WALLS IN MILNOR FIBER COMPLEXES

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ABSTRACT. For a real reflection group the reflecting hyperplanes cut out on the unit sphere a simplicial complex called the Coxeter complex. Abramenko showed that each reflecting hyperplane meets the Coxeter complex in another Coxeter complex if and only if the Coxeter diagram contains no subdiagram of type D_4, F_4, or H_4. The present paper extends Abramenko's result to a wider class of complex reflection groups. These groups have a Coxeter-like presentation and a Coxeter-like complex called the Milnor fiber complex. Our first main theorem classifies the groups whose reflecting hyperplanes meet the Milnor fiber complex in another Milnor fiber complex. To understand better the walls that fail to be Milnor fiber complexes we introduce *Milnor walls*. Our second main theorem generalizes Abramenko's result in a second way. It says that each wall of a Milnor fiber complex is a Milnor wall if and only if the diagram contains no subdiagram of type D_4, F_4, or H_4.

1. Introduction

For a real reflection group the reflecting hyperplanes cut out on the unit sphere a simplicial complex called the Coxeter complex. Abramenko [1] showed that each reflecting hyperplane meets the Coxeter complex in another Coxeter complex if and only if the Coxeter diagram contains no subdiagram of type D_4, F_4, or H_4.

The present paper extends Abramenko's result to a wider class of complex reflection groups. These groups have a Coxeter-like presentation and a Coxeter-like complex called the Milnor fiber complex. Our first main theorem (Theorem 1) classifies the groups whose reflecting hyperplanes meet the Milnor fiber complex in another Milnor fiber complex.

To understand better the walls that fail to be Milnor fiber complexes we introduce *Milnor walls*. These are walls with a type-selected set of chambers generating a Milnor fiber complex. Milnor walls in Coxeter complexes are walls that are Coxeter complexes. Our second main theorem (Theorem 2) thus generalizes Abramenko's result in a second way: it says each wall of a Milnor fiber complex is a Milnor wall if and only if the diagram contains no subdiagram of type D_4, F_4, or H_4.

As a benefit of Theorem 2 we find that Abramenko's result extends to give yet another equivalent condition in a curious classification [5, Theorem 14] with already 11 equivalent conditions coming from invariant theory, cohomology, combinatorics, and some group characters related to adding random numbers.

1.1. Fix a nonnegative integer $n \geq 0$ and a finite group G of the form

$$\langle\, r_1, r_2, \ldots, r_n \mid r_i^{p_i} = 1, \quad r_i r_j r_i \ldots = r_j r_i r_j \ldots \quad i \neq j \,\rangle \tag{1}$$

where $p_i \geq 2$, the number of terms on both sides of the braid relation is $m_{ij} = m_{ji} \geq 2$, and p_i equals p_j if m_{ij} is odd[1]. The empty set (when $n = 0$) generates the trivial group.

[1]For m_{ij} odd the braid relation says $(r_i r_j)^{(m_{ij}-1)/2} r_i = r_j (r_i r_j)^{(m_{ij}-1)/2}$ so that r_i is conjugate to r_j.

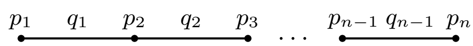
1.2. Preliminaries. Call a group presentation of the form (1) *admissible*. Koster [3] classified admissible presentations and found that the groups are precisely the finite direct products of finite irreducible Coxeter groups and Shephard groups. The classification implies no group has two different admissible presentations. Write $R = \{r_1, r_2, \ldots, r_n\}$ and call n the *rank* of G.

1.2.1. The classification uses a graphical notation for admissible presentations. The *diagram* Γ of (1) has for each r_i a vertex labeled p_i, and for each pair r_i, r_j with $m_{ij} > 2$ an edge labeled m_{ij} that connects r_i and r_j. We agree to suppress the minimal labels (2's on vertices and 3's on edges). We say Γ is *connected* if it has exactly one connected component; the diagram with no vertices is not connected. By *subdiagram of* Γ we mean a diagram gotten from Γ by removing any number of vertices and all their incident edges.

1.2.2. Write $G = G(\Gamma)$ and let $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$ be the connected components of Γ. Then

$$G = G_1 \times G_2 \times \ldots \times G_N, \quad G_i = G(\Gamma_i) \tag{2}$$

where the empty product is the trivial group. It follows that admissible diagrams are the unions of connected ones. Koster classified the connected ones. Table 1 lists them. The groups are the finite irreducible Coxeter groups and Shephard groups. Each comes from just one diagram. Finite irreducible Coxeter groups are the ones with all vertices 2. Shephard groups are the ones with *linear diagram*



(3)

The *symbol* $p_1[q_1]p_2[q_2]p_3 \ldots p_{n-1}[q_{n-1}]p_n$ (unique up to reversing term order) is shorthand for the linear diagram (3).

1.2.3. G has a representation analogous to the canonical one for Coxeter groups. Fix a vector space V over $\mathbf{C}$ of dimension n. A *reflection* in $\mathrm{GL}(V)$ is an element of finite order whose fixed space $V^r = \ker(r-1)$ is a hyperplane, and a *finite reflection group* is a finite group generated by reflections. Finite Coxeter groups have a canonical representation as a real reflection group that we view as a reflection group by extending the base field. In general the diagram Γ encodes a canonical faithful representation of G as a reflection group $G \subset \mathrm{GL}(V)$ in which each $r \in R$ is a reflection [3]. With this identification the reflections in G are precisely the non-identity elements that are conjugate to a power of a generator $r \in R$. Call G *irreducible* if the $\mathbf{C}G$-module V is irreducible. This happens if and only if Γ is connected. Shephard groups are irreducible; the trivial group is not.

1.2.4. Shephard and Todd classified the finite irreducible reflection groups and named *exceptional* ones $G_4, G_5, \ldots, G_{37}$. Not all of them are Coxeter or Shephard groups. The Coxeter ones have another set of names that we also use. For example H_3 and G_{23} both refer to the same Shephard group in Table 1.

1.2.5. Finite reflection groups on V are also the finite groups acting linearly on V whose algebra of invariant polynomial functions P on V (with respect to $gP(v) = P(g^{-1}v)$) is generated by $n = \dim V$ homogeneous algebraically independent polynomials P_i. The *basic degrees* $d_i = \deg P_i$ are unique and numbered so that $d_1 \le d_2 \le \ldots \le d_n$. If G is irreducible, then by the classification (see Table 1):

(i) $d_1 \ge 2$ with equality if and only if G is a Coxeter group.
(ii) $d_1 < d_2$ so that P_1 is unique up to a constant factor.

If G is irreducible, then $F_G = P_1^{-1}(1)$ is called *the Milnor fiber of* G.

1.2.6. According to details in §2.1 there is a unique (up to G-isomorphism) abstract simplicial complex $\Delta = \Delta(G, R)$ with simplices (labeled by) the cosets gG_I of standard parabolic subgroups $G_I = \langle I \rangle$ $(I \subset R)$ with face relation "gG_I is a face of hG_J" $\Leftrightarrow$ $gG_I \supset hG_J$, and with G acting on Δ by left translation $g.hG_I = ghG_I$. This is the classical abstract description of the Coxeter complex when G is a Coxeter group [10]. Call Δ *the Milnor fiber complex of* G. It has an explicit geometric realization in V that is G-homeomorphic to an equivariant strong deformation retract of the Milnor fiber F_G if G is irreducible [9]. In general it is described in §2.1 as the join of the Milnor fiber complexes of the irreducible factors G_i of G.

Maximal simplices in Δ have dimension $n-1$ and any two can be connected by a sequence of them in which consecutive terms share a face of codimension 1 so that Δ is a *chamber complex* and maximal simplices are called *chambers*. A general simplex $gG_{R\setminus I}$ has vertex set $\{gG_{R\setminus\{r\}} : r \in I\}$ and dimension $|I| - 1$. The set $I \subset R$ is uniquely determined by $gG_{R\setminus I}$. Call I the *type* of $gG_{R\setminus I}$ and write $\operatorname{type}(gG_{R\setminus I}) = I$.

1.2.7. The simplices of Δ that are fixed pointwise by a reflection of G form a subcomplex we call a *wall*. Since an element $g \in G$ fixing a simplex $hG_{R\setminus I} \in \Delta$ effects a type-preserving permutation of the vertices $hG_{R\setminus\{r\}}$ $(r \in I)$ the simplex is in fact fixed pointwise by g and so the walls of Δ are

$$\Delta^r = \{\sigma \in \Delta : r\sigma = \sigma\}, \quad r \in G \text{ a reflection.} \tag{4}$$

1.3. Our first theorem extends Abramenko's result to Milnor fiber complexes.

Theorem 1. *Each wall of the Milnor fiber complex Δ is again a Milnor fiber complex if and only if the diagram of G contains no subdiagram of type D_4, F_4, H_4, G_{25}, or G_{26}.*

We prove Theorem 1 in Section 2 by first reducing to the case where G is irreducible and then using the classification together with some enumerative and topological results about Δ that relate to the invariant theory of G.

1.3.1. We recover Abramenko's result from Theorem 1 by Proposition 10, which tells us that for Coxeter complexes all walls that are Milnor fiber complexes must be Coxeter complexes.

Corollary (Abramenko)**.** *Each wall of a Coxeter complex is again a Coxeter complex if and only if the diagram contains no subdiagram of type D_4, F_4, or H_4.* □

1.4. Our second theorem generalizes Abramenko's result in another way. The observation is that walls in Milnor fiber complexes can still hold Milnor fiber complexes of the same dimension in the sense that certain types of chambers in the wall generate a Milnor fiber complex. We make this precise with the definition of *Milnor wall* in Section 3, and then we prove the following theorem which also implies Abramenko's result.

Theorem 2. *Each wall of the Milnor fiber complex Δ is a Milnor wall if and only if the diagram of G contains no subdiagram of type D_4, F_4, or H_4.*

Remark 1. In [5, Theorem 14] we proved that if G is irreducible, then the diagram of G contains no subdiagram of type D_4, F_4, or H_4 if and only if the *Foulkes characters* $\phi_0, \phi_1, \ldots, \phi_n$ for G depend only on fixed-space dimension in the sense that $\phi_i(g) = \phi_i(h)$ whenever $\dim V^g = \dim V^h$. See [5, Theorem 14] for 9 other equivalent conditions.

2. Milnor fiber complexes

In this section we prove Theorem 1 after some preliminaries. §2.1 defines the Milnor fiber complex. §2.2 defines walls. §2.3 develops some topological results. §2.4 develops some enumerative results. §2.5 gives a combinatorial description of the Milnor fiber complex for the full monomial groups. Then in §2.6 we prove Theorem 1.

Recall that the connected components Γ_i of Γ partition R into disjoint sets R_i so that

$$G = G_1 \times G_2 \times \ldots \times G_N, \quad G_i = \langle R_i \rangle. \tag{5}$$

Let n_i and δ_i be the rank and smallest basic degree of G_i, so that $n_i = |R_i| \geq 1$, $n = n_1 + n_2 + \ldots + n_N$, and $\delta_i \geq 2$ with equality if and only if G_i is a Coxeter group.

2.1. The Milnor fiber complex. We define the *Milnor fiber complex of G* to be the complex Δ described by the following theorem. The definition is in terms of cosets gG_I of standard parabolic subgroups $G_I = \langle I \rangle$ $(I \subset R)$. If G is a Coxeter group, then the definition is the standard abstract one for the Coxeter complex of G and the properties that we list are well known, see [10]. The geometric construction of the Coxeter complex was generalized to include Shephard groups by Orlik [9]. He called this more general complex the Milnor fiber complex. The abstract definition of the Coxeter complex was later shown to hold for the Milnor fiber complex in the Shephard case [6]. For details about the extension of the definition and the properties to the Shephard case see [4]. The general case of the following theorem follows from the Coxeter and Shephard cases.

Theorem 3. *The standard parabolic cosets gG_I $(g \in G, I \subset R)$ with face relation*

$$\text{“}gG_I \text{ is a face of } hG_J\text{”} \Leftrightarrow gG_I \supset hG_J$$

is a simplicial complex Δ. Moreover Δ is a chamber complex with the following structure:

(i) *G acts on Δ by left translation $g.hG_I = ghG_I$.*
(ii) *Δ has a G-invariant type function $\Delta \to \{$subsets of $R\}$ given by* $\operatorname{type}(hG_{R\setminus I}) = I$.
(iii) *There exists a type-preserving G-equivariant isomorphism*

$$\Delta \cong \Delta_1 * \Delta_2 * \ldots * \Delta_N, \quad \Delta_k = \Delta(G_k, R_k) \tag{6}$$

where $g.(h_1G_{I_1} * \ldots * h_NG_{I_N}) = g_1h_1G_{I_1} * \ldots * g_Nh_NG_{I_N}$ for $g = g_1g_2\ldots g_N$, $g_i \in G_i$, $I_i \subset R_i$, and where $\operatorname{type}(h_1G_{R_1\setminus I_1} * \ldots * h_NG_{R_N\setminus I_N}) = I_1 \cup \ldots \cup I_N$.

Proof. For the case where G is a Coxeter or Shephard group see [10] and [4]. For the general case it suffices to prove (iii). To this end note that the mapping

$$h_1G_{I_1} * h_2G_{I_2} * \ldots * h_NG_{I_N} \mapsto h_1h_2\ldots h_NG_{I_1\cup I_2\cup\ldots\cup I_N}$$

takes $\Delta_1 * \Delta_2 * \ldots * \Delta_N$ bijectively onto Δ in a type-preserving fashion, and is compatible with the face relation and G-action. In particular Δ is a simplicial complex. □

We require the following lemma [4, Lemma 3.14 with $T = R \setminus U$] which tells us that each link in a Milnor fiber complex is again a Milnor fiber complex.

Proposition 4. *The link of a simplex gG_I in Δ is isomorphic to $\Delta(G_I, I)$.* □

2.2. The walls of the Milnor fiber complex. The simplices of Δ that are fixed pointwise by a reflection of G form a subcomplex we call a *wall*. The following proposition says that a simplex $\sigma \in \Delta$ is fixed pointwise by $g \in G$ if and only if $g\sigma = \sigma$. Write

$$\Delta^g = \{\sigma \in \Delta : g\sigma = \sigma\}, \quad g \in G. \tag{7}$$

Proposition 5. $\Delta^g = \{\sigma \in \Delta : \sigma \text{ fixed pointwise by } g\}$.

Proof. An element $g \in G$ fixing a simplex $hG_{R\setminus I} \in \Delta$ effects a type-preserving permutation of the vertices $hG_{R\setminus\{r\}}$ $(r \in I)$ of the simplex. Since no two of these vertices have the same type, the element g must fix each of the vertices. □

Proposition 5 gives the following description of walls.

Proposition 6. *The walls of* Δ *are*

$$\Delta^r = \{\sigma \in \Delta : r\sigma = \sigma\}, \quad r \in G \text{ a reflection.} \tag{8}$$

Equivalently, the walls of Δ *are (up to isomorphism)*

$$\Delta(G_1, R_1) * \ldots * \Delta(G_i, R_i)^t * \ldots * \Delta(G_N, R_N), \quad t \in G_i \text{ a reflection.} \tag{9}$$

Proof. The first part is by Proposition 5. The second part follows by Theorem 3(iii). □

As a benefit of (9) we have the following result that reduces the problem of determining when the walls of Δ are Milnor fiber complexes to the case when G is irreducible.

Proposition 7. *Each wall of* $\Delta(G, R)$ *is a Milnor fiber complex if and only if each wall of each* $\Delta(G_i, R_i)$ *is a Milnor fiber complex.*

Proof. If the wall in (9) is a Milnor fiber complex, then it follows from Proposition 4 that the wall $\Delta(G_i, R_i)^t$ of $\Delta(G_i, R_i)$ is a Milnor fiber complex. Conversely, if the wall $\Delta(G_i, R_i)^t$ of $\Delta(G_i, R_i)$ is a Milnor fiber complex $\Delta(G_i', R_i')$, then the wall in (9) is the Milnor fiber complex of $G_1 \times \ldots \times G_i' \times \ldots \times G_N$. □

2.3. Topological results. The following theorem tells us the homotopy type of the subcomplex Δ^g for any element $g \in G$. In particular, it tells us the homotopy type of any wall Δ^r. It is due to Orlik [9] and appears in the proof of his Theorem 4.1 on p. 145 where he observes that Δ^g is a deformation retract of the intersection of the fixed space V^g and the Milnor fiber of G, which has an isolated critical point at the origin.

Theorem (Orlik)**.** *If* G *is irreducible and* $g \in G$*, then the subcomplex* Δ^g *is homotopy equivalent to a bouquet of* $(d_1 - 1)^p$ *many* $(p-1)$*-spheres, where* $p = \dim V^g$*.*

If G is reducible, then the homotopy type of a given Δ^g is read off from Orlik's result and (6). We highlight the case $g = 1$. This case is used in the proof of Theorem 1 to help determine if a wall Δ^r is a Milnor fiber complex $\Delta(W, S)$ for some possibly reducible W.

Proposition 8. *Let* n_i *and* δ_i *be the rank and smallest basic degree of the irreducible factor* G_i*, so that* $n = n_1 + n_2 + \ldots + n_N$*. Then the Milnor fiber complex of* G *is homotopy equivalent to a bouquet of* $(\delta_1 - 1)^{n_1}(\delta_2 - 1)^{n_2} \ldots (\delta_N - 1)^{n_N}$ *many* $(n-1)$*-spheres.*

Proof. The Milnor fiber complex is the join $\Delta_1 * \Delta_2 * \ldots * \Delta_N$ of the complexes Δ_i of the irreducible factors G_i, and Orlik's theorem with $g = 1$ tells us that Δ_i is a bouquet of $(\delta_i - 1)^{n_i}$ many $(n_i - 1)$-spheres. Hence the result. □

From Proposition 8 we get the following characterization of Coxeter complexes as Milnor fiber complexes that are spheres.

Proposition 9. *A Milnor fiber complex is a Coxeter complex if and only if it is a sphere.*

Proof. Proposition 8 says Δ is a bouquet of $(\delta_1 - 1)^{n_1}(\delta_2 - 1)^{n_2} \ldots (\delta_N - 1)^{n_N}$ many $(n-1)$-spheres. If $N = 0$, then Δ is a single (-1)-sphere and G is the trivial Coxeter group $\langle \emptyset \rangle$. If $N > 0$, then the inequalities $n_i \geq 1$ and $\delta_i \geq 2$ imply that the number of spheres equals 1 if and only if each δ_i equals 2. Hence by §1.2.5(i) the Milnor fiber complex Δ is a single sphere if and only if each G_i is a Coxeter group, i.e., if and only if G is a Coxeter group. □

As a corollary of Proposition 9 we have the following.

Proposition 10. *A wall of a Coxeter complex is a Milnor fiber complex if and only if it is a Coxeter complex.*

Proof. Consider a wall of a Coxeter complex. If it is a Coxeter complex, then it is a Milnor fiber complex. If it is a Milnor fiber complex, then because it is also a sphere (being the equator of a sphere) Proposition 9 tells us that it is a Coxeter complex. □

2.4. Enumerative results. We require some enumerative results about the number of chambers of a wall. Denote by $f_k(\Sigma)$ the number of k-simplices in a complex Σ so that

$$f_k(\Sigma) = \#\{\sigma \in \Sigma : \dim \sigma = k\}.$$

The chambers of Δ are indexed by elements of G, and the number of elements of G equals the product of basic degrees $d_1, d_2, \dots, d_n$. Hence $f_{n-1}(\Delta)$ equals $d_1 d_2 \dots d_n$. Suppose for the rest of §2.4 that G is irreducible. Then Eq. (13) below tells us that $f_{n-2}(\Delta^r)$ equals $d_1 d_2 \dots d_{n-1}$ for any reflection r in G. It is natural to wonder then if a similar formula holds for elements g where $\dim V^g$ equals $n-2$, $n-3$, and so on. Remarkably this turns out to be the case if and only if the diagram of G does not contain any subdiagram of type D_4, F_4, or H_4. This is Theorem 11 below. It is a consequence of a collection of observations from [5] about Orlik–Solomon coexponents.

Continue to suppose that G is irreducible. Let L be the collection of all fixed spaces V^g ordered by reverse inclusion, so that V is at the bottom. This is the same as the lattice of intersections of reflecting hyperplanes. For μ the Möbius function of L and $X \in L$ define $B_X(t) \in \mathbf{Z}[t]$ by $B_X(t) = (-1)^{\dim X} \sum_{Y \geq X} \mu(X,Y)(-t)^{\dim Y}$. Then Orlik [9] (following Orlik–Solomon in the Coxeter case) showed that

$$f_{k-1}(\Delta^g) = \sum B_Y(d_1 - 1) \tag{10}$$

where the sum is over all k-dimensional subspaces Y that lie above V^g in L. In particular

$$f_{p-1}(\Delta^g) = B_X(d_1 - 1) \tag{11}$$

for $X = V^g$ and $p = \dim X$. Furthermore for $X \in L$ of dimension p there exist positive integers $b_1^X \leq b_2^X \leq \dots \leq b_p^X$ such that

$$B_X(t) = (t + b_1^X)(t + b_2^X)\dots(t + b_p^X). \tag{12}$$

Orlik and Solomon determined the b_i^X's for all irreducible Coxeter and Shephard groups. The tables in [7, 8] list the b_i^X's for each $X \in L$ when G is an exceptional group of rank at least 3. Following [5] we make the following remarkable observation.

Theorem 11. *If G is an irreducible Coxeter or Shephard group, then*

$$f_{n-2}(\Delta^r) = d_1 d_2 \dots d_{n-1} \tag{13}$$

for any reflection $r \in G$, and the following are equivalent:

(i) $f_{n-3}(\Delta^g) = d_1 d_2 \dots d_{n-2}$ *for $g \in G$ such that* $\dim V^g = n - 2$.
(ii) $f_{p-1}(\Delta^g) = d_1 d_2 \dots d_p$ *for $g \in G$ and $p = \dim V^g$.*
(iii) *The diagram of G contains no subdiagram of type D_4, F_4, or H_4.*

Proof. As stated above, this is a collection of observations from [5] about Orlik–Solomon coexponents (see [5, Prop. 13(ii), Thm. 14(a)(k), Pf. of Thm. 14]) as we now explain. For irreducible Coxeter groups, Eq. (13) was observed by Orlik and Solomon [8, p. 271]. In general [5, Thm. 14(g)(k)] is the equivalence (ii)⇔(iii). So it remains to show that (i) fails for F_4, H_4, E_6, E_7, E_8, and D_n ($n \geq 4$). This is implicit in the proof of [5, Thm. 14]. The exceptional cases F_4, H_4, E_6, E_7, and E_8 follow from (11) together with the tables of [8], and as explained in the proof of [5, Thm. 14], (i) fails for D_n since there is a certain fixed pace Y defined by $x_1 = x_2 = x_3$ such that $B_Y(d_1 - 1) < d_1 d_2 \dots d_{n-2}$. □

2.5. Another description of the Milnor fiber complex of the full monomial group. Fix an integer $m > 1$. The *full monomial group* $G(m, 1, n)$ is the group of all n-by-n monomial matrices (one nonzero entry in each row and column) whose nonzero entries are m-th roots of unity. Let ζ be a primitive m-th root of unity and denote by e_k the standard column vector in $\mathbf{C}^n$ with 1 in the i-th spot and 0 elsewhere. In cycle notation, the standard generators $r_1, r_2, \ldots, r_n$ of $G(m, 1, n)$ are the adjacent transpositions $(1\ 2), (2\ 3), \ldots, (n-1\ n)$, together with $(n\ \zeta n)$, where for example $(n\ \zeta n)$ is short for the n-by-n matrix whose i-th column is ζe_i if $i = n$ and e_i otherwise. In general, a reflection is conjugate to a power of a generating reflection. This gives a total of m conjugacy classes of reflections: one indexed by $(n-1\ n)$ and the others by $(n\ \zeta^k n)$ for $1 \le k \le m-1$. The Milnor fiber complex Δ of $G(m, 1, n)$ is realized (see [9]) as the union $\Delta = \cup gC$ of all translates gC of the simplex

$$C = \{\alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_n b_n : \alpha_1 + \ldots + \alpha_n = 1,\ \alpha_i \text{ nonnegative real}\} \tag{14}$$

where $b_k = (e_1 + \ldots + e_k)/k$. This leads to the following convenient description of the Milnor fiber complex. The flag complex $\Delta(P)$ of a finite poset P is the simplicial complex with elements of P for vertices and flags $\{x_1 < x_2 < \ldots < x_k : x_i \in P\}$ for simplices.

Proposition 12. *Let Δ_n be the Milnor fiber complex of $G(m, 1, n)$. Let r be a reflection in $G(m, 1, n)$. Let P_n be the collection of sets $\{\alpha_1 e_{i_1}, \ldots, \alpha_k e_{i_k}\}$ ordered by inclusion, where the α_i's are m-th roots of unity, $i_1 < \ldots < i_k$, and $1 \le k \le n-1$. Then*

(i) *Δ_n is equivariantly isomorphic to the flag complex of P_n.*
(ii) *The subposet $P_n^r = \{X \in P_n : rX = X\}$ is isomorphic to P_{n-1}.*

In particular, each wall of Δ_n is isomorphic to Δ_{n-1}.

Proof. As an abstract simplicial complex Δ_n is generated by the translates of the chamber $C = \{b_1, \ldots, b_n\}$. Consider the mapping that takes $b_k = (e_1 + \ldots + e_k)/k$ to the set $\{e_1, \ldots, e_k\}$, so that a face $\{b_{i_1}, \ldots, b_{i_k}\}$ of C (where $i_1 < i_2 < \ldots < i_k$) goes to the flag

$$\{e_1, \ldots, e_{i_1}\} \subset \{e_1, \ldots, e_{i_2}\} \subset \ldots \subset \{e_1, \ldots, e_{i_k}\}.$$

Extend by the action of $G(m, 1, n)$ to an isomorphism from Δ_n onto $\Delta(P_n)$. Hence (i).

The reflection r is conjugate to either $t = (n-1\ n)$ or $s = (n\ \xi n)$ for some root of unity $\xi \ne 1$. Hence P_n^r is isomorphic to either P_n^s or P_n^t. We show that P_n^s and P_n^t are isomorphic to P_{n-1}. Consider $X \in P_n$ and write $X = \{\alpha_1 e_{i_1}, \ldots, \alpha_k e_{i_k}\}$. The reflection t fixes X if and only if either X contains both αe_{n-1} and αe_n for some $\alpha \in \mathbf{C}$, or X contains neither αe_{n-1} nor αe_n for any $\alpha \in \mathbf{C}$. Hence the following is an isomorphism:

$$P_n^t \to P_{n-1} \text{ given by } X \mapsto X \setminus \mathbf{C}e_n.$$

The set X is fixed by s if and only if $X = X \setminus \mathbf{C}e_n$, so P_n^s is isomorphic to P_{n-1} as well. Hence (ii). The last statement follows: $\Delta_n^r \cong \Delta(P_n^r) \cong \Delta(P_{n-1}) \cong \Delta_{n-1}$. $\square$

2.6. Proof of Theorem 1. It suffices to assume that G is irreducible by Proposition 7.

Coxeter diagram with no subdiagram of type D_4, F_4, or H_4. In this case, Abramenko tells us that each wall is a Coxeter complex, and hence a Milnor fiber complex.

Coxeter diagram with a subdiagram of type D_4, F_4, or H_4. Since Δ is a Coxeter complex, Proposition 10 tells us that any wall of Δ that is a Milnor fiber complex must be a Coxeter complex. But Abramenko tells us that not all walls of Δ are Coxeter complexes. Hence not all walls of Δ are Milnor fiber complexes.

Full monomial groups $G(m, 1, n)$ $(m \ge 2)$. Proposition 12 says that any wall of the Milnor fiber complex of $G(m, 1, n)$ is isomorphic to the Milnor fiber complex of $G(m, 1, n-1)$.

Groups of rank 1 and 2. This case is clear.

The remaining exceptional groups: G_{25}, G_{26}, G_{32}. Here we use Orlik's theorem (see §2.3), Proposition 8, and the cell counts of Theorem 11.

Group G_{25}. This is the rank-3 group with symbol 3[3]3[3]3. The basic degrees are $6, 9, 12$. Fix a wall Δ^r. It is 1-dimensional with 54 chambers by (13), and it has the homotopy type of a 5^2-fold bouquet of 1-spheres by Orlik's theorem. If the wall is a Milnor fiber complex $\Delta(W, S)$, then because $\Delta(W, S)$ has dimension $|S| - 1$ with chambers indexed by the elements of W, the group W must be a rank-2 group with 54 elements. If W is reducible, then it must therefore be of the form $Z_j \times Z_k$ for some $j, k \in \mathbf{N}$ such that $jk = 54$ and $(j-1)(k-1) = 5^2$ by Proposition 8. This is impossible. So W must be irreducible with basic degrees d_1, d_2 satisfying $d_1 d_2 = 54$ and $(d_1 - 1)^2 = 5^2$. But from the classification (see Table 1) we find that there is no irreducible Coxeter or Shephard group of rank 2 whose basic degrees are 6 and 9. So no wall of the Milnor fiber complex of G_{25} is again a Milnor fiber complex.

Group G_{26}. This is the rank-3 group with symbol 3[3]3[4]2. The basic degrees are $6, 12, 18$. Consider the wall Δ^{r_1} cut out by the generator r_1 that commutes with the one of order 2. It is 1-dimensional with 72 chambers and the homotopy type of a 5^2-fold bouquet of 1-spheres. If it is a Milnor fiber complex $\Delta(W, S)$, then W is a rank-2 group with 72 elements. W must be irreducible because otherwise it is of the form $Z_j \times Z_k$ and there are no integers $j, k \in \mathbf{N}$ such that $jk = 72$ and $(j-1)(k-1) = 5^2$. Thus W is an irreducible rank-2 group whose basic degrees d_1, d_2 satisfy $d_1 d_2 = 72$ and $(d_1 - 1)^2 = 5^2$. Hence $d_1 = 6$ and $d_2 = 12$. According to the classification (see Table 1) there are only two such groups: the group G_5 whose symbol is 3[4]3, and the group $G(6, 1, 2)$ whose symbol is 2[4]6.

Vertices in $\Delta(W, S)$ are cosets $w\langle s_i \rangle$ in W of the cyclic groups $\langle s_i \rangle$ for $s_i \in S$. Edges of $\Delta(W, S)$ are the cosets $\{w\}$ ($w \in W$) and the incidence relation is containment. So $\Delta(W, S)$ has $|W|/|\langle s_i \rangle|$ vertices of degree $|\langle s_i \rangle|$ for $i = 1, 2$, and this accounts for all vertices. It follows from this discussion that the Milnor fiber complex of G_5 is 3-regular (all vertices have degree 3), and the Milnor fiber complex of $G(6, 1, 2)$ has 12 vertices of degree 6 and 36 vertices of degree 2. We claim that these vertex-degree distributions are different from the vertex-degree distribution in Δ^{r_1}, and in turn Δ^{r_1} is not a Milnor fiber complex. To this end it is enough to show that there is a vertex of degree 4 in Δ^{r_1}.

Consider the vertex of Δ indexed by $H = \langle r_1, r_2 \rangle$. This vertex is fixed (under left multiplication) by r_1 and therefore belongs to Δ^{r_1}. We claim that it has degree 4 in Δ^{r_1}. In Δ the edges incident to H are the cosets of $\langle r_2 \rangle$ and $\langle r_1 \rangle$ in H. The number of these cosets fixed by r_1 therefore equals the degree of H as a vertex in Δ^{r_1}. Since H is the group G_4 with symbol 3[3]3 whose basic degrees are 4 and 6, Eq. (13) tells us that the number of these cosets in H fixed by r_1 equals 4. This concludes the present case.

Group G_{32}. This is the rank-4 group with symbol 3[3]3[3]3[3]3 and basic degrees $12, 18, 24, 30$. Fix a wall. It is 2-dimensional with 5184 chambers and the homotopy type of a 11^3-fold bouquet of 2-spheres. Suppose that the wall is a Milnor fiber complex $\Delta(W, S)$. Then W must be a rank-3 group with 5184 elements. It can not be a product of 3 rank-1 groups $Z_i \times Z_j \times Z_k$ because no $i, j, k \in \mathbf{N}$ satisfy $ijk = 5184$ and $(i-1)(j-1)(k-1) = 11^3$. And it can not be the product of a rank-1 group Z_k and an irreducible rank-2 group H because then for d_1 the smallest basic degree of H we would have that $k|H| = 5184$ and $(k-1)(d_1 - 1)^3 = 11^3$, so that $k = 12$, $d_1 = 12$, and $|H| = 432$, while the only irreducible rank-2 Coxeter or Shephard group with 432 elements is a dihedral group whose smallest basic degree equals 2. Hence W must be an irreducible rank-3 Coxeter or Shephard group. Therefore the smallest basic degree d_1 of W must satisfy $(d_1 - 1)^3 = 11^3$. But Table 1 shows that no irreducible rank-3 Coxeter or Shephard group has 5184 elements and smallest basic degree d_1 equal to 12. □

3. Milnor walls

Write $\overline{\Omega} = \cup_{\sigma\in\Omega}\{\tau \in \Delta : \tau \subset \sigma\}$ for the simplicial complex generated by a family of simplices $\Omega \subset \Delta$. Let $\binom{R}{k}$ be the set of all k-element subsets of R, so that

$$\binom{R}{k} = \{\operatorname{type}\sigma : \sigma \in \Delta,\ \dim\sigma = k-1\}.$$

Definition 1. A wall Δ^r of Δ is a *Milnor wall* if for some $\mathcal{F} \subset \binom{R}{n-1}$ the subcomplex

$$(\Delta^r)_{\mathcal{F}} = \overline{\{\sigma \in \Delta^r : \operatorname{type}\sigma \in \mathcal{F}\}} \tag{15}$$

is a Milnor fiber complex of dimension $n-2$. A *proper Milnor wall* is a Milnor wall that is not a Milnor fiber complex.

Any wall Δ^r can be written as $(\Delta^r)_{\mathcal{F}}$ for $\mathcal{F} = \binom{R}{n-1}$. Therefore walls that are Milnor fiber complexes are Milnor walls. These are the non-proper Milnor walls. They are the only kind of Milnor wall found in Coxeter complexes. This is the following lemma.

Lemma 13. *Coxeter complexes have no proper Milnor walls.*

Proof. A wall in an $(n-1)$-dimensional Coxeter complexes is an $(n-2)$-sphere, and an $(n-2)$-dimensional Milnor fiber complex is a bouquet of $(n-2)$-spheres. Therefore, removing any chambers from a wall of an $(n-1)$-dimensional Coxeter complex gives something that is not a Milnor fiber complex of dimension $n-2$. ☐

Proposition 14. *A wall of a Coxeter complex is a Milnor wall if and only if the wall is a Coxeter complex.*

Proof. Suppose Σ is a Milnor wall of a Coxeter complex. Then Lemma 13 implies that Σ is a Milnor fiber complex. Since Σ is a wall of a Coxeter complex, it follows from Proposition 10 that Σ is a Coxeter complex. The other direction is clear: a wall of a Coxeter complex that is itself a Coxeter complex is a (non-proper) Milnor wall. ☐

Proposition 15. *Each wall of $\Delta(G, R)$ is a Milnor wall if and only if each wall of each $\Delta(G_i, R_i)$ is a Milnor wall.*

Proof. Consider a wall $\Delta^r = \Delta(G_1, R_1) * \ldots * \Delta(G_i, R_i)^t * \ldots * \Delta(G_N, R_N)$, so that $t \in G_i$ is a reflection, and write $n_i = |R_i|$.

Suppose $\Delta(G_i, R_i)^t$ is a Milnor wall, so that $(\Delta(G_i, R_i)^t)_{\mathcal{F}_i}$ is a Milnor fiber complex of dimension $n_i - 2$ for some $\mathcal{F}_i \subset \binom{R_i}{n_i-1}$. Put $\mathcal{F} = \{(R \setminus R_i) \cup S : S \in \mathcal{F}_i\}$, so that

$$(\Delta^r)_{\mathcal{F}} = \Delta(G_1, R_1) * \ldots * (\Delta(G_i, R_i)^t)_{\mathcal{F}_i} * \ldots * \Delta(G_N, R_N). \tag{16}$$

Since Theorem 3(iii) tells us that a join of Milnor fiber complexes is again a Milnor fiber complex, it follows that $(\Delta^r)_{\mathcal{F}}$ is a Milnor fiber complex of dimension $n-2$. Hence Δ^r is a Milnor wall.

Now suppose Δ^r is a Milnor wall. Then $(\Delta^r)_{\mathcal{F}}$ is an $(n-2)$-dimensional Milnor fiber complex for some $\mathcal{F} \subset \binom{R}{n-1}$. The subcomplex $\Delta(G_i, R_i)^t$ has dimension $|R_i| - 2$, and hence no simplex of type R_i. So the wall Δ^r has no simplex of type $R \setminus \{s\}$ for $s \in R_j$, $j \neq i$. Hence $\mathcal{F} \subset \{R \setminus \{s\} : s \in R_i\}$. But then for $\mathcal{F}_i = \{S \cap R_i : S \in \mathcal{F}\}$ we have (16). Since $(\Delta^r)_{\mathcal{F}}$ is an $(n-2)$-dimensional Milnor fiber complex in which $(\Delta(G_i, R_i)^t)_{\mathcal{F}_i}$ is the link of any simplex of type $R \setminus R_i$, it follows from Proposition 4 that $(\Delta(G_i, R_i)^t)_{\mathcal{F}_i}$ is a Milnor fiber complex of dimension $n_i - 2$. So $\Delta(G_i, R_i)^t$ is a Milnor wall. ☐

3.1. Proof of Theorem 2. It suffices to assume that G is irreducible by Proposition 15.

Suppose that the diagram of G contains no subdiagram of type D_4, F_4, or H_4, and that G is not G_{25}, G_{26}, or G_{32}. Then Theorem 1 tells us that each wall of Δ is a Milnor fiber complex. Hence each wall of Δ is a Milnor wall.

Suppose now the diagram of G contains a subdiagram of type D_4, F_4, or H_4. By the irreducibility of G and the classification (Table 1), G must be a Coxeter group. Then Abremenko's result tells us that not all walls of Δ are Coxeter complexes, and Proposition 14 tells us that the Milnor walls of Δ are the walls of Δ that are again Coxeter complexes. So in this case we conclude that not all walls of Δ are Milnor walls.

Finally suppose G is G_{25}, G_{26}, or G_{32}. The object is to show that all walls are Milnor walls. The remainder of the proof explains how this claim follows from Orlik's original construction of the Milnor fiber complex Δ together with some observations made by Coxeter about the regular complex polytopes associated to G_{25}, G_{26}, and G_{32}.

A regular complex polytope [2, p. 115] is a certain collection $\mathscr{P}$ of affine subspaces in $\mathbf{C}^n$ with incidence relation given by proper inclusion subject to some conditions. The symmetry group of a regular complex polytope is a Shephard group G and the Milnor fiber complex of the Shephard group is constructed in [9] as a geometric realization of the flag complex of the poset of simplices of $\mathscr{P}$, whose k-simplices are the flags $\mathscr{F} = (F^{(0)} \subsetneq F^{(1)} \subsetneq \ldots \subsetneq F^{(k)})$ of faces $F^{(i)} \in \mathscr{P}$; see [9, Thm. 5.1]. Index the generators of the Shephard group starting with 0 instead of 1, so that $R = \{r_0, r_1, \ldots, r_{n-1}\}$. Then a flag $\mathscr{F}$ corresponds in the Milnor fiber complex to a coset of $\langle R \setminus \{r_{\dim F} : F \in \mathscr{F}\}\rangle$, whose type is $\{r_{\dim F} : F \in \mathscr{F}\}$; see [4].

Coxeter considered the *section of $\mathscr{P}$ by a reflecting hyperplane H*. This is the set

$$\{F \in \mathscr{P} : F \subset F' \subset H \text{ for some } F' \in \mathscr{P} \text{ such that } \dim F' = n-2\}. \tag{17}$$

He observed the following [2, pp. 123, 132]: if $G = G_{25}$, then each section of $\mathscr{P}$ by a reflecting hyperplane is a regular complex polytope for $G(3,1,2)$; if $G = G_{26}$, then each section of $\mathscr{P}$ by a reflecting hyperplane is a regular complex polytope for either $G(3,1,2)$ or $G(6,1,2)$; and if $G = G_{32}$, then each section of $\mathscr{P}$ by a reflecting hyperplane is a regular complex polytope for G_{26}. These observations translate into the following statements about $\Sigma = (\Delta(G,R)^r)_{\mathcal{F}}$ for $\mathcal{F} = \{R \setminus \{r_{n-1}\}\}$ and $r \in G$ a reflection.

a. If $G = G_{25}$, then Σ is isomorphic to the Milnor fiber complex of $G(3,1,2)$.
b. If $G = G_{26}$, then Σ is isomorphic to the Milnor fiber complex of $G(3,1,2)$ or $G(6,1,2)$.
c. If $G = G_{32}$, then Σ is isomorphic to the Milnor fiber complex of G_{26}.

Hence if G is G_{25}, G_{26}, or G_{32}, then all walls of $\Delta(G,R)$ are Milnor walls. □

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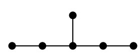
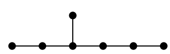
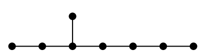
G	Symbol/diagram	Basic degrees
Z_m	m	m
$I_2(2m)$	$2[2m]2$	$2, 2m$
$I_2(2m-1)$	$2[2m-1]2$	$2, 2m-1$
G_4	$3[3]3$	$4, 6$
G_5	$3[4]3$	$6, 12$
G_6	$3[6]2$	$4, 12$
G_8	$4[3]4$	$8, 12$
G_9	$4[6]2$	$8, 24$
G_{10}	$4[4]3$	$12, 24$
G_{14}	$3[8]2$	$6, 24$
G_{16}	$5[3]5$	$20, 30$
G_{17}	$5[6]2$	$20, 60$
G_{18}	$5[4]3$	$30, 60$
G_{20}	$3[5]3$	$12, 30$
G_{21}	$3[10]2$	$12, 60$
G_{23} (H_3)	$2[3]2[5]2$	$2, 6, 10$
G_{25}	$3[3]3[3]3$	$6, 9, 12$
G_{26}	$3[3]3[4]2$	$6, 12, 18$
G_{28} (F_4)	$2[3]2[4]2[3]2$	$2, 6, 8, 12$
G_{30} (H_4)	$2[3]2[3]2[5]2$	$2, 12, 20, 30$
G_{32}	$3[3]3[3]3[3]3$	$12, 18, 24, 30$
G_{35} (E_6)		$2, 5, 6, 8, 9, 12$
G_{36} (E_7)		$2, 6, 8, 10, 12, 14, 18$
G_{37} (E_8)		$2, 8, 12, 14, 18, 20, 24, 30$
A_n	$2[3]2 \ldots 2[3]2[3]2$	$2, 3, \ldots, n+1$
D_{n+1}		$2, 4, \ldots, 2n, n+1$
$G(m, 1, n)$	$2[3]2 \ldots 2[3]2[4]m$	$m, 2m, \ldots, nm$

TABLE 1. $(m \geq 2)$ The finite irreducible Coxeter and Shephard groups, their diagrams, and their basic degrees.

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