for certain $B_{n_k}$, so

$$U = \pi(\pi^{-1}(U)) = \bigcup_{k\in\mathbb{N}} \pi(B_{n_k}),$$

which, since $\pi$ is open, demonstrates that $\{\pi(B_n) \mid n \in \mathbb{N}\}$ is a basis for the topology of $M'$. □

**15.11 Remark.** Let $M/\rho$ be a quotient manifold of $M$ and $\pi : M \to M/\rho$ the quotient map. Then the fibers of $\pi$ are precisely the equivalence classes of points in $M$: $[p] = \pi^{-1}(\pi(p))$ $(p \in M)$.

**15.12 Definition.** *Let $\rho$ be an equivalence relation on a manifold $M$ and let $f : M \to N$. Then $f$ is called an* invariant *of $\rho$ if $m_1 \sim_\rho m_2 \Rightarrow f(m_1) = f(m_2)$. Therefore $f$ induces a unique map $\tilde{f}$ on $M/\rho$ such that $f = \tilde{f} \circ \pi$. $\tilde{f}$ is called the* projection *of $f$.*

**15.13 Proposition.** *Let $M' := M/\rho$ be a quotient manifold of $M$. If $f : M \to N$ is a smooth invariant, then also the corresponding projection $\tilde{f} : M' \to N$ is smooth.*

**Proof.** We have $f = \tilde{f} \circ \pi$ and $\pi$ is a submersion, so the claim follows from Remark 13.2 (iii). □

# 16 Transformation groups

**16.1 Definition.** *A* transformation *of a manifold $M$ is a diffeomorphism $M \to M$. A group $G$ acts on $M$ as a* transformation group *(on the left) if there exists a map $\Phi : G \times M \to M$ satisfying:*

*(i)* $\forall g \in G$: $\Phi_g := m \mapsto \Phi(g, m)$ *is a transformation of $M$.*

*(ii)* $\forall g, h \in G$: $\Phi_g \circ \Phi_h = \Phi_{gh}$.

*In particular, $\Phi_e = \mathrm{id}_M$.*[2] *$G$ acts* effectively *on $M$ if $\Phi_g(m) = m$ for all $m$ implies $g = e$. It acts* freely *on $M$ if $\Phi_g(m) = m$ for some $g$ and some $m$ implies $g = e$.*

**16.2 Example.** (i) $G = \mathrm{SO}(n, \mathbb{R})$ acts effectively on $\mathbb{R}^n$ via $\Phi(T, x) = Tx$ (rotations around 0). Since 0 is a fixed point, the action is not free.

(ii) $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathrm{S}(n, \mathbb{R})$ (the set of all symmetric matrices) via $\Phi : (T, A) \mapsto TAT^t$. This action is not effective because $\Phi_I = \Phi_{-I} = \mathrm{id}_{\mathrm{S}(n,\mathbb{R})}$.

Let $G$ be a transformation group on $M$ and $K := \{k \in G \mid \Phi_k = \mathrm{id}_M\}$. Then clearly $K$ is a subgroup of $G$. Let $g \in G$ and $k \in K$. Then

$$\Phi_{gkg^{-1}} = \Phi_g \circ \Phi_k \circ \Phi_{g^{-1}} = \Phi_g \circ \Phi_{g^{-1}} = \mathrm{id}_M,$$

[2] Let $m \in M$ and $m' := \Phi_e^{-1}(m)$. Then $\Phi_e(m) = \Phi_e(\Phi_e(m')) = \Phi_{ee}(m') = \Phi_e(m') = m$.

so $gkg^{-1} \in K$, i.e., $K$ is a normal subgroup of $G$.

The quotient group $G/K$ acts naturally on $M$ via

$$\begin{aligned}\tilde{\Phi} : G/K \times M &\to M \\ (gK, m) &\mapsto \Phi(g, m).\end{aligned}$$

This is well-defined because $\Phi(gk, m) = \Phi_g(\Phi_k(m)) = \Phi_g(m) = \Phi(g, m)$.

**16.3 Proposition.** *The quotient group $G/K$ acts effectively on $M$.*

**Proof.** Let $\tilde{\Phi}_{gK}(m) = \Phi(g, m) = m$ for all $m \in M$. Then $\Phi_g = \mathrm{id}_M$, so $g \in K$ and thereby $gK = eK$. □

So far we have only considered left actions. In this case one often simply writes $gm$ for $\Phi(g, m)$. Then Definition 16.1 (ii) reads $g(hm) = (gh)m$ for all $m$, $g, h$.

Analogously, we say that $G$ acts as a transformation group on the right if there exists a map $\Phi : M \times G \to M$ such that the maps $\Phi_g : m \mapsto \Phi(m, g)$ satisfy

$$\text{(ii')} \qquad \forall g, h \in G : \ \Phi_g \circ \Phi_h = \Phi_{hg}.$$

Often we simply write $mg$ for $\Phi_g(m)$. Then (ii') reads:

$$(mh)g = m(hg)$$

Now let $G$ be a transformation group acting on $M$ from the left. We call two points $m_1$, $m_2$ equivalent if there exists some $g \in G$ with $m_2 = gm_1$. This defines an equivalence relation on $M$ and we write $G\backslash M$ for the resulting quotient. We want to find out when $G\backslash M$ can be equipped with a quotient manifold structure. In general, this need not be the case:

**16.4 Example.** Let $G = (\mathbb{R} \setminus \{0\}, \cdot)$ and $\Phi : G \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $\Phi(a, x) := a \cdot x$. The equivalence classes of this action, i.e., the elements of $G\backslash\mathbb{R}^{n+1}$ then are straight lines through 0 minus the point 0, and the point 0 itself. The former are not discrete, so by Proposition 15.5 (i), if $G\backslash M$ were a quotient manifold of $M$, then $\dim(G\backslash M)$ would have to be strictly smaller than $n+1$. In this case any fiber of $\pi$, hence by Remark 15.11 any equivalence class would be a manifold of dimension $\geq 1$. This leads to a contradiction in the case of the class $\{0\}$.

In many important cases, $G\backslash M$ permits the structure of a quotient manifold. We first consider the following special case: Let $G\backslash M$ be a quotient manifold of $M$ with $\dim(G\backslash M) = \dim M$ and such that $G$ acts freely on $M$. Then the quotient map $\pi : M \to G\backslash M$ is a local diffeomorphism, hence any point $m \in M$ has a neighborhood $U$ such that $\pi|_U$ is injective. From this it follows that $U \cap gU = \varnothing$ for $g \neq e$: if $u_1 = gu_2$, then $\pi(u_1) = \pi(u_2)$, so $u_1 = u_2$, and thereby $u_1 = gu_1$, implying $g = e$ because the action is free.

**16.5 Definition.** *A transformation group $G$ that acts freely on $M$ is called* discontinuous *if*

$$\forall m \in M \ \exists \textit{neighborhood } U \textit{ of } m : \ \forall g \neq e : U \cap gU = \varnothing. \tag{16.1}$$

This condition is also sufficient:

**16.6 Proposition.** *Let $G$ be a discontinuous transformation group on $M$. Then $G\backslash M$ possesses a differentiable structure as a quotient manifold of $M$ of the same dimension as $M$.*

**Proof.** Let $U$ be an open set as in (16.1) and assume in addition that $U$ is so small that there exists a chart $\varphi$ of $M$ with domain $U$. Then $\pi$ is injective on $U$: if $u_1, u_2 \in U$ with $\pi(u_1) = \pi(u_2)$, then there exists some $g \in G$ with $gu_1 = u_2$. Since $gU \cap U \neq \varnothing$, $g = e$, so $u_1 = u_2$.

Set $\chi := (\pi|_U)^{-1}$, $\chi : V := \pi(U) \to U$. Then $\psi := \varphi \circ \chi$ is bijective, $\psi : V \to \varphi(U)$ ($\subseteq \mathbb{R}^n$, open).

$$\begin{tikzcd} M \supseteq U \arrow[d, shift right, "\pi"'] \arrow[dr, "\varphi"] & \\ G\backslash M \supseteq \pi(U) = V \arrow[u, shift right, "\chi"'] \arrow[r, "\psi"'] & \varphi(U) \end{tikzcd}$$

We are going to show that the family of all such $(\psi, V)$ forms a $\mathcal{C}^\infty$-atlas for $G\backslash M$.

Clearly the $V = \pi(U)$ form a cover of $G\backslash M$. It remains to prove that for any two such charts $(\psi_1, V_1)$, $(\psi_2, V_2)$ with $V_1 \cap V_2 \neq \varnothing$, the transition function

$$\psi_2 \circ \psi_1^{-1} = \varphi_2 \circ \chi_2 \circ \chi_1^{-1} \circ \varphi_1^{-1}$$

is a smooth map between open sets. We have $\chi_2 \circ \chi_1^{-1} : \chi_1(V_1 \cap V_2) \to \chi_2(V_1 \cap V_2)$. Now let $m_1 \in \chi_1(V_1 \cap V_2)$ and set $m_2 := \chi_2(\chi_1^{-1}(m_1))$. Then

$$\pi(m_2) = \chi_2^{-1}(m_2) = \chi_1^{-1}(m_1) = \pi(m_1),$$

so $m_1 \sim m_2$ under $G$. Since $G$ acts freely, there is a unique $g \in G$ with $m_2 = gm_1$. Consequently, $U' := U_1 \cap g^{-1}U_2$ is an open neighborhood of $m_1 \in M$ and we claim that

$$U' \subseteq \chi_1(V_1 \cap V_2). \tag{16.2}$$

Indeed, let $m' \in U'$, then $m' \in U_1 = \chi_1(V_1)$, say $m' = \chi_1(v_1)$. Then $\pi(m') = v_1$ and due to $m' \in g^{-1}U_2$ we have

$$v_1 = \pi(m') \in \pi(g^{-1}U_2) = \pi(U_2) = V_2 \Rightarrow v_1 \in V_1 \cap V_2,$$

so $\pi(m') \in V_1 \cap V_2$, and, finally, $m' \in \chi_1(V_1 \cap V_2)$. This shows that $\chi_1(V_1 \cap V_2)$ is open in $M$.

Now let $m' \in U'$. Then by (16.2), $\pi(m') \in V_1 \cap V_2$, so in particular $\pi(m') \in V_1 = \pi(U_1) \Rightarrow \chi_1^{-1}(m') = \pi(m')$. Also, $\pi : U_2 \to \pi(U_2) = V_2$ is bijective, so there is a unique $m'' \in U_2$ with $\pi(m'') = \pi(m')$. Thus $m'' = \chi_2(\chi_1^{-1}(m'))$. On the other hand, by the definition of $U'$ we have $gm' \in U_2$ and $\pi(gm') = \pi(m') = \pi(m'')$. Recalling that $\pi|_{U_2}$ is injective, we conclude that $gm' = m''$. Hence $\chi_2 \circ \chi_1^{-1}(m') = gm'$, i.e., $\chi_2 \circ \chi_1^{-1}|_{U'} = \Phi_g|_{U'}$ and in particular it is smooth near $m_1$. Since $m_1$ was arbitrary, $\chi_2 \circ \chi_1^{-1}$ is a smooth map between open sets and we obtain a smooth atlas for $G\backslash M$ of dimension $n = \dim(M)$.

Finally, let $U$ be as above, $(\varphi, U)$ a chart of $M$ and $\psi = \varphi \circ \chi$ the corresponding chart of $G\backslash M$. Then the local representation of $\pi$ with respect to these charts reads

$$\varphi \circ \chi \circ \pi \circ \varphi^{-1} = \mathrm{id}|_{\varphi(U)}.$$

Therefore, $\mathrm{rk}(\pi) = \dim G\backslash M$, so $\pi$ is a submersion. This means that $G\backslash M$ is indeed a quotient manifold of $M$. □

A quotient manifold of a $T_2$-manifold need not be $T_2$ itself. To secure the Hausdorff property, we need to add another condition on $G$:

$$m_1 \not\sim m_2 \textit{ under } G \Rightarrow \exists \textit{ nbhds } U \textit{ of } m, U' \textit{ of } m' : \; U \cap gU' = \varnothing \;\; \forall g \in G. \tag{16.3}$$

**16.7 Lemma.** *Let $G$ be a transformation group on $M$ that satisfies* (16.1) *and* (16.3)*. Then both $M$ and $G\backslash M$ are Hausdorff.*

**Proof.** Let $m \neq m'$ be points in $M$. If $\pi(m) = \pi(m')$, then $m$ and $m'$ can be openly separated as in (16.1). If, on the other hand, $m \not\sim m'$ (i.e., $\pi(m) \neq \pi(m')$) then pick $U$, $U'$ as in (16.3). Setting $g = e$ in (16.3) it follows that $U \cap U' = \varnothing$, so $M$ is $T_2$. Moreover, since $\pi$ is a submersion, $\pi(U)$ and $\pi(U')$ are open by Corollary 15.3. Since $U \cap gU' = \varnothing$ for all $g \in G$, $\pi(U) \cap \pi(U') = \varnothing$, so these sets openly separate $\pi(m)$ and $\pi(m')$. □

If a transformation group $G$ acts from the right on $M$, then in complete analogy we obtain an equivalence relation and we denote the resulting quotient space by $M/G$. If $G$ acts freely then as in Definition 16.5 we can define what it means for $G$ to act discontinuously on $M$. Also in this case, $M/G$ is a quotient manifold of $M$ of the same dimension as $M$.

# 17 Distributions and the Frobenius theorem

In this section we always assume $M$ to be Hausdorff.

**17.1 Definition.** *A $k$-dimensional (geometric) distribution on an $n$-dimensional manifold $M$ is a map $\Omega$ that assigns to any point $m \in M$ a $k$-dimensional subspace $\Omega(m)$ of $T_mM$ and that is smooth in the following sense: For each $m \in M$ there exists a neighborhood $V$ of $m$ and $X_1, \dots, X_k \in \mathfrak{X}(V)$ such that $(X_1(p), \dots, X_k(p))$ is a basis of $\Omega(p)$ for each $p \in V$. Such vector fields $X_1, \dots, X_k$ are called a local basis (or a local frame) of $\Omega$ at $m$.*

**17.2 Examples.** (i) The only $n$-dimensional distribution on $M$ is $m \mapsto T_mM$.

(ii) If $X \in \mathfrak{X}(M)$ is such that $X(m) \neq 0$ for each $m \in M$, then $\Omega : m \mapsto \operatorname{span}(X(m))$ is a 1-dimensional distribution.

**17.3 Definition.** *A chart $\varphi = (x^1, \dots, x^n)$ of $M$ is called* flat *for a distribution $\Omega$ on $M$ if the vector fields $\frac{\partial}{\partial x^i}$ $(i = 1, \dots, k)$ form a local basis for $\Omega$.*

Before we continue we need to collect some preparations concerning Lie derivatives of and flows of vector fields.

**17.4 Definition.** *Let $X \in \mathfrak{X}(M)$ and $f \in \mathcal{C}^\infty(M)$. The Lie derivative of $f$ in direction $X$ is*

$$L_X f(p) := \frac{d}{dt}\Big|_0 f(\mathrm{Fl}_t^X(p)) = \frac{d}{dt}\Big|_0 [(\mathrm{Fl}_t^X)^* f](p).$$

**17.5 Lemma.** $L_X(f) = X(f)$.

**Proof.**

$$\frac{d}{dt}\Big|_0 f \circ \mathrm{Fl}_t^X(p) = T_p f\Big(\frac{d}{dt}\Big|_0 \mathrm{Fl}_t^X(p)\Big) = T_p f(X(p)) = X(f)(p).$$

□

**17.6 Definition.** *Let $X, Y \in \mathfrak{X}(M)$. The Lie derivative of $Y$ along $X$ is the vector field*

$$L_X Y(p) := \frac{d}{dt}\Big|_0 ((\mathrm{Fl}_t^X)^* Y)(p) \qquad ((\mathrm{Fl}_t^X)^* Y = T\mathrm{Fl}_{-t}^X \circ Y \circ \mathrm{Fl}_t^X).$$

**17.7 Proposition.** *Let $X, Y \in \mathfrak{X}(M)$. Then*

*(i)* $L_X Y = [X, Y]$.

*(ii)* $\frac{d}{dt}(\mathrm{Fl}_t^X)^* Y = (\mathrm{Fl}_t^X)^* L_X Y$.

**Proof.** See [5, 2.3.13]. □

From these preparations we can conclude that the Lie bracket of two vector fields can be seen as an obstruction to the commuting of the corresponding flows. To be precise, we say that the flows of two vector fields $X, Y \in X(M)$ commute if for any $p \in M$ we have: whenever $I$ and $J$ are open intervals containing 0 such that one of the expressions $\mathrm{Fl}_t^X \circ \mathrm{Fl}_s^Y$ or $\mathrm{Fl}_s^Y \circ \mathrm{Fl}_t^X$ is defined for all $(s,t) \in I \times J$, then both are defined and are equal. With this understanding, we have:

**17.8 Corollary.** *Let $X, Y \in \mathfrak{X}(M)$. TFAE:*

*(i)* $L_X Y = [X, Y] = 0$.

*(ii)* $(\mathrm{Fl}_t^X)^* Y = Y$*, wherever the left hand side exists.*

*(iii) The flows of $X$ and $Y$ commute.*

**Proof.** See [5, 2.3.18]. □

Our next aim is to find a necessary and sufficient condition for the existence of flat charts.

**17.9 Definition.** *A local vector field $X \in \mathfrak{X}(U)$ ($U \subseteq M$ open) is said to belong to a distribution $\Omega$ if $X(m) \in \Omega(m)$ for each $m \in U$. We then write $X \in \Omega$. The distribution $\Omega$ is called* involutive *if $\forall X, Y \in \Omega$: $[X, Y] \in \Omega$.*

**17.10 Example.** Let $X_1, \dots, X_k \in \mathfrak{X}(M)$ be such that $X_1(m), \dots, X_k(m)$ are linearly independent for each $m \in M$. Then the $X_i$ form a basis for a distribution $\Omega$ which is involutive if and only if each $[X_i, X_j]$ is a $\mathcal{C}^\infty(M)$-linear combination of the $X_l$ ($i, j, l \in \{1, \dots, k\}$).

The following result gives the desired local characterization:

**17.11 Theorem.** *(Frobenius) Let $M$ be an $n$-dimensional Hausdorff manifold and $\Omega$ a $k$-dimensional distribution on $M$. Then the following are equivalent:*

*(i) $\Omega$ is involutive.*

*(ii) Around any point in $M$ there exists a chart that is flat for $\Omega$.*

**Proof.** (ii)⇒(i): Let $X \in \mathfrak{X}(V_1)$, $Y \in \mathfrak{X}(V_2)$, $X, Y \in \Omega$ and let $m \in V_1 \cap V_2$. Also, let $(\varphi = (x^1, \dots, x^n), U)$ be a flat chart for $\Omega$ around $m$ with $U \subseteq V_1 \cap V_2$. Then we can write $X|_U = \sum_{i=1}^k X(x^i)\frac{\partial}{\partial x^i}$ and $Y|_U = \sum_{i=1}^k Y(x^i)\frac{\partial}{\partial x^i}$, so

$$[X, Y]|_U = \sum_{i=1}^k (X(Y(x^i)) - Y(X(x^i)))\frac{\partial}{\partial x^i},$$

implying that $[X, Y](p) \in \Omega(p)$ for all $p \in U$. Hence $[X, Y] \in \Omega$.
(i)⇒(ii): Let $m \in M$ and let $X_1, \dots, X_k \in \mathfrak{X}(\tilde{U})$ be a local basis of $\Omega$ on an open neighborhood $\tilde{U}$ of $m$. Without loss of generality let $\tilde{U}$ be the domain of a chart

$\tilde\varphi$ of $M$ with $\tilde\varphi(m)=0$, $\tilde\varphi=(\tilde x^1,\dots,\tilde x^n)$. Let $\partial_i := \frac{\partial}{\partial \tilde x^i}$ $(1\le i\le n)$, then there exist smooth functions $f^i_j : \tilde U \to \mathbb{R}$ $(i=1,\dots,n, j=1,\dots,k)$ such that $X_j = \sum_{i=1}^n f^i_j \partial_i$.

Since $\{X_1(p),\dots,X_k(p)\}$ is linearly independent for each $p\in\tilde U$, the matrix $(f^i_j(m))_{i,j}$ has rank $k$, hence possesses $k$ linearly independent rows. Renumbering if necessary we may assume that these are the first $k$ rows. Thus $\det(f^i_j(m))^k_{i,j=1}\ne 0$, and by continuity we may shrink $\tilde U$ such that indeed $\det(f^i_j(p))^k_{i,j=1}\ne 0$ for each $p\in\tilde U$.

For $p\in\tilde U$ let $(g^i_j(p))^k_{i,j=1}$ be the matrix inverse to $(f^i_j(p))^k_{i,j=1}$. By the formula for matrix inversion, the functions $g^i_j$ are smooth on $\tilde U$. For $i=1,\dots,k$ set $Y_i := \sum_{j=1}^k g^j_i X_j$. Since $(g^i_j)^k_{i,j=1}$ is invertible, also $(Y_1,\dots,Y_k)$ is a basis of $\Omega$ on $\tilde U$. For $1\le i\le k$ we have

$$Y_i = \sum_{j=1}^k g^j_i X_j = \sum_{j=1}^k\sum_{l=1}^n g^j_i f^l_j \partial_l = \partial_i + \sum_{l=k+1}^n h^l_i \partial_l, \tag{17.1}$$

where $h^l_i \in \mathcal{C}^\infty(\tilde U)$ $(i=1,\dots,k, l=k+1,\dots,n)$.

We claim that $[Y_i,Y_j]=0$ for all $i,j\in\{1,\dots,k\}$. Indeed, since $\Omega$ is involutive and $Y_i\in\Omega$ for all $i$, $[Y_i,Y_j]\in\Omega$ for all $i,j$, so there exist $c^l_{ij}\in\mathcal{C}^\infty(\tilde U)$ such that

$$[Y_i,Y_j] = \sum_{l=1}^k c^l_{ij} Y_l \underset{(17.1)}{=} \sum_{l=1}^k c^l_{ij}\partial_l + \sum_{l=k+1}^n d^l_{ij}\partial_l \qquad (d^l_{ij}\in\mathcal{C}^\infty(\tilde U)). \tag{17.2}$$

On the other hand, by inserting for $Y_i, Y_j$ from (17.1) we obtain smooth functions $r^l_{ij}$ such that

$$[Y_i,Y_j] = \sum_{l=k+1}^n r^l_{ij}\partial_l,$$

implying that all $c^l_{ij}$ must vanish. Thus by (17.2) the $Y_i$ commute.

The vectors $T_m\tilde\varphi(Y_1(m)),\dots,T_m\tilde\varphi(Y_k(m))$ generate a $k$-dimensional subspace of $\mathbb{R}^n$. We compose $\tilde\varphi$ with a linear isomorphism (and keep denoting the resulting chart by $\tilde\varphi$) such that this subspace is transformed into $\mathbb{R}^k\times\{0\}\subseteq\mathbb{R}^n$. Now choose open neighborhoods $V$ of 0 in $\mathbb{R}^k$ and $W$ of 0 in $\mathbb{R}^{n-k}$ such that $V\times W\subseteq\tilde\varphi(\tilde U)$ and such that

$$F(t^1,\dots,t^k,a) := \mathrm{Fl}^{Y_1}_{t^1}\circ\cdots\circ\mathrm{Fl}^{Y_k}_{t^k}\circ\tilde\varphi^{-1}(0,a)$$

is defined for $(t^1,\dots,t^k)\in V$ and $a\in W$. Then since the $Y_i$ commute, by Corollary 17.8 we get $\frac{\partial}{\partial t^i}F(t,a) = Y_i(F(t,a))$ for $i=1,\dots,k$. In addition, $F(0,a)=\tilde\varphi^{-1}(0,a)$, so

$$\frac{\partial}{\partial a^j}\Big|_0 F(0,a) = T_0\tilde\varphi^{-1}(e^j) = (T_m\tilde\varphi)^{-1}(e^j)$$

($e^j$ the $j$-th unit vector, $k+1\le j\le n$). By the above we have that

$$(T_m\tilde\varphi(Y_1(m)),\dots,T_m\tilde\varphi(Y_k(m)),e^{k+1},\dots,e^n)$$

is a basis of $\mathbb{R}^n$, so $(Y_1(m),\dots,Y_k(m),(T_m\tilde\varphi)^{-1}(e^{k+1}),\dots,(T_m\tilde\varphi)^{-1}(e^n))$ is a basis of $T_mM$. This implies that $F$ is a diffeomorphism in a neighborhood of 0. Shrinking $V$ and $W$ we may achieve that $F(V\times W) := U\subseteq\tilde U$ is an open neighborhood of $m$ and $\varphi := F^{-1}|_U$ is a chart of $M$. By construction, for $\varphi=(x^1,\dots,x^n)$ we have

$$\frac{\partial}{\partial x^i}\Big|_p = T_{\varphi(p)}F(e^i) = \frac{\partial}{\partial t^i}F(\varphi(p)) = Y_i(F(\varphi(p))) = Y_i(p)$$

for each $p\in U$ and each $i=1,\dots,k$. This means that $(\varphi,U)$ is a flat chart for $\Omega$ around $m$. $\square$

**17.12 Corollary.** *(Straightening out theorem) Let $X \in \mathfrak{X}(M)$, $m \in M$ and $X(m) \neq 0$. Then there exists a chart around $m$ in which $X = \frac{\partial}{\partial x^1}$.*

**Proof.** Since $X$ is continuous there exists an open neighborhood $\tilde{U}$ of $m$ such that $X(p) \neq 0$ for each $p \in \tilde{U}$. Then $\Omega : p \mapsto \operatorname{span}(X(p))$ is a one-dimensional distribution on $\tilde{U}$ that is (trivially) involutive. Without loss of generality we may assume that there exists a chart $\tilde{\varphi}$ with domain $\tilde{U}$. Now set $Y_1 := X$ in the proof of Theorem 17.11, (i)⇒(ii). Then $F(t^1, a) = \mathrm{Fl}^X_{t^1} \circ \tilde{\varphi}^{-1}(0, a)$ is a local diffeomorphism and for $\varphi := F^{-1}$ we get $\frac{\partial}{\partial x^1} = X$. □

**17.13 Corollary.** *Let $\Omega$ be a one-dimensional distribution on $M$. Then around any point of $M$ there exists a flat chart for $\Omega$.*

**Proof.** Apply Corollary 17.12 to a local basis of $\Omega$. □

Let $M'$ be an immersive submanifold of $M$ with inclusion $j : M' \hookrightarrow M$. If $m \in M'$ then $T_m j : T_m M' \hookrightarrow T_m M$ is injective. $T_m j(T_m M')$ is called the subspace of $T_m M$ *tangential to* $M'$. A vector in $T_m M$ is called tangential to $M'$ if it lies in $T_m j(T_m M')$. A vector field $X \in \mathfrak{X}(M)$ is called tangential to $M'$ if $X(m)$ is tangential to $M'$ for each $m \in M'$.

**17.14 Proposition.** *Let $X \in \mathfrak{X}(M)$ and let $j : M' \hookrightarrow M$ be an immersion. The following are equivalent:*

*(i) $X$ is tangential to $M'$.*

*(ii) There exists some $X' \in \mathfrak{X}(M')$ such that $X'$ is $j$-related to $X$ ($X' \sim_j X$).*

**Proof.** (i)⇒(ii): For all $m \in M'$, $T_m j : T_m M' \to T_m j(T_m M')$ is bijective. We may therefore define $X' : M' \to TM'$ by $X'(m) := (T_m j)^{-1}(X(m))$. Then $X'$ is a section of $TM'$ and $T_m j \circ X'(m) = X(j(m))$ for each $m \in M'$, i.e., $X' \sim_j X$. So it only remains to show that $X'$ is smooth. To this end we apply Remark 13.2 (i) to pick charts $(\varphi = (x^1, \dots, x^n), U)$ resp. $(\psi = (y^1, \dots, y^l), V)$ around $m$ in $M$ resp. $M'$ with $V \subseteq U \cap M'$ and such that $\varphi \circ j \circ \psi^{-1} = y \mapsto (y, 0)$, i.e., $y^i = x^i \circ j$ for $1 \leq i \leq l$. Then

$$X'(y^i) = X'(x^i \circ j) = Tj \circ X'(x^i) = X(x^i) \qquad (1 \leq i \leq l),$$

so $X'(y^i) = X(x^i)|_V = X(x^i) \circ j|_V$, and since $X(x^i) \in \mathcal{C}^\infty(U)$ and $j \in \mathcal{C}^\infty(V, U)$ it follows that $X'(y^i) \in \mathcal{C}^\infty(V)$. Therefore, $X'$ is smooth.

(ii)⇒(i): For $m \in M'$ we have $X(m) = X \circ j(m) = T_m j(X'(m)) \in T_m j(T_m M')$. □

**17.15 Definition.** *Let $\Omega$ be a $k$-dimensional distribution on $M$. An immersive submanifold $M'$ of $M$ is called an* integral manifold *of $\Omega$ if, for all $m \in M'$, $\Omega(m) = T_m j(T_m M')$.*

In particular, $\dim(M') = k$ and precisely the tangent vectors over $M'$ that are tangential to $M'$ belong to $\Omega$. Let us first consider the one-dimensional case:

**17.16 Proposition.** *Let $c : I \to M$ ($I$ an open interval) be an integral curve of $X \in \mathfrak{X}(M)$ with $c'(t) \neq 0$ for each $t \in I$. Then $C := c(I)$ can be endowed with a natural manifold structure such that $C$ is a one-dimensional immersive submanifold of $M$. If $X(m) \neq 0$ for all $m \in M$, then (by Example 17.2 (ii)) it spans a distribution $\Omega$ on $M$, and $C$ is an integral manifold of $\Omega$.*

**Proof.** Let $j : C \hookrightarrow M$ be the inclusion. Since $c'(t) \neq 0$ for all $t \in I$, $c$ is an immersion. By the remark following Definition 13.7, any $t \in I$ has a neighborhood

$U \subseteq I$ such that $c : U \to M$ is an embedding (hence injective). We then define a chart of $C$ around $c(t)$ by $\varphi := c^{-1} \circ j : c(U) \to \mathbb{R}$. Let $\varphi_\alpha, \varphi_\beta$ be two such charts with $c(U_\alpha) \cap c(U_\beta) \neq \emptyset$, then we have to show that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is a smooth map between open subsets of $\mathbb{R}$. Thus let $x$ be an element of the domain of $\varphi_\beta \circ \varphi_\alpha^{-1}$ and set $y := \varphi_\beta \circ \varphi_\alpha^{-1}(x)$. Then $c(x) = c(y) =: m$. Both curves $t \mapsto c(x+t)$ and $t \mapsto c(y+t)$ are integral curves of $X$ that start at $m$, so they must agree for small values of $|t|$. Setting $t := s - x$, it follows that for $s$ near $x$ we have $\varphi_\alpha^{-1}(s) = c(s) = c(x + (s-x)) = c(y + (s-x)) = \varphi_\beta^{-1}(s + y - x)$, i.e., $\varphi_\beta \circ \varphi_\alpha^{-1} = s \mapsto s + y - x$ near $x$. Hence $\varphi_\beta \circ \varphi_\alpha^{-1}$ is defined on a neighborhood of $x$ and is smooth. So indeed we obtain a $\mathcal{C}^\infty$-atlas for $C$.

Now let $m \in C$ and let $(\varphi, c(U))$ be a chart of $C$ around $m$. Then $c \circ \varphi = j$ on $c(U)$, hence $j : C \to M$ is smooth. Moreover, for $m = c(t)$ we have

$$T_m j \left( \frac{\partial}{\partial x}\Big|_m \right) = T_t c \circ T_m \varphi \left( \frac{\partial}{\partial x}\Big|_m \right) = T_t c \left( \frac{\partial}{\partial t} \right) = c'(t) = X(c(t)) = X(m).$$

Since $X(m) \neq 0$, $j$ is an immersion, and our claim follows. □

**17.17 Remark.** Our next aim is to show that any involutive distribution possesses integral manifolds. To this end we define a new $\mathcal{C}^\infty$-structure on $M$ that turns $M$ into a $k$-dimensional manifold, where $k = \dim \Omega$. Suppose first that $k < n = \dim M$. For $p \in M$, by Theorem 17.11 there exists a flat chart $\varphi : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ around $p$. Let $a := \mathrm{pr}_2(\varphi(p))$ and $U_a := \varphi^{-1}(\mathbb{R}^k \times \{a\})$. Then $u := \mathrm{pr}_1 \circ \varphi|_{U_a} : U_a \to \mathbb{R}^k$ is injective and because $\varphi(U_a) = \varphi(U) \cap (\mathbb{R}^k \times \{a\})$ it follows that $u(U_a)$ is open in $\mathbb{R}^k$. Thus $(u, U_a)$ is a $k$-dimensional chart of the *set* $M$ and it remains to verify that the chart transition functions are smooth.

So let $(v, V_b)$ be another such chart, defined via a flat chart $(\psi = (y^1, \dots, y^n), V)$ around $q \in M$ with $b = \mathrm{pr}_2(\psi(q))$ and let $U_a \cap V_b \neq \emptyset$. Picking any $m \in U_a \cap V_b$, we have to show that $u \circ v^{-1}$ is defined on an open neighborhood of $v(m)$ and is smooth there.

Both $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial y^i}$ $(1 \leq i \leq k)$ are bases of $\Omega$ on $U \cap V$. Consequently,

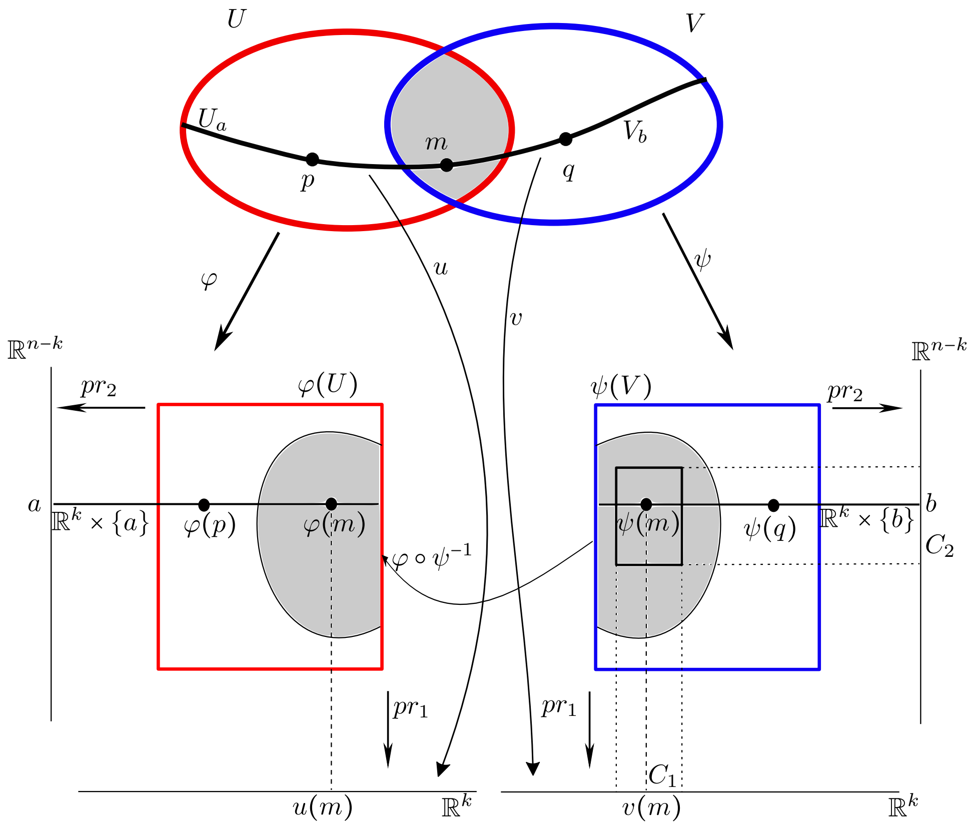
$$\frac{\partial}{\partial y^i} = \sum_{j=1}^n \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j} \in \mathrm{span}\left( \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k} \right),$$

implying that $\frac{\partial x^j}{\partial y^i} = 0$ for $i = 1, \dots k$, $j = k+1, \dots, n$. Let $\chi := \varphi \circ \psi^{-1}$. Then

$$\partial_i \chi^j = 0 \qquad i = 1, \dots k, \; j = k+1, \dots, n. \tag{17.3}$$

Let $C_1 \subseteq \mathbb{R}^k$, $C_2 \subseteq \mathbb{R}^{n-k}$ be connected open neighborhoods of $v(m)$ and $b$, respectively, such that $C_1 \times C_2$ is contained in the domain of $\chi$, see the following figure.[3]

[3] I am greatly indebted to Sarah Irène Ampe for supplying me with this picture.




We then have:

(i) $\mathrm{pr}_1(\psi(m)) = v(m)$ and $\mathrm{pr}_2(\psi(m)) = b$ (since $m \in V_b$). Therefore $\psi(m) = (v(m), b)$ and we conclude that, for each $z \in C_1$,

$$\mathrm{pr}_2 \circ \chi(z,b) \underset{(17.3)}{=} \mathrm{pr}_2 \circ \chi(v(m),b) = \mathrm{pr}_2 \circ \varphi \circ \psi^{-1}(v(m),b) = \mathrm{pr}_2(\varphi(m)) = a.$$

(ii) $C_1 \subseteq \mathrm{domain}(u \circ v^{-1}) = v(U_a \cap V_b)$.

*Proof.* Let $z \in C_1$. Then $v(\psi^{-1}(z,b)) = \mathrm{pr}_1 \circ \psi(\psi^{-1}(z,b)) = z$. Hence $v^{-1}(z) = \psi^{-1}(z,b) \in V_b$, because $\mathrm{pr}_2 \circ \psi(\psi^{-1}(z,b)) = b$. Moreover, $\psi^{-1}(z,b) \in U_a$ since

$$\mathrm{pr}_2 \circ \varphi(\psi^{-1}(z,b)) = \mathrm{pr}_2 \circ \chi(z,b) \underset{(i)}{=} a,$$

implying that $\psi^{-1}(z,b) \in U_a \cap V_b$, so $z \in v(U_a \cap V_b)$. □

(iii) $u \circ v^{-1}|_{C_1} = \mathrm{pr}_1 \circ \chi(\,.\,,b)$, hence is $\mathcal{C}^\infty$.

*Proof.* Let $z \in C_1$. Then by (ii) it follows that $z \in v(V_b)$, so $v^{-1}(z) = \psi^{-1}(z,b)$. Thus

$$u \circ v^{-1}(z) = \mathrm{pr}_1 \circ \varphi \circ \psi^{-1}(z,b) = \mathrm{pr}_1 \circ \chi(z,b).$$

□

Altogether, we have shown that $u \circ v^{-1}$ is smooth on its open domain of definition. Thus we obtain a $k$-dimensional atlas. We denote the resulting $k$-dimensional manifold (with the same underlying set $M$) by $M(\Omega)$.

Finally, if $k = n$ we set $M(\Omega) := M$.

**17.18 Example.** Let $\Omega$ be the distribution on $\mathbb{R}^2$ that is generated by $\frac{\partial}{\partial x^1}$. Then the global chart id is flat for $\Omega$. The corresponding chart for $\mathbb{R}^2(\Omega)$ around a point $(a,b)$ is then given by $u = \mathrm{pr}_1|_{\mathbb{R}\times\{b\}}$ (cf. Example 14.3).

**17.19 Proposition.** *Let $\Omega$ be an involutive distribution on $M$. Then $M(\Omega)$ is an integral manifold of $\Omega$.*

**Proof.** For $k = \dim M = n$ this holds by definition, so let $k < n$. By Theorem 17.11, for any $m \in M$ we can pick a flat chart $\varphi = (x^1, \dots, x^n)$ for $\Omega$ around $m$. Let $(u, U_a)$ be the corresponding chart around $m$ in $M(\Omega)$ (according to Remark 17.17), where $a = \mathrm{pr}_2(\varphi(m))$. If $j : M(\Omega) \to M$ is the identity map, then since $u = \mathrm{pr}_1 \circ \varphi|_{U_a}$, on $U_a$ we have:

$$x^i \circ j = \begin{cases} u^i & (1 \le i \le k) \\ a^{i-k} & (k+1 \le i \le n). \end{cases} \tag{17.4}$$

With respect to the charts $\varphi$ and $u$, $j$ therefore has the local representation $\varphi \circ j \circ u^{-1} = x \mapsto (x, a)$ and consequently is an immersion. Hence $M(\Omega)$ is an immersive submanifold of $M$. Finally, for $1 \le i \le k$ we have by ([5, (2.1.5)] and) (17.4):

$$T_m j\Big(\frac{\partial}{\partial u^i}\Big|_m\Big) = \sum_{s=1}^{n} \frac{\partial(x^s \circ j)}{\partial u^i}\Big|_m \frac{\partial}{\partial x^s}\Big|_m = \frac{\partial}{\partial x^i}\Big|_m,$$

showing that $M(\Omega)$ is an integral manifold of $\Omega$. □

**17.20 Proposition.** *Let $\Omega$ be an involutive distribution on $M$ and let $M'$ be a subset of $M$. The following are equivalent:*

*(i) $M'$ is an integral manifold of $\Omega$.*

*(ii) $M'$ is an open submanifold of $M(\Omega)$.*

**Proof.** Assume first that $k = n := \dim M$. Then (i) implies that $M'$ is an immersive submanifold of $M$ of the same dimension as $M$, so by Proposition 14.1 it is an open submanifold of $M = M(\Omega)$. Conversely, if (ii) holds and $m \in M$, then with $j : M' \hookrightarrow M$ we have that $T_m j : T_m M' \to T_m M = T_m M(\Omega)$ is an isomorphism, so $T_m j(T_m M') = T_m M = \Omega(m)$.

Thus from now on we assume that $k < n$.

(i)⇒(ii): Since $\dim M' = k = \dim M(\Omega)$, by Proposition 14.1 it suffices to show that $M'$ is an immersive submanifold of $M(\Omega)$, i.e., that for each $m \in M'$ we have that $j' : M' \hookrightarrow M(\Omega)$ is smooth and of rank $k$. So let $m \in M'$ and let $(\varphi = (x^1, \dots, x^n), U)$ be a chart around $m$ in $M$ that is flat for $\Omega$ (Theorem 17.11). $M'$ is an immersive submanifold of $M$, so $j : M' \hookrightarrow M$ is smooth, hence in particular continuous. Consequently, there exists a chart $(w, W)$ of $M'$ around $m$ with $W = j(W) \subseteq U$ and such that $W$ is connected. By (i), for each $p \in W$,

$$T_p j\bigg(\mathrm{span}\Big(\frac{\partial}{\partial w^1}\Big|_p, \dots, \frac{\partial}{\partial w^k}\Big|_p\Big)\bigg) = \Omega(p) = \mathrm{span}\Big(\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^k}\Big|_p\Big).$$

Since $T_p j\big(\frac{\partial}{\partial w^i}\big|_p\big) = \sum_{s=1}^n \frac{\partial(x^s \circ j)}{\partial w^i}\big|_p \frac{\partial}{\partial x^s}\big|_p$ it follows that

$$\frac{\partial(x^s \circ j)}{\partial w^i} = 0 \ \text{ on } W \text{ for } 1 \le i \le k,\ k+1 \le s \le n. \tag{17.5}$$

Since $W$ is connected, it follows that $x^s \circ j$ is constant on $W$ $(k+1 \le s \le n)$. Choosing $a$ such that $x^s \circ j \equiv a^{s-k}$ on $W$, this gives $j(W) \subseteq U_a$, where $(u, U_a)$ is a chart of $M(\Omega)$ as above. Thus

$$\mathrm{pr}_1 \circ \varphi \circ j|_W = (\mathrm{pr}_1 \circ \varphi)|_{U_a} \circ j|_W = u \circ j'|_W. \tag{17.6}$$

This shows that $u \circ j'$ is smooth on $W$, implying that $j' : M' \hookrightarrow M(\Omega)$ is $\mathcal{C}^\infty$ around $m$. Finally, since $\dim(M') = k$, we obtain

$$k = \mathrm{rk}\left(\frac{\partial(x^s \circ j)}{\partial w^i}\right)_{\substack{s=1,\dots,n\\ i=1,\dots,k}} \underset{(17.5)}{=} \mathrm{rk}\left(\frac{\partial(x^s \circ j)}{\partial w^i}\right)_{\substack{s=1,\dots,k\\ i=1,\dots,k}} \underset{(17.6)}{=} \mathrm{rk}\left(\frac{\partial(u^s \circ j')}{\partial w^i}\right)_{\substack{s=1,\dots,k\\ i=1,\dots,k}},$$

so $\mathrm{rk}(j') = k$.

(ii)⇒(i): In this case, $j'$ is a diffeomorphism onto its image, so $T_m j'(T_m M') = T_m(M(\Omega))$ for each $m \in M'$. Now for $\mathrm{id} : M(\Omega) \to M$, by Proposition 17.19 we have $T_m\mathrm{id}(T_m(M(\Omega))) = \Omega(m)$.

$$\begin{tikzcd} M' \arrow[r, "j"] \arrow[d, "j'"'] & M \\ M(\Omega) \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

Altogether, $j = \mathrm{id} \circ j' : M' \to M$ is an immersion and $T_m j(T_m M') = \Omega(m)$, i.e., $M'$ is an integral manifold of $\Omega$. □

**17.21 Definition.** *A distribution $\Omega$ on $M$ is called* integrable *if each point $m \in M$ is contained in an integral manifold of $\Omega$.*

**17.22 Theorem.** *Let $\Omega$ be a distribution on $M$. The following are equivalent:*

*(i) $\Omega$ is involutive.*

*(ii) $\Omega$ is integrable.*

**Proof.** (i)⇒(ii): This is immediate from Proposition 17.19.

(ii)⇒(i): Let $X, Y \in \Omega$ be local vector fields and let $m$ lie in the domain of $[X, Y]$. Let $M'$ be an integral manifold of $\Omega$ through $m$. Then both $X$ and $Y$ are tangential to $M'$ (near $m$). Hence by Proposition 17.14 there exist local vector fields $X', Y'$ on $M'$ that are $j$-related to $X$ and $Y$, respectively. Then by Lemma 4.4,

$$[X', Y'] \sim_j [X, Y] \Rightarrow [X, Y](m) = T_m j([X', Y'](m)) \in \Omega(m)$$

since $M'$ is an integral manifold of $\Omega$. Thus $[X, Y] \in \Omega$, so $\Omega$ is involutive. □

**17.23 Definition.** *Let $\Omega$ be an integrable distribution on $M$. If $m \in M$, we call the connected component of $m$ in $M(\Omega)$ the* leaf *of $\Omega$ through $m$.*

Since $M(\Omega)$ is a manifold, the leaf $L_m$ through $m$ is open in $M(\Omega)$. The collection of all leaves of $\Omega$ is a so-called *foliation* of $M$.

**17.24 Remark.** The leaf $L_m$ through $m$ is the maximal connected integral manifold containing $m$. Indeed, since $L_m$ is open in $M(\Omega)$, it is an integral manifold of $\Omega$ by Proposition 17.20. If $L' \supseteq L_m$ is another connected integral manifold of $\Omega$, then again by Proposition 17.20 $L'$ is also an open submanifold of $M(\Omega)$ and therefore carries the trace topology of $M(\Omega)$. Consequently, $L_m = L'$.

**17.25 Examples.** (i) Let $X$ be a nowhere vanishing vector field on $M$ and let $\Omega$ be the distribution generated by $X$. If $c : I \to M$ is a maximal integral curve of $X$ then by Proposition 17.16, $C = c(I)$ is an integral manifold of $\Omega$. We now show that these $C$ are precisely the leaves of $\Omega$. Let $L$ be a leaf of $\Omega$ and let $m \in L$. Let $C_m$ be the image of the maximal integral curve of $X$ through $m$. By the above and Proposition 17.20, $C_m$ is an open (and connected) submanifold of $M(\Omega)$ and since $L$ is the leaf through $m$, we have $C_m \subseteq L$. It follows that $C_m$ is open in $L$ and $L = \bigcup_{p \in L} C_p$. If $m, m' \in L$, then either $C_m = C_{m'}$ or $C_m \cap C_{m'} = \emptyset$. Since $L$ is connected we conclude that $L = C_m$.

(ii) Let $\dim(M) = n$, $\dim(M') = n - k$ $(0 < k < n)$ and $f : M \to M'$ a submersion. Then $\Omega : m \mapsto \ker T_m f$ is an integrable distribution: for any $m \in M$, by Remark 13.2 (i) there exist charts $\psi = (y^1, \dots y^{n-k})$ around $f(m)$ in $M'$ and $\varphi = (x^1, \dots, x^n)$ around $m$ in $M$ such that $y^{s-k} \circ f = x^s$ for $k + 1 \leq s \leq n$. Hence $T_p f\left(\frac{\partial}{\partial x^i}\big|_p\right) = 0$ for $1 \leq i \leq k$, so $\left\{\frac{\partial}{\partial x^i} \mid 1 \leq i \leq k\right\}$ forms a basis for $\Omega$ near $m$, i.e., $\varphi$ is flat for $\Omega$.

By Proposition 15.5 the fiber $F_m := f^{-1}(f(m))$ is a $k$-dimensional submanifold of $M$. With $j : F_m \hookrightarrow M$ the inclusion, Proposition 15.5 (ii) shows that

$$T_m j(T_m F_m) = \ker T_m f = \Omega(m),$$

so any fiber of $f$ is an integral manifold of $\Omega$, hence by Proposition 17.20 is open in $M(\Omega)$. Any two fibers are either disjoint or they coincide ($p \in F_m \cap F_{m'} \Rightarrow f(p) = f(m) = f(m') \Rightarrow F_m = F_{m'}$), so the fibers form a disjoint open cover of $M(\Omega)$. Since the leaves are connected open subsets of $M(\Omega)$, any leaf must lie entirely within some fiber of $f$.

If $f : M_1 \to M$ is smooth and $f(M_1) \subseteq M'$, where $M'$ is a submanifold of $M$, then also $f : M_1 \to M'$ is smooth (see Remark 13.5). However, if $M'$ is merely an immersive submanifold of $M$ then this need no longer be the case. Nevertheless we have:

**17.26 Proposition.** *Let $f : M_1 \to M$ be smooth and $f(M_1) \subseteq M'$, where $M'$ is an immersive submanifold of $M$. If $M'$ is an integral manifold of an integrable distribution $\Omega$ on $M$ and if $M'$ is second countable, then also the induced map $f : M_1 \to M'$ is smooth.*

**Proof.** We write $f'$ for the induced map $f' : M_1 \to M'$. If $\dim \Omega = n$, then $M'$ is open in $M$ by Proposition 17.20, so the claim follows from Remark 13.5. We may therefore suppose that $\dim \Omega = k < n$.

Let $m_1 \in M_1$ and let $(\varphi = (x^1, \dots, x^n), U)$ be a chart of $M$ around $m := f(m_1)$ that is flat for $\Omega$. Since $f$ is continuous there exists a connected neighborhood $W$ of $m_1$ in $M_1$ such that $f(W) \subseteq U$. Let $c \in \mathbb{R}^{n-k}$ and set $U_c := \varphi^{-1}(\mathbb{R}^k \times \{c\})$. By Remark 17.17, such sets form chart neighborhoods of $M(\Omega)$ and for $c_1 \neq c_2$ we have $U_{c_1} \cap U_{c_2} = \emptyset$. By Proposition 17.20, $M'$ is open in $M(\Omega)$, hence $\{M' \cap U_c \mid c \in \mathbb{R}^{n-k}\}$ is a family of disjoint open sets in $M'$. Since $M'$ is second countable, at most countably many $M' \cap U_c$ are non-empty, say $\{M' \cap U_{c_i} \mid i \in \mathbb{N}\}$. Since $U = \bigcup_{c \in \mathbb{R}^{n-k}} U_c$ we have

$$U \cap M' = \bigcup_{c \in \mathbb{R}^{n-k}} U_c \cap M' = \bigcup_{i \in \mathbb{N}} U_{c_i} \cap M'.$$

Since $f(W) \subseteq U \cap M'$ and $x^s|_{U_c} \equiv c^s$ for $k + 1 \leq s \leq n$, it follows that $x^s \circ f(W) \subseteq \mathbb{R}$ is at most countable for these values of $s$. However, $x^s \circ f$ is continuous and $W$ is connected, so $x^s \circ f(W)$ is connected as well, and thereby can only consist of a single point in $\mathbb{R}$. Thus $f(W) \subseteq U_a$, where $a = \mathrm{pr}_2(\varphi(m))$, leading to

$$u^i \circ j' \circ f'|_W = x^i \circ f|_W \qquad (i = 1, \dots, k),$$

where, as in Remark 17.17, $u := \mathrm{pr}_1 \circ \varphi|_{U_a}$ and $j' : M' \hookrightarrow M(\Omega)$. Since $u$ is a chart of $M(\Omega)$, $u \circ j'$ is a chart of the open submanifold $M'$ of $M(\Omega)$. Furthermore, $x^i \circ f$ is smooth on $W$, so also $u^i \circ j' \circ f'$ is smooth near $m_1$, implying that $f'$ is smooth near $m_1$. □

In Example 17.25 we have seen that any submersion $M \to M'$ generates a distribution on $M$. For applications to Lie groups we need to find a characterization of those distributions that come about in this way.

Let $\Omega$ be a $k$-dimensional integrable distribution on an $n$-dimensional manifold $M$ and let $k < n$. As in Remark 17.17 let $(\varphi = (x^1, \dots, x^n), U)$ be a flat chart for $\Omega$ and set, for $a \in \mathrm{pr}_2(\varphi(U))$, $U_a := \varphi^{-1}(\mathbb{R}^k \times \{a\})$. The set $U_a$ is called a *slice* (or *plaque*) of $\varphi$. Being a chart domain in $M(\Omega)$, $U_a$ is an open submanifold of $M(\Omega)$, hence by Proposition 17.20 an integral manifold of $\Omega$. Any connected component of a slice is an open and connected submanifold of $M(\Omega)$, hence is entirely contained in a leaf of $\Omega$. A leaf $L$ of $\Omega$ may intersect $U$ in one or several slices.

**17.27 Definition.** *A flat chart $(\varphi, U)$ for $\Omega$ is called* regular *if for every leaf $L$ of $\Omega$ with $L \cap U \neq \varnothing$ there exists a unique $a \in \mathbb{R}^{n-k}$ with $U \cap L = U_a$. A distribution $\Omega$ is called regular if any point of $M$ lies in the domain of a chart that is regular for $\Omega$.*

**17.28 Proposition.** *Let $f : M \to M'$ be a submersion,* $\dim M = n$, $\dim M' = n - k$, $0 < k < n$. *Then the $k$-dimensional distribution $\Omega : m \mapsto \ker T_m f$ is regular.*

**Proof.** By Example 17.25 (ii), $\Omega$ is a $k$-dimensional integrable distribution, so it only remains to show that any $m \in M$ lies in the domain of a regular chart for $\Omega$. Let $(\varphi = (x^1, \dots, x^n), U)$ be a chart around $m$ in $M$ and $(\psi = (y^1, \dots, y^{n-k}), V)$ a chart around $f(m)$ in $M'$ such that $y^{s-k} \circ f = x^s$ for $k + 1 \le s \le n$. We have seen in Example 17.25 that $(\varphi, U)$ is flat for $\Omega$ and we now proceed to show that it indeed can be chosen to be regular for $\Omega$. Without loss of generality let $\varphi(U)$ be a cube in $\mathbb{R}^n$. Then any slice $U_a$ of $(\varphi, U)$ is a connected open subset of $M(\Omega)$ (indeed a chart domain), so $U_a$ lies entirely within one leaf of $\Omega$ (recall the discussion preceding Definition 17.27).$(*)$

We have $f(U_a) = f \circ \varphi^{-1}(\mathbb{R}^k \times \{a\}) = \psi^{-1}(a)$. Thus if $a \neq b$ then $U_a$ and $U_b$ lie in different fibers (namely $f^{-1}(\psi^{-1}(a))$ and $f^{-1}(\psi^{-1}(b))$) of $f$. From Example 17.25 (ii) we know that any leaf lies within one fiber, so any leaf can at most contain one slice.$(**)$

Now let $L$ be a leaf that intersects $U$ and let $m \in L \cap U$. Choose $a$ such that $m \in U_a$. By $(*)$, $U_a$ is contained in some leaf $L'$, so $m \in L' \cap L$, implying $L' = L$, and thereby $U_a \subseteq L$. Due to $U = \bigcup_{b \in \mathbb{R}^{n-k}} U_b$ and $(**)$ we finally obtain that $U \cap L = U_a$, so $(\varphi, U)$ is regular. □

We now want to show that, conversely, any regular distribution $\Omega$ generates a submersion whose fibers are precisely the leaves of $\Omega$ (hence are in particular connected). To this end, let $\Omega$ be an integrable distribution and define a map $\omega$ by

$$m \mapsto \text{leaf of } \Omega \text{ through } m.$$

Then $\omega : M \to M' := \{L \mid L \text{ is a leaf of } \Omega\}$. The fibers of $\omega$ then are the sets $\omega^{-1}(\omega(m)) = \{p \mid \text{leaf of } p = \text{ leaf of } m\} = \{p \mid p \in \omega(m)\}$, which is precisely the leaf of $\Omega$ through $m$. In this situation we have:

**17.29 Proposition.** *Let $\Omega$ be a regular distribution. Then $M'$ can be endowed with a $\mathcal{C}^\infty$-structure with respect to which $\omega : M \to M'$ is a submersion. If $k = \dim \Omega$, then $\dim M' = n - k$.*

**Proof.**[4] Let $(\varphi = (x^1, \dots, x^n), U)$ be a regular chart and let $U' := \omega(U)$. Since $\varphi$ is regular, any leaf that intersects $U$ does so in some $U_a$. Hence the map $\varphi' : U' \to$

[4]It is recommended to look at the picture from Remark 17.17 when studying this proof

$\mathbb{R}^{n-k}$, $m' \mapsto \mathrm{pr}_2(\varphi(m))$ $(= a)$, where $m$ is an arbitrary element of $\omega^{-1}(m') \cap U$, is well-defined. $\varphi'$ is injective: Let $\varphi'(m'_1) = \varphi'(m'_2)$, then $\mathrm{pr}_2(\varphi(m_1)) = \mathrm{pr}_2(\varphi(m_2))$, with $m_i \in \omega^{-1}(m'_i) \cap U_{a_i}$ $(i = 1, 2)$. Thus $a_1 = a_2$, and so $\omega^{-1}(m'_1) \cap U = U_{a_1} = U_{a_2} = \omega^{-1}(m'_2) \cap U$. Hence $\omega^{-1}(m'_1) \cap \omega^{-1}(m'_2) \neq \varnothing$. As two leaves with non-empty intersection conincide, this gives $\omega^{-1}(m'_1) = \omega^{-1}(m'_2)$, and thereby $m'_1 = m'_2$. Also, $\varphi'(U') = \mathrm{pr}_2(\varphi(U))$ is open in $\mathbb{R}^{n-k}$, so $(\varphi', U')$ is a chart for $M'$.

We have to show that the corresponding chart transition functions are diffeomorphisms around any $m'$ in the intersection of their domains. We do this first in the following special case: Let $(\varphi, U)$ as above, let $(\psi = (y^1, \dots, y^n), V)$ be another regular chart, and suppose that

$$U \cap V \cap L \neq \varnothing, \tag{17.7}$$

where $L$ is the leaf $\omega^{-1}(m')$.[5] Pick $m \in U \cap V \cap L$ (so $m' = \omega(m) \in U' \cap V'$) and set $\psi(m) =: (b, \psi'(m'))$ for a suitable $b \in \mathbb{R}^k$.

Let $C_1, C_2$ be open in $\mathbb{R}^k$ resp. $\mathbb{R}^{n-k}$ such that $\psi(m) \in C_1 \times C_2 \subseteq \psi(U \cap V)$. Let $w \in C_2$ and set $p := \psi^{-1}(b, w)$. Then $p \in \omega^{-1}({\psi'}^{-1}(w))$ (because $\psi'(\omega(\psi^{-1}(b, w))) = w$), so by definition of $\varphi'$ we obtain

$$\varphi'({\psi'}^{-1}(w)) = \mathrm{pr}_2(\varphi(p)) = \mathrm{pr}_2 \circ \varphi \circ \psi^{-1}(b, w).$$

This implies that $\varphi' \circ {\psi'}^{-1}$ is smooth on $C_2$, i.e., near $\psi'(m')$. By symmetry, also $\psi' \circ {\varphi'}^{-1}$ is smooth near $\varphi'(m')$.

In the general case, not assuming (17.7), let $m' \in U' \cap V'$ and set $L := \omega^{-1}(m')$. Then $m' \in U'$ for some $(\varphi, U)$ if and only if $U \cap L \neq \varnothing$. Let

$$S := \{(\varphi, U) \text{ regular chart } \mid U \cap L \neq \varnothing\}.$$

We call $(\varphi, U)$, $(\psi, V) \in S$ equivalent if $\varphi' \circ {\psi'}^{-1}$ is a local diffeomorphism near $\psi'(m')$. This defines an equivalence relation on $S$. If $S'$ is an equivalence class of this relation, then let

$$\nu(S') := \bigcup\{U \cap L \mid (\varphi, U) \in S'\}.$$

It follows that $\nu(S') \neq \varnothing$ and it is an open subset of the integral manifold $L$ of $\Omega$. If $\nu(S') \cap \nu(S'') \neq \varnothing$, then there exist charts $(\varphi, U) \in S'$, $(\psi, V) \in S''$ with $U \cap V \cap L \neq \varnothing$, i.e., (17.7) is satisfied. By what we have shown in the special case above, $(\varphi, U) \sim (\psi, V)$, so $S' = S''$ and therefore $\nu(S') = \nu(S'')$. Consequently, the $\nu(S')$ form a disjoint open cover of $L$. But $L$ is connected, so there can be only one $\nu(S')$ (and $\nu(S') = L$). This means there is only one equivalence class, i.e., any two elements of $S$ are equivalent. It follows that for any two charts $\varphi, \psi$ from $S$ whose domains intersect the transition function $\varphi' \circ {\psi'}^{-1}$ is smooth near $\psi'(m')$, verifying that we indeed obtain a $\mathcal{C}^\infty$-atlas.

Finally, $\omega$ is a submersion: Let $m \in M$ and let $(\varphi = (x^1, \dots, x^n), U)$ be a regular chart for $\Omega$ around $m$. Let $\varphi'$ be the corresponding chart for $M'$. Then by definition of $\varphi'$ we have $\varphi' \circ \omega = \mathrm{pr}_2 \circ \varphi$, so $\omega$ is smooth and $\mathrm{rk}(\omega) = n - k$. □

**17.30 Corollary.** *The leaves of a regular distribution are submanifolds of $M$.*

**Proof.** By Proposition 17.29 and the remark preceding it, these leaves are precisely the fibers of the submersion $\omega$. Hence the claim follows from Proposition 15.5 (i). □

[5] Note that $U' \cap V' \neq \varnothing$ could also be true if $U \cap V = \varnothing$.

**17.31 Example.** Let $\Omega$ be a $k$-dimensional integrable distribution on $M$ and suppose that $k < n = \dim M$. If a leaf $L$ of $\Omega$ is dense in $M$ then there do not exist any regular charts for $\Omega$: Let $(\varphi = (x^1, \dots, x^n), U)$ be flat for $\Omega$. Since $\bar{L} = M$, $U \cap L \neq \varnothing$. So let $m \in L \cap U$ and let $U_a$ be the slice of $\varphi$ through $m$. Because $k < n$ there exists some $V \subseteq U$ open with $V \cap U_a = \varnothing$. Again by denseness of $L$ we also have $V \cap L \neq \varnothing$. Thus $(U \setminus U_a) \cap L \neq \varnothing$, so that $(\varphi, U)$ is not regular.

To conclude this section we combine some of the above results into a general version of the Frobenius theorem:

**17.32 Theorem.** *(Frobenius) Let $M$ be an $n$-dimensional $T2$-manifold and let $\Omega$ be a $k$-dimensional distribution on $M$. Then the following are equivalent:*

*(i)* *$\Omega$ is involutive.*

*(ii)* *Around any point of $M$ there exists a chart that is flat for $\Omega$.*

*(iii)* *$\Omega$ is integrable.*

*(iv)* *Any point $m \in M$ lies in the domain of a chart $(\varphi = (x^1, \dots, x^n))$ centered at $m$ that is cubic ($\varphi(U) = (-c, c)^n$ for some $c > 0$) such that the slices $U_a = \varphi^{-1}(\mathbb{R}^k \times \{a\})$ are integral manifolds of $\Omega$. If $M'$ is a connected integral manifold of $\Omega$ with $M' \subseteq U$ then $M'$ lies entirely in one such slice.*

**Proof.** (i)$\Leftrightarrow$(ii): See Theorem 17.11.

(i)$\Leftrightarrow$(iii): This is Theorem 17.22.

(iv)$\Rightarrow$(iii) is clear.

(ii)$\Rightarrow$(iv): Let $(\varphi, U)$ be flat for $\Omega$, centered at $m$, and cubic (without loss of generality). Then the $U_a$ are chart domains, hence open submanifolds of $M(\Omega)$ and thereby integral manifolds of $\Omega$ by Proposition 17.20. Finally, let $M'$ be a connected integral manifold of $\Omega$ with $M' \subseteq U$. Then $M'$ is open in $M(\Omega)$ by Proposition 17.20 again and (due to $U_a \cap U_{a'} = \varnothing$ for $a \neq a'$):

$$M' = M' \cap U = \dot{\bigcup_{a \in \mathbb{R}^{n-k}}} (M' \cap U_a).$$

The $M' \cap U_a$ thereby form an open (in $M(\Omega)$) partition of $M'$, and since $M'$ is connected there is a unique $a \in \mathbb{R}^{n-k}$ with $M' \cap U_a \neq \varnothing$. Thus $M' = M' \cap U_a$, i.e., $M' \subseteq U_a$. □

# 18 Lie subgroups

**18.1 Definition.** *A Lie group $H$ that is a subgroup of the Lie group $G$ and an immersive submanifold of $G$ is called a* Lie subgroup *of $G$.*

**18.2 Proposition.** *If $H$ is a subgroup of $G$ that is a submanifold of $G$, then $H$ is a Lie subgroup of $G$.*

**Proof.** We need to show that $H$ is a Lie group. Let $j : H \hookrightarrow G$ be the inclusion. Then $\mu \circ (j \times j) : H \times H \to G$ is smooth and takes values in $H$. By Remark 13.5 (i) therefore also $\mu_H = \mu \circ (j \times j) : H \times H \to H$ is smooth, giving the claim. □

A Lie subgroup $H$ is called connected if it has this property when endowed with its natural manifold topology.

**18.3 Proposition.** *The connected component $G_e$ of $e$ in $G$ is a connected Lie subgroup of $G$.*

**Proof.** By Remark 2.2 $G_e$ is a connected subgroup of $G$ that is open in $G$, hence is an open submanifold of $G$. The result therefore follows from Proposition 18.2. □

**18.4 Proposition.** *Any connected component of a Lie group $G$ is an open submanifold that is second countable. Any two such connected components are diffeomorphic.*

**Proof.** By Remark 2.2, the connected components of $G$ are precisely the sets $L_g(G_e)$. Hence they are open submanifolds of $G$. Moreover, $L_g : G_e \to L_g(G_e)$ is smooth by Remark 13.5 (i), and analogously for $L_{g^{-1}} : L_g(G_e) \to G_e$. By Proposition 2.8, $G_e$ and therefore also $L_g(G_e)$ is second countable. □

The following result provides us with a rich source of examples for Lie subgroups:

**18.5 Proposition.** *Let $G$ be an $N$-dimensional Lie group and let $M$ be a manifold of dimension $n < N$. Let $\phi : G \to M$ be smooth and of rank $n$ at $e$ and set $H := \phi^{-1}(\phi(e))$. Finally, suppose that $\phi \circ L_h = \phi$, i.e., $\phi(hg) = \phi(g)$ for all $h \in H$ and all $g \in G$. Then $H$ can be equipped with the structure of a Lie subgroup of $G$ of dimension $N - n$.*

**Proof.** First, $H$ is a subgroup of $G$ since for $h_1, h_2 \in H$ we have $\phi(h_1 h_2) = \phi(h_2) = \phi(e)$ and $\phi(e) = \phi(h_1 h_1^{-1}) = \phi(h_1^{-1})$, showing that $h_1 h_2$ and $h_1^{-1}$ are elements of $H$. By Proposition 18.2 it suffices to show that $H$ is a submanifold of $G$. According to Proposition 15.5 (ii) this is the case if $\mathrm{rk}_h(\phi) = n$ for each $h \in H$. Now

$$\mathrm{rk}_h \phi = \mathrm{rk}_e(\phi \circ L_h) = \mathrm{rk}_e \phi = n,$$

so the claim follows. □

**18.6 Examples.** (i) $\mathrm{SL}(n, \mathbb{R})$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ $(n > 1)$: To see this, let $\phi : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$. $\phi(A) := \det(A)$. Then

$$\phi^{-1}(\phi(I)) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) = 1\} = \mathrm{SL}(n, \mathbb{R}).$$

Using Proposition 18.5 we can now show that $\mathrm{SL}(n, \mathbb{R})$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ of dimension $n^2 - 1$. We have $\phi \circ L_B(A) = \det(BA) = \det(B)\det(A) = \det(A) = \phi(A)$ for all $A \in \mathrm{GL}(n, \mathbb{R})$ and all $B \in \mathrm{SL}(n, \mathbb{R})$. Due to $\mathrm{rk}(\det) = 1$ on $\mathrm{GL}(n, \mathbb{R})$ also the rank condition is satisfied.

(ii) $\mathrm{O}(n, \mathbb{R})$ is a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ of dimension $\frac{1}{2}n(n-1)$: To see this, let $\mathrm{S}(n, \mathbb{R})$ be the space of symmetric $n \times n$ matrices (whose dimension is $\frac{1}{2}n(n+1)$). $\mathrm{S}(n, \mathbb{R})$ is a submanifold of $\mathrm{M}(n, \mathbb{R})$ with global chart $(a_{ij}) \mapsto (a_{ij})_{i \le j}$. Now let

$$\begin{aligned} \phi : \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{S}(n, \mathbb{R}) \\ A &\mapsto A^t A. \end{aligned}$$

Then $\mathrm{O}(n, \mathbb{R}) = \phi^{-1}(\phi(I))$ and $\phi \circ L_B(A) = (BA)^t(BA) = A^t B^t B A = \phi(A)$ for all $B \in \mathrm{O}(n, \mathbb{R})$. To calculate the rank of $\phi$, note that

$$T_I\phi(A) = \frac{d}{ds}\Big|_0 \phi(I + sA) = \frac{d}{ds}\Big|_0 (I + sA^t)(I + sA) = A^t + A,$$

so $T_I\phi : \mathrm{M}(n, \mathbb{R}) \to T_I S(n, \mathbb{R}) \cong \mathrm{S}(n, \mathbb{R})$ is surjective. Hence $\mathrm{rk} T_I \phi = \dim \mathrm{S}(n, \mathbb{R}) = \frac{1}{2}n(n+1)$, and so by Proposition 18.5 $\mathrm{O}(n, \mathbb{R})$ is a Lie subgroup of dimension $n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$.

Up to now, all our examples of Lie subgroups were in fact submanifolds. This, however, need not necessarily be the case:

**18.7 Example.** $\mathbb{T}^2 = S^1 \times S^1$ is a Lie group by Example 1.3 (v). Here,

$$\mu : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$$
$$((e^{ia}, e^{ib}), (e^{ic}, e^{id})) \mapsto (e^{i(a+c)}, e^{i(b+d)}).$$

Now let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and set $c : \mathbb{R} \to \mathbb{T}^2$, $t \mapsto (e^{i\alpha t}, e^{it})$. Then $H := c(\mathbb{R})$ can be turned into a manifold via the global chart $(e^{i\alpha t}, e^{it}) \mapsto t$ and with this structure $c(\mathbb{R})$ becomes an immersive submanifold of $\mathbb{T}^2$. Then $H$ is a subgroup of $\mathbb{T}^2$ and the product $\mu_H$ on it is smooth because in terms of the above global chart it is of the form $(r, s) \mapsto r + s$. Consequently, $H$ is a Lie subgroup of $\mathbb{T}^2$. However, since $\alpha$ is irrational, $H$ is dense in $\mathbb{T}^2$, so by Proposition 14.2 it is not a submanifold of $\mathbb{T}^2$.

# 19 Lie subgroups and Lie algebras

**19.1 Definition.** *A distribution $\Omega$ on a Lie group $G$ is called* left invariant *if*

$$\forall g, h \in G : \quad \Omega(gh) = \Omega(L_g h) = T_h L_g(\Omega(h)).$$

Thus left invariant distributions are completely determined by their value at $e$. A Lie subalgebra of a Lie algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that for any $v, w \in \mathfrak{h}$ also $[v, w] \in \mathfrak{h}$.

**19.2 Proposition.** *Let $S$ be a vector subspace of $T_e G$. Then $\Omega : g \mapsto T_e L_g(S)$ is a left invariant distribution on $G$ with $\Omega(e) = S$. $\Omega$ is integrable if and only if $S$ is a Lie subalgebra of $\mathfrak{g} = T_e G$.*

**Proof.** We first note that $\Omega$ is indeed a distribution: Let $(v_1, \dots, v_k)$ be a basis of $S$, then for each $g \in G$, $L^{v_i}(g) = T_e L_g(v_i)$ $(i = 1, \dots, k)$ is a basis of $\Omega(g)$. Also, $\Omega$ is left invariant because

$$\Omega(gh) = T_e L_{gh}(S) = T_h L_g \circ T_e L_h(S) = T_h L_g(\Omega(h)).$$

By Theorem 17.32, $\Omega$ is integrable if and only if it is involutive. Now if $\Omega$ is involutive, then for all $i, j$ we have $[L^{v_i}, L^{v_j}] \in \Omega$ and therefore

$$[v_i, v_j] = L^{[v_i, v_j]}(e) = [L^{v_i}, L^{v_j}](e) \in \Omega(e) = S,$$

showing that $S$ is a Lie subalgebra of $\mathfrak{g}$. Conversely, if $S$ is a Lie subalgebra of $\mathfrak{g}$, then for all $i, j$ we have $[L^{v_i}, L^{v_j}] = L^{[v_i, v_j]} \in \Omega$, so $\Omega$ is involutive. □

**19.3 Proposition.** *Let $H$ be a Lie subgroup of $G$ and denote by $j : H \hookrightarrow G$ the inclusion. Then $T_e j$ is a Lie algebra isomorphism from $\mathfrak{h}$ onto a Lie subalgebra of $\mathfrak{g}$. $H$ is an integral manifold of the left invariant distribution $\Omega$ on $G$ with $\Omega(e) = T_e j(\mathfrak{h})$.*

**Proof.** $j$ is an immersion, hence $T_e j : \mathfrak{h} \to T_e j(\mathfrak{h})$ is a linear isomorphism. We need to show that $T_e j(\mathfrak{h})$ is a Lie subalgebra of $\mathfrak{g}$. Let $v \in \mathfrak{h}$, $w := T_e j(v)$, and denote by $\tilde{L}_h$ the left translation on $H$. Then

$$L_h \circ j = j \circ \tilde{L}_h. \tag{19.1}$$

Thus

$$
\begin{aligned}
Tj \circ \tilde{L}^v(h) = Tj(T_e\tilde{L}_h(v)) = T_e(j \circ \tilde{L}_h)(v) &\underset{(19.1)}{=} T_e(L_h \circ j)(v) \\
= T_eL_h(w) = L^w(h) = L^w \circ j(h),
\end{aligned}
$$

meaning that $\tilde{L}^v \sim_j L^w$. If also $v_1 \in \mathfrak{h}$ and $w_1 = T_ej(v_1)$, then by Lemma 4.4 we have $[\tilde{L}^v, \tilde{L}^{v_1}] \sim_j [L^w, L^{w_1}]$ and therefore

$$[w, w_1] = [L^w, L^{w_1}](e) = T_ej([\tilde{L}^v, \tilde{L}^{v_1}](e)) = T_ej([v, v_1]) \in T_ej(\mathfrak{h}).$$

By Proposition 19.2, $S := T_ej(\mathfrak{h})$ defines an integrable and left invariant distribution $\Omega$ on $G$. For $h \in H$ we obtain

$$\Omega(h) = T_eL_h(T_ej(\mathfrak{h})) \underset{(19.1)}{=} T_hj \circ T_e\tilde{L}_h(\mathfrak{h}) = T_hj(T_hH),$$

so $H$ is an integral manifold of $\Omega$. □

**19.4 Example.** By Example 18.6 (ii), $\mathrm{O}(n,\mathbb{R})$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$ of dimension $\frac{1}{2}n(n-1)$. Setting

$$
\begin{aligned}
\phi : \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{S}(n,\mathbb{R}) \\
A &\mapsto A^tA,
\end{aligned}
$$

$\mathrm{O}(n,\mathbb{R}) = \phi^{-1}(\phi(I))$ is a fiber of the submersion $\phi$. By Proposition 15.5 (ii) it follows that the Lie algebra of $\mathrm{O}(n,\mathbb{R})$ is given by $\mathfrak{o}(n,\mathbb{R}) := T_I\mathrm{O}(n,\mathbb{R}) = \ker(T_I\phi) = \{A \mid A + A^t = 0\}$, i.e., by the space of skew symmetric matrices.

Proposition 19.3 shows that given a Lie subgroup $H$ of $G$, the Lie algebra $\mathfrak{h}$ of $H$ corresponds to a Lie subalgebra of $\mathfrak{g}$. It consists of the left invariant vector fields that are tangential to $H$ at $e$. For connected Lie subgroups also the converse is true:

**19.5 Theorem.** *If $G$ is a Lie group and $S \neq \{0\}$ is a Lie subalgebra of $\mathfrak{g}$, then there exists a unique connected Lie subgroup $H$ of $G$ such that $T_ej : \mathfrak{h} \to S$ is a Lie algebra isomorphism (with $j : H \hookrightarrow G$).*

**Proof.** Let $\Omega := g \mapsto T_eL_g(S)$. By Proposition 19.2, $\Omega$ is an integrable left invariant distribution on $G$. Let $H$ be the leaf of $\Omega$ that contains $e$.

For any $g \in G$, $L_g \circ j$ is an injective immersion. By declaring $L_g \circ j$ to be a diffeomorphism we may therefore induce on $L_g \circ j(H) = gH$ a manifold structure such that $gH$ becomes an immersive submanifold of $G$: with $j' : gH \hookrightarrow G$ the inclusion we have that $j' \circ (L_g \circ j) = L_g \circ j : H \to G$ is an immersion, so also $j'$ is one.

Let us show that $gH$ is an integral manifold of $\Omega$. For $gh \in gH$ we have:

$$
\begin{aligned}
T_{gh}j'(T_{gh}(gH)) = T_{gh}j'(T_h(L_g \circ j)(T_hH)) &= T_h(j' \circ L_g \circ j)(T_hH) \\
&= T_hL_g(T_hj(T_hH)) = T_hL_g(\Omega(h)) = \Omega(gh).
\end{aligned}
$$

Since $gH$ is diffeomorphic to $H$, it is connected in its natural manifold topology. Also, $gH$ is an integral manifold of $\Omega$, so by Proposition 17.20 it is an open submanifold of $G(\Omega)$ and thereby carries the trace topology of $G(\Omega)$, hence is connected in $G(\Omega)$. It also contains $g$, so $gH \subseteq K$, where $K$ is the leaf of $\Omega$ through $g$.

We show that in fact $gH = K$. Suppose, to the contrary, that $gH$ were strictly contained in $K$. Then we equip $g^{-1}K$ as above with a manifold structure diffeomorphic to $K$ and also as above we conclude that $g^{-1}K$ is a connected integral manifold of $\Omega$ containing $e$. By assumption, however, $H \subsetneq g^{-1}K$, contradicting the fact that $H$ is a leaf of $\Omega$.

In particular, for $g \in H$ (and thereby $H = K$) we conclude that $gH = H$ (hence also $g^{-1}H = H$). Thus for $g, h \in H$ we have $gh \in H$ and since $e \in H$ also $g^{-1} \in H$. This shows that $H$ is a subgroup of $G$ whose left cosets are precisely the leaves of $\Omega$.

Being a leaf of $\Omega$, $H$ is an immersive submanifold of $G$, and so it only remains to show that $H$ itself is a Lie group, i.e., that $\mu_H : H \times H \to H$ is smooth. Now $j \circ \mu_H = \mu_G \circ (j \times j)$. Since $H$ is a connected immersive submanifold, $H$ is connected as a subset of $G$ by Lemma 14.4, so $H \subseteq G_e$. By Lemma 13.6, $H$ is an immersive submanifold of $G_e$. Since $G_e$ is second countable by Proposition 2.8, so is $H$ due to Proposition 14.7.

Now $\mu_G \circ (j \times j)$ is smooth and takes values in $H$, where $H$ is an integral manifold of an integrable distribution that in addition is second countable. Therefore Proposition 17.26 establishes that $\mu_H$ is smooth, so $H$ is indeed a Lie subgroup of $G$.

Since $H$ is an integral manifold of $\Omega$, $T_e j(\mathfrak{h}) = T_e j(T_e H) = \Omega(e) = S$, so $T_e j$ is a linear isomorphism of $\mathfrak{h}$ onto $S$. Indeed, $T_e j$ is a Lie algebra isomorphism by Proposition 19.3.

Finally, we show uniqueness: Let $\tilde{H}$ be another connected Lie subgroup of $G$ with inclusion $\tilde{j} : \tilde{H} \hookrightarrow G$ such that $T_e\tilde{j} : \tilde{\mathfrak{h}} \to S$ is a Lie algebra isomorphism. By Proposition 19.3 then also $\tilde{H}$ is a connected integral manifold of $\Omega$ containing $e$. Since $H$ is the leaf of $\Omega$ through $e$, this gives $\tilde{H} \subseteq H$. By Proposition 17.20, both $H$ and $\tilde{H}$ are open submanifolds of $G(\Omega)$, hence both carry the trace topology of $G(\Omega)$. Consequently, $\tilde{H}$ is an open neighborhood of $e$ in $H$. But then by Proposition 2.5 $\tilde{H}$ generates $H$, i.e., $H = \tilde{H}$. □

**19.6 Remark.** (i) In particular, for $S = \mathfrak{g}$ we obtain $H = G_e$.

(ii) Note that the proof of Theorem 19.5 makes use of most of the major results we had derived previously.

The following result is proved in the theory of Lie algebras (cf., e.g., [7, Ch. V]):

**19.7 Theorem.** *(Ado's Theorem) Any finite dimensional Lie algebra is isomorphic to a Lie subalgebra of the space of $n \times n$ matrices (with bracket the commutator of matrices).*

Theorems 19.5 and 19.7 together now imply:

**19.8 Theorem.** *(Lie's third fundamental Theorem) If $\mathfrak{g}$ is a finite dimensional (non-trivial) Lie algebra then there exists a Lie group $G$ with Lie algebra $\mathfrak{g}$. $G$ can be chosen as a Lie subgroup of* $\mathrm{GL}(n, \mathbb{R})$ *(resp.* $\mathrm{GL}(n, \mathbb{C})$*).*

**19.9 Proposition.** *Let $H$ be a Lie subgroup of $G$ and let $j : H \hookrightarrow G$ be the inclusion. Setting $j' := T_e j$, we have:*

$$\begin{tikzcd} \mathfrak{h} \arrow[r, "j'"] \arrow[d, "\exp^H"'] & \mathfrak{g} \arrow[d, "\exp^G"] \\ H \arrow[r, "j"] & G \end{tikzcd}$$

**Proof.** This is immediate from Theorem 8.8 (i). □

Hence if $v \in \mathfrak{g}$ is tangential to $H$ (i.e., if $v \in T_e j(\mathfrak{h})$) then the curve $t \mapsto \exp^G(tv)$ is contained in $H$. If $H$ is second countable then also the converse is true:

**19.10 Proposition.** *Let $H$ be a second countable Lie subgroup of $G$ and let $c : I \to G$ be a smooth curve with $c(I) \subseteq H$. Then $c'(t)$ is tangential to $H$ for each $t \in I$.*

**Proof.** By Proposition 19.3 $H$ is an integral manifold of an integrable distribution on $G$. By Proposition 17.26 we have $c = j \circ \tilde{c}$ with $\tilde{c} : I \to H$ smooth, so

$$c'(t) = T_{c(t)} j(\tilde{c}'(t)) \in T_{c(t)} j(T_{c(t)} H) \quad \forall t \in I.$$

□

In Remark 8.5 we used exp to construct specific charts for $G$ (canonical coordinates of first and second kind). For our study of Lie subgroups we need yet another type of coordinates: Let $S$ be a proper subspace of $\mathfrak{g} = T_e G$ and let $(v_1, \dots, v_n)$ be a basis of $\mathfrak{g}$ such that $(v_1, \dots, v_k)$ is a basis of $S$. Let

$$\begin{aligned} \chi : \mathfrak{g} &\to G \\ \sum_{i=1}^{n} y^i v_i &\mapsto \exp\Big(\sum_{i=k+1}^{n} y^i v_i\Big) \cdot \exp\Big(\sum_{i=1}^{k} y^i v_i\Big). \end{aligned}$$

Then $T_0\chi(v_i) = \frac{d}{dt}\big|_0 \chi(tv_i) = \frac{d}{dt}\big|_0 \exp(tv_i) = v_i$, so $T_0\chi = \mathrm{id}$ and thereby $\chi$ is a diffeomorphism around 0, so that $\varphi := \chi^{-1}$ is a chart of $G$ around $e$.

Suppose now in addition that $S$ is a Lie subalgebra of $\mathfrak{g}$. By Proposition 19.2 $S$ defines a left invariant distribution $\Omega$ on $G$. By Theorem 19.5 the leaf $H$ of $\Omega$ through $e$ is a connected (hence second countable by Proposition 2.8) Lie subgroup of $G$ and all other leaves of $\Omega$ are precisely the cosets of $H$. In this situation we have:

**19.11 Lemma.** *The chart $\varphi$ is flat for $\Omega$.*

**Proof.** Let $\varphi = (x^1, \dots, x^n)$. Since $\dim \Omega = k$ it suffices to show that $\frac{\partial}{\partial x^i} \in \Omega$ for $1 \le i \le k$. Setting $V := \mathrm{span}(v_{k+1}, \dots, v_n)$ we have $\mathfrak{g} = S \oplus V$. Let $U := \mathrm{dom}(\varphi)$ and let $g \in U$. Then there exists a unique $(v, s) \in V \times S$ such that $g = \exp(v) \cdot \exp(s)$. For $1 \le i \le k$ we consider the coordinate line $\gamma_i := t \mapsto \exp(v)\exp(s + tv_i)$. Then $\gamma_i'(0) = \frac{\partial}{\partial x^i}\big|_g$, so it remains to show that $\gamma_i'(0) \in \Omega(g)$. Now $c(t) := \exp(s)^{-1} \cdot \exp(s + tv_i)$ is a smooth curve in $G$ and $\exp(s + tv_i) = \exp(s) \cdot c(t)$. By Proposition 19.9 the curve $t \mapsto \exp(s + tv_i)$ $(1 \le i \le k)$ lies entirely in $H$, hence the same is true for $c$. As $H$ is second countable, Proposition 19.10 implies that $c'(t)$ is tangential to $H$ for all $t$. Since $H$ is an integral manifold of $\Omega$ this means that $c'(t) \in \Omega(c(t))$ for each $t$. Now $\gamma_i(t) = g \cdot c(t)$ and $\Omega$ is left invariant, so we conclude that

$$\gamma_i'(0) = T_{c(0)} L_g(c'(0)) \in T_e L_g(\Omega(e)) = \Omega(g). \tag{19.2}$$

□

# 20 Closed connected Lie subgroups

Any subgroup $H$ of a Lie group acts as a transformation group on the right on $G$ via

$$\begin{aligned} \Phi : G \times H &\to G \\ (g, h) &\mapsto gh. \end{aligned}$$

The quotient $G/H$ then consists of the left cosets $gH$ of $H$. In this section we derive conditions under which $G/H$ can be endowed with the structure of a quotient manifold of $G$.

**20.1 Remark.** (i) If $H$ is open, then so is any $gH$. Thus any point in $G/H$ is open in the quotient topology (because $\pi^{-1}(\pi(g)) = gH$), which is therefore discrete. This means that $G/H$ cannot be turned into a quotient manifold (cf. Proposition 15.9).

(ii) Let $G/H$ be a quotient manifold of $G$ and let $\pi : G \to G/H$ be the quotient map. Then $H = \pi^{-1}(\pi(e))$ is closed, being the inverse image of a point (recall that any manifold is $T_1$).

(iii) It follows from (i) and (ii) that a necessary condition for the existence of a quotient manifold structure on $G/H$ is that $H$ is closed but not open in $G$. We will show that this condition is also sufficient if $H$ is a Lie subgroup of $G$.

**20.2 Lemma.** *Let $V$ be a finite dimensional vector space and let $(v_k)$ be a sequence in $V$ with $v_k \neq 0$ for all $k$ and $v_k \to 0$ $(k \to \infty)$. Then there exists a subsequence $(v_{k_l})$ of $(v_k)$, a sequence of numbers $a_l \to 0$, $a_l > 0$ and vectors $w_l$ that converge to some $w \neq 0$ such that $v_{k_l} = a_l w_l$ for all $l \in \mathbb{N}$.*

**Proof.** Without loss of generality let $V = \mathbb{R}^n$. Write $v_k = b_k s_k$ with $s_k \in S^{n-1}$. As the latter is compact, there exists a subsequence $w_l := s_{k_l} \to w \in S^{n-1}$. Since $v_{k_l} \to 0$ we necessarily have $a_l := b_{k_l} \to 0$. □

**20.3 Lemma.** *Let $H$ be a closed subgroup of a Lie group $G$. Let $v_m \in \mathfrak{g}$, $v_m \to v$, $0 \neq a_m \in \mathbb{R}$, $a_m \to 0$. If $\exp(a_m v_m) \in H$ for all $m$, then $\exp(tv) \in H$ for all $t \in \mathbb{R}$.*

**Proof.** Fix $t \in \mathbb{R}$. Then for all $m \in \mathbb{N}$ there exists some $n_m \in \mathbb{Z}$ with $|t - n_m a_m| < a_m$. Thus $t = \lim n_m a_m$. Consequently,

$$tv = \lim n_m a_m v_m \Rightarrow \exp(tv) = \lim \exp(n_m a_m v_m) = \lim \exp(a_m v_m)^{n_m} \in H$$

because $H$ is closed. □

**20.4 Corollary.** *Let $H$ be a closed subgroup of $G$ and let $c : I \to G$ be a smooth curve with $c(I) \subseteq H$ and $c(0) = e$. Then with $v := c'(0)$ we have $\exp(tv) \in H$ for each $t \in I$.*

**Proof.** For $0 < |s| < \varepsilon$ ($\varepsilon$ sufficiently small) let $k(s) := \frac{1}{s}\exp^{-1}(c(s))$, i.e., $c(s) = \exp(sk(s))$, $k : (-\varepsilon, \varepsilon) \to \mathfrak{g}$. Let $a_m \in (-\varepsilon, \varepsilon)$, $a_m \neq 0$, $a_m \to 0$.

We claim that $v_m := k(a_m) \to c'(0)$. Indeed, let $\exp^{-1} = \varphi = (x^1, \dots, x^n)$ locally around $e$ (with respect to some basis in $\mathfrak{g}$). Then $k(s) = \sum_{i=1}^n k^i(s) \frac{\partial}{\partial x^i}\big|_e$, with $k_i$ smooth. Therefore $(\varphi \circ c)^i(s) = x^i(\exp(sk(s))) = sk^i(s)$, so

$$\lim_{s \to 0} k^i(s) = \lim_{s \to 0} \frac{(\varphi \circ c)^i(s)}{s} = \frac{d}{ds}\Big|_0 (\varphi \circ c)^i(s).$$

Thus

$$\lim v_m = \lim k(a_m) = \lim_{s \to 0} k(s) = T_e\varphi(c'(0)) = c'(0).$$

By assumption, $\exp(a_m v_m) = c(a_m) \in H$ for each $m$, so the claim follows from Lemma 20.3. □

**20.5 Theorem.** *Let $H$ be a closed, non-open, connected Lie subgroup of $G$. Then $G/H$ can be equipped with a $\mathcal{C}^\infty$-structure as a quotient manifold of $G$.*

**Proof.** By Propositions 19.2 and 19.3, $H$ is an integral manifold of the left invariant distribution on $G$ with $\Omega(e) = T_e j(\mathfrak{h})$. Since $H$ is connected, the proof of Theorem 19.5 shows that the leaves of $\Omega$ are precisely the left cosets $gH$ of $H$, so $G/H$ is the set of all leaves of $\Omega$. If we can show that $\Omega$ is regular, then the result will follow from Proposition 17.29. To this end, let $\varphi$ be the chart from Lemma 19.11 (since $H$ is not open, $k = \dim\Omega < \dim G = n$). We are going to show that a suitable restriction of $\varphi$ is regular for $\Omega$.

Let $S := \Omega(e)$ and pick $V$ as in Lemma 19.11 (so $\mathfrak{g} = S \oplus V$). We first restrict $\varphi$ such that its image is of the form $y(S' + V')$, where $y : \sum_{i=1}^n y^i v_i \mapsto (y^1, \dots, y^n)$ is the coordinate isomorphism and $S'$, $V'$ are neighborhoods of 0 in $S$ and $V$, respectively. Denote this restriction by $(\varphi', U')$. The slices of $\varphi'$ then are of the form $U'_a = (\exp a) \cdot (\exp S')$, where $a \in V'$ is fixed. By the remark following Proposition 19.9, we have

$$U'_0 = \exp S' \subseteq H. \tag{20.1}$$

We now claim that by suitably shrinking $V'$ we can achieve that

$$U' \cap H = \exp(V') \cdot \exp(S') \cap H = U'_0. \tag{20.2}$$

Here, $\supseteq$ is clear. For the other direction, suppose to the contrary that there exist neighborhoods $V'_m$ of 0 such that $V'_m \to \{0\}$ $(m \to \infty)$ in $V$ and for each $m \in \mathbb{N}$ there exists some $v_m \neq 0$, $v_m \in V'_m$ as well as some $s_m \in S'$ such that $(\exp v_m)\cdot(\exp s_m) \in H$. Since $v_m \in V'_m$ we have $v_m \to 0$. By Lemma 20.2 and 20.3 there exists some $0 \neq v \in V$ such that $\exp(tv) \in H$ for each $t \in \mathbb{R}$ (note that $\exp(v_m) \in H$ because $\exp(s_m) \in H$ and $\exp(v_m) \cdot \exp(s_m) \in H$). Now $H$ is a connected Lie group, hence is second countable by Proposition 2.8. Then Proposition 19.10 shows that

$$v = \frac{d}{dt}\Big|_0 \exp(tv) \in T_e j(\mathfrak{h}) = S,$$

contradicting $S \cap V = \{0\}$, and thereby proving (20.2).

Now let $(\tilde{\varphi} = (x^1, \dots, x^n), U)$ be a further restriction of $\varphi$ such that $y(S'' + V'') = \tilde{\varphi}(U)$, where $S''$, $V''$ are connected neighborhoods of 0 in $S'$ and $V'$, respectively and such that $U^{-1} \cdot U \subseteq U'$.

We next claim that $(\tilde{\varphi}, U)$ is regular for $\Omega$ at $e$. Note first that by what we discussed before Definition 17.27, any slice of $\tilde{\varphi}$ lies in a leaf of $\Omega$, hence in a coset of $H$. It therefore suffices to show that, conversely, any coset of $H$ intersects $U$ in at most one slice. So let $g = \exp(v)\exp(s)$ and $g_1 = \exp(v_1)\exp(s_1)$ be elements of $U$ that belong to the same coset $\tilde{g}H$ of $H$ $(v, v_1 \in V'',\ s, s_1 \in S'')$. By (20.1), $\exp(s), \exp(s_1) \in H$, so $\exp(v), \exp(v_1) \in \tilde{g}H$ and therefore $\exp(v_1)^{-1}\exp(v) \in H$. Furthermore, $\exp(v) = \exp(v) \cdot \exp(0) \in U$ and also $\exp(v_1) \in U$, so by (20.1) and (20.2) we obtain

$$\underbrace{(\exp v_1)^{-1}}_{\in U^{-1}} \underbrace{(\exp v)}_{\in U} \in H \cap U' = \exp S'.$$

Thus $\exp v = \exp(v_1)\cdot\exp(s')$ for some $s' \in S'$. Now any point in $U'$ can uniquely be written as $\exp(a)\cdot\exp(b)$ with $a \in V'$ and $b \in S'$, so $v_1 = v$ (and $s' = 0$). Consequently, $g$ and $g_1$ lie in the same slice of $\tilde{\varphi}$ (namely in $\exp(v) \cdot \exp(S'')$).

Since the leaves of $\Omega$ are the left cosets of $H$, any $L_g$ maps leaves to leaves. If $g \in G$, then $\psi := \tilde{\varphi} \circ L_{g^{-1}}$ is a chart of $G$ around $g$ and this chart is regular for $\Omega$: First, $(\psi =: (z^1, \dots, z^n), gU)$ is flat for $\Omega$ at $g$: As $\tilde{\varphi} = (x^1, \dots, x^n)$ is flat for $\Omega$ around $e$, $\Omega(g') = \operatorname{span}\Big(\frac{\partial}{\partial x^1}\Big|_{g'}, \dots, \frac{\partial}{\partial x^k}\Big|_{g'}\Big)$ for any $g' \in U$. Therefore, for any element $gg'$ of $gU$ we have

$$\Omega(gg') = T_e L_g(\Omega(g')) = \operatorname{span}\Big(T_e L_g\Big(\frac{\partial}{\partial x^1}\Big|_{g'}\Big), \dots, T_e L_g\Big(\frac{\partial}{\partial x^k}\Big|_{g'}\Big)\Big).$$

Since

$$T_e L_g\Big(\frac{\partial}{\partial x^i}\Big|_{g'}\Big) = T_e L_g \circ (T_{g'}\tilde{\varphi})^{-1}(e_i) = (T_{gg'}(\tilde{\varphi} \circ L_{g^{-1}}))^{-1}(e_i) = \frac{\partial}{\partial z^i}\Big|_{gg'},$$

the claim follows.

Finally, $\mathrm{dom}(\psi) = L_g(U) = g \cdot U$ and $\psi(gU) = \tilde{\varphi}(U) = y(S'' + V'')$. A slice of $\psi$ therefore is of the form $\psi^{-1}(\{a\} \times S'')$. Now if $g_1 \cdot H$ is any leaf of $\Omega$, then since $(\tilde{\varphi}, U)$ is regular for $\Omega$ at $e$, for a suitable $a$ we get

$$\begin{aligned} g_1 H \cap gU &= g(g^{-1}g_1 H \cap U) = L_g(\tilde{\varphi}^{-1}(\{a\} \times S'')) \\ &= (\tilde{\varphi} \circ L_{g^{-1}})^{-1}(\{a\} \times S'') = \psi^{-1}(\{a\} \times S''), \end{aligned}$$

showing that $(\psi, gU)$ is regular for $\Omega$ at $g$. Consequently, $\Omega$ is regular. □

**20.6 Corollary.** *Let $H$ be a closed and connected Lie subgroup of $G$. Then $H$ is a submanifold of $G$.*

**Proof.** This is clear if $H$ is open. Otherwise, the proof of Theorem 20.5 shows that $H$ is a leaf of a regular distribution on $G$, so the claim follows from Corollary 17.30. □

# 21 Closed subgroups of a Lie group

Our aim in this section is to show that a subgroup $H$ of a Lie group $G$ that is a closed subset of $G$ is either discrete or can be endowed with a manifold structure in such a way as to become a Lie subgroup of $G$.

We first assume that $H$ is closed and connected and define a vector subspace $S$ of $\mathfrak{g}$ which will turn out to be a Lie subalgebra. The corresponding Lie subgroup then is precisely $H$.

**21.1 Lemma.** *Let $S := \{v \in \mathfrak{g} \mid \exp(tv) \in H \ \forall t \in \mathbb{R}\}$. Then $S$ is a linear subspace of $\mathfrak{g}$.*

**Proof.** Let $v, w \in S$ and let $c : \mathbb{R} \to G$, $c(t) := \exp(tv)\exp(tw)$. Then by Lemma 1.5 we have

$$c'(0) = T_{(e,e)}\mu\Big(\frac{d}{dt}\Big|_0 \exp(tv), \frac{d}{dt}\Big|_0 \exp(tw)\Big) = T_e L_e(w) + T_e R_e(v) = v + w.$$

Since $c(\mathbb{R}) \subseteq H$, Corollary 20.4 implies that $v + w \in S$. Moreover, $\exp(t(\lambda v)) = \exp((t\lambda)v)$, so also $\lambda v \in S$ for each $\lambda \in \mathbb{R}$. □

**21.2 Lemma.** *$S$ is a Lie subalgebra of $\mathfrak{g}$.*

**Proof.** For $S = 0$ or $S = \mathfrak{g}$ there is nothing to do. Otherwise the claim will follow from Proposition 19.2 once we know that the distribution $\Omega : g \mapsto T_e L_g(S)$ is integrable. To this end we first show that the chart $\varphi$ from Lemma 19.11 is flat for $\Omega$ at $e$. Using the notation introduced there, we again have to show that $\gamma_i'(0) \in \Omega(g)$ for $1 \leq i \leq k$. By definition of $S$ and because $S$ is a vector space by Lemma 21.1, we know that $\exp(s + tv_i) \in H$ for each $t \in \mathbb{R}$ and $1 \leq i \leq k$. For $c(t) := \exp(s)^{-1} \cdot \exp(s + tv_i)$ this implies $c(\mathbb{R}) \subseteq H$. By Corollary 20.4 we conclude that $\exp(tc'(0)) \in H$ for all $t \in \mathbb{R}$, so by definition of $S$ we get $c'(0) \in S = \Omega(e)$. From this, as in the proof of Lemma 19.11 (see (19.2)) we conclude $\gamma_i'(0) \in \Omega(g)$ (where $g = \exp(v)\exp(s) \in \mathrm{dom}(\varphi)$). For general $g \in G$ it now follows verbatim as in the proof of Theorem 20.5 that $\psi := \varphi \circ L_g^{-1}$ is a chart that is flat for $\Omega$ at $g$. Thus $\Omega$ is integrable. □

Using this we can now show:

**21.3 Proposition.** *Let $H$ be a closed and connected subgroup of a Lie group $G$. Then either $H = \{e\}$ or $H$ can be endowed with a manifold structure to become a connected Lie subgroup of $G$.*

**Proof.** When equipped with the trace topology of $G$, $H$ is a connected topological group. By Lemma 21.2, $S := \{v \in \mathfrak{g} \mid \exp(tv) \in H \ \forall t \in \mathbb{R}\}$ is a Lie subalgebra of $\mathfrak{g}$. We distinguish the following cases:

1.) $S = \mathfrak{g}$: Since $\exp(\mathfrak{g})$ is connected, $\exp(\mathfrak{g}) \subseteq G_e$. Let $S' \subseteq \mathfrak{g}$ be an open neighborhood of 0 in $\mathfrak{g}$ with $\exp : S' \to \exp(S')$ a diffeomorphism. By Proposition 2.5 $G_e$ is generated by $\exp(S')$. Since $\exp(S') \subseteq \exp(S) \subseteq H$ and $H$ is connected we obtain $H = G_e$. It therefore allows a manifold structure as an open Lie subgroup of $G$.

2.) $0 \neq S \neq \mathfrak{g}$: As in Lemma 19.11 let $V \subseteq \mathfrak{g}$ be complementary to $S$, $\mathfrak{g} = S \oplus V$, and let $\varphi$ be the corresponding chart of $G$. We restrict $\varphi$ such that its image is of the form $y(S' \oplus V')$, where $S'$ and $V'$ are connected open neighborhoods of 0 in $S$ and $V$, respectively.

We claim that for $V'$ sufficiently small we have for $\varphi : U' \to y(S' \oplus V')$:

$$U' \cap H = \exp(S'). \tag{21.1}$$

In fact, otherwise there would exist neighborhoods $V'_m$ of 0 in $V$ with $V'_m \to \{0\}$ such that for each $m \in \mathbb{N}$ there exist $0 \neq v_m \in V'_m$ and $s_m \in S'$ such that $\exp(v_m)\cdot\exp(s_m) \in H$, so $\exp v_m \in H$ for each $m$. As $v_m \to 0$, Lemmas 20.2 and 20.3 show the existence of some $0 \neq w \in V$ with $\exp(tw) \in H$ for each $t \in \mathbb{R}$. But this means $w \in S$, contradicting $S \cap V = \{0\}$.

By Theorem 19.5 there exists a unique connected Lie subgroup $H'$ of $G$ such that for its Lie algebra $\mathfrak{h}'$ we have $T_e j(\mathfrak{h}') = S$ (with $j : H' \hookrightarrow G$). The set $S'' := (T_e j)^{-1}(S')$ is an open neighborhood of 0 in $\mathfrak{h}'$ and by Proposition 19.9 we have

$$\exp(S') = \exp \circ T_e j(S'') = j \circ \exp^{H'}(S'') = \exp^{H'}(S'') \subseteq H'.$$

By Theorem 8.2 there exists an open neighborhood $W \subseteq S''$ of 0 in $\mathfrak{h}'$ such that $\exp^{H'}(W)$ is open in $H'$. According to Proposition 2.5 $\exp^{H'}(W)$, so in particular $\exp^{H'}(S'') = \exp(S')$ generates $H'$. On the other hand, $U' \cap H$ is a neighborhood of $e$ in the connected topological group $H$, so it also generates $H$. By (21.1) we conclude that $H = H'$, so $H$ possesses a manifold structure as a Lie subgroup of $G$.

3.) $S = \{0\}$: Let $V'$ be an open neighborhood of 0 in $T_eG$ such that exp is a diffeomorphism on $V'$. We show that $V'$ can be chosen so small that $\exp(V') \cap H = \{e\}$. Indeed, otherwise there would be a sequence $V'_m \subseteq V'$ of 0-neighborhoods with $V'_m \to \{0\}$ and for each $m \in \mathbb{N}$ some $0 \neq v_m \in V'_m$ with $\exp(v_m) \in H$. As in the proof of (21.1) above we conclude from this the existence of some $0 \neq w \in S$, contradicting the fact that $S = \{0\}$. Thus $\{e\}$ is an open subset of $H$, hence generates $H$, meaning that $H = \{e\}$. □

Our next aim is to generalize Theorem 20.5 to general (not necessarily connected) closed subgroups $H$ of a Lie group $G$. We want to show that $G/H$ can be turned into a quotient manifold of $G$. By Remark 20.1 we again have to suppose that $H$ is not open.

When equipped with the trace topology, $H$ is a topological group, hence by Proposition 2.4 $H_e$ (the connected component of $e$ in $H$) is a normal subgroup of $H$. $H_e$ is closed and connected in $H$ and therefore (since $H$ is closed) also closed and connected in $G$. By Proposition 21.3 we therefore either have $H_e = \{e\}$ or we can equip $H_e$ with the structure of a closed connected Lie subgroup of $G$.

In the proof of Proposition 21.3 we have seen that if $S = \mathfrak{g}$, then $H_e = G_e$. But then by Proposition 2.5 also

$$H = \bigcup_{h \in H} hH_e = \bigcup_{h \in H} hG_e,$$

so $H$ is open in $G$, which means we can exclude this case. Consequently, $\dim H_e < \dim G$, and Theorem 20.5 shows that $G/H_e$ can be turned into a quotient manifold of $G$. For $G/H$ itself we have:

**21.4 Proposition.** *Let $H$ be a closed, non-open subgroup of a Lie group. Then $H/H_e$ acts freely on the manifold $G/H_e$ as a discontinuous transformation group and the corresponding quotient set is $G/H$.*

**Proof.** Let $K$ be a left coset of $H_e$ in $H$. Then the map

$$\begin{aligned} \varphi_K : G &\to G/H_e \\ g &\mapsto gK \end{aligned}$$

is smooth: for $h \in K$ we have $K = hH_e$, so $\varphi_K = g \mapsto gh \mapsto (gh)H_e$. Also, $\varphi_K$ is an invariant of the equivalence relation defined on $G$ by $H_e$: Let $K = hH_e$ with $h \in H$ and suppose that $g_1 \sim g_2$, i.e., $g_1H_e = g_2H_e$, so that $g_1 = g_2h'$ with $h' \in H_e$. Then since $H_e$ is a normal subgroup of $H$,

$$\varphi_K(g_1) = g_1K = g_1hH_e = g_2h'hH_e = g_2h\underbrace{h^{-1}h'hH_e}_{=H_e} = g_2hH_e = g_2K = \varphi_K(g_2).$$

By Proposition 15.13 therefore also the induced map (projection)

$$\begin{aligned} \phi_K : G/H_e &\to G/H_e \\ gH_e &\mapsto (gh)H_e \qquad (K = hH_e) \end{aligned}$$

is well-defined and smooth. Let $K^{-1}$ be the inverse of $K = hH_e$ in $H/H_e$, i.e., $K^{-1} = h^{-1}H_e$. Then $\phi_{K^{-1}} = (\phi_K)^{-1}$, so $\phi_K$ is a transformation of the manifold $G/H_e$.

We now show that $H/H_e$ is a transformation group on $G/H_e$ that acts freely on the right via the map

$$\begin{aligned} \phi : G/H_e \times H/H_e &\to G/H_e \\ (gH_e, K) &\mapsto \phi_K(gH_e). \end{aligned}$$

According to Definition 16.1 this means verifying that

(i) For $K$ fixed $\phi(., K) = \phi_K$ is a transformation by the above.

(ii) For $K_1 = h_1H_e$, $K_2 = h_2H_e$ we have $K_2 \cdot K_1 = (h_2h_1)H_e$, so

$$\phi_{K_1}(\phi_{K_2}(gH_e)) = \phi_{K_1}(gh_2H_e) = gh_2h_1H_e = \phi_{K_2 \cdot K_1}(gH_e).$$

The action is free: Let $\phi_K(gH_e) = gH_e$ for some $g \in G$, $K = hH_e$. Then $ghH_e = gH_e$, so $hH_e = H_e \Rightarrow h \in H_e \Rightarrow K = H_e$.

Moreover, $H/H_e$ acts discontinuously: to see this, by Definition 16.5 we have to show that for each $gH_e \in G/H_e$ there exists a neighborhood $W$ of $gH_e$ in $G/H_e$ with $W \cap \phi_K(W) = \varnothing$ unless $K = H_e$. Now $H_e$ is open in the trace topology of $G$ on $H$: the proof of Proposition 21.3 shows (even without the assumption that $H$ is connected) that there exists some $U'$ open in $G$ with $\exp(S') = U' \cap H$, so $U' \cap H = \exp(S') \subseteq H_e$ because $S'$ can be chosen connected. $U' \cap H$ is open in $H$, hence also in $H_e$, and thereby generates it (see Proposition 2.5). Thus $H_e = \bigcup_{n \in \mathbb{N}}(U' \cap H)^n$, implying that $H_e$ is open in the trace topology of $G$ on $H$. Therefore there exists some $U \subseteq G$ open with $H_e = H \cap U$. Let $V$ be an open neighborhood of $e$ in $G$ with $V^{-1}V \subseteq U$ and let $W := \pi(gV)$, where $\pi : G \to G/H_e$ is the quotient map. Since $\pi^{-1}(\pi(gV)) = gVH_e$ is open in $G$, $W$ is an open neighborhood of $gH_e$ in $G/H_e$.

Suppose that $W \cap \phi_K(W) \neq \varnothing$. Then there exist $a, b \in V$ such that

$$(ga)H_e = \pi(ga) = \phi_K(\pi(gb)) = \phi_K(gbH_e) = (gbh)H_e.$$

But then

$$(gbh)^{-1}(ga) = h^{-1}b^{-1}a \in H_e \Rightarrow b^{-1}a \in hH_e \subseteq H$$

and

$$b^{-1}a \in V^{-1}V \subseteq U \Rightarrow b^{-1}a \in H \cap U = H_e \Rightarrow h \in H_e \Rightarrow K = H_e.$$

It remains to show that the quotient set induced by $\phi$ is precisely $G/H$. Let $g_1H_e, g_2H_e$ be equivalent under $\phi$. Then there is some $K = hH_e \in H/H_e$ with $g_1hH_e = \phi_K(g_1H_e) = g_2H_e$, so

$$g_2^{-1}g_1h \in H_e \Rightarrow g_2^{-1}g_1 \in H_eh^{-1} \subseteq H \Rightarrow g_1H = g_2H.$$

Conversely, if $g_1H = g_2H$ then there exists some $h \in H$ with $g_1h = g_2$. Then setting $K := hH_e$ we have $\phi_K(g_1H_e) = g_1hH_e = g_2H_e$. Summing up, $[gH_e] \mapsto gH$ is a bijection from $(G/H_e)/(H/H_e)$ onto $G/H$. □

Using this result we can now show:

**21.5 Theorem.** *Let $H$ be a closed subgroup of a Lie group $G$. Then either*

*(i) $H$ is open in $G$ and the quotient topology on $G/H$ is discrete, or*

*(ii) $G/H$ can be endowed with the structure of a quotient manifold of $G$ (of the same dimension as $G/H_e$).*

**Proof.** If $H$ is open then (i) follows from Remark 20.1 (i). So suppose that $H$ is not open and let $H_e$ be the connected component of $e$ in $H$. From the Remark preceding Proposition 21.4 we know that either $H_e = \{e\}$ or $H_e$ can be equipped with the structure of a closed Lie subgroup of $G$ whose dimension is smaller than that of $G$. By Theorem 20.5 therefore $G/H_e$ possesses the structure of a quotient manifold of $G$, so the quotient map $\pi : G \to G/H_e$ is a submersion.

Proposition 21.4 shows that $H/H_e$ acts freely as a discontinuous transformation group on $G/H_e$ and $(G/H_e)/(H/H_e) = G/H$. Due to Proposition 16.6 $G/H$ therefore possesses the structure of a quotient manifold of $G/H_e$ of the same dimension as $G/H_e$, so the quotient map $\rho : G/H_e \to (G/H_e)/(H/H_e) = G/H$ is a submersion. Now $\rho \circ \pi(g) = \rho(gH_e) = gH$ (cf. the proof of Proposition 21.4), so also the quotient map $\tilde{\pi} : G \to G/H$, being the composition of two submersions is a submersion itself. □

In case $H$ is a normal subgroup and thereby $G/H$ is a group we can say more:

**21.6 Corollary.** *Let $H$ be a closed, non-open normal subgroup of a Lie group $G$. Then the quotient manifold $G/H$ is a Lie group.*

**Proof.** It only remains to show that $G/H$ is a Lie group. So let $\mu : G \times G \to G$ be the multiplication on $G$ and let $\pi : G \to G/H$ be the quotient map. By Theorem 21.5, $\pi \times \pi : G \times G \to G/H \times G/H$ is a surjective submersion and since $H$ is normal the map $\pi \circ \mu : G \times G \to G/H$ is an invariant of the corresponding equivalence relation on $G \times G$: Let $(g_1, g_2) \sim (g_1', g_2')$, i.e., $g_1H = g_1'H$ and $g_2H = g_2'H$. Then

$$\pi \circ \mu(g_1, g_2) = \pi(g_1g_2) = g_1g_2H = (g_1H)(g_2H) = (g_1'H)(g_2'H) = \pi \circ \mu(g_1', g_2').$$

By Proposition 15.13 the corresponding projection $\tilde{\mu}$ of $\pi \circ \mu$ is smooth:

$$\begin{aligned} \tilde{\mu} : G/H \times G/H &\to G/H \\ (g_1H, g_2H) &\mapsto (g_1g_2H) = (g_1H) \cdot (g_2H). \end{aligned}$$

Hence $\tilde{\mu}$ is precisely the multiplication on $G/H$, yielding that $G/H$ is a Lie group. □

**21.7 Theorem.** *(Cartan) Let $H$ be a closed subgroup of a Lie group $G$. Then either $H$ is discrete or there is a unique smooth structure on it that turns it into a submanifold of $G$. With this structure it then is a Lie subgroup of $G$. If $H$ is neither discrete nor open then* $\dim(G/H) = \dim G - \dim H$.

**Proof.** By Theorem 21.5 $H$ is either open in $G$ (hence can be viewed as an open submanifold of $G$) or $G/H$ is a quotient manifold of $G$. In the second case $H$ is a fiber of the quotient map $\tilde{\pi} : G \to G/H$, $H = \tilde{\pi}^{-1}(eH)$. By Proposition 15.5 $H$ therefore is either discrete or it possesses the structure of a submanifold of $G$ of dimension $\dim G - \dim(G/H)$. By Remark 13.5 (ii) the submanifold structure of $H$ is uniquely determined. Finally, Proposition 18.2 shows that $H$ is a Lie subgroup of $G$. □

Conversely, we have:

**21.8 Theorem.** *Let $H$ be a Lie subgroup of $G$ that is a submanifold of $G$ (i.e., that carries the trace topology of $G$). Then $H$ is closed in $G$.*

**Proof.** Let $h_i \in H$, $h_i \to h$ in $G$. We have to show that $h \in H$. Since $H$ is a submanifold of $G$ there exists an adapted chart $(\varphi, U)$ around $e$: $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$, $\varphi(U \cap H) = \varphi(U) \cap (\mathbb{R}^k \times \{0\})$. Pick open neighborhoods $V, W$ of $e$ such that $V^{-1} \cdot V \subseteq \overline{W} \subseteq U$. Let $i_0$ be such that $h_i \in hV$ for $i \geq i_0$. Then for $i, j \geq i_0$ we have $h_i^{-1} h_j \in V^{-1} \cancel{h}^{-1} \cancel{h} V \subseteq \overline{W}$, so

$$\varphi(h_i^{-1} h_j) \in \varphi(\overline{W} \cap H) = \varphi(\overline{W}) \cap (\mathbb{R}^k \times \{0\}) \subseteq \varphi(U) \cap (\mathbb{R}^k \times \{0\}).$$

For $j \to \infty$ we obtain $\varphi(h_i^{-1} h) \in \varphi(U) \cap (\mathbb{R}^k \times \{0\}) = \varphi(U \cap H)$, so $h_i^{-1} h \in U \cap H \subseteq H$ and, finally, $h \in h_i H \subseteq H$. □

**21.9 Theorem.** *Let $H$ be a subgroup of a Lie group $G$. Then the following are equivalent:*

*(i) When equipped with the trace topology of $G$, $H$ can be turned into a Lie group.*

*(ii) $H$ is non-discrete and closed in $G$.*

*If these equivalent conditions hold, $H$ is a Lie subgroup of $G$.*

**Proof.** (ii)⇒(i) as well as the last claim follow from Theorem 21.7.

(i)⇒(ii) Let $j : H \hookrightarrow G$ be the inclusion. Then $j$ is a continuous homomorphism of Lie groups, hence is smooth by Theorem 8.6. By Theorem 8.8, $j'(v) = 0$ implies $j \circ \exp^H(tv) = \{e\}$, so since $j$ is injective $\exp^H(tv) = e$ for all $t \in \mathbb{R}$, and choosing $t$ so small that $tv$ lies in a neighborhood where $\exp^H$ is injective, $v = 0$. It follows that $j' = T_e j$ is injective. The fact that, for any $h_0 \in H$, $j = L^G_{j(h_0)} \circ j \circ L^H_{h_0^{-1}}$, implies that indeed each $T_{h_0} j$ is injective, i.e., $j$ is an immersion. Thus $H$ is a Lie subgroup of $G$. Since it carries the trace topology of $G$, it is even a submanifold and thereby closed due to Theorem 21.8. □

**21.10 Example.** Matrix groups like $\mathrm{SL}(n, \mathbb{R})$ or $\mathrm{O}(n)$ are closed subgroups of $\mathrm{GL}(n, \mathbb{R})$ and for $n > 1$ they are not discrete. By Theorem 21.7 they can therefore uniquely be turned into submanifolds of $\mathrm{GL}(n, \mathbb{R})$ and then become Lie subgroups. Being an open subset of $\mathbb{R}^{n^2}$, $\mathrm{GL}(n, \mathbb{R})$ is second countable, so the same is true for its submanifolds $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{O}(n)$.

**21.11 Remark.** (i) Let $H$ be a subgroup of the Lie group $G$. Then $H$ can be turned in at most one way into a Lie subgroup with a second countable topology.

Indeed, suppose that $H'$ is another such structure on $H$. Then both $j : H \hookrightarrow G$ and $j' : H' \hookrightarrow G$ are smooth and we want to show that $\mathrm{id} : H \to H'$ is smooth (since by symmetry so then will be its inverse, implying $H = H'$). Now $\mathrm{id} = j$ when considered as a map from $H \to H'$. By Proposition 19.3 $H'$ is an integral manifold of an integrable distribution on $G$. Therefore $\mathrm{id} = j : H \to H'$ is smooth by Proposition 17.26 because $H'$ is second countable.

(ii) Let $H$ be a closed, non-discrete subgroup of a Lie group $G$. By Theorem 21.7 $H$ admits a unique structure as a submanifold of $G$ and is a Lie subgroup with this structure. It may however happen that there is another manifold structure on $H$ that turns it into an immersive submanifold such that it is a Lie subgroup as well. This further structure must, however, have a dimension different from the previous one: Denote by $H'$ the new structure and suppose that $\dim H = \dim H'$. Then $j : H' \hookrightarrow G$ is smooth and $j(H') \subseteq H$, so $j = \mathrm{id} : H' \to H$ is smooth by Remark 13.5 (i). Let $i : H \hookrightarrow G$. Then $i$ is an immersion and $j = i \circ \mathrm{id}_{H',H}$, so $\mathrm{rk}(\mathrm{id}) = \mathrm{rk}(j) = \dim H' = \dim H$, showing that id is a local diffeomorphisms. Since it also is injective, it is a diffeomorphism, i.e., $H = H'$.

To conclude this section we want to consider another class of connected (but not necessarily closed) subgroups of Lie groups that are always Lie subgroups. In the theory of principal fiber bundles this result is used to show that holonomy groups are Lie subgroups of the structure group.

**21.12 Theorem.** *Let $G$ be a Lie group and let $H$ be a subgroup of $G$ such that for each $h \in H$ there exists a piecewise smooth curve $\gamma : I \to G$ with $\gamma(I) \subseteq H$ that connects $e$ with $h$. Then either $H = \{e\}$ or it possesses a smooth structure as a Lie subgroup of $G$.*

**Proof.** Let

$$S := \{v \in \mathfrak{g} \mid \exists b > 0\ \exists \gamma : [0,b] \to G \text{ pw. } \mathcal{C}^\infty, \gamma(0) = e,\ \gamma'(0) = v,\ \gamma([0,b]) \subseteq H\}.$$

We show that $S$ is a Lie subalgebra of $\mathfrak{g}$. Let $v \in S$, $\lambda \in \mathbb{R}$ and $\gamma_v : [0,b] \to G$ as in the definition of $S$. Then $z(t) := \gamma_v(\lambda t)$ satisfies $z'(0) = \lambda v$, so $\lambda v \in S$. If $v, w \in S$ with corresponding $\gamma_v, \gamma_w$, we have $\gamma_v'(0) = v$ and $\gamma_w'(0) = w$. Setting $z(t) : \gamma_v(t)\gamma_w(t)$ on the intersection of the domains, we have $\mathrm{im}(z) \subseteq H$ and $z'(0) = v + w$, so $v + w \in S$, which therefore is a linear subspace.

It remains to show that $[v,w] \in S$. To see this, let $h \in H$ and set $\alpha(t) := h\gamma_w(t)h^{-1} = \mathrm{conj}_h(\gamma_w(t))$. Then $\mathrm{im}(\alpha) \subseteq H$, $\alpha$ is piecewise smooth, $\alpha(0) = e$ and $\alpha'(0) = \mathrm{Ad}(h)(w)$, so for all $h \in H$ and all $w \in S$ we get $\mathrm{Ad}(h)(w) \in S$. Consequently, $\mathrm{Ad}(\gamma_v(t))w \in S$ for all $t$, and thereby

$$S \ni \frac{d}{dt}\Big|_0 \mathrm{Ad}(\gamma_v(t))w = \mathrm{ad}(v)(w) = [v,w].$$

So indeed $S$ is a Lie subalgebra of $\mathfrak{g}$. By our assumption on $H$, $S = \{0\}$ is only possible if there is no nontrivial element in $H$, i.e., if $H = \{e\}$, Otherwise, by Theorem 19.5 there exists a unique connected Lie subgroup $K$ of $G$ such that, with $j : K \hookrightarrow G$, $T_e j : \mathfrak{k} \to S$ is a Lie algebra isomorphism. We show that $H = K$:

$H \subseteq K$: Let $h \in H$. Then there is a piecewise smooth curve $\gamma : [0,1] \to G$, $\mathrm{im}(\gamma) \subseteq H$, $\gamma(0) = e$, $\gamma(1) = h$. Fix $s \in [0,1]$ and set $\alpha_s(t) := \gamma(s)^{-1}\gamma(t)$. Then $\alpha_s'(s) \in S$ by definition of $S$ (set $\sigma(t) := \alpha_s(s+t)$, then $\sigma'(0) = \alpha_s'(s)$). Since $\alpha_s'(s) = T_{\gamma(s)} L_{\gamma(s)^{-1}}(\gamma'(s))$ we obtain $\gamma'(s) \in T_e L_{\gamma(s)} S$ (note that this also holds for one-sided derivatives, $\gamma$ is only piecewise smooth). According to Theorem 19.5 $K$ is an integral manifold of the left invariant distribution $\Omega : g \mapsto S_g := T_e L_g(S)$. This implies that $\gamma([0,1]) \subseteq K$: Let $t_0 \in [0,1]$, then there exists an interval $J$ around $t_0$ in $[0,1]$ and a flat chart $(U, \varphi = (x^1, \dots, x^n))$ around $\gamma(t_0)$ with $\gamma(J) \subseteq U$. Then for all $t \in J$:

$$\gamma'(t) \in S_{\gamma(t)} = \operatorname{span}\Big(\frac{\partial}{\partial x^1}\Big|_{\gamma(t)}, \dots, \frac{\partial}{\partial x^k}\Big|_{\gamma(t)}\Big),$$

so $\gamma_i'(t) = 0$ for $k+1 \le i \le n$. This means that $\gamma|_J$ is entirely contained in a slice $U_a = \varphi^{-1}(\mathbb{R}^k \times \{a\})$ and therefore lies in some leaf of $\Omega$. If $\gamma(t_0)$ lies in two flat charts then the corresponding leaves intersect, hence coincide. Covering $\gamma([0,1])$ by such flat charts it follows that $\gamma$ lies entirely in one leaf. Since $\gamma(0) = e \in K$, this leaf is $K$. In particular, $h = \gamma(1) \in K$.

$K \subseteq H$: For any $v \in S$ there exists a $C^1$ curve $c_v : (-\varepsilon, \varepsilon) \to G$, $c_v((-\varepsilon,\varepsilon)) \subseteq H$, $c_v(0) = e$, $c_v'(0) = v$: let $\gamma_v$ as in the definition of $S$ and let $\nu : g \to g^{-1}$. Then set

$$c_v(t) := \begin{cases} \gamma_v(t) & t \ge 0 \\ \nu \circ \gamma_v(-t) & t < 0 \end{cases}$$

Because $T_e\nu = -\mathrm{id}_{\mathfrak{g}}$, this $c_v$ does the job (correcting for the fact that $\gamma_v$ is only one-sided differentiable at 0). Now let $(v_1, \dots, v_k)$ be a basis of $S$ and let $c_{v_i} : (-\varepsilon, \varepsilon) \to G$ be $C^1$, $\operatorname{im}(c_{v_i}) \subseteq H$, $c_{v_i}(0) = e$, $c_{v_i}'(0) = v_i$ $(1 \le i \le k)$. Set

$$f(t_1, \dots, t_k) := c_{v_1}(t_1) \cdots c_{v_k}(t_k) \in H \subseteq K.$$

Then $f : (-\varepsilon, \varepsilon)^k \to G$ is $C^1$ (for suitable $\varepsilon > 0$). By Proposition 14.7 $K$ is second countable (because $K \subseteq G_e$, which is second countable). The proof of Proposition 17.26 therefore shows that $f$ is also $C^1$ when considered as a map $(-\varepsilon,\varepsilon)^k \to K$. Moreover, $\partial_i f(0) = v_i$, so $f$ is a $C^1$ diffeomorphism from some neighborhood $U$ of 0 in $\mathbb{R}^k$ onto a neighborhood $V = f(U)$ of $e$ in $K$. Since $K$ is connected, $V$ generates $K$. But $V = f(U) \subseteq H$, so $K \subseteq H$. □

# 22 Classification of Lie groups

In Theorem 19.8 we saw that any abstract Lie algebra can be realized as the Lie algebra of some Lie group. In this section we want to classify all Lie groups with prescribed Lie algebra, following [2]. We first consider some further properties of Lie group homomorphisms.

Let $\varphi : G \to H$ be a Lie group homomorphism. Then $\ker\varphi$ is a closed normal subgroup of $G$, hence by Theorem 21.7 it is (if not discrete) a Lie subgroup and a submanifold of $G$.

**22.1 Lemma.** *Let $\varphi : G \to H$ be a Lie group homomorphism with $\ker\varphi$ non-discrete. The Lie algebra of $\ker\varphi$ is $\ker\varphi'$ (where $\varphi' = T_e\varphi$). Consequently, it is an ideal in $\mathfrak{g}$.*

**Proof.** By Section 21, the Lie algebra $\mathrm{Lie}(\ker\varphi)$ of $\ker\varphi$ is given by

$$\{v \in \mathfrak{g} \mid \exp(tv) \in \ker(\varphi)\ \forall t \in \mathbb{R}\}.$$

(Indeed, for any Lie subgroup $K$, $\mathrm{Lie}(K) = \mathrm{Lie}(K_e)$, and by Proposition 2.4 for $K$ closed in $G$, $K_e$ is a closed connected Lie subgroup of $G$. Hence the claim follows from Proposition 21.3.) Using Theorem 8.8 it follows that $v \in \mathrm{Lie}(\ker\varphi)$ if and only if

$$\forall t:\ e = \varphi \circ \exp^G(tv) = \exp^H(\varphi'(tv)) = \exp^H(t\varphi'(v)) \Leftrightarrow \varphi'(v) = 0,$$

where the last equivalence follows by picking $t$ so small that $t\varphi'(v)$ lies in a neighborhood where $\exp^H$ is a diffeomorphism. Finally, $\ker\varphi'$ is an ideal: Let $v \in \ker\varphi'$, $w \in \mathfrak{g}$. Then

$$\varphi'([v,w]) = [\varphi'(v), \varphi'(w)] = 0 \Rightarrow [v,w] \in \ker\varphi'.$$

□

A generalization of the last assertion in the previous result is as follows:

**22.2 Proposition.** *Let $H$ be a Lie subgroup of $G$, and suppose that both $H$ and $G$ are connected. Let $j : H \hookrightarrow G$. Then the following are equivalent:*

*(i) $H$ is a normal subgroup of $G$.*

*(ii) $j'(\mathfrak{h})$ is an ideal in $\mathfrak{g}$.*

**Proof.** (i)⇒(ii): $H$ is normal in $G$ if and only if $\mathrm{conj}_g(h) \in H$ for all $g \in G$ and all $h \in H$. Fix $g \in G$ and let $v \in \mathfrak{h}$, $t \in \mathbb{R}$. Then (using Theorem 8.8):

$$\exp(t\mathrm{Ad}(g)(j'(v))) \underset{9.2}{=} \mathrm{conj}_g(\underbrace{\exp^G(tj'(v))}_{\in H}) \in H$$

Since $H$ is connected, it is second countable by Proposition 2.8. Then Proposition 19.10 implies that $\mathrm{Ad}(g)(j'(v)) \in j'(\mathfrak{h})$. Now let $w \in \mathfrak{g}$ and set $g := \exp(tw)$. Then $\mathrm{Ad}(\exp(tw))(j'(v)) \in j'(\mathfrak{h})$ for all $t$. Differentiating at $t = 0$ then gives

$$\mathrm{ad}(w)(j'(v)) \underset{9.2}{=} [w, j'(v)] \in j'(\mathfrak{h}),$$

so $j'(\mathfrak{h})$ is an ideal in $\mathfrak{g}$.
(ii)⇒(i): For all $w \in \mathfrak{g}$ and all $v \in \mathfrak{h}$, $\mathrm{ad}(w)(j'(v)) = [w, j'(v)] \in j'(\mathfrak{h})$. Hence

$$\mathrm{ad}(w)(j'(\mathfrak{h})) \subseteq j'(\mathfrak{h}) \Rightarrow \mathrm{Ad}(\exp(w))(j'(\mathfrak{h})) \underset{9.2}{=} e^{\mathrm{ad}(w)}(j'(\mathfrak{h})) \subseteq j'(\mathfrak{h}).$$

Since $G$ is connected and Ad is a group homomorphism it follows (via Proposition 2.5 and Theorem 8.2) that $\mathrm{Ad}(g)(j'(\mathfrak{h})) \subseteq j'(\mathfrak{h})$ for each $g \in G$. Finally, for any $v \in \mathfrak{h}$ we have

$$\mathrm{conj}_g(j(\exp^H(v))) \underset{8.8}{=} \mathrm{conj}_g(\exp^G(j'(v))) \underset{9.2}{=} \exp^G(\mathrm{Ad}(g)(j'(v))),$$

which lies in $H$ by Proposition 19.9. Since $H$ is connected and $\mathrm{conj}_g$ is a group homomorphism, $\mathrm{conj}_g(H) \subseteq H$. □

From Corollary 21.6 we know:

$$\pi : G \to G/\ker\varphi \quad \text{is a surjective submersion and a LG-homomorphism} \tag{22.1}$$

(if $\ker\varphi$ is not open).

**22.3 Remark.** (Reminder on coverings) A continuous map $p : E \to X$ between topological spaces is called a *covering* if for any $x \in X$ there exists an open neighborhood $U_x$ of $x$ and for any $\tilde{x} \in p^{-1}(x)$ there is an open neighborhood $U_{\tilde{x}}$ of $\tilde{x}$ such that:

(i) $U_{\tilde{x}} \cap U_{\tilde{x}'} = \varnothing$ for all $\tilde{x} \neq \tilde{x}' \in p^{-1}(x)$

(ii) $p^{-1}(U_x) = \dot{\bigcup}_{\tilde{x} \in p^{-1}(x)} U_{\tilde{x}}$.

(iii) $\forall \tilde{x} \in p^{-1}(x)$: $p|_{U_{\tilde{x}}} : U_{\tilde{x}} \to U_x$ is a homeomorphism.

Then $U_x$ is called *trivializing* for $p$. If $E$, $X$ are manifolds then one additionally requires $p|_{U_{\tilde{x}}}$ to be a diffeomorphism for each $\tilde{x} \in p^{-1}(x)$.

**22.4 Theorem.** *Let $G, H$ be connected Lie groups and let $\varphi : G \to H$ be a Lie group homomorphism. Then*

*(i) If $\varphi'$ is injective, then $\ker\varphi$ is a discrete normal subgroup that is contained in the center $Z(G) := \{g \in G \mid g'g = gg' \ \forall g' \in G\}$.*

*(ii) If $\varphi'$ is surjective, then so is $\varphi$ and the map $\hat{\varphi}$ from*

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & H \\ G/\ker\varphi \arrow[ur, "\hat{\varphi}"'] & \end{tikzcd}$$

*is a Lie group isomorphism.*

*(iii) If $\varphi'$ is bijective, then $\varphi : G \to H$ is a covering map and a local diffeomorphism.*

**Proof.** (i) We have $\ker\varphi' = \{0\} \subseteq \mathfrak{g}$. Then $\ker\varphi$ must be discrete, since otherwise by Theorem 21.7 it would be a Lie subgroup with Lie algebra $\ker\varphi' \neq \{0\}$ (see Lemma 22.1). Now let $v \in \mathfrak{g}$ and $g \in \ker\varphi$. Consider the smooth curve

$$c(t) := \exp(tv) \cdot g \cdot \exp(tv)^{-1}.$$

Since $\ker\varphi$ is normal, $c(t) \in \ker\varphi$ for all $t$. But $\ker\varphi$ is discrete so $c(t) \equiv c(0) = g$ and thereby $\exp(tv)g = g\exp(tv)$ for all $t$ and $v$. Since $G$ is connected, hence generated by $\exp\mathfrak{g}$, this implies that $g \in Z(G)$.

(ii) By Theorem 8.8 (i) we have

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\varphi'"] \arrow[d, "\exp^G"'] & \mathfrak{h} \arrow[d, "\exp^H"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

Since $\varphi'(\mathfrak{g}) = \mathfrak{h}$ it follows that $\exp^H(\mathfrak{h}) \subseteq \varphi(G)$. Also, $H$ is connected, so $\exp^H(\mathfrak{h})$ generates $H$. This implies that $H \subseteq \varphi(G)$, so $\varphi$ is surjective.

Therefore there exists a unique $\hat{\varphi}$ as in (ii) and $\hat{\varphi}$ is a bijective group homomorphism. Now note that $\ker\varphi$ is not open, for if it were, it would be an open neighborhood of $e$, hence $\ker\varphi = G$ and $\varphi \equiv e$, meaning that $\varphi' = 0$. But $\varphi'$ is surjective, so $\mathfrak{h} = 0$, contradicting the fact that $H$ is a Lie group. Thus we can apply (22.1) to infer that $\pi$ is a surjective submersion. By Remark 13.2 (iii) this implies that $\hat{\varphi}$ is smooth. Also, $\hat{\varphi}' \circ \pi' = \varphi'$, so that $\hat{\varphi}'$ is surjective and

$$v \in \ker\pi' \Rightarrow \varphi'(v) = \hat{\varphi}'(\pi'(v)) = 0 \Rightarrow \ker\pi' \subseteq \ker\varphi'.$$

Let $n = \dim G$ and $k = \dim\ker\varphi$. Then by Theorem 21.7, $\dim(G/\ker\varphi) = n - k$. Hence $\dim\ker\pi' = n - (n-k) = k$, showing that $\ker\pi' = \ker\varphi'$.

Now let $\pi'(v)$ be an arbitrary element of $T_e(G/\ker\varphi)$ with $\hat{\varphi}'(\pi'(v)) = 0$. Then $\varphi'(v) = 0$ and so also $\pi'(v) = 0$. Thus $\hat{\varphi}'$ is also injective, hence bijective. It follows that $\hat{\varphi}$ is a diffeomorphism on a neighborhood of $e$, hence on a neighborhood of any point ($\hat{\varphi} = L_{\hat{\varphi}(g)} \circ \hat{\varphi} \circ L_{g^{-1}}$). Since it is also bijective, it is a global diffeomorphism and thereby a Lie group isomorphism.

(iii) Since $\hat{\varphi}$ is a diffeomorphism by (ii), it suffices to show that $\pi$ is a covering and a local diffeomorphism. By (i), $\ker\varphi$ is discrete, so there exists an open neighborhood $U$ of $e$ in $G$ such that $U \cap \ker\varphi = \{e\}$. Now pick an open neighborhood $V \subseteq U$ of $e$ with $V^{-1}V \subseteq U$. Let $V_g := R_g(V) = Vg$. Then for $v_1, v_2 \in V$ and $g_1, g_2 \in \ker\varphi$ we have

$$v_1 g_1 = v_2 g_2 \Rightarrow v_2^{-1} v_1 = g_2 g_1^{-1} \in U \cap \ker\varphi = \{e\} \Rightarrow v_1 = v_2,\ g_1 = g_2. \tag{22.2}$$

Using this we can now show:

1.) $\pi : V_g \cdot g' \to \pi(V_g)$ is bijective for each $g \in G$ and each $g' \in \ker\varphi$: First, $\pi(V_g \cdot g') = \pi(V_g)$. Moreover,

$$v_1 g g' \ker\varphi = v_2 g g' \ker\varphi \Rightarrow v_1 g = v_2 g \underbrace{\tilde{g}}_{\in \ker\varphi} \Rightarrow v_1 = v_2 \cdot \underbrace{\mathrm{conj}_g(\tilde{g})}_{\in \ker\varphi} \underset{(22.2)}{\Rightarrow} v_1 = v_2.$$

2.) If $g_1 \neq g_2$ are elements of $\ker\varphi$ and $g \in G$, then by (22.2) we have $V_g g_1 \cap V_g g_2 = \emptyset$. Therefore, for each $g \in G$:

$$\pi^{-1}(\pi(V_g)) = \dot{\bigcup_{g' \in \ker\varphi}} V_g \cdot g'.$$

Indeed, $\pi(V_g g') = Vg\ker\varphi = \pi(V_g)$ for all $g' \in \ker\varphi$ and conversely, $\pi(h) \in \pi(V_g)$ implies $h\ker\varphi \in V_g \cdot \ker\varphi$, so $h \in V_g \cdot \ker\varphi$.

By Theorem 21.5, $\dim(G/\ker\varphi) = \dim G$ and $\pi$ is a submersion, hence a local diffeomorphism. Together with 1.) this shows that $\pi : V_g g' \to \pi(V_g)$ is a diffeomorphism for all $g \in G$ and all $g' \in \ker\varphi$.

Consequently, for all $g \in G$ the set $\pi(V_g)$ is trivializing for $\pi(g) = g \cdot \ker\varphi \in G/\ker\varphi$, so $\pi : G \to G/\ker\varphi$ is a covering. □

**22.5 Corollary.** *Let $G$ be a connected abelian Lie group. Then there exist $k, m \in \mathbb{N}_0$ such that $G \cong \mathbb{T}^k \times \mathbb{R}^m$.*

**Proof.** Let $v, w \in \mathfrak{g}$ and $N \in \mathbb{N}$. Since $G$ is abelian,

$$\exp(v) \cdot \exp(w) = \exp\Big(\frac{v}{N}\Big)^N \exp\Big(\frac{w}{N}\Big)^N = (\exp\Big(\frac{v}{N}\Big)\exp\Big(\frac{w}{N}\Big))^N.$$

Now for $N$ large, we may apply the Baker–Campbell–Hausdorff formula from Theorem 12.1 to this, and since the bracket on $\mathfrak{g}$ vanishes by Proposition 5.5 (ii), $C(v/N, w/N) = v/N + w/N + 0$, so

$$\exp(v) \cdot \exp(w) = \exp\Big(\frac{1}{N}(v + w)\Big)^N = \exp(v + w).$$

Thus $\exp : \mathfrak{g} \to G$ is a Lie group homomorphism. Now $\exp' = \mathrm{id}_{\mathfrak{g}}$ is bijective, so by Theorem 22.4 (i) and (ii), $\exp : \mathfrak{g} \to G$ is surjective, $G \cong \mathfrak{g}/\ker(\exp)$ and $\ker(\exp)$ is a discrete normal subgroup in $\mathfrak{g}$. But then $\mathfrak{g}/\ker(\exp) \cong \mathbb{T}^k \times \mathbb{R}^m$. □

We now want to derive conditions under which there exists a Lie group homomorphism $\varphi$ with prescribed $\varphi'$. Globally, this might not be possible: For example, the map $\psi : (\mathbb{R}, +) \to \mathrm{U}(1)$, $t \mapsto e^{it}$ is a group homomorphism with $\psi' = i \cdot \mathrm{id}$, which is a Lie algebra isomorphism. However, there does not exist a Lie group homomorphism $\varphi : \mathrm{U}(1) \to \mathbb{R}$ with $\varphi' = (\psi')^{-1}$, since otherwise $\varphi(\mathrm{U}(1))$ would be a non-trivial compact subgroup of $(\mathbb{R}, +)$, but such groups do not exist. We therefore define:

**22.6 Definition.** *Let $G, H$ be Lie groups. A local Lie group homomorphism from $G$ to $H$ is a smooth map $\varphi : U \to H$, $U$ an open neighborhood of $e$ such that $\varphi(e) = e$ and $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in U$ with $gh \in U$. If, in addition, $\varphi$ is a diffeomorphism $U \to \varphi(U)$ then $\varphi$ is called a local isomorphism and $G$ and $H$ are called locally isomorphic.*

We shall need the following result from algebraic topology (cf., e.g., [3, III.4.22]):

**22.7 Lemma.** *If $E$ is connected and $X$ is simply connected and locally path connected, then any covering $p : E \to X$ is a homeomorphism.*

**22.8 Theorem.** *Let $G, H$ be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$ and let $f : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then there exists a local Lie group homomorphism $\varphi : G \supseteq U \to H$ with $\varphi' = f$. If $G$ is simply connected then $\varphi$ can even be chosen as a global Lie group homomorphism.*

**Proof.** $G \times H$ is a Lie group with Lie algebra $\mathfrak{g} \times \mathfrak{h}$ (see Example 4.6 (ii)). Consider the graph of $f$, $\Gamma_f := \{(v, f(v)) \mid v \in \mathfrak{g}\} \subseteq \mathfrak{g} \times \mathfrak{h}$. Since $f$ is linear, this is a linear subspace of $\mathfrak{g} \times \mathfrak{h}$. Moreover,

$$[(v, f(v)), (w, f(w))] = ([v, w], [f(v), f(w)]) = ([(v, w)], f([v, w])) \in \Gamma_f,$$

so $\Gamma_f$ is a Lie algebra. Hence by Theorem 19.5 there exists a connected Lie subgroup $K$ of $G \times H$ with Lie algebra $\mathfrak{k} = \Gamma_f$. The projections $\mathrm{pr}_1 : G \times H \to G$ and $\mathrm{pr}_2 : G \times H \to H$ are Lie group homomorphisms, so $\pi := \mathrm{pr}_1|_K : K \to G$ is a Lie group homomorphism as well.

We have $\pi'(v, f(v)) = T_e\mathrm{pr}_1(v, f(v)) = v$, so $\pi' : \mathfrak{k} \to \mathfrak{g}$ is a linear isomorphism. Hence by Theorem 22.4 (iii), $\pi : K \to G_e$ is a covering map and a local diffeomorphism. Now let $V$ and $U$ be open neighborhoods of $e$ in $K$ and $G$, respectively, such that $\pi : V \to U$ is a diffeomorphism and let $\varphi : U \to H$, $\varphi := \mathrm{pr}_2 \circ (\pi|_V)^{-1}$.

$$\begin{tikzcd}
V \arrow[r, hook, "\subseteq"] \arrow[dd, "\pi|_V"'] & K \arrow[r, hook, "\subseteq"] \arrow[d, "\pi"] & G \times H \arrow[d, "\mathrm{pr}_2"] \arrow[dl, "\mathrm{pr}_1"] \\
& G & H \\
U \arrow[ur, "\subseteq"] \arrow[urr, "\varphi = \mathrm{pr}_2 \circ (\pi|_V)^{-1}"'] & &
\end{tikzcd}$$

Then $\varphi$ is a local Lie group homomorphism because $\pi$ is a Lie group homomorphism. In addition, for any $w \in \mathfrak{g}$,

$$\varphi'(w) = \mathrm{pr}_2((\pi')^{-1}(w)) = \mathrm{pr}_2(w, f(w)) = f(w),$$

so $\varphi' = f$.

Finally, if $G$ is simply connected, then $G = G_e$ and by Lemma 22.7, $\pi : K \to G$ is a homeomorphism and a local diffeomorphism, hence a Lie group isomorphism. Consequently, $\varphi := \mathrm{pr}_2 \circ \pi^{-1}$ is a global Lie group homomorphism with $\varphi' = f$. □

As a consequence, we obtain the first and second fundamental theorem of S. Lie:

**22.9 Theorem.**

*(i) Lie's first fundamental theorem: Let $G, H$ be locally isomorphic Lie groups. Then their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are isomorphic.*

(ii) *Lie's second fundamental theorem: If the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ of the Lie groups $G$, $H$ are isomorphic, then $G$ and $H$ are locally isomorphic.*

**Proof.** (i) Let $\varphi : G \supseteq U \to V \subseteq H$ be a local isomorphism. Then $\varphi' = T_e\varphi$ is a Lie algebra isomorphism by (the proof of) Proposition 5.5.

(ii) Let $f : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra isomorphism. By Theorem 22.8, there exists some local Lie group homomorphism $\varphi : U \to H$ with $\varphi' = f$. Since $\varphi'$ is invertible, we may assume without loss of generality that $\varphi$ is a diffeomorphism from $U$ onto $\varphi(U)$, which is thereby open in $H$. Thus $\varphi$ is the desired local Lie group isomorphism. □

For the proof of the classification theorem we need one further result from algebraic topology (cf. [3]):

**22.10 Remark.** Let $X$ be path connected, locally path connected and semi locally simply connected. Then there exists a covering $p : \tilde{X} \to X$ with $\tilde{X}$ simply connected. $\tilde{X}$ is called the *universal cover* of $X$. Then $\tilde{X}$ covers any path connected covering of $X$: If $\pi : E \to X$ is a covering of $E$ and $E$ is path connected, then there exists a covering $q : \tilde{X} \to E$ such that $p = \pi \circ q$. In particular, $\tilde{X}$ is unique up to isomorphism.

If $Y$ is simply connected and $f : Y \to X$ is continuous then there exists a continuous lift $\tilde{f} : Y \to \tilde{X}$ with $p \circ \tilde{f} = f$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

The lift $\tilde{f}$ is uniquely determined by prescribing its value in any one point. If $p : M' \to M$ is a (topological) cover of a smooth manifold $M$ then there is a unique $\mathcal{C}^\infty$-structure on $M'$ such that $p$ is a local diffeomorphism. Since any manifold is locally contractible, any connected manifold is path connected, locally path connected and semi-locally simply connected. Thus there exists a unique universal cover $\tilde{M}$, which is a smooth manifold when endowed with the smooth structure just described.

Using this we can now show:

**22.11 Theorem.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then there exists a (unique up to Lie group isomorphism) simply connected Lie group $\tilde{G}$ with Lie algebra $\mathfrak{g}$. Any other connected Lie group $\hat{G}$ with Lie algebra (isomorphic to) $\mathfrak{g}$ is isomorphic to a quotient of $\tilde{G}$ by a discrete normal subgroup $H \subseteq \tilde{G}$ with $H \subseteq Z(\tilde{G})$.*

**Proof.** By Theorem 19.8 there exists a Lie group $G$ with Lie algebra $\mathfrak{g}$. Let $p : \tilde{G} \to G$ be the universal cover of $G$. Then $\tilde{G}$ is a simply connected smooth manifold and $p$ is a local diffeomorphism. Let $\tilde{e}$ be an element of $\tilde{G}$ with $p(\tilde{e}) = e$. Since also $\tilde{G} \times \tilde{G}$ is simply connected, by Remark 22.10 the smooth map $\mu \circ (p \times p) : \tilde{G} \times \tilde{G} \to G$ possesses a unique smooth (since $p$ is a local diffeomorphism) lift $\tilde{\mu} : \tilde{G} \times \tilde{G} \to \tilde{G}$ with $\tilde{\mu}(\tilde{e}, \tilde{e}) = \tilde{e}$. Analogously, there exists a unique smooth lift $\tilde{\nu} : \tilde{G} \to \tilde{G}$ of $\nu \circ p : \tilde{G} \to G$ with $\tilde{\nu}(\tilde{e}) = \tilde{e}$.

Then both $\tilde{\mu} \circ (\mathrm{id} \times \tilde{\mu})$ and $\tilde{\mu} \circ (\tilde{\mu} \times \mathrm{id})$ are lifts of $\mu \circ (\mathrm{id} \times \mu) \circ (p \times (p \times p)) : \tilde{G} \times \tilde{G} \times \tilde{G} \to G$:

$$\begin{aligned} p \circ \tilde{\mu} \circ (\mathrm{id} \times \tilde{\mu}) &= \mu \circ (p \times p) \circ (\mathrm{id} \times \tilde{\mu}) = \mu \circ (p \times (p \circ \tilde{\mu})) = \mu \circ (p \times (\mu \circ (p \times p))) \\ &= \mu \circ (\mathrm{id} \times \mu) \circ (p \times (p \times p)) = \mu \circ (\mu \times \mathrm{id}) \circ ((p \times p) \times p) \\ &= \cdots = p \circ \tilde{\mu} \circ (\tilde{\mu} \times \mathrm{id}) \end{aligned}$$

by the associativity of $\mu$. Moreover, both maps take the value $\tilde{e}$ at $(\tilde{e}, \tilde{e}, \tilde{e})$, hence they coincide. This shows the associativity of the multiplication on $\tilde{G}$. Analogously, one verifies all the other properties of $\tilde{\mu}$ and $\tilde{\nu}$, so $\tilde{G}$ is a Lie group.

The covering map $p : \tilde{G} \to G$ is a Lie group homomorphism because by definition we have

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \xrightarrow{\tilde{\mu}} & \tilde{G} \\ {\scriptstyle p\times p}\downarrow & & \downarrow{\scriptstyle p} \\ G \times G & \xrightarrow{\mu} & G \end{array}$$

Since $p$ is a local diffeomorphism, it also is a local Lie group isomorphism, so $p' : \tilde{\mathfrak{g}} \to \mathfrak{g}$ is an isomorphism of Lie algebras (cf. Theorem 22.9 (i)). Thus $\tilde{\mathfrak{g}} \cong \mathfrak{g}$, so $\tilde{G}$ indeed has Lie algebra $\mathfrak{g}$.

Now let $\hat{G}$ be another connected Lie group with Lie algebra (isomorphic to) $\mathfrak{g}$. Since $\tilde{G}$ is simply connected, Theorem 22.8 shows the existence of a Lie group homomorphism $\varphi : \tilde{G} \to \hat{G}$ with $\varphi' = \mathrm{id}_{\mathfrak{g}}$ (resp. equal to the given Lie algebra isomorphism). By Theorem 22.4 (iii) $\varphi$ is a covering map and a local diffeomorphism. Moreover, by Theorem 22.4 (ii) $\hat{\varphi} : \tilde{G}/\ker\varphi \to \hat{G}$ is a Lie group isomorphism and $\ker\varphi$ is a discrete normal subgroup with $\ker\varphi \subseteq Z(\tilde{G})$ by Theorem 22.4 (i).

Finally, if $\hat{G}$ is simply connected then $\varphi$ is a homeomorphism by Lemma 22.7. Since it is a local diffeomorphism it is therefore even a global diffeomorphism, hence a Lie group isomorphism. □

# 23 Representation theory of compact Lie groups

In this and the following sections we study some basics of the representation theory of compact Lie groups, following [11]. Recall from Definition 9.1 that a representation of a Lie group $G$ on a finite dimensional (which we will henceforth always tacitly assume) vector space $V$ is a homomorphism of Lie groups $\varphi : G \to \mathrm{GL}(V)$. When the map $\varphi$ is clear from the context we often simply write $g \cdot v$ or $gv$ for $(\varphi(g))(v)$ $(g \in G,\ v \in V)$. Two representations will be called equivalent if they are the same up to change of basis, more precisely:

**23.1 Definition.** *Let $(\varphi, V)$ and $(\psi, W)$ be representations of a Lie group $G$.*

*(i) $T \in L(V, W)$ is called an* intertwining operator *or $G$-map if $T \circ \varphi = \psi \circ T$.*

*(ii) The set of all $G$-maps is denoted by* $\mathrm{Hom}_G(V, W)$.

*(iii) The representations $V$ and $W$ are* equivalent*, denoted by $V \cong W$, if there exists a bijective $G$-map from $V$ to $W$.*

**23.2 Example.** The *standard representation* of any matrix Lie group (subgroup of $\mathrm{GL}(n)$) on $\mathbb{C}^n$ is $\varphi(g) := v \mapsto g \cdot v$, i.e., matrix multiplication on the left.

Standard operations from (multi-)linear algebra can be transferred to representations of Lie groups. For the following result, note that given a vector space $V$, by $V^*$ we denote its dual and by $\bar{V}$ its *conjugate*. The latter has the same additive structure as $V$, but has a new scalar multiplication structure $\lambda \cdot' v := \bar{\lambda} v$.

**23.3 Definition.** *Let $V$ and $W$ be finite-dimensional representations of a Lie group $G$. Then $G$ acts on*

*(i) $V \oplus W$ by $g(v, w) = (gv, gw)$.*

*(ii) $V \otimes W$ by $g \sum v_i \otimes w_j = \sum gv_i \otimes gw_j$.*

*(iii) $L(V, W)$ by $(gT)(v) = g\big(T(g^{-1}v)\big)$.*

*(iv)* $\bigotimes^k V$ *by* $g \sum v_{i_1} \otimes \cdots \otimes v_{i_k} = \sum (gv_{i_1}) \otimes \cdots \otimes (gv_{i_k})$.

*(v)* $\bigwedge^k V$ *by* $g \sum v_{i_1} \wedge \cdots \wedge v_{i_k} = \sum (gv_{i_1}) \wedge \cdots \wedge (gv_{i_k})$.

*(vi)* $V^*$ *by* $(gT)(v) = T(g^{-1}v)$.

*(vii)* $\bar{V}$ *by the same action it has on* $V$.

It is straightforward to check that each of these maps indeed provides a representation. For example, for (iii) we calculate:

$$[g_1(g_2T)](v) = g_1[(g_2T)(g_1^{-1}v)] = g_1g_2[T(g_2^{-1}g_1^{-1}v)] = [(g_1g_2)T](v).$$

A main goal of representation theory is to obtain a classification of possible representations. A first step is to single out smallest building blocks:

**23.4 Definition.** *Let $V$ be a representation of $G$.*

*(i) A subspace $U$ of $V$ is called $G$-invariant (or a submodule of $G$) if $gU \subseteq U$ for each $g \in G$. Thus $U$ itself is also a representation of $G$.*

*(ii) A nonzero representation $V$ is called* irreducible *if the only $G$-invariant subspaces are $\{0\}$ and $V$ itself. It is called* reducible *if there exists a non-trivial proper $G$-invariant subspace of $V$.*

Thus $V$ is irreducible if and only if $V = \operatorname{span}_{\mathbb{C}}\{gv \mid g \in G\}$ for each nonzero $v \in V$: violation of this condition for one $v$ is equivalent to the existence of a proper $G$-invariant subspace.

**23.5 Theorem.** *(Schur's Lemma) If representations $V$ and $W$ of $G$ are irreducible, then*

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases}$$

**Proof.** Suppose that there is some $0 \neq T \in \operatorname{Hom}_G(V, W)$. Then $\ker T$ is not all of $V$ and is $G$-invariant, so by irreducibility we must have $\ker T = 0$, i.e., $T$ is injective. Analogously, the image of $T$ is nonzero and $G$-invariant, hence must equal $W$, implying that $T$ is a bijection. Hence there exists a nonzero $T \in \operatorname{Hom}_G(V, W)$ if and only if $V \cong W$.

Suppose now that $V \cong W$ and fix a bijective $T_0 \in \operatorname{Hom}_G(V, W)$. If $T \in \operatorname{Hom}_G(V, W)$, then $T \circ T_0^{-1} \in \operatorname{Hom}_G(W, W)$. Since $W$ is a finite-dimensional complex vector space, there exists an eigenvalue $\lambda$ of $T \circ T_0^{-1}$. Also, $\ker(T \circ T_0^{-1} - \lambda I)$ is nonzero and $G$-invariant, so by irreducibility we must have $T \circ T_0^{-1} = \lambda I$, implying that $\operatorname{Hom}_G(V, W) = \mathbb{C}T_0$. □

Note that, in particular,

$$\operatorname{Hom}_G(V, V) = \mathbb{C}I \tag{23.1}$$

for any irreducible $V$.

**23.6 Definition.** *Let $V$ be a representation of $G$. A form $(\,.\,,\,.\,) : V \times V \to \mathbb{R}$ (resp. $V \times V \to \mathbb{C}$) is called $G$-invariant if $(gv, gv') = (v, v')$ for all $g \in G$ and $v, v' \in V$. $V$ is called* Euclidean *resp.* unitary *if there exists a $G$-invariant Euclidean (resp. Hermitian) inner product on $V$.*

We will follow the convention that Hermitian inner products are conjugate linear in the second factor.

**23.7 Theorem.** *Let $G$ be a compact Lie group and let $\rho$ be a representation of $G$ on a real or complex vector space $V$. Then $\rho$ is Euclidean or unitary.*

**Proof.** We have to show that there exists a $G$-invariant inner product on $V$. Let $v_1, \dots, v_n$ be a basis of $\mathfrak{g}$, with corresponding right invariant vector fields $R^{v_i} \in \mathfrak{X}_R(G)$ $(1 \leq i \leq n)$. Denote by $\omega^i \in \Omega^1(G)$ the corresponding dual one forms, i.e., $\omega^i(R^{v_j}) = \delta^i_j$. Then the $\omega^i$ are right invariant as well, $R_g^* \omega^i = \omega^i$ for all $g \in G$ and all $i$: To see this it suffices to verify that also $R_g^* \omega^i(R^{v_j}) = \delta^i_j$. Indeed, for any $h \in G$ we have

$$R_g^* \omega^i(R^{v_j})|_h = \omega^i(R_g(h))(T_h R_g(R^{v_j}(h))) = \omega^i(hg)(R^{v_j}(hg)) = \omega^i(R^{v_j})(hg) = \delta^i_j.$$

Consequently, $\omega := \omega^1 \wedge \cdots \wedge \omega^n$ is a right invariant volume form on $G$. Now let $\langle\, .\, ,\, .\, \rangle$ be a Euclidean resp. Hermitian scalar product on $V$ and for $v, w \in V$ put

$$(v, w) := \int_G \langle \rho(g)v, \rho(g)w \rangle \omega_g.$$

Since $G$ is compact, this defines a new Euclidean resp. Hermitian scalar product on $V$ (with definiteness following since for $v \neq 0$ we have $\langle \rho(g)v, \rho(g)v \rangle > 0$ for all $g$). In addition, $(\, .\, ,\, .\, )$ is $G$-invariant, because

$$\begin{aligned}(\rho(h)v, \rho(h)w) &= \int_G \langle \rho(gh)v, \rho(gh)w \rangle \omega_g = \int_G R_h^*(\langle \rho(g)v, \rho(g)w \rangle) \underbrace{\omega_g}_{=R_h^* \omega_g} \\ &\underset{Subst.}{=} \int_G \langle \rho(g)v, \rho(g)w \rangle \omega_g = (v, w).\end{aligned}$$

□

**23.8 Remark.** In the previous proof we constructed a right invariant volume form on any compact Lie group. Analogously we can construct a left invariant volume form $\omega \in \Omega^n(G)$. If we additionally require that $\int_G 1 dg \equiv \int_G \omega = 1$, then the corresponding Borel measure is uniquely determined and is called the *Haar* measure on $G$. For the general construction of the Haar measure on locally compact groups see [4].

It follows that representations of compact Lie groups in fact take values in the (orthogonal or) unitary group on $V$. Concerning the relation between $G$-invariant scalar products and the derivative of the underlying representation we have:

**23.9 Theorem.** *Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a (not necessarily compact) Lie group $G$ and let $(\, .\, ,\, .\, )$ be a $G$-invariant scalar product on $V$, i.e.,*

$$(\rho(g)v, \rho(g)w) = (v, w) \qquad \forall g \in G, v, w \in V. \tag{23.2}$$

*Then for $\rho' \equiv T_e \rho$ we have*

$$(\rho'(X)v, w) + (v, \rho'(X)w) = 0 \qquad \forall X \in \mathfrak{g}, v, w \in V. \tag{23.3}$$

*If $G$ is connected, then* (23.2) *and* (23.3) *are equivalent.*

**Proof.** Let $X \in \mathfrak{g}$ and $v, w \in V$. Then by (23.2) and Theorem 8.8 we have

$$(v, w) = (\rho(\exp(tX))v, \rho(\exp(tX))w) = (e^{t\rho'(X)}v, e^{t\rho'(X)}w)$$

for each $t$, which gives (23.3) upon differentiating at $t = 0$.

Conversely, suppose that $G$ is connected and that (23.3) holds. Then $G$ is generated by any neighborhood of $e$, and so since exp is a local diffeomorphism and $\rho$ is a homomorphism, it suffices to show that

$$(\rho(\exp(tX))v, \rho(\exp(tX))w) = (v, w) \tag{23.4}$$

for any $X \in \mathfrak{g}$ and $v, w \in V$. Fixing $X, v, w$, for $t \in \mathbb{R}$ set

$$F(t) := (\rho(\exp(tX))v, \rho(\exp(tX))w) = (e^{t\rho'(X)}v, e^{t\rho'(X)}w).$$

Then

$$F'(t) = (\rho'(X)e^{t\rho'(X)}v, e^{t\rho'(X)}w) + (e^{t\rho'(X)}v, \rho'(X)e^{t\rho'(X)}w).$$

Now (23.3) implies that $F'(t) = 0$ for each $t$, i.e., $F(t) \equiv F(0) = (v, w)$, establishing (23.4). □

**23.10 Definition.** *A representation of a Lie group is called* completely reducible *if it is a direct sum of irreducible representations.*

**23.11 Corollary.** *Any finite dimensional representation of a compact Lie group is completely reducible.*

**Proof.** Suppose that $V$ is a reducible representation of a compact Lie group $G$, and pick a $G$-invariant scalar product on $V$. Then there exists a proper $G$-invariant subspace $W$ of $V$ and we can decompose $V$ into $W \oplus W^\perp$. Now also $W^\perp$ is a $G$-invariant proper subspace because

$$(gw', w) = (w', g^{-1}w) = 0 \qquad (w \in W,\ w' \in W^\perp).$$

Since $V$ is finite dimensional, the claim follows by induction. □

It follows that any representation of a compact Lie group can be written as

$$V \cong \bigoplus_{i=1}^{N} n_i V_i, \tag{23.5}$$

where $V_i$ $(1 \leq i \leq N)$ is a collection of inequivalent irreducible representations of $G$ and $n_i V_i := V_i \oplus \cdots \oplus V_i$ ($n_i$ copies).

**23.12 Corollary.** *A finite dimensional representation $V$ of a compact Lie group $G$ is irreducible if and only if* $\dim \operatorname{Hom}_G(V, V) = 1$.

**Proof.** If $V$ is irreducible, the claim follows from (23.1). Conversely, suppose that $V$ is reducible, $V = W \oplus W'$ for proper submodules $W$, $W'$ of $V$. Then $\operatorname{Hom}_G(V, V)$ contains at least the projections onto each of these subspaces, hence is of dimension $\geq 2$. □

**23.13 Corollary.** *Let $V$ be a representation of the compact Lie group $G$.*

*(i)* $\bar{V} \cong V^*$.

*(ii) If $V$ is irreducible, then the $G$-invariant scalar product on $V$ is unique up to a positive scalar multiple.*

**Proof.** (i) Let $(\,.\,.\,)$ be a $G$-invariant scalar product on $V$ and define the bijective linear map $T : \bar{V} \to V^*$ by $Tv := (\,.\,, v)$. Then by Definition 23.3 (vi),

$$g(Tv) = (g^{-1}\,.\,, v) = (\,.\,, gv) = T(gv),$$

so $T$ is a $G$-map.

(ii) Let $V$ be irreducible and suppose that $(\,.\,,\,.\,)'$ is another $G$-invariant inner product on $V$. Define a bijective linear map $T' : \bar{V} \to V^*$ by $T'v = (\,.\,,v)'$. Then $T'$ is a $G$-map. Schur's Lemma (Theorem 23.5) implies that $\dim \mathrm{Hom}_G(\bar{V}, V^*) = 1$. Now $T, T' \in \mathrm{Hom}_G(\bar{V}, V^*)$, so there exists some $c \in \mathbb{C}$ with $T' = cT$. But then $(\,.\,,v)' = c(\,.\,,v)$ for all $v$, and obviously $c$ must be positive. $\square$

**23.14 Corollary.** *Let $(\,.\,,\,.\,)$ be a $G$-invariant inner product on $V$. If $V_1, V_2$ are inequivalent irreducible submodules of $V$, then $V_1 \perp V_2$.*

**Proof.** Let $W_1 := \{v_1 \in V_1 \mid (v_1, V_2) = 0\}$. The invariance of $(\,.\,,\,.\,)$ entails that $W_1$ is a submodule of $V_1$. If $(V_1, V_2) \neq 0$, i.e., if $W_1 \neq V_1$, then by irreducibility we must have $W_1 = \{0\}$. Analogously we can argue with $W_2 := \{v_2 \in V_2 \mid (V_1, v_2) = 0\}$, so $(\,.\,,\,.\,)$ is a nondegenerate pairing of $V_1$ and $V_2$. Therefore, $\phi := v_1 \mapsto (\,.\,, v_1)$ is an injective map from $V_1$ into $V_2^*$. Analogously, $V_2$ can be injectively mapped into $V_1^*$, so $\dim V_1 = \dim V_2$, and $\phi$ is in fact an equivalence $\overline{V_1} \cong V_2^*$. By Corollary 23.13 then $V_1 \cong \overline{V_2^*} \cong V_2$. $\square$

**23.15 Definition.** *Let $G$ be a compact Lie group.*

*(i) We denote the set of equivalence classes of irreducible (unitary) representations of $G$ by $\hat{G}$. For any class $[\rho] \in \hat{G}$ we may, when needed, pick a representative $(\rho, E_\rho)$.*

*(ii) Let $V$ be a finite dimensional representation of $G$. For $[\rho] \in \hat{G}$ denote by $V_{[\rho]}$ the largest subspace of $V$ that is a direct sum of irreducible submodules equivalent to $E_\rho$. Then $V_{[\rho]}$ is called the $\rho$-*isotypic* component of $V$.*

*(iii) The* multiplicity *of $\rho$ in $V$, $m_\rho$, is*

$$m_\rho := \frac{\dim V_{[\rho]}}{\dim E_\rho},$$

*i.e., $V_{[\rho]} \cong m_\rho E_\rho$.*

The following Lemma shows that $V_{[\rho]}$ is well-defined and that $V_{[\rho]}$ is the sum of *all* submodules of $V$ that are equivalent to $E_\rho$.

**23.16 Lemma.** *If $V_1$, $V_2$ are direct sums of irreducible submodules isomorphic to $E_\rho$ then so is $V_1 + V_2$.*

**Proof.** Let $W_1, \dots, W_k$ be irreducible submodules isomorphic to $E_\rho$ and suppose first that $V_1 = W_1$ and $V_2 = W_2 \oplus \cdots \oplus W_k$. Then if $W_1 \subseteq W_2 \oplus \cdots \oplus W_k$ there is nothing to do. Otherwise, $W_1 \cap (W_2 \oplus \cdots \oplus W_k)$ is a $G$-invariant proper subspace of $W_1$, hence must equal $\{0\}$ by irreducibility of $W_1$, giving the claim. Finally, if $V_1$ is a non-trivial direct sum we can argue as before for each summand in turn. $\square$

If $V, W$ are representations of $G$ and $V \cong W \oplus W$, then this decomposition is not canonical: If $c \in \mathbb{C} \setminus \{0\}$, then also $W' := \{(w, cw) \mid w \in W\}$ and $W'' := \{(w, -cw) \mid w \in W\}$ are submodules, both equivalent to $W$ and $V \cong W' \oplus W''$. The following result shows how to obtain a canonical decomposition:

**23.17 Theorem.** *Let $V$ be a finite dimensional representation of a compact Lie group $G$.*

*(i) There is a $G$-intertwining isomorphism*

$$\iota_\rho : \mathrm{Hom}_G(E_\rho, V) \otimes E_\rho \to V_{[\rho]}$$

*induced by the map $T \otimes v \mapsto T(v)$ for $T \in \mathrm{Hom}_G(E_\rho, V)$ and $v \in E_\rho$. In particular, the multiplicity of $\rho$ is*

$$m_\rho = \dim \mathrm{Hom}_G(E_\rho, V).$$

*(ii) There is a $G$-intertwining isomorphism*

$$\bigoplus_{[\rho]\in\hat{G}} \mathrm{Hom}_G(E_\rho, V) \otimes E_\rho \to V = \bigoplus_{[\rho]\in\hat{G}} V_{[\rho]}.$$

**Proof.** (i) Let $0 \neq T \in \mathrm{Hom}_G(E_\rho, V)$. Then since $E_\rho$ is irreducible, $\ker T = \{0\}$. Therefore, $T$ is an equivalence of $E_\rho$ with $T(E_\rho)$, so $T(E_\rho) \subseteq V_{[\rho]}$. Now $G$ acts trivially on $\mathrm{Hom}_G(E_\rho, V)$: Let $w \in E_\rho$ and $T \in \mathrm{Hom}_G(E_\rho, V)$. Then by Definitions 23.1 and 23.3 (iii) we have

$$(gT)(w) = g(T(g^{-1}w)) = T(gg^{-1}w) = T(w).$$

It follows that $g(T \otimes v) = T \otimes gv$, so

$$\iota_\rho(g(T \otimes v)) = T(gv) = gT(v) = g\iota_\rho(T \otimes v),$$

showing that $\iota_\rho$ is intertwining. It is also surjective: Let $V_1 \cong E_\rho$ be a direct summand in $V_{[\rho]}$, with corresponding equivalence $T : E_\rho \to V_1$. Then $T \in \mathrm{Hom}_G(E_\rho, V)$ and for any $v_1 \in V_1$ we can pick $v \in E_\rho$ with $T(v) = v_1$. Thus $\iota_\rho(T \otimes v) = T(v) = v_1$, showing that $V_1$ lies in the image of $\iota_\rho$.

To see that in fact $\iota_\rho$ is bijective we show that the dimensions of its domain and target coincide. Let $V_{[\rho]} = V_1 \oplus \cdots \oplus V_{m_\rho}$, with $V_i \cong E_\rho$ for all $i$. Then from Schur's Lemma (Theorem 23.5) we obtain

$$\begin{aligned}\dim \mathrm{Hom}_G(E_\rho, V) &= \dim \mathrm{Hom}_G(E_\rho, V_{[\rho]}) = \dim \mathrm{Hom}_G(E_\rho, V_1 \oplus \cdots \oplus V_{m_\rho}) \\ &= \sum_{i=1}^{m_\rho} \dim \mathrm{Hom}_G(E_\rho, V_i) = m_\rho.\end{aligned}$$

Therefore, $\dim \mathrm{Hom}_G(E_\rho, V) \otimes E_\rho = m_\rho \dim E_\rho = \dim V_{[\rho]}$.

(ii) Due to (i) it only remains to show that $V = \bigoplus_{[\rho]\in\hat{G}} V_{[\rho]}$. Now by (23.5), $V = \sum_{[\rho]\in\hat{G}} V_{[\rho]}$, and Corollary 23.14 shows that the sum is direct. □

Note that we thereby also have derived a formula for the $n_i$ in (23.5).

# 24 Matrix coefficients

In this section we study function spaces on a compact Lie group $G$ in terms of unitary representations of $G$.

Let $(\rho, V)$ be a finite dimensional unitary representation of a compact Lie group $G$ with $G$-invariant inner product $( . , . )$. Fix some basis $v_i$ $(1 \leq i \leq n)$ of $V$ and let $v_j^*$ be the corresponding dual basis, $(v_i, v_j^*) = \delta_{ij}$ (i.e., we identify $V^*$ with $V$ by means of the given scalar product). Then the linear map $\rho(g) : V \to V$ has the matrix representation $(gv_j, v_i^*)$ with respect to the basis $v_i$, because

$$gv_i = \sum_{j=1}^{n} (gv_i, v_j^*) v_j.$$

The smooth functions $g \mapsto (gv_j, v_i^*)$, $G \to \mathbb{C}$ motivate the following definition.

**24.1 Definition.** *Any function on a compact Lie group of the form* $f^V_{u,v}(g) = (gu, v)$ *for a finite dimensional unitary representation* $V$ *of* $G$ *with* $u, v \in V$ *and* $G$*-invariant inner product* $(\,.\,,\,.\,)$ *is called a* matrix coefficient *of* $G$*. The set of all matrix coefficients is denoted by* $\mathrm{MC}(G)$*.*

**24.2 Lemma.** $\mathrm{MC}(G)$ *is a subalgebra of* $\mathcal{C}^\infty(G,\mathbb{C})$ *containing the constant functions. If* $[\rho] \in \hat{G}$ *and* $(v_1^\rho, \dots, v_{n_\rho}^\rho)$ *is a basis for* $E_\rho$*, then*

$$\{f^{E_\rho}_{v_i^\rho, v_j^\rho} \mid [\rho] \in \hat{G},\ 1 \le i, j \le n_\rho\}$$

*spans* $\mathrm{MC}(G)$*.*

**Proof.** Let $V, V'$ be unitary representations of $G$ with inner products $(\,.\,,\,.\,)_V$ and $(\,.\,,\,.\,)_{V'}$. Then $V \oplus V'$ is unitary with respect to the inner product

$$((u,u'),(v,v'))_{V\oplus V'} := (u,v)_V + (u',v')_{V'}$$

and $V \otimes V'$ is unitary with respect to the inner product

$$\Big(\sum_i u_i \otimes u_i', \sum_j v_j \otimes v_j'\Big)_{V\otimes V'} := \sum_{i,j} (u_i, v_j)_V (u_i', v_j')_{V'}.$$

Therefore, $cf^V_{u,v} + f^{V'}_{u',v'} = f^{V\oplus V'}_{(cu,u'),(v,v')}$ and $f^V_{u,v} f^{V'}_{u',v'} = f^{V\otimes V'}_{u\otimes u', v\otimes v'}$, so $\mathrm{MC}(G)$ is indeed an algebra. The constant functions are attained as matrix coefficients of the trivial representation.

As in the proof of Corollary 23.11 we decompose $V$ into a direct sum $V = \bigoplus_i V_i$ of irreducible and mutually perpendicular submodules $V_i \cong E_{\rho_i}$. Then any $u, v \in V$ can be written as $u = \sum_i u_i$, $v = \sum_i v_i$ with $u_i, v_i \in V_i$, so $f^V_{u,v} = \sum_i f^{V_i}_{u_i,v_i}$. If $T_i : V_i \to E_{\rho_i}$ is an intertwining isomorphism, then $(T_i u_i, T_i v_i)_{E_{\rho_i}} := (u_i, v_i)_{V_i}$ defines a unitary structure on $E_{\rho_i}$ such that $f^V_{u,v} = \sum_i f^{E_{\rho_i}}_{T_i u_i, T_i v_i}$. Then expanding $T_i u_i$ and $T_i v_i$ into the basis for $E_{\rho_i}$ proves the last claim. ◻

Concerning the $L^2$ inner products of matrix coefficients with respect to irreducible representations we have:

**24.3 Theorem.** *(Schur orthogonality relations) Let* $(\rho_U, U)$*,* $(\rho_V, V)$ *be irreducible finite dimensional unitary representations of a compact Lie group* $G$ *with invariant inner products* $(\,.\,,\,.\,)_U$ *and* $(\,.\,,\,.\,)_V$*. If* $u_i \in U$ *and* $v_i \in V$*, then*

$$\int_G (gu_1, u_2)_U \overline{(gv_1, v_2)_V}\, dg = \begin{cases} 0 & \text{if } U \ncong V, \\ \frac{1}{\dim V}(u_1, v_1)_V \overline{(u_2, v_2)_V} & \text{if } U = V. \end{cases}$$

**Proof.** Given $u \in U$ and $v \in V$, set $T_{u,v} : U \to V$, $T_{u,v}(\,.\,) := (\,.\,, u)_U \cdot v$. After choosing bases, for any $g \in G$ we may consider the map $g \mapsto \rho_V(g) \circ T_{u,v} \circ \rho_U^{-1}(g)$ as a matrix valued function. Integrating coordinate-wise, we define $\tilde{T}_{u,v} : U \to V$ by

$$\tilde{T}_{u,v} := \int_G \rho_V(g) \circ T_{u,v} \circ \rho_U(g)^{-1}\, dg.$$

Since the Haar measure is left invariant we get for any $h \in G$

$$\begin{aligned} \rho_V(h) \circ \tilde{T}_{u,v} &= \int_G \rho_V(hg) \circ T_{u,v} \circ \rho_U(g)^{-1}\, dg \\ &= \int_G \rho_V(g) \circ T_{u,v} \circ \rho_U(h^{-1}g)^{-1}\, dg = \tilde{T}_{u,v} \circ \rho_U(h), \end{aligned}$$

showing that $\tilde{T}_{u,v} \in \mathrm{Hom}_G(U,V)$. By Schur's Lemma (Theorem 23.5) it follows that $\tilde{T}_{u,v} = cI$, where $c = c(u,v) \in \mathbb{C}$ if $U = V$, and $c = 0$ if $U \ncong V$. We have

$$\begin{aligned} c(u_2,v_2)\cdot(u_1,v_1)_V = (\tilde{T}_{u_2,v_2}u_1,v_1)_V &= \int_G (gT_{u_2,v_2}g^{-1}u_1,v_1)_V\,dg \\ &= \int_G ((g^{-1}u_1,u_2)_U gv_2,v_1)_V\,dg \underset{g\mapsto g^{-1}}{=} \int_G (gu_1,u_2)_U(g^{-1}v_2,v_1)_V\,dg \\ &= \int_G (gu_1,u_2)_U(v_2,gv_1)_V\,dg = \int_G (gu_1,u_2)_U\overline{(gv_1,v_2)_V}\,dg. \end{aligned}$$

This already implies the claim if $U \ncong V$. Next, let $U = V$. Then we need to calculate $c(u_2,v_2)$. To do this, take the trace of $c(u_2,v_2)I = \tilde{T}_{u_2,v_2}$:

$$c(u_2,v_2)\dim V = \mathrm{tr}\tilde{T}_{u_2,v_2} = \int_G \mathrm{tr}(g\circ T_{u_2,v_2}\circ g^{-1})\,dg = \int_G \mathrm{tr}(T_{u_2,v_2})\,dg = \mathrm{tr}(T_{u_2,v_2}).$$

To determine $\mathrm{tr}(T_{u_2,v_2})$ for nonzero $u_2$, choose an orthonormal basis for $U = V$ of the form $(w_1,\dots,w_n)$, where $n = \dim V$ and $w_1 = v_2$ (noting that without loss of generality we can suppose that $v_2$ is a unit vector). Then since $T_{u_2,v_2}(\,.\,) = (\,.\,,u_2)_V v_2$ we obtain

$$\mathrm{tr}(T_{u_2,v_2}) = \sum_{i=1}^n (T_{u_2,v_2}w_i,w_i)_V = \sum_{i=1}^n (v_2,w_i)_V(w_i,u_2)_V = (v_2,u_2)_V.$$

Thus, finally, $c(u_2,v_2) = \frac{1}{n}\overline{(u_2,v_2)_V}$. □

If, instead of $U = V$ we merely have $U \cong V$ and if $T: U \to V$ is a $G$-intertwining isomorphism, then due to Corollary 23.13 there exists some $\lambda > 0$ such that $(u_1,u_2)_U = \lambda\cdot(Tu_1,Tu_2)_V$. Then $(gu_1,u_2)_U = \lambda(Tgu_1,Tu_2)_V = \lambda(gTu_1,Tu_2)_V$, so Theorem 24.3 shows that the Schur orthogonality relation in this case reads

$$\int_G (gu_1,u_2)_U\overline{(gv_1,v_2)_V}\,dg = \frac{\lambda}{\dim V}(Tu_1,v_1)_V\overline{(Tu_2,v_2)_V}.$$

# 25 Infinite dimensional representations

We now want to also consider representations of compact Lie groups on infinite dimensional spaces. Ultimately, it will turn out that the resulting theory can be reduced to the finite dimensional case. To see this, we first have to extend the required definitions to a more general setting.

A topological vector space is a vector space endowed with a topology such that the vector space operations are continuous. If $V, V'$ are topological vector spaces we write $\mathrm{Hom}(V,V')$ for the vector space of *continuous* linear maps from $V$ to $V'$, and $\mathrm{GL}(V,V')$ for the subset of (continuously) invertible elements of $\mathrm{Hom}(V,V')$.

**25.1 Definition.**

*(i) A* representation *of a Lie group $G$ on a topological vector space $V$ is a pair $(\rho,V)$, where $\rho: G\to \mathrm{GL}(V)$ is a homomorphism and the map $G\times V\to V$, $(g,v)\mapsto \rho(g)v$ is continuous.*

*(ii) If $(\rho,V)$, $(\rho',V')$ are representations on topological vector spaces, $T\in\mathrm{Hom}(V,V')$ is called an* intertwining operator *or $G$-map if $T\circ\rho = \rho'\circ T$.*

*(iii) The set of all $G$-maps is denoted by $\mathrm{Hom}_G(V,V')$.*

*(iv) The representations $V$ and $V'$ are called* equivalent, *$V\cong V'$, if there exists a bijective $G$-map from $V$ to $V'$, whose inverse is also a $G$-map.*

*(v) A subspace $U \subseteq V$ is called $G$-invariant if $gU \subseteq U$ for each $g \in G$. If $U$ is closed then the resulting representation on $U$ is called a* submodule *or* subrepresentation.

*(vi) A nonzero representation $V$ is called* irreducible *if the only closed $G$-invariant subspaces are $\{0\}$ and $V$ itself. A nonzero representation is called* reducible *if there is a proper closed $G$-invariant subspace of $V$.*

The most interesting cases will be unitary representations on Hilbert spaces, although many results are in fact applicable to Hausdorff topological vector spaces, and in particular to Fréchet spaces (completely metrizable locally convex vector spaces).

**25.2 Example.** The Lie group $S^1$ acts on $L^2(S^1)$ via $(\rho(e^{i\varphi})f)(e^{i\alpha}) := f(e^{i(\alpha-\varphi)})$.

Before we continue we need to recall some basic facts about vector valued integration, referring to [10] for details. Let $V$ be a Hausdorff locally convex vector space and let $f : G \to V$ be continuous. Then there exists a unique element in $V$, called $\int_G f(g)\,dg$ such that

$$T\Big(\int_G f(g)\,dg\Big) = \int_G T(f(g))\,dg$$

for each $T \in \mathrm{Hom}(V,\mathbb{C})$. If $V$ is a Fréchet space, then $\int_G f(g)\,dg$ is a limit of elements of the form

$$\sum_{i=1}^{n} f(g_i)\,dg(\Delta_i),$$

where $\{\Delta_i\}_{i=1}^n$ is a Borel partition of $G$ (recall that $G$ is compact) and $g_i \in \Delta_i$.

A linear map $T$ on a Hilbert space $V$ is *positive* if $(Tv,v) \ge 0$ for all $v \in V$ and strictly greater than 0 for some $v$. It is called *compact* if the unit ball is mapped by $T$ to a relatively compact set. The set of all compact operators is a closed left and right ideal under composition with bounded operators.

**25.3 Lemma.** *Let $(\rho,V)$ be a unitary representation of a compact Lie group $G$ on a Hilbert space $V$. Then there exists a nonzero finite dimensional (hence closed) $G$-invariant subspace of $V$.*

**Proof.** Let $T_0 \in \mathrm{Hom}(V,V)$ be any self-adjoint positive compact operator (e.g., any finite rank projection). Then set (using vector valued integration)

$$T := \int_G \rho(g) \circ T_0 \circ \rho(g)^{-1}\,dg. \tag{25.1}$$

As noted above, $T$ is a limit (in $\mathrm{Hom}(V,V)$, hence with respect to the operator norm) of operators of the form $\sum dg(\Delta_i)\rho(g_i) \circ T_0 \circ \rho(g_i)^{-1}$ with $g_i \in \Delta_i \subseteq G$, so $T$ is still a compact operator. Moreover, $T$ is $G$-invariant since $dg$ is left invariant.

Since $T_0$ is positive, there exists some $v \in V$ with $(T_0 v, v) > 0$. Therefore, using the $G$-invariant inner product $(\,.\,,\,.\,)$ on $V$,

$$(Tv,v) = \int_G (\rho(g)T_0\rho(g)^{-1}v, v)\,dg = \int_G (T_0\rho(g)^{-1}v, \rho(g)^{-1}v)\,dg > 0,$$

so $T$ is nonzero. Since $V$ is a unitary representation, the adjoint of $\rho(g)$ is $\rho(g)^{-1}$. Using (25.1) and the fact that $T_0$ is self-adjoint, it follows that $T$ is self-adjoint as well.

Now the spectral theorem for compact self-adjoint operators (cf., e.g., [10]) implies the existence of an eigenvalue $\lambda \ne 0$ of $T$ with nonzero finite dimensional eigenspace

$U = \ker(T - \lambda I)$. This $U$ is the desired finite dimensional $G$-invariant subspace. $\square$

If $\{V_\alpha \mid \alpha \in A\}$ is a family of Hilbert spaces with inner products $( . , . )_\alpha$ then its *Hilbert space direct sum* is

$$\widehat{\bigoplus_{\alpha \in A}} V_\alpha := \{(v_\alpha) \mid v_\alpha \in V_\alpha,\ \sum_{\alpha \in A} \|v_\alpha\|_\alpha^2 < \infty\}.$$

Then $\widehat{\bigoplus}_\alpha V_\alpha$ is a Hilbert space with inner product

$$((v_\alpha), (v'_\alpha)) := \sum_{\alpha \in A} (v_\alpha, v'_\alpha)_\alpha.$$

Moreover, $\bigoplus_\alpha V_\alpha$ is a dense subspace and $V_\alpha \perp V_\beta$ for $\alpha \neq \beta$.

**25.4 Definition.** *Given a representation $V$ of a compact Lie group $G$ on a topological vector space, the set of $G$-finite vectors is the set of all $v \in V$ such that $Gv$ generates a finite dimensional subspace:*

$$V_{G-\text{fin}} := \{v \in V \mid \dim(\operatorname{span}\{gv \mid g \in G\}) < \infty\}.$$

The following result shows that infinite dimensional representations on Hilbert spaces in fact do not generate information beyond the finite dimensional case:

**25.5 Theorem.** *Let $(\rho, V)$ be a unitary representation of a compact Lie group $G$ on a Hilbert space. Then there exists a family $\{V_\alpha\}$ of finite dimensional irreducible $G$-submodules $V_\alpha \subseteq V$ such that*

$$V = \widehat{\bigoplus}_\alpha V_\alpha$$

*In particular, the irreducible unitary representations of $G$ are all finite dimensional. Moreover, the set of $G$-finite vectors is dense in $V$.*

**Proof.** Consider the collection of all sets $\{V_\alpha \mid \alpha \in A\}$ with the following properties:

(i) Each $V_\alpha$ is finite dimensional, $G$-invariant, and irreducible.

(ii) $V_\alpha \perp V_\beta$ for $\alpha \neq \beta \in A$.

This set is partially ordered by inclusion. Furthermore, every chain (totally ordered subset) has an upper bound, namely the union of the sets in the chain. Thus Zorn's Lemma implies the existence of a maximal element $\{V_\alpha \mid \alpha \in A\}$.

Now if $V \neq \widehat{\bigoplus}_\alpha V_\alpha$, then $\left(\widehat{\bigoplus}_\alpha V_\alpha\right)^\perp$ is closed, non-empty and $G$-invariant, so it constitutes a unitary Hilbert space representation in its own right. Therefore, Lemma 25.3 and Corollary 23.11 imply that there exists a finite dimensional $G$-invariant irreducible submodule $V_\beta \subseteq \left(\widehat{\bigoplus}_\alpha V_\alpha\right)^\perp$, contradicting maximality.

For the final claim, it suffices to note that $\bigoplus_\alpha V_\alpha$ consists of $G$-finite vectors. $\square$

Given a compact Lie group $G$, the set $C(G)$ of continuous ($\mathbb{C}$-valued) functions on $G$ is a Banach space with respect to the norm $\|f\|_\infty := \sup_{g \in G} |f(g)|$. Also, the space $L^2(G)$ of square integrable functions on $G$ is a Hilbert space with respect to the norm $\|f\|_2 := \left(\int_G |f(g)|^2 \, dg\right)^{1/2}$. On both spaces we have left and right actions defined by

$$(l_g f)(h) := f(g^{-1}h)$$
$$(r_g f)(h) := f(hg).$$

As the next result shows, they are representations, called the *left/right regular representation*.

**25.6 Lemma.** *The left and right actions of a compact Lie group $G$ on $C(G)$ and $L^2(G)$ are representations and preserve the norms.*

**Proof.** The only non-obvious property from Definition 25.1 is continuity of the map $(g, f) \mapsto l_g f$ (and analogously for $r_g$). In $C(G)$, we have

$$\begin{aligned}|f_1(g_1^{-1}h) - f_2(g_2^{-1}h)| &\le |f_1(g_1^{-1}h) - f_1(g_2^{-1}h)| + |f_1(g_2^{-1}h) - f_2(g_2^{-1}h)| \\ &\le |f_1(g_1^{-1}h) - f_1(g_2^{-1}h)| + \|f_1 - f_2\|_\infty.\end{aligned}$$

Since $g \mapsto g^{-1}$ is uniformly continuous on the compact group $G$ and $f_1$ is continuous, it follows that $\|l_{g_1} f_1 - l_{g_2} f_2\|_\infty$ can be made arbitrarily small for $g_1$ near $g_2$ in $G$ and $\|f_1 - f_2\|_\infty$ sufficiently small.

Next, for $f_i \in L^2(G)$ and $f \in C(G)$ we get

$$\begin{aligned}\|l_{g_1} f_1 - l_{g_2} f_2\|_2 &= \|f_1 - l_{g_1^{-1}g_2} f_2\|_2 \le \|f_1 - f_2\|_2 + \|f_2 - l_{g_1^{-1}g_2} f_2\|_2 \\ &= \|f_1 - f_2\|_2 + \|l_{g_1} f_2 - l_{g_2} f_2\|_2 \\ &\le \|f_1 - f_2\|_2 + \|l_{g_1} f_2 - l_{g_1} f\|_2 + \|l_{g_1} f - l_{g_2} f\|_2 + \|l_{g_2} f - l_{g_2} f_2\|_2 \\ &= \|f_1 - f_2\|_2 + 2\|f_2 - f\|_2 + \|l_{g_1} f - l_{g_2} f\|_2 \\ &\le \|f_1 - f_2\|_2 + 2\|f_2 - f\|_2 + \|l_{g_1} f - l_{g_2} f\|_\infty\end{aligned}$$

Since $C(G)$ is dense in $L^2(G)$ and we already know that $G$ acts continuously on $C(G)$, the claim for $L^2(G)$ follows as well. $\square$

**25.7 Theorem.** *Let $G$ be a compact Lie group.*

*(i) The set of $G$-finite vectors of $C(G)$ with respect to $l_g$ coincides with that of the $G$-finite vectors of $C(G)$ with respect to $r_g$.*

*(ii) $C(G)_{G-\mathrm{fin}} = \mathrm{MC}(G)$.*

**Proof.** We begin by showing that $C(G)_{G-\mathrm{fin}}$ with respect to $l_g$ is the set of matrix coefficients. Thus let $f^V_{u,v}(g) = (gu, v)$ be a matrix coefficient for a finite-dimensional unitary representation $V$ of $G$, where $u, v \in V$ and $( . , . )$ is a $G$-invariant inner product. Then

$$(l_g f^V_{u,v})(h) = (g^{-1}hu, v) = (hu, gv),$$

so $l_g f^V_{u,v} = f^V_{u,gv}$. Consequently,

$$\{l_g f^V_{u,v} \mid g \in G\} \subseteq \{f^V_{u,v'} \mid v' \in V\}.$$

Since $V$ is finite dimensional, so is the right hand side here. Therefore, $f^V_{u,v} \in C(G)_{G-\mathrm{fin}}$, so $\mathrm{MC}(G) \subseteq C(G)_{G-\mathrm{fin}}$.

Conversely, given $f \in C(G)_{G-\mathrm{fin}}$, by definition there exists a finite dimensional submodule $V \subseteq C(G)$ (with respect to the left action) so that $f \in V$. Since $\overline{gf} = g\bar{f}$, also $\overline{V} = \{\bar{v} \mid v \in V\}$ is a finite-dimensional submodule of $C(G)$. Writing $( . , . )$ for the $L^2$-inner product on $\overline{V}$, the evaluation functional at $e$ is an element of the dual space of $\overline{V}$, so there exists some $\overline{v_0} \in \overline{V}$ with $\bar{v}(e) = (\bar{v}, \overline{v_0})$ for all $\bar{v} \in \overline{V}$. Therefore,

$$\bar{f}(g) = l_{g^{-1}}\bar{f}(e) = (l_{g^{-1}}\bar{f}, \overline{v_0}) = (\bar{f}, l_g\overline{v_0}) = \overline{(l_g\overline{v_0}, \bar{f})} = \overline{f^{\overline{V}}_{\overline{v_0},\bar{f}}(g)}.$$

Consequently,

$$f = f^{\overline{V}}_{\overline{v_0},\bar{f}} \in \mathrm{MC}(G). \tag{25.2}$$

Therefore also $C(G)_{G-\mathrm{fin}} \subseteq \mathrm{MC}(G)$, proving (ii) for $l_g$.

Turning to (i), let $f$ be a left $G$-finite vector. By what we have just shown, $f$ is a matrix coefficient, $f = f^V_{u,v}$. Therefore, $r_g f(h) = (hgu, v)$, so $r_g f = f^V_{gu,v}$. Since $\{gu \mid g \in G\}$ is contained in the finite dimensional space $V$, it follows that the space of left $G$-finite vectors (and thereby $\mathrm{MC}(G)$) is contained in the space of right $G$-finite vectors.

Now let $f$ be a right $G$-finite vector. Again pick a finite dimensional submodule $V \subseteq C(G)$ with respect to $r_g$ with $f \in V$ and let $( . , . )$ be the $L^2$-inner product restricted to $V$. As above, since evaluation at $e$ is in the dual of $V$ there exists some $v_0 \in V$ such that $v(e) = (v, v_0)$ for all $v \in V$. In particular,

$$f(g) = r_g f(e) = (r_g f, v_0).$$

Consequently, $f = f^V_{f,v_0} \in \mathrm{MC}(G)$. Combined with the above, we arrive at MC(G) $= C(G)_{G-\mathrm{fin}}$ also with respect to $r_g$. □

For the proof of the Peter-Weyl theorem below we recall the following fundamental approximation result (cf., e.g., [10]).

**25.8 Theorem.** *(Stone–Weierstrass) Let $S$ be a compact Hausdorff space and let $A$ be a subalgebra of $C(S)$ that is closed under complex conjugation, separates points of $S$ and such that for any $p \in S$ there is some $f \in A$ with $f(p) \neq 0$. Then $A$ is dense in $C(S)$ (with respect to uniform convergence).*

Using this we can now show:

**25.9 Theorem.** *(Peter–Weyl Theorem) Let $G$ be a compact Lie group. Then $C(G)_{G-\mathrm{fin}}$ is dense in both $C(G)$ and $L^2(G)$.*

**Proof.** Since $C(G)$ is dense in $L^2(G)$ and since uniform convergence implies $L^2$-convergence, it suffices to prove the first claim. Note first that by Lemma 24.2 and Theorem 25.7, $C(G)_{G-\mathrm{fin}}$ is a subalgebra of $C(G)$ that contains 1 and is closed under complex conjugation (see (25.2)). By the Stone–Weierstrass theorem 25.8 it therefore suffices to show that $C(G)_{G-\mathrm{fin}}$ separates points. Using left-translation this in turn reduces to showing that for any $g_0 \neq e$ there exists some $f \in C(G)_{G-\mathrm{fin}}$ with $f(g_0) \neq f(e)$.

For this, first choose an open neighborhood $U$ of $e$ such that $U \cap g_0 U = \emptyset$. Then the characteristic function $\chi_U$ of $U$ belongs to $L^2(G)$. Also, $l_{g_0}\chi_U = \chi_{g_0 U}$, so $(l_{g_0}\chi_U, \chi_U) = 0$. Since $(\chi_U, \chi_U) > 0$, $l_{g_0}$ is not the identity operator on $L^2(G)$. By Theorem 25.5, with respect to the left action of $G$ on $L^2(G)$ there exist finite dimensional irreducible $G$-modules $V_\alpha \subseteq L^2(G)$ such that

$$L^2(G) = \widehat{\bigoplus_{\alpha}} V_\alpha.$$

In particular, there must be some $\alpha_0$ such that $l_{g_0}$ does not act as the identity on $V_{\alpha_0}$. Hence for some $x \in V_{\alpha_0}$ we have $l_{g_0}x \neq x$, and a foritiori there is some $y \in V_{\alpha_0}$ such that $(l_{g_0}x, y) \neq (x, y)$. Consequently, the matrix coefficient $f = f^{V_{\alpha_0}}_{x,y}$ then satisfies

$$f^{V_{\alpha_0}}_{x,y}(g_0) = (g_0 x, y) \neq (x, y) = f^{V_{\alpha_0}}_{x,y}(e),$$

as desired. □

# Appendices

## A Submanifolds

**Proof of Proposition 14.7**

We first need two auxiliary results:

**A.1 Lemma.** *Let $M$ be second countable and $M'$ an immersive submanifold. Then there exists a countable family $\mathcal{C}$ of open subsets of $M$ with the following property: Any point of $M'$ has a coordinate neighborhood which, for some $W \in \mathcal{C}$, is a connected component of the open submanifold $M' \cap W$ of $M'$.*

**Proof.** Let $\dim M = m$, $\dim M' = l$ and $j : M' \hookrightarrow M$. For $p \in M'$ there exist charts $(\varphi, U)$ and $(\psi, V)$ around $p$ in $M$ resp. $M'$ such that $\varphi \circ j \circ \psi^{-1} = x \mapsto (x, 0)$ and $V \subseteq U$ (see Remark 13.2). Choose a neighborhood $U'$ of $\varphi(p)$ in $\mathbb{R}^m$ with $U' \subseteq \varphi(U)$ and

$$U' \cap (\mathbb{R}^l \times \{0\}) \subseteq \psi(V) \times \{0\}. \tag{A.1}$$

Let $\mathcal{B}$ be a countable basis for the topology of $M$. Choose $W \in \mathcal{B}$ such that $p \in W \subseteq \varphi^{-1}(U')$. Then since $\varphi$ is injective,

$$\varphi(V \cap W) = \varphi(V) \cap \varphi(W) \subseteq (\mathbb{R}^l \times \{0\}) \cap \varphi(W).$$

Conversely, if $x \in (\mathbb{R}^l \times \{0\}) \cap \varphi(W)$, then $x = \varphi(q)$ for some $q \in W$, so $\varphi(q) \in U' \cap (\mathbb{R}^l \times \{0\}) \subseteq \psi(V) \times \{0\}$ by (A.1). By the above, $\varphi \circ \psi^{-1} : \psi(V) \to \varphi(V)$, $x \mapsto (x, 0)$, so $\varphi(V) = \varphi \circ \psi^{-1}(\psi(V)) = \psi(V) \times \{0\}$. It follows that $\varphi(q) \in \varphi(V)$, and so $x = \varphi(q) \in \varphi(V) \cap \varphi(W) = \varphi(V \cap W)$. Altogether,

$$\varphi(V \cap W) = (\mathbb{R}^l \times \{0\}) \cap \varphi(W). \tag{A.2}$$

Since $j$ is continuous, $M' \cap W = j^{-1}(W)$ is open in $M'$, hence is an open submanifold of $M'$ and thereby is itself an immersive submanifold of $M$. The set $\varphi^{-1}(\mathbb{R}^m \setminus (\mathbb{R}^l \times \{0\}))$ is open in $M$, hence intersects the immersive submanifold $M' \cap W$ in an open set. We now claim that

$$(M' \cap W) \cap \varphi^{-1}(\mathbb{R}^m \setminus (\mathbb{R}^l \times \{0\})) = (M' \cap W) \setminus (V \cap W).$$

Indeed, if there were some $q$ is in the l.h.s. such that $q \in V \cap W$, then by (A.2) $\varphi(q) \in \mathbb{R}^l \times \{0\}$, so $q \in \varphi^{-1}(\mathbb{R}^l \times \{0\})$, a contradiction. Conversely, if there were some $q$ in the r.h.s. such that $\varphi(q) \in \mathbb{R}^l \times \{0\}$, then $\varphi(q) \in (\mathbb{R}^l \times \{0\}) \cap \varphi(W) = \varphi(V \cap W)$ by (A.2), so $q \in V \cap W$, again a contradiction.

It follows that $V \cap W$ is closed in $M' \cap W$. On the other hand, $V \cap W = V \cap (M' \cap W)$ is also open in the manifold $M' \cap W$. Therefore $V \cap W$ is a union of connected components of $M' \cap W$ (If $X$ is any topological space and $A \subseteq X$ is both open and closed then it is a union of connected components). Let $C$ be the connected component that contains $p$. Then $C$ is open in $M' \cap W$, and thereby also in $M'$. Since $C \subseteq V$, $C$ additionally is a coordinate neighborhood of $p$ in $M'$. Now define $\mathcal{C}$ to be the set of all $W$ as above. Then $\mathcal{C} \subseteq \mathcal{B}$, hence is itself countable. $\square$

**A.2 Lemma.** *Let $M$ be a connected manifold and let $\mathcal{U} := \{U_\alpha \mid \alpha \in A\}$ be a covering of $M$ by coordinate neighborhoods. If for any $\alpha \in A$ the number of all $\beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ is countable, then $M$ is second countable.*

**Proof.** It suffices to show that from $\mathcal{U}$ we can extract a countable sub-covering (which then is a countable atlas for $M$, giving the claim by [5, 1.3.7]). We construct such a covering as follows: Let $B_1 \in \mathcal{U}$ be arbitrary and inductively define $B_n$ for $n = 2, 3, \dots$ as the union of all sets from $\mathcal{U}$ that intersect $B_{n-1}$. Any $B_n$ and thereby also $B := \bigcup_{n=1}^{\infty} B_n$ is a countable union of elements of $\mathcal{U}$. We have:

$$U \in \mathcal{U} \text{ and } U \cap B \neq \varnothing \Rightarrow U \subseteq B. \tag{A.3}$$

Indeed, there exists some $n$ such that $U \cap B_n \neq \varnothing$, so $U \subseteq B_{n+1} \subseteq B$. To conclude the proof we show that $B = M$. Since $M$ is connected and $\varnothing \neq B$ is open it suffices to show that $M \setminus B$ is open. Thus let $p \in M \setminus B$, then there exists some $U \in \mathcal{U}$ with $p \in U$. Due to (A.3) we then have $U \cap B = \varnothing$, so $U \subseteq M \setminus B$. □

Based on these Lemmas, we can now turn to Proposition 14.7:

**Proof.** Let $\mathcal{V} = \{V_\alpha \mid \alpha \in A\}$ be the covering of $M'$ by coordinate neighborhoods constructed in Lemma A.1. By Lemma A.2 it suffices to show that any element of $\mathcal{V}$ can only intersect countably many further elements of $\mathcal{V}$.

It follows from Lemma A.1 that any $V_\alpha \in \mathcal{V}$ is a connected component of some set $M' \cap W$, where $W \in \mathcal{C}$. Therefore it suffices to show that any $V_\alpha$ can intersect at most countably many connected components of any fixed $M' \cap W'$ (with $W' \in \mathcal{C}$): Indeed, there are only countably many $W' \in \mathcal{C}$. Once we have shown that only countably many connected components of any fixed $W' \cap M'$ meet the given $V_\alpha$ it follows that $V_\alpha$ can in total only intersect countably many connected components of the countably many $W' \cap M'$, i.e., only countably many $V_\beta$.

Now the nontrivial intersections of such connected components with $V_\alpha$ are open subsets of $V_\alpha$, which are necessarily disjoint. Since $V_\alpha$, being a coordinate neighborhood, is homeomorphic to an open subset of $\mathbb{R}^l$ (with $l = \dim M'$), there can indeed only be at most countably many such sets. □

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