DENSE PROPORTIONS OF ZEROS IN CHARACTER VALUES

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Abstract. Proportions of zeros in character tables of finite groups are dense in $[0, 1]$.

For any finite group G, denote by $\check{G}$ a complete set of class representatives,

$P_I(G)$ the proportion of pairs (χ, g) in $\mathrm{Irr}(G) \times G$ with $\chi(g) = 0$,

$P_{II}(G)$ the proportion of pairs (χ, g) in $\mathrm{Irr}(G) \times \check{G}$ with $\chi(g) = 0$,

so $P_{II}(G)$ is the proportion of zeros in the character table of G. Fixing a choice P of P_I or P_{II}, Burnside's result on the existence of zeros for nonlinear irreducible characters [1] gives $P(G) > 0$ if and only if G is nonabelian.

The purpose of this note is to show:

Theorem 1. *The set of proportions* $\{P(G) : |G| < \infty\}$ *is dense in* $[0, 1]$.

For any two sequences $a_n \in [0, 1]$ *and* $\varepsilon_n \in (0, \infty)$, *and any prime* p, *there is an ascending chain of* p*-groups* $G_1 < G_2 < \ldots$ *with* $|a_n - P(G_n)| < \varepsilon_n$ *for each* n.

In particular, for each $L \in [0, 1]$, *there is a chain of* p*-groups* G_n *with* $P(G_n) \to L$.

Lemma 1. *For any finite nonabelian group* G, *we have* $P(G^n) \to 1$ *as* $n \to \infty$.

Proof. For any two finite groups X and Y, we have

$$P(X \times Y) = P(X) + (1 - P(X))P(Y), \tag{1}$$

since for any $\chi \times \psi \in \mathrm{Irr}(X \times Y)$ we have $(\chi \times \psi)(x, y) = 0$ if and only if $\chi(x) = 0$ or both $\chi(x) \neq 0$ and $\psi(y) = 0$. So for any finite group G, the sequence $P(G^n)$ satisfies

$$P(G^{n+1}) = P(G^n) + (1 - P(G^n))P(G),$$

making $P(G^n)$ monotonic, bounded, and thus convergent with limit L satisfying $L = L + (1 - L)P(G)$, from which the result follows by Burnside. □

Proof of Theorem 1. Fix a chain $H_1 < H_2 < \ldots$ with H_n extraspecial of order p^{2n+1} for each n, so H_n has $p^{2n} + p - 1$ irreducible characters, of which $p - 1$ are nonlinear, and each nonlinear one vanishes off the center of order p, giving

$$P_I(H_n) = \frac{(p-1)(p^{2n+1} - p)}{(p^{2n} + p - 1)p^{2n+1}} \to 0 \tag{2}$$

and

$$P_{II}(H_n) = \frac{(p-1)(p^{2n} - 1)}{(p^{2n} + p - 1)^2} \to 0. \tag{3}$$

Let $a \in (0,1)$, $\varepsilon > 0$, and $G = H_{s_1} \times H_{s_2} \times \ldots \times H_{s_k}$ with $k \geq 1$. It suffices to show that $|a - P(G')| < \varepsilon$ for some $G' > G$ which is also a product of H_i's.

Put $H = H_s^k$ for some $s > \max_i s_i$ such that $P(H_s) < a/k$. Then $H > G$ and by (1),

$$P(H) \leq kP(H_s) < a.$$

Writing $x = P(H)$, let l be such that

$$P(H_l) < \min\left\{\frac{\varepsilon}{1-x}, \frac{a-x}{1-x}\right\}.$$

Then the sequence $P(H_l^n)$ starts below $(a-x)/(1-x)$ and tends monotonically to 1 with steps of size $< \varepsilon/(1-x)$ by Lemma 1 and the fact that

$$0 < P(H_l^{n+1}) - P(H_l^n) = (1 - P(H_l^n))P(H_l) < \frac{\varepsilon}{1-x}.$$

So for some m,

$$\frac{a-x}{1-x} - \frac{\varepsilon}{1-x} < P(H_l^m) < \frac{a-x}{1-x},$$

or equivalently, $a - \varepsilon < P(H \times H_l^m) < a$. □

There is also an interesting consequence of Lemma 1 for Young subgroups

$$S_\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_n} \leq S_n$$

with λ drawn uniformly at random from the partitions of n:

Theorem 2. *The expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$.*

Proof. Fix an integer $k > 2$, and let $m_k(\lambda)$ denote the multiplicity of k in any given partition λ. Using (1), we have

$$P(S_\lambda) \geq P(S_k^{m_k(\lambda)}) \geq P(S_k^m) \quad \text{whenever} \quad m_k(\lambda) \geq m,$$

so for any integer $m \geq 0$, the expected value of $P(S_\lambda)$ is at least

$$\text{Prob}(m_k(\lambda) \geq m)P(S_k^m).$$

By [2, Thm. 2.1], $\lim_{n\to\infty} \text{Prob}(m_k(\lambda) \geq m) = 1$ for any m, and by Lemma 1, $P(S_k^m) \to 1$ as $m \to \infty$, hence the expected value of $P(S_\lambda)$ tends to 1 as $n \to \infty$. □

References

[1] W. Burnside, On an arithmetical theorem connected with roots of unity, and its application to group-characteristics. *Proc. London. Math. Soc.* **1** (1904) 112–116.

[2] B. Fristedt, The structure of random partitions of large integers. *Trans. Amer. Math. Soc.* **337** (1993) 703–735.

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