CHARACTER RESTRICTIONS AND REFLECTION GROUPS

EUGENIO GIANNELLI AND ALEXANDER R. MILLER

Abstract. Recent results of Ayyer–Prasad–Spallone and Isaacs–Navarro–Olsson–Tiep on odd-degree character restrictions for symmetric groups are extended to reflection groups $G(r, p, n)$.

Introduction

Fix some integers $m \geq 0$ and $n \geq 1$. Recently Ayyer, Prasad, and Spallone [1, Theorem 1] proved that if χ is an odd-degree irreducible character of the symmetric group $\mathfrak{S}_n$, then the restriction of χ to $\mathfrak{S}_{n-1}$ contains a unique odd-degree irreducible constituent. Isaacs, Navarro, Olsson, and Tiep [4] proved a stronger result which incorporates multiplicities. They proved [4, Theorem A] that if $n \geq 2^m$ and χ is an odd-degree irreducible character of $\mathfrak{S}_n$, then the restriction of χ to $\mathfrak{S}_{n-2^m}$ contains a unique odd-degree irreducible constituent of odd multiplicity.

The object of the present paper is to extend these recent results to reflection groups $G(r, p, n)$. This includes Coxeter groups of type A_{n-1}, B_n, D_n, which are $G(1, 1, n)$, $G(2, 1, n)$, $G(2, 2, n)$. Denote by $\mathrm{Irr}_{2'}(G)$ the set of odd-degree irreducible characters of a finite group G, and let

$$k(m, p) = \begin{cases} 2^m & \text{if } p \text{ is odd,} \\ 2^{m+1} + 2^m + 1 & \text{if } p \text{ is even.} \end{cases}$$

Theorem A. *If $\chi \in \mathrm{Irr}_{2'}(G(r, p, n))$ and $n \geq k(m, p)$, then the restriction of χ to $G(r, p, n-2^m)$ contains a unique odd-degree irreducible constituent of odd multiplicity.*

Theorem B. *If $\chi \in \mathrm{Irr}_{2'}(G(r, p, n))$ and $n \geq k(0, p)$, then the restriction of χ to $G(r, p, n-1)$ contains a unique odd-degree irreducible constituent.*

Remark. The inequalities $n \geq k(m, p)$ and $n \geq k(0, p)$ can not be relaxed, for if $n = 3 \times 2^m$ and p is even, then there exists an odd-degree irreducible character of $G(r, p, n)$ whose restriction to $G(r, p, n - 2^m)$ contains at least three odd-degree irreducible constituents of multiplicity 1.

Theorems A and B are proved in §2 after some preliminaries in §1 which includes extensions of [1, Theorem 1] and [4, Theorem A] to wreath products $G \wr \mathfrak{S}_n$ (Theorems 1.5 and 1.6). §3 remarks on a surjectivity result for the map $\mathrm{Irr}_{2'}(G(r, p, n)) \to \mathrm{Irr}_{2'}(G(r, p, n - 2^m))$ which sends χ to the unique odd-degree irreducible constituent of odd multiplicity in the restriction of χ to $G(r, p, n - 2^m)$.

2010 *Mathematics Subject Classification.* 20C15.

Key words and phrases. odd-degree characters, reflection groups.

Supported by Trinity Hall, University of Cambridge.

Supported by a grant from the Austrian Science Foundation.

1. Preliminaries

1.1. **Partitions and binomials.**

1.1.1. Denote by $\mathrm{Par}(n)$ the set of integer partitions λ of n, viewed as Young diagrams, and write $|\lambda| = n$. By $\mu \prec \lambda$ we mean that μ is obtained from λ by removing a single box. By $\mu \le \lambda$ we mean that μ is obtained from λ by removing a collection of boxes. Denote by $\mathrm{Par}_\lambda(k)$ the set of all $\mu \in \mathrm{Par}(k)$ such that $\mu \le \lambda$.

Given a positive integer r we denote by $\mathrm{Par}_r(n)$ the set of all r-tuples $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_r)$ where each λ_i is a partition and $\sum |\lambda_i| = n$. By $\boldsymbol{\mu} \prec \boldsymbol{\lambda}$ we mean that $\boldsymbol{\mu} \in \mathrm{Par}_r(n-1)$ is obtained from $\boldsymbol{\lambda}$ by removing a single box from some λ_i. By $\boldsymbol{\mu} \le \boldsymbol{\lambda}$ we mean that $\boldsymbol{\mu}$ is obtained from $\boldsymbol{\lambda}$ by removing collections of boxes from the partitions λ_i in $\boldsymbol{\lambda}$. Finally let $\mathrm{Par}_{\boldsymbol{\lambda}}(k) = \{\boldsymbol{\mu} \in \mathrm{Par}_r(k) : \boldsymbol{\mu} \le \boldsymbol{\lambda}\}$.

1.1.2. For $\boldsymbol{\lambda} \in \mathrm{Par}_r(n)$ and $\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-k)$ $(0 \le k \le n)$ we define multinomial coefficients

$$c_{\boldsymbol{\lambda}} = \binom{n}{|\lambda_1|\,,\, |\lambda_2|\,,\, \dots\,,\, |\lambda_r|}, \quad c_{\boldsymbol{\lambda}\boldsymbol{\mu}} = \binom{k}{|\lambda_1| - |\mu_1|\,,\, |\lambda_2| - |\mu_2|\,,\, \dots\,,\, |\lambda_r| - |\mu_r|}.$$

Recall $\binom{n}{n_1, n_2, \dots, n_r} = n!/n_1! n_2! \dots n_r!$ where the n_i's are nonnegative integers that add up to n. Two nonnegative integers a and b are said to be 2-*disjoint* if there is no common summand in their 2-adic decompositions. This is the same as saying that no carries occur when a is added to b in base 2. The following well-known result goes back to Kummer [6, p. 116].[1]

Lemma 1.1. *Let k be the largest nonnegative integer such that 2^k divides $c_{\boldsymbol{\lambda}}$. Then k equals the number of carries that occur when adding up the $|\lambda_i|$'s in base 2. In particular, $c_{\boldsymbol{\lambda}}$ is odd if and only if the $|\lambda_i|$'s are 2-disjoint.* □

Lemma 1.2. *If $c_{\boldsymbol{\lambda}}$, $c_{\boldsymbol{\mu}}$, $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ are odd for $\boldsymbol{\lambda} \in \mathrm{Par}_r(n)$ and $\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$, then $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ differ in exactly one component, say the k-th component. Moreover, k is determined by $\boldsymbol{\lambda}$ and m. Let 2^s be the smallest term in the 2-adic decomposition of n such that $2^s \ge 2^m$. Then k is the unique index where 2^s appears in the 2-adic decomposition of $|\lambda_k|$.*

Proof. The first part follows from $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ being odd and Lemma 1.1. The second part follows from Lemma 1.1 and the first part by noting that at the level of the 2-adic decomposition of $|\lambda_k|$, subtracting 2^m from $|\lambda_k|$ replaces the smallest term $2^s \ge 2^m$ by $2^{s-1} + 2^{s-2} + \dots + 2^m$. □

1.2. **Symmetric groups.**

The irreducible characters of $\mathfrak{S}_n$ are indexed by partitions of n [5]. Denote by χ^λ the character indexed by λ in the usual way. When we talk of $\mathfrak{S}_{n-k}$ as a subgroup of $\mathfrak{S}_n$ we mean the pointwise stabilizer of k points in $\{1, 2, \dots, n\}$. The restriction of χ^λ to $\mathfrak{S}_{n-k}$ is denoted by $\chi^\lambda|_{\mathfrak{S}_{n-k}}$. The branching rule for restricting from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$ is given by $\chi^\lambda|_{\mathfrak{S}_{n-1}} = \sum_{\mu \prec \lambda} \chi^\mu$.

1.3. **Wreath products.**

Fix a finite group G. Then the wreath product $G \wr \mathfrak{S}_n$ has elements $(g_1, g_2, \dots, g_n; \sigma)$ where $g_i \in G$ and $\sigma \in \mathfrak{S}_n$, and multiplication is given by

$$(g_1, g_2, \dots, g_n; \sigma).(h_1, h_2, \dots, h_n; \tau) = (g_1 h_{\sigma^{-1}(1)}, g_2 h_{\sigma^{-1}(2)}, \dots, g_n h_{\sigma^{-1}(n)}; \sigma\tau).$$

---

[1] Kummer [6, p. 116] proved that if p is a prime and N is the largest nonnegative integer such that p^N divides $\binom{a+b}{b}$, then N equals the number of carries that occur when a is added to b in base p. The extension to multinomial coefficients and r-fold sums follows by writing $\binom{n}{n_1, n_2, \dots, n_r} = \prod_{i=1}^r \binom{n_1 + n_2 + \dots + n_i}{n_i}$.

1.3.1. Fix a numbering $\gamma_1, \gamma_2, \ldots, \gamma_r$ of the irreducible characters of G. Then the irreducible characters $X^{\boldsymbol{\lambda}}$ of $G \wr \mathfrak{S}_n$ are indexed by r-tuples $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathrm{Par}_r(n)$ as follows [8]. Let W_i be the $\mathbf{C}G$-module affording γ_i, and let V_{λ_i} be the $\mathbf{C}\mathfrak{S}_{|\lambda_i|}$-module affording χ^{λ_i}. Let $\mathfrak{S}_{\boldsymbol{\lambda}} = \mathfrak{S}_{|\lambda_1|} \times \mathfrak{S}_{|\lambda_2|} \times \ldots \times \mathfrak{S}_{|\lambda_r|} \leq \mathfrak{S}_n$, and define $\mathbf{C}(G \wr \mathfrak{S}_{\boldsymbol{\lambda}})$-modules

$$V_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \ldots \otimes V_{\lambda_r}, \quad W_{\boldsymbol{\lambda}} = W_1^{\otimes|\lambda_1|} \otimes W_2^{\otimes|\lambda_2|} \otimes \ldots \otimes W_r^{\otimes|\lambda_r|},$$

where for $g_i \in G$ and $\sigma = \sigma_1\sigma_2 \ldots \sigma_r \in \mathfrak{S}_{\boldsymbol{\lambda}}$

$$(g_1, g_2, \ldots, g_n; \sigma)(v_1 \otimes v_2 \otimes \ldots \otimes v_r) = \sigma_1 v_1 \otimes \sigma_2 v_2 \otimes \ldots \otimes \sigma_r v_r,$$
$$(g_1, g_2, \ldots, g_n; \sigma)(w_1 \otimes w_2 \otimes \ldots \otimes w_n) = g_1 w_{\sigma^{-1}(1)} \otimes g_2 w_{\sigma^{-1}(2)} \otimes \ldots \otimes g_n w_{\sigma^{-1}(n)}.$$

Then $X^{\boldsymbol{\lambda}}$ is the irreducible character of $G \wr \mathfrak{S}_n$ afforded by $\mathbf{C}(G \wr \mathfrak{S}_n) \otimes_{\mathbf{C}(G\wr\mathfrak{S}_{\boldsymbol{\lambda}})} (V_{\boldsymbol{\lambda}} \otimes W_{\boldsymbol{\lambda}})$.

1.3.2. The branching rule [8, Theorem 4.1] says

$$X^{\boldsymbol{\lambda}}|_{G\wr\mathfrak{S}_{n-1}} = \sum_{i=1}^{r} \sum_{\mu_i \prec \lambda_i} \gamma_i(1) X^{(\lambda_1, \ldots, \lambda_{i-1}, \mu_i, \lambda_{i+1}, \ldots, \lambda_r)}. \tag{1}$$

Iterating the rule gives a general one which is well known.

Theorem 1.3. *$X^{\boldsymbol{\lambda}}|_{G\wr\mathfrak{S}_{n-k}} = \sum c_{\boldsymbol{\lambda}\boldsymbol{\mu}} m_{\boldsymbol{\lambda}\boldsymbol{\mu}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}} X^{\boldsymbol{\mu}}$ over all $\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-k)$, where*

$$m_{\boldsymbol{\lambda}\boldsymbol{\mu}} = \prod_{1 \leq i \leq r} \langle \chi^{\lambda_i}|_{\mathfrak{S}_{|\mu_i|}}, \chi^{\mu_i} \rangle, \quad d_{\boldsymbol{\lambda}\boldsymbol{\mu}} = \prod_{1 \leq i \leq r} \gamma_i(1)^{|\lambda_i| - |\mu_i|}.$$

Proof. Repeatedly apply (1) in order to restrict $X^{\boldsymbol{\lambda}}$ from $G \wr \mathfrak{S}_n$ down to $G \wr \mathfrak{S}_{n-k}$, and note that $c_{\boldsymbol{\lambda}\boldsymbol{\mu}} m_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ counts the number of ways to go from $\boldsymbol{\lambda}$ to $\boldsymbol{\mu}$ by successively removing boxes. □

1.3.3. The construction of $X^{\boldsymbol{\lambda}}$ implies the following basic fact.

Lemma 1.4. *$X^{\boldsymbol{\lambda}}(1) = c_{\boldsymbol{\lambda}} \prod_{i=1}^{r} \gamma_i(1)^{|\lambda_i|} \chi^{\lambda_i}(1)$ for $\boldsymbol{\lambda} \in \mathrm{Par}_r(n)$. In particular $X^{\boldsymbol{\lambda}}(1)$ is odd if and only if the $|\lambda_i|$'s are 2-disjoint, the degrees $\chi^{\lambda_i}(1)$ are odd, and $|\lambda_i| = 0$ when $\gamma_i(1)$ is even.* □

1.4. **Odd-degree character restrictions and wreath products.** Here we extend [1, Theorem 1] and [4, Theorem A] from $\mathfrak{S}_n$ to $G \wr \mathfrak{S}_n$. Recall [4, Theorem A] says that if $\chi^{\lambda} \in \mathrm{Irr}_{2'}(\mathfrak{S}_n)$ and $n \geq 2^m$, then there is a unique $\chi^{\lambda^\star} \in \mathrm{Irr}_{2'}(\mathfrak{S}_{n-2^m})$ of odd multiplicity in $\chi^{\lambda}|_{\mathfrak{S}_{n-2^m}}$.

Theorem 1.5. *If $X \in \mathrm{Irr}_{2'}(G \wr \mathfrak{S}_n)$ and $n \geq 2^m$, then the restriction of X to $G \wr \mathfrak{S}_{n-2^m}$ contains a unique odd-degree irreducible constituent $X^{\boldsymbol{\mu}}$ of odd multiplicity. Moreover $\boldsymbol{\mu}$ equals $(\lambda_1, \ldots, \lambda_{k-1}, \lambda_k^\star, \lambda_{k+1}, \ldots, \lambda_r)$ where k is the number determined by $\boldsymbol{\lambda}$ and m in Lemma 1.2.*

Proof. Assume $X^{\boldsymbol{\lambda}}(1)$ is odd. Then there exists an odd-degree constituent $X^{\boldsymbol{\mu}}$ of odd multiplicity in the restriction of $X^{\boldsymbol{\lambda}}$ to $G \wr \mathfrak{S}_{n-2^m}$. The aim is to show that $\boldsymbol{\mu}$ is as claimed.

Theorem 1.3 tells us that $c_{\boldsymbol{\lambda}\boldsymbol{\mu}} m_{\boldsymbol{\lambda}\boldsymbol{\mu}} d_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is the multiplicity of $X^{\boldsymbol{\mu}}$ in $X^{\boldsymbol{\lambda}}|_{G\wr\mathfrak{S}_{n-2^m}}$. Since $c_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is odd, and since both $c_{\boldsymbol{\lambda}}$ and $c_{\boldsymbol{\mu}}$ are odd by Lemma 1.4, we conclude from Lemma 1.2 that there is a unique index k (as described in the statement) such that $|\mu_k| = |\lambda_k| - 2^m$ and $|\mu_i| = |\lambda_i|$ for $i \neq k$. Since $\boldsymbol{\mu} \leq \boldsymbol{\lambda}$ it follows that $\mu_i = \lambda_i$ for $i \neq k$. Hence $m_{\boldsymbol{\lambda}\boldsymbol{\mu}} = \langle \chi^{\lambda_k}|_{\mathfrak{S}_{n-2^m}}, \chi^{\mu_k} \rangle$. Since $m_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is odd, and both $\chi^{\lambda_k}(1)$ and $\chi^{\mu_k}(1)$ are odd (by Lemma 1.4), it follows from [4, Theorem A] that $\mu_k = \lambda_k^\star$. □

Theorem 1.6. *If $X \in \mathrm{Irr}_{2'}(G \wr \mathfrak{S}_n)$, then the restriction of X to $G \wr \mathfrak{S}_{n-1}$ contains a unique odd-degree irreducible constituent.*

Proof. Assume $X^{\boldsymbol{\lambda}}(1)$ is odd. Then by (1) and Lemma 1.4, all irreducible constituents of the restriction $X^{\boldsymbol{\lambda}}|_{G\wr\mathfrak{S}_{n-1}}$ have odd multiplicity. It follows by Theorem 1.5 that $X^{\boldsymbol{\lambda}}|_{G\wr\mathfrak{S}_{n-1}}$ has a unique odd-degree irreducible constituent. $\square$

1.5. **The reflection groups $\mathbf{G(r,p,n)}$.** Fix positive integers r and p such that p divides r. Then $G(r,1,n) \leq \mathrm{GL}(n,\mathbf{C})$ is the group of all n-by-n monomial matrices (one nonzero entry in each row and column) with r-th roots of unity for nonzero entries. The normal subgroup $G(r,p,n)$ consists of all $x \in G(r,1,n)$ for which the product of the nonzero entries is an r/p-th root of unity. The quotient $G(r,1,n)/G(r,p,n)$ is cyclic of order p. When we speak of $G(r,p,n-k)$ as a subgroup of $G(r,p,n)$ we mean the pointwise stabilizer in $G(r,p,n)$ of k of the column vectors $e_1, e_2, \ldots, e_n$, where e_i is the standard column vector in $\mathbf{C}^n$ with 1 in the i-th spot and 0's elsewhere. For the purposes of the present paper, we define $G(r,p,0)$ to be the trivial group.

1.5.1. Let Z_r be the cyclic group of r-th roots of unity. Then $\varphi : Z_r \wr \mathfrak{S}_n \to \mathrm{GL}(n,\mathbf{C})$ defined by $\varphi((z_1, z_2, \ldots, z_n; \sigma))e_i = z_{\sigma(i)}e_{\sigma(i)}$ $(0 \leq i \leq n)$ takes $Z_r \wr \mathfrak{S}_n$ isomorphically onto $G(r,1,n)$. From this identification $G(r,1,n) \simeq Z_r \wr \mathfrak{S}_n$ and the construction of the irreducible characters $X^{\boldsymbol{\lambda}}$ of a general wreath product $G \wr \mathfrak{S}_n$ we obtain the irreducible characters $\chi^{\boldsymbol{\lambda}}$ of $G(r,1,n)$. In particular, the characters $\chi^{\boldsymbol{\lambda}}$ of $G(r,1,n)$ are indexed by the r-tuples $\boldsymbol{\lambda} \in \mathrm{Par}_r(n)$.

By Theorem 1.3 we have the following well-known branching rule for $G(r,1,n)$:

$$\chi^{\boldsymbol{\lambda}}|_{G(r,1,n-k)} = \sum_{\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-k)} c_{\boldsymbol{\lambda}\boldsymbol{\mu}} m_{\boldsymbol{\lambda}\boldsymbol{\mu}} \chi^{\boldsymbol{\mu}}, \quad m_{\boldsymbol{\lambda}\boldsymbol{\mu}} = \prod_{1 \leq i \leq r} \langle \chi^{\lambda_i}|_{\mathfrak{S}_{|\mu_i|}}, \chi^{\mu_i} \rangle. \tag{2}$$

By Lemma 1.4 we also have the following formula for $\chi^{\boldsymbol{\lambda}}(1)$.

Lemma 1.7. $\chi^{\boldsymbol{\lambda}}(1) = c_{\boldsymbol{\lambda}} \chi^{\lambda_1}(1)\chi^{\lambda_2}(1)\ldots\chi^{\lambda_r}(1)$ *for* $\boldsymbol{\lambda} \in \mathrm{Par}_r(n)$. *In particular* $\chi^{\boldsymbol{\lambda}}(1)$ *is odd if and only if the* $|\lambda_i|$*'s are* 2*-disjoint and the degrees* $\chi^{\lambda_i}(1)$ *are odd.* $\square$

1.5.2. Write $r = dp$ and consider the cyclic group $C_p = \langle \omega^d \rangle$ where ω is the r-cycle $(1\,2\,3\,\ldots\,r)$. C_p acts on $\mathrm{Par}_r(n)$ by permuting coordinates

$$\sigma.(\lambda_1, \lambda_2, \ldots, \lambda_r) = (\lambda_{\sigma^{-1}(1)}, \lambda_{\sigma^{-1}(2)}, \ldots, \lambda_{\sigma^{-1}(r)})$$

and we denote by $\overline{\boldsymbol{\lambda}}$ the orbit of $\boldsymbol{\lambda}$ under the action of C_p. Define

$$\mathrm{Par}_{d,p}(n) = \{\overline{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \mathrm{Par}_r(n)\}, \quad \mathrm{Aut}(\boldsymbol{\lambda}) = \{\sigma \in C_p : \sigma\boldsymbol{\lambda} = \boldsymbol{\lambda}\} \leq C_p.$$

Then the irreducible characters of $G(r,p,n)$ are indexed by the pairs $(\overline{\boldsymbol{\lambda}}, i)$ where $\overline{\boldsymbol{\lambda}} \in \mathrm{Par}_{d,p}(n)$ and $0 \leq i \leq |\mathrm{Aut}(\boldsymbol{\lambda})| - 1$, and the indexing is such that [3, 7]

$$\chi^{\boldsymbol{\lambda}}|_{G(r,p,n)} = \sum_{0 \leq i \leq |\mathrm{Aut}(\boldsymbol{\lambda})|-1} \chi^{(\overline{\boldsymbol{\lambda}},i)} \quad \text{for} \quad \chi^{\boldsymbol{\lambda}} \in \mathrm{Irr}(G(r,1,n)). \tag{3}$$

In particular, the summands $\chi^{(\overline{\boldsymbol{\lambda}},i)}$ are all conjugate and of degree

$$\chi^{(\overline{\boldsymbol{\lambda}},i)}(1) = |\mathrm{Aut}(\boldsymbol{\lambda})|^{-1}\, \chi^{\boldsymbol{\lambda}}(1) = |\mathrm{Aut}(\boldsymbol{\lambda})|^{-1}\, c_{\boldsymbol{\lambda}}\, \chi^{\lambda_1}(1)\chi^{\lambda_2}(1)\ldots\chi^{\lambda_r}(1). \tag{4}$$

The branching rule for $G(r,p,n)$ says that [7, Proposition 3.2]

$$\langle \chi^{(\overline{\boldsymbol{\lambda}},i)}|_{G(r,p,n-1)}, \chi^{(\overline{\boldsymbol{\mu}},j)} \rangle = |\mathrm{Aut}(\boldsymbol{\lambda})|^{-1} \times |\{\boldsymbol{\nu} \prec \boldsymbol{\lambda} : \overline{\boldsymbol{\nu}} = \overline{\boldsymbol{\mu}}\}|. \tag{5}$$

1.5.3. We end with some basic observations. Recall that $n \geq 1$.

Lemma 1.8. *Let $\boldsymbol{\lambda} \in \mathrm{Par}_r(n)$.*

(i) $|\mathrm{Aut}(\boldsymbol{\lambda})|$ *divides p.*
(ii) *If $|\mathrm{Aut}(\boldsymbol{\lambda})| = k$ and $\lambda_i \in \boldsymbol{\lambda}$, then λ_i appears $k \times l$ times in $\boldsymbol{\lambda}$ for some $l \geq 1$.*
(iii) *If $\chi^{\boldsymbol{\lambda}}(1)$ is odd , then $|\mathrm{Aut}(\boldsymbol{\lambda})| = 1$.*

Proof. (i) and (ii) follow from $\mathrm{Aut}(\boldsymbol{\lambda})$ being a subgroup of C_p. Consider (iii). If $|\mathrm{Aut}(\boldsymbol{\lambda})| \neq 1$, then by (ii) the $|\lambda_i|$'s are not 2-disjoint, and therefore $\chi^{\boldsymbol{\lambda}}(1)$ is even by Lemma 1.7. □

2. Proofs of Theorems A and B

After two results in §2.1 we prove Theorems A and B in §2.2 and §2.3. Note that Theorems A and B for $G(r,1,n)$ are consequences of Theorems 1.5 and 1.6 with $G = \mathbf{Z}/r\mathbf{Z}$.

2.1. The following general branching rule for $G(r,p,n)$ will be useful in the sequel.

Theorem 2.1. *Let $0 \leq k < n$. Let $\boldsymbol{\lambda} \in \mathrm{Par}_r(n)$ and $\boldsymbol{\mu} \in \mathrm{Par}_r(k)$. Write $a_{\boldsymbol{\lambda}\boldsymbol{\nu}} = \langle \chi^{\boldsymbol{\lambda}}|_{G(r,1,k)}, \chi^{\boldsymbol{\nu}} \rangle$ for $\boldsymbol{\nu} \in \mathrm{Par}_r(k)$. Then for all i and j,*

$$\langle \chi^{(\overline{\boldsymbol{\lambda}},i)}|_{G(r,p,k)}, \chi^{(\overline{\boldsymbol{\mu}},j)} \rangle = \frac{1}{|\mathrm{Aut}(\boldsymbol{\lambda})|} \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\lambda}\boldsymbol{\nu}} \tag{6}$$

where the sum is over all $\boldsymbol{\nu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(k)$ such that $\overline{\boldsymbol{\nu}} = \overline{\boldsymbol{\mu}}$. Equivalently,

$$\chi^{(\overline{\boldsymbol{\lambda}},i)}|_{G(r,p,k)} = \frac{1}{|\mathrm{Aut}(\boldsymbol{\lambda})|} \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\lambda}\boldsymbol{\nu}} \chi^{\boldsymbol{\nu}}|_{G(r,p,k)} = \frac{1}{|\mathrm{Aut}(\boldsymbol{\lambda})|} \sum_{\boldsymbol{\nu}} \sum_{j} a_{\boldsymbol{\lambda}\boldsymbol{\nu}} \chi^{(\overline{\boldsymbol{\nu}},j)} \tag{7}$$

where the first two sums are over $\boldsymbol{\nu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(k)$ and the third sum goes from 0 to $|\mathrm{Aut}(\boldsymbol{\nu})| - 1$.

Proof. The right-hand side of the branching rule (5) does not depend on i, and so the restriction $\chi^{(\overline{\boldsymbol{\lambda}},i)}|_{G(r,p,k)} = \chi^{(\overline{\boldsymbol{\lambda}},i)}|_{G(r,p,n-1)}|_{G(r,p,k)}$ does not depend on i. Hence by (3)

$$\chi^{\boldsymbol{\lambda}}|_{G(r,p,k)} = \chi^{\boldsymbol{\lambda}}|_{G(r,p,n)}|_{G(r,p,k)} = \sum_{t} \chi^{(\overline{\boldsymbol{\lambda}},t)}|_{G(r,p,k)} = |\mathrm{Aut}(\boldsymbol{\lambda})| \cdot \chi^{(\overline{\boldsymbol{\lambda}},i)}|_{G(r,p,k)}$$

where t goes from 0 to $|\mathrm{Aut}(\boldsymbol{\lambda})| - 1$. On the other hand $\chi^{\boldsymbol{\lambda}}|_{G(r,p,k)} = \chi^{\boldsymbol{\lambda}}|_{G(r,1,k)}|_{G(r,p,k)}$ implies

$$\chi^{\boldsymbol{\lambda}}|_{G(r,p,k)} = \sum_{\boldsymbol{\nu}} a_{\boldsymbol{\lambda}\boldsymbol{\nu}} \chi^{\boldsymbol{\nu}}|_{G(r,p,k)} = \sum_{\boldsymbol{\nu}} \sum_{j} a_{\boldsymbol{\lambda}\boldsymbol{\nu}} \chi^{(\overline{\boldsymbol{\nu}},j)}$$

where $\boldsymbol{\nu}$ runs over $\mathrm{Par}_{\boldsymbol{\lambda}}(k)$ and j runs from 0 to $|\mathrm{Aut}(\boldsymbol{\nu})| - 1$. Hence

$$\chi^{(\overline{\boldsymbol{\lambda}},i)}|_{G(r,p,k)} = \frac{1}{|\mathrm{Aut}(\boldsymbol{\lambda})|} \sum_{\boldsymbol{\nu}} \sum_{j} a_{\boldsymbol{\lambda}\boldsymbol{\nu}} \chi^{(\overline{\boldsymbol{\nu}},j)}$$

were the sums are as before. This completes the proof. □

The next result determines the odd-degree irreducible characters of $G(r,p,n)$ in terms of the odd-degree characters of $G(r,1,n)$ and $\mathfrak{S}_{2^s}$.

Proposition 2.2. *Let $\boldsymbol{\lambda} \in \mathrm{Par}_r(n)$ and consider an irreducible constituent $\chi^{(\overline{\boldsymbol{\lambda}},i)}$ of $\chi^{\boldsymbol{\lambda}}|_{G(r,p,n)}$. Then $\chi^{(\overline{\boldsymbol{\lambda}},i)}(1)$ is odd if and only if either*

(i) *$\chi^{\boldsymbol{\lambda}}(1)$ is odd, or*
(ii) *$\chi^{\boldsymbol{\lambda}}(1)$ is even, p is even, $|\mathrm{Aut}(\boldsymbol{\lambda})| = 2$, and $\boldsymbol{\lambda} = (\varnothing, \ldots, \varnothing, \lambda, \varnothing, \ldots, \varnothing, \lambda, \varnothing, \ldots, \varnothing)$ for some λ such that $\chi^{\lambda}(1)$ is odd and $|\lambda| = 2^s$ for some $s \geq 0$.*

Proof. If (i) holds, then $|\mathrm{Aut}(\boldsymbol{\lambda})| = 1$ by Lemma 1.8(iii). Hence by (3) we have $i = 0$ and in turn $\chi^{(\overline{\boldsymbol{\lambda}},i)}(1) = \chi^{\boldsymbol{\lambda}}(1)$ is odd. If (ii) holds, then $\chi^{(\overline{\boldsymbol{\lambda}},i)}(1)$ is odd by (4) and Lemma 1.1.

Suppose $\chi^{(\overline{\boldsymbol{\lambda}},i)}(1)$ is odd and $\chi^{\boldsymbol{\lambda}}(1)$ is even. Then by comparing degrees in (4) and Lemma 1.7, $|\mathrm{Aut}(\boldsymbol{\lambda})| = 2k$ for some positive integer k. Fix some nonempty $\lambda \in \boldsymbol{\lambda}$. Lemma 1.8(ii) tells us that λ must occur at least $2k$ times in $\boldsymbol{\lambda}$. Let $a = |\lambda|$. At least 1 carry occurs when adding a to a in base 2, and in turn at least 2 carries occur when adding up 4 copies of a in base 2, and so on. In general, at least k carries occur in adding up $2k$ copies of a in base 2. Hence $c_{\boldsymbol{\lambda}}$ is a multiple of 2^k by Lemma 1.1. Since

$$\chi^{(\overline{\boldsymbol{\lambda}},i)}(1) = (2k)^{-1} c_{\boldsymbol{\lambda}} \chi^{\lambda_1}(1)\chi^{\lambda_2}(1)\dots\chi^{\lambda_r}(1) \equiv 1 \pmod 2 \tag{8}$$

and $2k \le 2^k$ with equality if and only if $k = 1$, we conclude that $|\mathrm{Aut}(\boldsymbol{\lambda})| = 2$. From (8) with $k = 1$, it follows that $c_{\boldsymbol{\lambda}}$ is divisible by 2 but not 4. By Lemma 1.1 this is the same as saying that exactly 1 carry occurs in adding up the $|\lambda_i|$'s. Since already $\lambda \neq \varnothing$ occurs at least twice in $\boldsymbol{\lambda}$, it follows that λ occurs exactly twice, $|\lambda| = 2^s$ for some $s \ge 0$, and the only other partition that occurs in $\boldsymbol{\lambda}$ is the empty one. Finally, p is even by Lemma 1.8(i). $\square$

2.2. We now prove Theorem A.

Proof of Theorem A. Fix $G(r,p,n)$ and recall that the integers m and n satisfy $m \ge 0$ and $n \ge 1$. Assume $n \ge k(m,p)$, so that n is at least 2^m if p is odd, and at least $3 \times 2^m + 1$ if p is even. Since the assertion is trivial if $n = 2^m$, we may further assume that $n > 2^m$ if p is odd. Fix $\chi^{\boldsymbol{\lambda}} \in \mathrm{Irr}(G(r,1,n))$ so that $\boldsymbol{\lambda} \in \mathrm{Par}_r(n)$. Fix an irreducible constituent $\chi^{(\overline{\boldsymbol{\lambda}},i)}$ of the restriction $\chi^{\boldsymbol{\lambda}}|_{G(r,p,n)}$ and abbreviate it to $\chi^{\overline{\boldsymbol{\lambda}}}$. The object is to show that if $\chi^{\overline{\boldsymbol{\lambda}}}(1)$ is odd, then the restriction $\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-2^m)}$ has a unique odd-degree irreducible constituent of odd multiplicity. Henceforth we call an r-tuple $\boldsymbol{\mu}$ *odd* if $\chi^{\boldsymbol{\mu}}(1)$ is odd, and *even* if $\chi^{\boldsymbol{\mu}}(1)$ is even. Assume that $\chi^{\overline{\boldsymbol{\lambda}}}(1)$ is odd.

Case 1. Suppose $\chi^{\boldsymbol{\lambda}}(1)$ is odd. Then $|\mathrm{Aut}(\boldsymbol{\lambda})| = 1$ by Lemma 1.8(iii). Hence by (7)

$$\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-2^m)} = \sum_{\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)} a_{\boldsymbol{\lambda}\boldsymbol{\mu}}\, \chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)} \quad \text{for} \quad a_{\boldsymbol{\lambda}\boldsymbol{\mu}} = \langle \chi^{\boldsymbol{\lambda}}|_{G(r,1,n-2^m)}, \chi^{\boldsymbol{\mu}} \rangle. \tag{9}$$

If $\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$ is even, then all irreducible constituents of $\chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)}$ are of even degree. Suppose otherwise. Then Proposition 2.2 tells us that $|\mathrm{Aut}(\boldsymbol{\mu})| = 2$,

$$\boldsymbol{\mu} = (\varnothing, \dots, \varnothing, \mu, \varnothing, \dots, \varnothing, \mu, \varnothing, \dots, \varnothing), \quad |\mu| = 2^s \text{ for some } s \ge 0,$$

and p is even. Since p is even, we assume $n \ge 3 \times 2^m + 1$. Since $n = 2 \times 2^s + 2^m$ we conclude that $s \ge m+1$. Since $\boldsymbol{\mu} \le \boldsymbol{\lambda} \in \mathrm{Par}_r(n)$ it follows that 2^s is a summand in two of the 2-adic decompositions of the $|\lambda_i|$'s. By Lemma 1.7, this contradicts $\boldsymbol{\lambda}$ being odd.

If $\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$ is odd, then $|\mathrm{Aut}(\boldsymbol{\mu})| = 1$ by Lemma 1.8(iii). Hence $\chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)} = \chi^{(\overline{\boldsymbol{\mu}},0)}$. We conclude by the previous paragraph that the odd-degree irreducible constituents of $\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-2^m)}$ are precisely the characters $\chi^{(\overline{\boldsymbol{\mu}},0)} = \chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)}$ where $\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$ is odd. It follows then from (6) that the multiplicity of such a constituent is given by

$$\langle \chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-2^m)}, \chi^{(\overline{\boldsymbol{\mu}},0)} \rangle = \sum a_{\boldsymbol{\lambda}\boldsymbol{\nu}}, \quad a_{\boldsymbol{\lambda}\boldsymbol{\nu}} = \langle \chi^{\boldsymbol{\lambda}}|_{G(r,1,n-2^m)}, \chi^{\boldsymbol{\nu}} \rangle, \tag{10}$$

where the sum is over all odd $\boldsymbol{\nu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$ such that $\overline{\boldsymbol{\nu}} = \overline{\boldsymbol{\mu}}$. By Theorem 1.5 applied to $G(r,1,n)$ there exists a unique odd $\boldsymbol{\mu}^\star \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$ such that $a_{\boldsymbol{\lambda}\boldsymbol{\mu}^\star}$ is odd. In particular, $a_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is even for all other odd $\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$. Therefore $\chi^{(\overline{\boldsymbol{\mu}^\star},0)}$ is by (10) the unique odd-degree irreducible constituent of odd multiplicity in the restriction of $\chi^{\overline{\boldsymbol{\lambda}}}$ to $G(r,p,n-2^m)$.

Case 2. Suppose $\chi^{\boldsymbol{\lambda}}(1)$ is even. Since $\chi^{\overline{\boldsymbol{\lambda}}}(1)$ is odd, Proposition 2.2 tells us that p is even,

$$|\mathrm{Aut}(\boldsymbol{\lambda})| = 2,\ \boldsymbol{\lambda} = (\varnothing,\dots,\varnothing,\lambda,\varnothing,\dots,\varnothing,\lambda,\varnothing,\dots,\varnothing),\ \chi^{\lambda}(1) \text{ odd},\ |\lambda| = 2^s \text{ with } s \geq 0. \quad (11)$$

Since p is even, we assume $n \geq 3\times 2^m+1$. Since (11) implies $n = 2^{s+1}$, it follows that $s \geq m+1$.

From (7) with $|\mathrm{Aut}(\boldsymbol{\lambda})| = 2$, we have

$$\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-2^m)} = \frac{1}{2}\sum_{\boldsymbol{\mu}\in\mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)} \langle \chi^{\boldsymbol{\lambda}}|_{G(r,1,n-2^m)}, \chi^{\boldsymbol{\mu}}\rangle \chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)}$$

where the $\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$ are precisely those sequences in $\mathrm{Par}_r(n-2^m)$ of the form

$$\boldsymbol{\mu} = (\varnothing,\dots,\varnothing,\nu_1,\varnothing,\dots,\varnothing,\nu_2,\varnothing,\dots,\varnothing), \quad \varnothing < \nu_1,\nu_2 \leq \lambda, \quad (12)$$

where ν_1 and ν_2 occupy the same two positions occupied by the λ's in $\boldsymbol{\lambda}$. Fix such a $\boldsymbol{\mu}$.

If $\boldsymbol{\mu}$ differs from $\boldsymbol{\lambda}$ in exactly two places, then we claim that all irreducible constituents of $\chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)}$ are of even degree. To this end, suppose $\boldsymbol{\mu}$ differs from $\boldsymbol{\lambda}$ in exactly two places. Then $|\nu_1| = 2^s - a$ and $|\nu_2| = 2^s-(2^m-a)$ for some $0 < a < 2^m$. Since $s \geq m+1$ it follows that $2^m, 2^{m+1},\dots,2^{s-1}$ are summands in the 2-adic decompositions of $|\nu_1|$ and $|\nu_2|$. It follows that if $|\mathrm{Aut}(\boldsymbol{\mu})| = 1$, then $\chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)}$ is irreducible of even degree by Lemma 1.7. Assume now $|\mathrm{Aut}(\boldsymbol{\mu})| = 2$. Then $a = 2^{m-1}$ and $|\nu_1| = |\nu_2| = 2^s - 2^{m-1} = 2^{s-1}+2^{s-2}+\ldots+2^{m-1}$. Hence $s-m+1$ carries occur when $|\nu_1|$ is added to $|\nu_2|$ in base 2. Since $s-m+1 \geq 2$, it follows from (4) and Lemma 1.1 that all irreducible constituents of $\chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)}$ are of even degree.

If $\boldsymbol{\mu}$ differs from $\boldsymbol{\lambda}$ in exactly one place, then we claim that $\chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)}$ is irreducible and

$$\langle \chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-2^m)}, \chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)}\rangle = \langle \chi^{\boldsymbol{\lambda}}|_{G(r,1,n-2^m)}, \chi^{\boldsymbol{\mu}}\rangle. \quad (13)$$

To this end, suppose $\boldsymbol{\mu}$ differs from $\boldsymbol{\lambda}$ in exactly one place. Then $|\mathrm{Aut}(\boldsymbol{\mu})| = 1$, and therefore $\chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)} = \chi^{(\overline{\boldsymbol{\mu}},0)}$. Hence by (6)

$$\langle \chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-2^m)}, \chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)}\rangle = \frac{1}{2}\sum_{\boldsymbol{\nu}}\langle \chi^{\boldsymbol{\lambda}}|_{G(r,1,n-2^m)}, \chi^{\boldsymbol{\nu}}\rangle$$

where the sum is over $\boldsymbol{\nu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$ such that $\overline{\boldsymbol{\nu}} = \overline{\boldsymbol{\mu}}$. Since there are exactly two summands and both equal $\langle \chi^{\boldsymbol{\lambda}}|_{G(r,p,n-2^m)}, \chi^{\boldsymbol{\mu}}\rangle$, we obtain the claimed equality (13).

We conclude that the odd-degree irreducible constituents of odd multiplicity in $\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-2^m)}$ are precisely the restrictions $\chi^{\boldsymbol{\mu}}|_{G(r,p,n-2^m)} = \chi^{(\overline{\boldsymbol{\mu}},0)}$ for $\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$ such that

(i) $\boldsymbol{\mu}$ differs from $\boldsymbol{\lambda}$ in exactly one place,
(ii) $\chi^{\boldsymbol{\mu}}(1)$ is odd,
(iii) $\langle \chi^{\boldsymbol{\lambda}}|_{G(r,1,n-2^m)}, \chi^{\boldsymbol{\mu}}\rangle$ is odd.

Recall $|\lambda| = 2^s$ and $m+1 \leq s$, so that $2^m < 2^s$. The $\boldsymbol{\mu} \in \mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$ that differ from $\boldsymbol{\lambda}$ in exactly one place are the sequences of the form

$$\boldsymbol{\mu_1}(\nu) = (\varnothing,\dots,\varnothing,\nu,\varnothing,\dots,\varnothing,\lambda,\varnothing,\dots,\varnothing),\ \boldsymbol{\mu_2}(\nu) = (\varnothing,\dots,\varnothing,\lambda,\varnothing,\dots,\varnothing,\nu,\varnothing,\dots,\varnothing),$$

where $\nu \in \mathrm{Par}_{\lambda}(2^s-2^m)$ and where ν and λ occupy the same positions occupied by the λ's in $\boldsymbol{\lambda}$. By Equation (2) we have

$$\langle \chi^{\boldsymbol{\lambda}}|_{G(r,1,n-2^m)}, \chi^{\boldsymbol{\mu_1}(\nu)}\rangle = \langle \chi^{\boldsymbol{\lambda}}|_{G(r,1,n-2^m)}, \chi^{\boldsymbol{\mu_2}(\nu)}\rangle = \langle \chi^{\lambda}|_{\mathfrak{S}_{n-2^m}}, \chi^{\nu}\rangle. \quad (14)$$

Since $\chi^{\lambda}(1)$ is odd and the numbers $|\nu| = 2^s-2^m$ and $|\lambda| = 2^s$ are 2-disjoint, by Lemma 1.7 we also have

$$\chi^{\boldsymbol{\mu_1}(\nu)}(1) \text{ is odd} \Leftrightarrow \chi^{\boldsymbol{\mu_2}(\nu)}(1) \text{ is odd} \Leftrightarrow \chi^{\nu}(1) \text{ is odd}. \quad (15)$$

Since $2^m < 2^s$ and $\chi^\lambda(1)$ is odd, [4, Theorem 1] says there exists a unique $\nu^\star \in \mathrm{Par}_\lambda(2^s - 2^m)$ such that $\chi^{\nu^\star}(1)$ and $\langle \chi^\lambda|_{\mathfrak{S}_{2^s-2^m}}, \chi^{\nu^\star}\rangle$ are odd. By (14) and (15) we conclude that $\boldsymbol{\mu_1}(\nu^\star)$ and $\boldsymbol{\mu_2}(\nu^\star)$ are the unique elements of $\mathrm{Par}_{\boldsymbol{\lambda}}(n-2^m)$ that satisfy (i)–(iii). Therefore the restrictions

$$\chi^{\boldsymbol{\mu_1}(\nu^\star)}|_{G(r,p,n-2^m)} = \chi^{(\overline{\boldsymbol{\mu_1}(\nu^\star)},0)} \quad \text{and} \quad \chi^{\boldsymbol{\mu_2}(\nu^\star)}|_{G(r,p,n-2^m)} = \chi^{(\overline{\boldsymbol{\mu_2}(\nu^\star)},0)}$$

are the only odd-degree irreducible constituents of odd multiplicity in $\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-2^m)}$. But the equality $|\mathrm{Aut}(\boldsymbol{\lambda})| = 2$ implies $\overline{\boldsymbol{\mu_1}(\nu^\star)} = \overline{\boldsymbol{\mu_2}(\nu^\star)}$ and hence $\chi^{(\overline{\boldsymbol{\mu_1}(\nu^\star)},0)} = \chi^{(\overline{\boldsymbol{\mu_2}(\nu^\star)},0)}$. So there is exactly one odd-degree irreducible constituent of odd multiplicity in $\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-2^m)}$. □

2.3. The proof of Theorem B is similar to the proof of Theorem A in the special case $m = 0$.

Proof of Theorem B. Fix $G(r,p,n)$. Since the assertion is trivial if $n = 1$, we may assume $n \geq 2$. Let $\chi^{(\overline{\boldsymbol{\lambda}},i)} \in \mathrm{Irr}_{2'}(G(r,p,n))$ so that $\chi^{(\overline{\boldsymbol{\lambda}},i)}$ is an irreducible constituent of $\chi^{\boldsymbol{\lambda}}|_{G(r,p,n)}$ for some fixed $\boldsymbol{\lambda} \in \mathrm{Par}_r(n)$. Abbreviate $\chi^{(\overline{\boldsymbol{\lambda}},i)}$ to $\chi^{\overline{\boldsymbol{\lambda}}}$.

Case 1. Suppose $\chi^{\boldsymbol{\lambda}}(1)$ is even. Then Proposition 2.2 tells us that $|\mathrm{Aut}(\boldsymbol{\lambda})| = 2$ and $\boldsymbol{\lambda}$ is of the form $(\varnothing,\ldots,\varnothing,\lambda,\varnothing,\ldots,\varnothing,\lambda,\varnothing,\ldots,\varnothing)$ for some λ. From Theorem 2.1 it follows that all irreducible constituents of $\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-1)}$ have multiplicity 1. Theorem A tells us that $\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-1)}$ has a unique odd-degree irreducible constituent of odd multiplicity. Hence $\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-1)}$ has a unique odd-degree irreducible constituent.

Case 2. Suppose $\chi^{\boldsymbol{\lambda}}(1)$ is odd. Then $\chi^{\overline{\boldsymbol{\lambda}}} = \chi^{\boldsymbol{\lambda}}|_{G(r,p,n)}$ by Lemma 1.8(iii). Hence

$$\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-1)} = \sum_{\boldsymbol{\mu} \prec \boldsymbol{\lambda}} \chi^{\boldsymbol{\mu}}|_{G(r,p,n-1)}.$$

By Theorem 1.6 applied to $G(r,1,n)$ there is a unique $\boldsymbol{\mu}^\star \prec \boldsymbol{\lambda}$ such that $\chi^{\boldsymbol{\mu}^\star}(1)$ is odd. Since $\boldsymbol{\mu}^\star$ is odd, $|\mathrm{Aut}(\boldsymbol{\mu}^\star)| = 1$ by Lemma 1.8(iii), and so $\chi^{(\overline{\boldsymbol{\mu}^\star},0)} = \chi^{\boldsymbol{\mu}^\star}|_{G(r,p,n-1)}$ is an odd-degree irreducible constituent of $\chi^{\overline{\boldsymbol{\lambda}}}|_{G(r,p,n-1)}$. To end, we claim that if $\boldsymbol{\mu} \prec \boldsymbol{\lambda}$ is even, then $\chi^{\boldsymbol{\mu}}|_{G(r,p,n-1)}$ has only even-degree irreducible constituents. Suppose otherwise, so that for some even $\boldsymbol{\mu} \prec \boldsymbol{\lambda}$ the restriction $\chi^{\boldsymbol{\mu}}|_{G(r,p,n-1)}$ has odd-degree irreducible constituents. Then Proposition 2.2 tells us that p is even, $|\mathrm{Aut}(\boldsymbol{\mu})| = 2$, and $\boldsymbol{\mu} = (\varnothing,\ldots,\varnothing,\mu,\varnothing,\ldots,\varnothing,\mu,\varnothing,\ldots,\varnothing)$ where $|\mu| = 2^s$ for some $s \geq 0$. Since p is even, our assumption is $n > 3$. Since $n = 2 \times 2^s + 1$, then $s > 0$. It follows that 2^s is a summand in two of the 2-adic decompositions of the $|\lambda_i|$'s. By Lemma 1.7 this contradicts $\boldsymbol{\lambda}$ being odd. □

3. Remarks

Here we remark on some surjectivity results of [4] and [2] and their extensions to $G \wr \mathfrak{S}_n$ and $G(r,p,n)$. The extensions are stated in Theorems 3.1 and 3.2.

Suppose that $n \geq 2^m$ and let $f : \mathrm{Irr}_{2'}(\mathfrak{S}_n) \to \mathrm{Irr}_{2'}(\mathfrak{S}_{n-2^m})$ be the map that takes an odd-degree irreducible character χ of $\mathfrak{S}_n$ to the unique odd-degree irreducible character $f(\chi)$ of odd multiplicity in the restriction of χ to $\mathfrak{S}_{n-2^m}$. Isaacs, Navarro, Olsson, and Tiep proved [4, Proposition 4.5] that if 2^m is a summand in the 2-adic decomposition of n, then f is a 2^m-to-1 surjection. Bessenrodt, Giannelli, and Olsson later proved [2, Theorem A] that f is surjective if and only if either $m > 0$ and at least one of the powers $2^m, 2^{m+1}$ is a summand in the 2-adic decomposition of n, or $m = 0$ and at least one of the powers $2^m, 2^{m+1}, 2^{m+2}$ is a summand in the 2-adic decomposition of n. Moreover [2, Theorem 3.5] if f is surjective, then f is 2^m-to-1 if 2^m is a summand in the 2-adic decomposition of n, and is 2-to-1 otherwise.

3.1. The following theorem extends [4, Proposition 4.5] and [2, Theorem 3.5] from $\mathfrak{S}_n$ to $G \wr \mathfrak{S}_n$. The proof is omitted as a straightforward application of Theorem 1.5, [4, Proposition 4.5], and [2, Theorem 3.5].

Theorem 3.1. *Suppose that $n \geq 2^m$ and $|G| \neq 1$. Let $F : \mathrm{Irr}_{2'}(G \wr \mathfrak{S}_n) \to \mathrm{Irr}_{2'}(G \wr \mathfrak{S}_{n-2^m})$ be the map where $F(X)$ is the unique odd-degree irreducible constituent of odd multiplicity in the restriction of X to $G \wr \mathfrak{S}_{n-2^m}$. Then the following hold.*

(i) *F is surjective if and only if 2^m or 2^{m+1} is a summand in the 2-adic decomposition of n.*

(ii) *If 2^m is a summand in the 2-adic decomposition of n, then*

$$F^{-1}(X^{\boldsymbol{\mu}}) = \{X^{\boldsymbol{\lambda}} : \boldsymbol{\lambda} = (\mu_1, \ldots, \mu_{i-1}, \lambda_i, \mu_{i+1}, \ldots, \mu_r),\ \chi^{\lambda_i} \in f^{-1}(\chi^{\mu_i}),\ \gamma_i(1) \text{ odd}\} \tag{16}$$

and F is a $2^m|\mathrm{Irr}_{2'}(G)|$-to-1 surjection.

(iii) *If 2^m is not a summand in the 2-adic decomposition of n and 2^{m+1} is a summand in the 2-adic decomposition of n, then F is a 2-to-1 surjection and*

$$F^{-1}(X^{\boldsymbol{\mu}}) = \{X^{\boldsymbol{\lambda}} : \boldsymbol{\lambda} = (\mu_1, \ldots, \mu_{k-1}, \lambda_k, \mu_{k+1}, \ldots, \mu_r),\ \chi^{\lambda_k} \in f^{-1}(\chi^{\mu_k})\} \tag{17}$$

where k is the index such that 2^m is a summand in the 2-adic decomposition of $|\mu_k|$. □

3.2. The next theorem extends [4, Proposition 4.5] and [2, Theorem 3.5] from $\mathfrak{S}_n$ to $G(r,p,n)$. The proof is omitted as a straightforward application of Theorem A, Theorem 3.1, [4, Proposition 4.5], and [2, Theorem 3.5].

Theorem 3.2. *Suppose that $n \geq k(m,p)$ and $r \neq 1$. Let $\Phi : \mathrm{Irr}_{2'}(G(r,p,n)) \to \mathrm{Irr}_{2'}(G(r,p,n-2^m))$ be the map where $\Phi(\chi)$ is the unique odd-degree irreducible constituent of odd multiplicity in the restriction of χ to $G(r,p,n-2^m)$. If p is even and $n-2^m$ is a power of 2, then Φ is not surjective. If it is not the case that both p is even and $n-2^m$ is a power of 2, then the following hold.*

(i) *Φ is surjective if and only if at least one of the powers 2^m, 2^{m+1} is a summand in the 2-adic decomposition of n, or $(n,m,r,p) = (4,0,2,2)$.*

(ii) *If 2^m is a summand in the 2-adic decomposition of n, then Φ is a $2^m r$-to-1 surjection.*

(iii) *If 2^m is not a summand in the 2-adic decomposition of n, and 2^{m+1} is a summand in the 2-adic decomposition of n, then Φ is a 2-to-1 surjection.*

(iv) *If $(n,m,r,p) = (4,0,2,2)$, then Φ is a 2-to-1 surjection.* □

Acknowledgements. The authors gratefully acknowledge the hospitality of MSRI where this collaboration began.

References

1. A. Ayyer, A. Prasad, and S. Spallone, Odd partitions in Young's lattice. *Sémin. Lothar. Comb.* **75** (2016) Article B75g.
2. C. Bessenrodt, E. Giannelli, and J. Olsson, Restriction of odd degree characters of $\mathfrak{S}_n$. *SIGMA Symmetry Integrability Geom. Methods Appl.* **13** (2017) 10 pp.
3. T. Halverson and A. Ram, Murnaghan–Nakayama rules for characters of Iwahori–Hecke algebras of the complex reflection groups $G(r,p,n)$. *Can. J. Math.* **50** (1998) 167–192.
4. I. M. Isaacs, G. Navarro, J. B. Olsson, and P. H. Tiep, Character restriction and multiplicities in symmetric groups. *J. Algebra* **478** (2017) 271–282.
5. G. D. James, *The representation theory of the symmetric groups*, Lecture Notes in Mathematics, vol. 682, Springer, Berlin, 1978.

6. E. E. Kummer, Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen. *J. reine angew. Math.* **44** (1852) 93–146.
7. I. Marin, Branching properties for the groups $G(de, e, n)$. *J. Algebra* **323** (2010) 966–982.
8. S. Okada, Wreath products by the symmetric groups and product posets of Young's lattices. *J. Combin. Theory Ser. A* **55** (1990) 14–32.

Trinity Hall, University of Cambridge, Trinity Lane, CB21TJ, United Kingdom
E-mail address: eg513@cam.ac.uk

Fakultät für Mathematik, Universität Wien, A-1090 Wien, Austria
E-mail address: alexander.r.miller@univie.ac.at