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BOUNDARY VALUES OF HOLOMORPHIC FUNCTIONS, SINGULAR UNITARY REPRESENTATIONS OF THE GROUPS $O(p,q)$, AND THEIR LIMITS AS $q \to \infty$

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Abstract. Let Ω be a bounded circular domain in $\mathbb{C}^N$, M be a submanifold in the boundary of Ω, and H be a Hilbert space of holomorphic functions in Ω. We show that, under certain conditions stated in terms of the reproducing kernel of the space H, the restriction operator to the submanifold M is well defined for all functions from H.

We apply this result to constructing a family of "singular" unitary representations of the groups $SO(p,q)$. The singular representations arise as discrete components of the spectrum in the decomposition of irreducible unitary highest weight representations of the groups $U(p,q)$ restricted to the subgroups $SO(p,q)$.

Another property of the singular representations is that they persist in the limit as $q \to \infty$.

Since the works of V. Bargmann [32] and I. M. Gelfand and Naimark [54], which initiated the study of infinite-dimensional representations, Hilbert spaces with scalar products of the form

$$\langle f, g\rangle = \int\limits_M \int\limits_M K(x,y) f(x)\overline{g(y)} dx\, dy, \tag{0.1}$$

where M is a manifold and $K(x,y)$ is a distribution on $M \times M$, are usual in the theory of unitary representations of semisimple groups. The Hermitian forms of type (0.1) a priori need not be positive definite. Thus, in each specific case, the verification of the positive definiteness is more or less difficult problem (sometimes very difficult).

The above function spaces did not fix the attention of analysts (although, some objects of this type were studied, e.g., the scalar product in Sobolev spaces is written in the form (0.1) with $M = \mathbb{R}^n$ and $K(x,y) = \|x-y\|^\lambda$). On the other hand, in representation theory, such Hilbert spaces are regarded as a "receptacle" of algebraic structure, but not as an object worth special study.

The aim of the present paper is to show that there are phenomena in noncommutative harmonic analysis and representation theory (some of them are usually regarded as algebraic phenomena of complicated nature) which are based on the assertions of the "trace theorem" type. We mean that in a function space $\mathcal{H}$ with a scalar product of the form (0.1) there is a well-defined restriction operator of a function f to a submanifold $N \subset M$

of sufficiently large dimension (and suitably located). We note that the value of f at an individual point $x \in N$ can be undefined. Such phenomena are known in analysis, e.g., the restriction to the boundary in Hardy spaces; the restriction to a submanifold in Sobolev spaces (see, e.g., [8], §3.2 and [26], §1.3); the Nagel–Rudin theorem about the restriction of a function from H^∞ in the ball to a suitably located curve on the sphere, see [25], §11.2; some questions in the theory of exceptional sets (the Salem–Zygmund theorem, see [65, 10]); the results of Stein and Fefferman on Fourier transform of functions $f \in L^p$ ([70]). The space $E(N)$ of all functions on N that are the restrictions of functions $f \in \mathcal{H}$ to N is a quotient space of $\mathcal{H}$. Therefore, $E(N)$ is a Hilbert space itself.

Dually, it can happen that $\mathcal{H}$ contains distributions with support lying in N. Then the space $E^\circ(N)$ of all such distributions is also a Hilbert space as a subspace of $\mathcal{H}$.

Now let a Lie group G act on M by the transformations $g : x \mapsto x^g$, and let N be a G-invariant submanifold of M. Assume that we are given a unitary representation ρ of G in $\mathcal{H}$ defined by the formula

$$\rho(g)f(x) = c(g,x)f(x^g).$$

Then we obtain a unitary representation of G in $E(N)$ (or in $E^\circ(N)$).

This method can be used in representation theory in order to present partially discrete spectra and to construct exotic unitary representations.

In [45], Pukansky studied the decomposition problem for the tensor product of irreducible representations of $\mathrm{SL}_2(\mathbb{R})$. Here the result turned out to be somewhat monotonous (e.g., the product is almost independent of the factors, the principal series can only entirely occur in the tensor product, and only as a direct continuous integral). However, the behavior of the complementary series turned out to be interesting. Namely, a representation of the complementary series cannot occur in the tensor product of irreducible representations with the exception of the case where the factors belong to the complementary series. In the latter case, the tensor product has at most one representation from the complementary series as a direct summand (for applications of the Pukansky theorem to infinite-dimensional groups, see [5, 17]). For $\mathrm{SL}_2(\mathbb{C})$, a similar result was obtained by Naimark [15]. Subsequently, such phenomena were found in different spectral problems, see, e.g., [33, 12], and [50]. As is shown in [16], all these phenomena can clearly be explained by the trace theorems.

Flensted–Jensen [36] studied decompositions of L^2 on symmetric pseudo-Riemannian spaces. It turns out that these decompositions have partially discrete spectra, and the discrete part of a spectrum consists of very strange unitary representations which did not fit in the constructions previously known (presently, there exists a difficult pure algebraic techniques [31] intended to handle such irreducible representations). After the paper of Flensted–Jensen, an extensive literature was devoted to a search of "exotic" unitary representation in various discrete spectra, see, e.g., Adams [30], Schlichtkrull [48], and Kobayashi [41, 63].

In [19] and [21], one of the authors obtained the classification of the unitary representations of $O(p,\infty)$, $U(p,\infty)$, and $\mathrm{Sp}\,(p,\infty)$ which are continuous in the weak topology. It turns out that, in the problems of restriction of unitary representations from $O(p,\infty)$ to $O(p,n)$, a partially discrete spectrum arises. This spectrum consists of strange unitary representations of $O(p,n)$ (the same for $U(p,n)$ and $\mathrm{Sp}\,(p,n)$). It also turns out that these spectra intersect with spectra from papers of Schlichtkrull [48] and Adams [30]. The present paper (partially announced in [16, 21] and [22]) is a result of the works of the authors undertaken with quite different purposes. One of the authors studied the approximation of the unitary representations of $O(p,\infty)$, $U(p,\infty)$, and $\mathrm{Sp}\,(p,\infty)$ by representations of finite-dimensional groups $O(p,n)$, $U(p,n)$, and $\mathrm{Sp}\,(p,n)$. The other author tried to understand the nature of phenomena connected with complementary series.

In §1 we review some standard facts about Hilbert spaces of holomorphic functions on domains $\Omega \subset \mathbb{C}^N$.

In §2 we present a very simple analytic construction. We show that, under some additional conditions, the restriction operators to some subsets M of the boundary of Ω are well defined in Hilbert spaces of holomorphic functions on $\Omega \subset \mathbb{C}^N$. We consider an illustrative example in detail. Namely, we consider the unit ball $B_q \subset \mathbb{C}^q$, and the boundary S^{2q-1} of B_q. We take the real equator of $S^{q-1} = \mathbb{R}^q \cap S^{2q-1}$ in the role of M. It turns out that for some Hilbert spaces of holomorphic functions on B_q, the restriction operator to S^{q-1} is well defined. In terms of representation theory this means that the representations of the complementary series of $O(1,q)$ embed into the representations of $U(1,q)$ with highest weight.

In §3 we consider a special case of our basic construction. Let $B_{p,q}$ be a matrix ball (the Cartan domain of the first kind), i.e., the set of all matrices of size $p \times q$ with norm < 1. We consider the Hilbert spaces of holomorphic functions in $B_{p,q}$ introduced by Berezin in [4]. As an analog of the real equator, we consider the Stiefel manifold which is the real part of the Shilov boundary of $B_{p,q}$. We show that (under some conditions on parameters) in Berezin spaces, the operator of restriction to M is defined. In terms of representation theory this means that we present an irreducible subrepresentation of $O(p,q)$ in the restriction of the representation of $U(p,q)$ with highest weight to the subgroup $O(p,q)$.

In §4 we extend the construction of §3 to arbitrary unitary representations of $U(p,q)$ with highest weight. This means that we turn from Hilbert spaces of scalar functions on M to spaces of vector-valued functions. Analytic estimates providing the existence of a restriction to the boundary remain the same, however, the algebraic structure becomes considerably complicated, and “singular” unitary representations of $O(p,q)$ arise in the spaces of boundary values of holomorphic functions on M. In §5 we discuss some properties of these “singular” representations (we almost do not use this section in the sequel).

In §6 we present explicit constructions of all irreducible unitary representations of the group $O(p,\infty)$ equipped with the weak topology (see [19]). For any such representation ρ, we construct a sequence of subspaces of H,

$$H_k \subset H_{k+1} \subset H_{k+2} \subset \dots,$$

such that H_j is invariant under $O(p,j)$ and the representation of $O(p,j)$ in H_j is irreducible (here k is sufficiently large and depends on ρ).

The inductive limits of classical noncompact Lie groups have an immense number of unitary representations. However, in rare cases these representations are inductive limits of irreducible representations. Actually, such sequences are only known for the representations of $O(p,\infty)$, $U(p,\infty)$, and $\mathrm{Sp}\,(p,\infty)$ continuous in the weak topology, and also in the (trivial) case of the representations of $\mathrm{Sp}\,(2\infty, \mathbb{R})$ and $U(\infty, \infty)$, $SO^*(2\infty)$ with highest weight. It seems to be an interesting question whether this is valid in some other cases.

In §7 we discuss several more examples where the trace theorem may be applied. First (Secs. 7.1–7.8, these sections don't depend on §4–6), we solve the problem of decomposition of tensor product of representations for the representations of the spherical principal series of the group $O(1, q)$. We also give a simple solution of the problem (solved by Boyer in [33]) concerning the restriction of these representations to the subgroup $O(1, q-1)$. Further, in Sec. 7.10 we observe that all asserted about the complementary series of $O(1, q)$ can easily be generalized to the "singular" representations of $O(p, q)$ constructed in §§3–5. In particular, in this case we also have partially discrete spectrum in the problems of decomposition of tensor products and of restriction to a subgroup. The existence of the Gelfand–Tsetlin basis is one more curious property of the singular representations. In the present paper, we do not discuss this question.

In this paper we mainly concern ourselves with representations of the group $O(p, q)$. In fact, the same is valid for $U(p, q)$ and $\mathrm{Sp}\,(p, q)$. All necessary remarks on this point are made in Secs. 7.12–7.14.

As noted above, the set of singular representations constructed in the present paper intersects with the representations constructed by Adams in the study of the dual pairs of type $(\mathrm{Sp}\,(2k, \mathbb{R}),\ O(p, q))$, and also with the representations constructed by Schlichtkrull in his study of decompositions of L^2 on the indefinite Stiefel manifold $O(p, q)/O(p-\alpha, q)$. The existence of such intersection has simple a priori reasons which we explain in §8. We note that our construction has a continuous parameter whereas the constructions of Adams, Schlichtkrull, and Kobayashi depend only on discrete parameters.

We also note that our construction is extremely simple both from the conceptual and technical viewpoints (the technical part involves estimation of some integral). We only use the part of the representation theory of semisimple groups (which is simple and self-contained) devoted to the representations with highest weight. Therefore, we had a chance to make our exposition almost self-contained and tried to attain this aim.

In conclusion, we want to thank V. F. Molchanov, H. Schlichtkrull, and G. Zuckerman for discussion of the subject.

§1. Hilbert spaces of holomorphic functions: preliminary remarks

1.1. Positive definite kernels (see [11]). Let X be a set. A *positive definite kernel* is a function $K(x, y)$ on $X \times X$ such that

(a) $K(x, y) = \overline{K(x, y)}$;

(b) for all $x_1, \dots, x_n \in X$, the matrix

$$\begin{pmatrix} K(x_1, x_1) & \dots & K(x_1, x_n) \\ \dots & \dots & \dots \\ K(x_n, x_1) & \dots & K(x_n, x_n) \end{pmatrix}$$

is nonnegative definite.

Let $x \mapsto \Psi_x$ be a mapping of X into a Hilbert space H. Then the function

$$K(x, y) = \langle \Psi_x, \Psi_y \rangle \tag{1.1}$$

is a positive definite kernel on X.

Conversely, let $K(x, y)$ be a positive definite kernel. Then there exists a Hilbert space $H = H_K(X)$ and a total system of vectors $\Psi_x \in H$ such that (1.1) is fulfilled. Moreover, the space H is unique in the sense that if H' is another such space and $\Psi'_x \in H'$ is the corresponding system of vectors, then there exists a unitary operator $U : H \to H'$ such that $U\Psi_x = \Psi'_x$ for all $x \in X$.

Let $h \in H_K(X)$. Then we have a function $(Ih)(x)$ on X defined by the formula

$$Ih(x) = \langle h, \Psi_x \rangle_{H_K(X)}.$$

Lemma 1.1. (a) *If $h_1 \neq h_2$, then $Ih_1 \neq Ih_2$.*

(b) *If $h_j \to h$ weakly in $H_K(X)$, then Ih_j tends to Ih pointwise.*

Proof is obvious.

From now on, we identify $H_K(X)$ with the corresponding space of functions on X. We note that the linear functional on $H_K(X)$ given by the formula

$$l_x(f) = f(x) = \langle f, \Psi_x \rangle$$

is continuous for every $x \in X$.

1.2. Hilbert spaces of holomorphic functions (see, e.g., [55], IX.2). Let Ω be a bounded open subset of $\mathbb{C}^N$. Let $\bar{\Omega}$ be the closure of Ω, and let $\partial\Omega$ be its boundary. We say that Ω is a *regular circular domain* if

1^0. For all $z \in \Omega$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, we have $\lambda z \in \Omega$.

2^0. For all $z \in \partial\Omega$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, we have $\lambda z \in \Omega$.

Below we always assume that Ω satisfies these conditions unless otherwise stated.

Remark. Recall that a complete circular domain is a domain in $\mathbb{C}^N$ satisfying only 1^0. Consider the bidisk $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1,\ |z_2| < 1\}$ without the annulus $\{(z, z) : \frac{1}{2} \leq |z| < 1\}$. This gives an example of a complete circular domain which is not a regular circular domain.

Recall (see [29], §I.3.8), that every holomorphic function f in a complete circular domain Ω can be expanded into a convergent series

$$f = \sum f_j$$

in homogeneous polynomials.

Let H be a linear subspace of the space of holomorphic functions on Ω. We say that H is a *Hilbert space of holomorphic functions* on Ω if H is equipped with a positive definite scalar product $\langle \cdot, \cdot \rangle$ such that

(a) H is a Hilbert space,

(b) for every $z \in \Omega$ the linear functional

$$l_z(f) = f(z)$$

is continuous on H,

(c) the operators

$$R_\varphi f(z) = f(e^{i\varphi} z)$$

are unitary in H and depend weakly continuously on $\varphi \in \mathbb{R}$.

For examples, see 1.4–1.5 below. Since the linear functionals l_u are continuous, there exists a system of functions $\psi_u \in H$, $u \in \Omega$, such that

$$h(u) = \langle h, \psi_u \rangle \quad \forall h \in H.$$

We define a function $K(z, \bar{u})$ (*"a reproducing kernel"*) on $\Omega \times \Omega$ by the formula

$$K(z, \bar{u}) = \langle \psi_u, \psi_z \rangle = \psi_u(z) = \overline{\psi_z(u)}. \tag{1.2}$$

It is obvious that

1^0. $K(z, \bar{u})$ is positive definite,

2^0. $K(z, \bar{u})$ is holomorphic in z and antiholomorphic in u,

3^0. $K(e^{i\varphi} z,\, e^{-i\varphi} \bar{u}) = K(z, \bar{u})$.

Proposition 1.1. (a) *Let $K(z, \bar{u})$ be a function on $\Omega \times \Omega$ satisfying 1^0–3^0. Then $H_K(\Omega)$ is a Hilbert space of holomorphic functions on Ω, and the system of functions $\psi_u \in H_K(\Omega)$ is given by the formula*

$$\psi_u(z) = K(z, \bar{u}).$$

(b) *For every $f \in H_K(\Omega)$ the following inequality holds:*

$$|f(z)| \le \|f\|_{H_k(\Omega)} \cdot \sqrt{K(z, \bar{z})}. \tag{1.3}$$

(c) *The operation of constructing the Hilbert space H from a kernel K is inverse to the operation of constructing K from H.*

(d) *If $h_j \to h$ weakly in $H_K(\Omega)$, then $h_j \to h$ uniformly on compact sets in Ω.*

Proof. Statement (b) follows from the Cauchy–Bunyakovsky inequality. Now we proceed to the proof of (a). By the definition of $H_K(\Omega)$, finite linear combinations of ψ_u are dense in $H_K(\Omega)$. Let θ_j be a sequence of such functions which converges to some function θ in $H_K(\Omega)$. Then θ_j converges pointwise (due to Lemma 1.1) and is bounded on compact sets due to (1.3) (one should take into account that $K(z, \bar{z})$ is continuous by the Hartogs theorem, see [29]). Therefore, θ is holomorphic due to the compactness principle, see [29]. The rest is obvious. $\square$

1.3. Polynomials. We denote by Pol the space of polynomials on $\mathbb{C}^N$, and by Pol_j the space of homogeneous polynomials of degree j.

Proposition 1.2. *Let $H = H_K(\Omega)$ be a Hilbert space of holomorphic functions in a regular circular domain Ω. Then*

$$H = \bigoplus_{j=0}^{\infty} (H \cap \mathrm{Pol}_j).$$

Remark. We emphasize that H may not contain Pol (this is true in some cases).

Proof. It is obvious that every holomorphic eigenfunction of the operator $f(z) \mapsto f(e^{i\varphi}x)$ is a polynomial. On the other hand, these operators form a compact Abelian group. Therefore, every $f \in H$ can be expanded into a convergent series in eigenvectors. $\square$

Remark. By the property 3^0 of Sec. 1.2, the kernel $K(z, \bar{u})$ can be expanded into a series of the form

$$K(z, \bar{u}) = \sum_{j=0}^{\infty} K_j(z, \bar{u}),$$

where K_j is a homogeneous polynomial of degree j in z and of degree j in $\bar{u}$. Then the Hilbert space of holomorphic functions $H_K(\Omega) \cap \mathrm{Pol}_j$ is defined by the positive definite kernel $K_j(z, \bar{u})$.

Remark. We show that $K(z, \bar{u})$ is defined if $z \in \Omega$ and $u \in \bar{\Omega}$. Indeed, since K_j are homogeneous, the kernel $K(z, \bar{u})$ is invariant under the transformations $(z, u) \mapsto (c^{-1}z, cu)$. Now we can choose a positive $c < 1$ such that $c^{-1}z \in \Omega$ and put

$$K(z, \bar{u}) := K(c^{-1}z, \overline{cu})$$

for all $z \in \Omega$ and $u \in \bar{\Omega}$.

1.4. Example: weight spaces. Let ν be a measure in $\mathbb{C}^N$ invariant under the transformations $z \mapsto e^{i\varphi}z$. Let the condition

$$0 < \int |P(z)|^2 \, d\nu(z) < \infty$$

hold for every nonzero polynomial $P(z)$.

Then the formula

$$\langle f, g \rangle = \int f(z)\overline{g(z)} \, d\nu(z)$$

defines a scalar product in Pol. Let $H(\nu)$ be the completion of Pol with respect to this scalar product. Let Ξ be the set of all $u \in \mathbb{C}^N$ such that the linear functional $P \mapsto P(u)$ on Pol can be extended to a continuous linear functional on $H(\nu)$. Let Ω be the interior of Ξ. If Ω is nonempty, then $H(\nu)$ is the Hilbert space of holomorphic functions on Ω (this follows from [25], 1.5.6).

Example 1. Let $N = 1$, and let ν be the Lebesgue measure on the circle $|z| = 1$. Then $H(\nu)$ is the Hardy space H^2 in the disk $|z| < 1$.

Example 2. Let the measure ν has density $e^{-|z|^2}$ with respect to Lebesgue measure on $\mathbb{C}^N$. Then $\Omega = \mathbb{C}^N$, and $H(\nu)$ is the boson Fock space (see Sec. 8.1 below).

Although the weight spaces are important in analysis, our main interest here is the spaces which do not fall into the scheme described above.

1.5. Example: Hilbert spaces of holomorphic functions in the ball. Denote by B_q the open ball $\sum_{k=1}^{q} |z_k|^2 < 1$ in $\mathbb{C}^q$.

First let $s > q$. Consider the space H_s of holomorphic functions in B_q satisfying the condition

$$\int\limits_{B_q} |f(z)|^2 \, (1-|z|^2)^{s-q-1} \, dz \, d\bar{z} < \infty.$$

We define a scalar product in the space H_s by the formula

$$\langle f, g\rangle = \frac{1}{\tau_{q,s}} \int\limits_{B_q} f(z)\overline{g(z)} \, (1-|z|^2)^{s-q-1} \, dz \, d\bar{z},$$

where $\tau_{q,s}$ is defined by $\langle 1, 1\rangle = 1$. It can easily be checked that H_s is a Hilbert space of holomorphic functions and that the corresponding reproducing kernel is given by the formula

$$K_s(z, \bar{u}) = \left(1 - \sum_{k=1}^{q} z_k \bar{u}_k\right)^{-s}. \tag{1.4}$$

Proposition 1.3. *The kernel $K_s(z, \bar{u})$ is positive definite for all $s \geq 0$.*

Proof. It is obvious that

$$K_s(z, \bar{u}) = \sum_{j=0}^{\infty} \frac{s(s+1)\dots(s+j-1)}{j!} \left(\sum z_k \bar{u}_k\right)^j. \tag{1.5}$$

All the coefficients of this series are positive, and the statement follows. □

Now let $s \geq 0$. We define $H_s = H_s(B_q)$ to be the Hilbert space of holomorphic functions in B_q defined by the kernel (1.4).

Remark. Let $f, g \in H_s$, and let $f = \sum f^{(j)}$ and $g = \sum g^{(j)}$ be the expansions of f and g, respectively, in homogeneous polynomials. Then, by (1.5) and [25], I.4.9, we have

$$\langle f, g\rangle_{H_s} = \sum \frac{s(s+1)\dots(s+j-1)}{(q+1)\dots(q+j)} \langle f^{(j)}, \, g^{(j)}\rangle_{L^2(B^q)}.$$

Remark. The space H_0 is one-dimensional and consists of constants.

Remark. The space $H_{q+1}(B_q)$ is a Bergmann space; the space $H_q(B_q)$ coincides with the Hardy space H^2; for $s > q$, the spaces $H_s(B_q)$ are discussed, e.g., in [1]. Our main concern is the case $0 < s < \frac{q-1}{2}$.

1.6. Distributions. From the viewpoint of "coherent states" (cf. [4, 23]), the system of functions ψ_u can naturally be regarded as a kind of a "basis" in $H_K(\Omega)$. We proceed to introduce an appropriate formalism.

Put, as usual,

$$\mathcal{D}_x^{(\alpha)} = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_N} x_N}.$$

Let Ω and $H_K(\Omega)$ be as above. Let $u \in \Omega$, and let $\psi_u^\alpha(z) = \mathcal{D}_{\bar{u}}^\alpha K(z, \bar{u})$.

Lemma 1.2. *The functions $\psi_u^{(\alpha)}(z)$ belong to $H_K(\Omega)$, and the following relations hold:*

$$\begin{aligned} \langle h, \psi_z^{(\alpha)}\rangle &= \mathcal{D}_z^{(\alpha)} h(z) \quad \forall h \in H_K(\Omega), \\ \langle \psi_u^{(\alpha)}, \psi_z^{(\beta)}\rangle &= \mathcal{D}_{\bar{u}}^{(\alpha)} \mathcal{D}_z^{(\beta)} K(z, \bar{u}). \end{aligned} \tag{1.6}$$

For simplicity of notation, we give the proof for a one-dimensional domain and the first order derivatives. We should check that the family of functions

$$\tau_\varepsilon(z) = \frac{1}{\varepsilon}(\psi_{u+\varepsilon}(z) - \psi_u(z))$$

is weakly convergent in $H_K(\Omega)$ as $\varepsilon \to 0$. The pointwise convergence is obvious. It remains to prove that $\tau_\varepsilon(z)$ is uniformly bounded. We have

$$\|\tau_\varepsilon\|^2 = \frac{1}{\varepsilon^2}(K(u+\varepsilon, \bar{u}+\varepsilon) - K(u, \bar{u}+\varepsilon) - K(u+\varepsilon, \bar{u}) + K(u, \bar{u})).$$

The right-hand side of this relation is $\frac{\partial^2}{\partial z\, \partial \bar{w}} K(z, \bar{w})$ at some point close to $(u, \bar{u})$. This proves that $\psi_u^{(\alpha)} \in H_K$, and also the first equation. The second equation is a special case of the first one. $\square$

Let A be a subset of $\mathbb{C}^N$. We denote by $\mathcal{D}'(A)$ the set of distributions on $\mathbb{C}^N$ with support in A.

Let $\chi \in \mathcal{D}'(\bar{\Omega})$. Consider the function

$$(\Xi\chi)(z) = \int K(z, \bar{u})\chi(u) du, \tag{1.7}$$

which is holomorphic in Ω.

Proposition 1.4. *If $\chi \in \mathcal{D}'(\Omega)$, then $\Xi\chi \in H_K(\Omega)$. Moreover, for all $h \in H_K(\Omega)$*

$$\langle h, \Xi\chi\rangle = \int h(z)\overline{\chi(z)}\, dz. \tag{1.8}$$

In particular, for all χ, $\widetilde{\chi} \in \mathcal{D}'(\Omega)$,

$$\langle \Xi\chi, \Xi\widetilde{\chi}\rangle = \int\int \overline{K(z, \bar{u})}\chi(z)\overline{\widetilde{\chi}(u)}\, dz\, du. \tag{1.9}$$

Proof. The support of the distribution χ lies in a compact set $M \subset \Omega$, and for $u \in M$ and $|\alpha| \leq K$, the functions $\|\psi_u^{(\alpha)}\|$ are uniformly bounded (see (1.6)). Therefore, $\Xi\chi$ is a barycenter of a finite measure concentrated on a bounded subset of $H_K(\Omega)$ (consisting of the functions of the form $\psi_u^{(\alpha)}$). The rest of the proof is obvious. $\square$

Now we proceed to discuss distributions with support in the closed domain $\bar{\Omega}$.

Lemma 1.3. *Let $\chi \in \mathcal{D}'(\overline{\Omega})$. Let $0 < c < 1$, and let*

$$(\Xi_c \chi)(z) = \int K(z, c\bar{u})\chi(u)du.$$

Then $\Xi_c\chi \in H_K(\Omega)$, and $\|\Xi_c\chi\|_{H_K(\Omega)}$ increases in the interval $0 < c < 1$.

Proof. Let $K = \sum K_j$ be the expansion of K in homogeneous polynomials. We define the vectors

$$\Xi^{(j)}\chi(z) = \int K_j(z, \bar{u})\chi(u)du \in \mathrm{Pol}_j \cap H_K(\Omega).$$

Then

$$\Xi_c\chi = \sum_{j=0}^{\infty} c^j (\Xi^{(j)}\chi), \tag{1.10}$$

and now the statement is obvious. $\square$

It also follows from (1.10) that

$$\begin{aligned} \|\Xi_c\chi\|^2 = \sum c^{2j}\|\Xi^{(j)}\chi\|^2 = \sum c^{2j} \int\int \overline{K^{(j)}(z, \bar{u})}\, \chi(z)\overline{\chi(u)}\, dz\, du \\ = \int\int \overline{K(cz, c\bar{u})}\chi(z)\overline{\chi(u)}\, dz\, du. \end{aligned} \tag{1.11}$$

Proposition 1.5. (a) *Let $\chi \in \mathcal{D}'(\bar{\Omega})$. Then the following conditions are equivalent:*

(i) *the quantity $\|\Xi_c\chi\|^2_{H_K(\Omega)}$ is bounded for $0 < c < 1$,*

(ii) $\Xi\chi \in H_K(\Omega)$,

(iii) *the linear functional on* Pol $\cap H_K(\Omega)$ *given by the distribution χ can be extended to a bounded linear functional on $H_K(\Omega)$.*

(b) *Let χ and $\widetilde{\chi}$ satisfy* (i)–(iii). *Then*

$$\langle \Xi\chi, \Xi\widetilde{\chi}\rangle = \lim_{c\to 1-0} \langle \Xi_c\chi, \Xi_c\widetilde{\chi}\rangle = \lim_{c\to 1-0} \int\int \overline{K(cz, c\bar{u})}\chi(z)\overline{\chi(u)}\, dz\, du. \tag{1.12}$$

Proof. All the statements obviously follow from Lemma 1.3 and formulas (1.10) and (1.11).

1.7. Invariant kernels. Now let a connected Lie group G act on Ω by biholomorphic transformations (actually, we are interested in Cartan domains; the case $\Omega = B_q$ is discussed in the next subsection). We denote by z^g the image of $z \in \Omega$ under $g \in G$. Denote by $j(g, z)$ the complex Jacobian of the map $z \mapsto z^g$. Let $\lambda \in \mathbb{R}$. Consider the action

$$\rho_\lambda(g)f(z) = j(g, z)^\lambda f(z^g) \tag{1.13}$$

of G on the space of holomorphic functions on Ω. For $\lambda \in \mathbb{Z}$ the following relation holds:

$$\rho_\lambda(g_1 g_2) = \rho_\lambda(g_1)\rho_\lambda(g_2), \tag{1.14}$$

i.e., ρ_λ is a linear representation of G.

The case $\lambda \notin \mathbb{Z}$ needs clarification. In this case, formula (1.13) can be interpreted in two ways.

A. Since $j(g,z)$ does not vanish, for a fixed g, the multi-valued function $j(g,z)^\lambda$ splits into a countable set of holomorphic single-valued branches on (simply connected) domain Ω. These branches differ by factors of the form $\exp(2\pi i\lambda k)$. Therefore, we may assume that, for each $g \in G$, there is an operator $\rho_\lambda(g)$ determined to within a factor. Then,

for all $g_1, g_2 \in G$, we have

$$\rho_\lambda(g_1 g_2) = \theta(g_1, g_2)\rho_\lambda(g_1)\rho_\lambda(g_2)$$

with $\theta(\cdot,\cdot) \in \mathbb{C}$ and $|\theta| = 1$. In other words, we have a projective representation of G.

B. Let $\widetilde{G}$ be the universal covering group of G. Then $j(g,z)^\lambda$ can be regarded as a single-valued function on $\widetilde{G} \times \Omega$, and formula (1.13) defines a linear representation of $\widetilde{G}$.

We are interested in scalar products invariant under the operators $\rho_\lambda(g)$.

Proposition 1.6. *Let $H_K(\Omega)$ be the Hilbert space of holomorphic functions. Fix $\lambda \in \mathbb{R}$. Let $H_K(\Omega)$ be invariant under the operators $\rho_\lambda(g)$, and let $\rho_\lambda(g)$ be unitary in $H_K(\Omega)$. Then*

(a) *the kernel $K(z,\bar{u})$ satisfies the relation*

$$K(z^g, \overline{u^g}) = j(g,z)^{-\lambda}\overline{j(g,u)}^{\,-\lambda}K(z,\bar{u}); \tag{1.15}$$

(b)

$$\rho_\lambda(g^{-1})\psi_z = j(g,z)^\lambda \psi_{z^g}. \tag{1.16}$$

Proof. We have

$$\langle h, \rho_\lambda(g^{-1})\psi_z\rangle = \langle \rho_\lambda(g)h, \psi_z\rangle = j(g,z)^\lambda h(z^g) = j(g,z)^\lambda \langle h, \psi_{z^g}\rangle.$$

Thus, we have proved (b). Now (a) follows from (b) and (1.2).

Now let the action of G on Ω be transitive. Then Eq. (1.15) for $K(z,\bar{u})$ can easily be solved. Namely, $K(z,\bar{u})$ is completely determined by its values on the diagonal, i.e., by the function $\varkappa_\lambda(z) := K(z,\bar{z})$. Owing to (1.15), this function satisfies the equation

$$\varkappa_\lambda(z^g) = |j(g,z)|^{-2\lambda}\varkappa_\lambda(z). \tag{1.17}$$

For each $z \in \Omega$, we choose an element g_z such that $z = 0^{g_z}$. Then

$$\varkappa_\lambda(z) = |j(g_z, 0)|^{-2\lambda}\varkappa_\lambda(0).$$

This means that a solution of Eq. (1.17), if it exists, is uniquely determined and can be given by an explicit formula. The dependence of $\varkappa_\lambda$ on λ has the form $\varkappa_\lambda(z) = a(z)^\lambda$.

Nevertheless, the question whether the resulting kernels are positive definite proves to be difficult. In the case of Cartan domains, the answer to this question has taken much efforts and was given in [4, 47, 51, 56] (see also Sec. 3.5 below).

1.8. Example: invariant kernels in B_q. We denote by $U(1,q)$ the group of block matrices of size $(1+q)\times(1+q)$ satisfying the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & \\ & 1_q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} -1 & \\ & 1_q \end{pmatrix}$$

where 1_q denotes the identity matrix of size $q\times q$. We denote by $SU(1,q)$ the group consisting of matrices $g\in U(1,q)$ with determinant equal to 1.

The pseudounitary group $U(1,q)$ acts on B_q by linear-fractional transformations of the form

$$z\mapsto z^g=(a+zc)^{-1}(b+zd) \tag{1.18}$$

where $z=(z_1\dots z_q)$ is regarded as a row matrix.

These transformations take B_q into itself, and all holomorphic transformations of B_q into itself have the form (1.18) (see, e.g., [25], Chap. 2). Since the scalar matrices $h=e^{i\theta}\cdot 1_{(1+q)}$ yield the identity mapping $z\mapsto z^h$, we may assume, without loss of generality, that $g\in SU(1,q)$ in (1.18). The action of $SU(1,q)$ on B_q is transitive, and the stabilizer of 0 consists of the linear unitary transformations of $\mathbb{C}^q$:

$$z\mapsto zh;\quad h\in U(q).$$

Therefore, for B_q regarded as a homogeneous space, we have

$$B_q\simeq SU(1,q)/U(q)\simeq U(1,q)/(U(1)\times U(q)).$$

Let $g=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in SU(1,q)$ (we emphasize that we consider $SU(1,q)$, not $U(1,q)$). The Jacobian of $z\mapsto z^g$ is given by the formula

$$j(g,z)=(a+zc)^{-(q+1)}.$$

Let $s>0$. Consider the projective representation ρ_s of $SU(1,q)$ in H_s (see Sec. 1.5) given by the formula

$$\rho_s(g)f(z)=(a+zc)^{-s}f(z^g)$$

(our notation slightly disagrees with (1.13)).

Proposition 1.7. *The operators $\rho_s(g)$ are unitary on H_s.*

Proof. It suffices to check the relation

$$\langle\rho_s(g^{-1})\psi_u,\rho_s(g^{-1})\psi_v\rangle_{H_s}=\langle\psi_u,\psi_v\rangle_{H_s}. \tag{1.19}$$

One easily sees that the functions ψ_u are transformed under the action of ρ_s in accordance with (1.16), namely,

$$\rho_s(g^{-1})\psi_u=(a+zc)^{-s}\psi_{u^g}.$$

Now we see that (1.19) is equivalent to (1.15), which can easily be proved in its turn. □

Remark. Strictly speaking, the proposition above motivates the definition of H_s. In terms of representation theory, the ρ_s are "spherical" representations with highest weight. If $s > q$, we have the Harish–Chandra discrete series; if $s = q$, we have the limit point of the discrete series; if $0 < s < q$, we have the analytic continuation of the discrete series.

1.9. Hilbert spaces of vector-valued holomorphic functions. The construction described above can be generalized (see [72], we do not use this generalization up to §4). Let T be a finite-dimensional Euclidean space, and let $\mathrm{End}\,(T)$ be the set of linear operators on T. Let H be a subspace of the space of holomorphic T-valued functions on Ω. We say that H is *a Hilbert space of holomorphic T-valued functions* if

(a) H is equipped with a scalar product, and H is complete.

(b) For every $u \in \Omega$, the linear operator

$$A_u f = f(u)$$

from H to T is continuous.

(c) The operators

$$R_\varphi f(z) = f(e^{i\varphi} z); \quad \varphi \in \mathbb{R}$$

are unitary and form a weakly continuous group.

We define the reproducing kernel as the $\mathrm{End}\,(T)$-valued function $K(z, \bar u)$ on $\Omega \times \Omega$ given by the formula

$$K(z, \bar u) = A_z A_u^* \tag{1.20}$$

and also define the $\mathrm{End}\,(T)$-valued functions ψ_u on Ω by the formula

$$\psi_u(z) = K(z, \bar u).$$

Then, for all $\xi \in T$, we have

$$\langle h, \psi_u \xi \rangle_H = \langle h(u), \xi \rangle_T. \tag{1.21}$$

The matrix-valued kernel $K(z, \bar u)$ possesses the following properties:

(a) The function

$$\widetilde{K}((z, \xi), (\bar u, \eta)) := \langle K(z, \bar u)\xi, \eta \rangle$$

is a positive definite kernel on $\Omega \times T$.

(b) The function $K(z, \bar u)$ is holomorphic in z and antiholomorphic in u.

(c) $K(e^{i\varphi} z, e^{-i\varphi} \bar u) = K(z, \bar u)$.

Conversely, for every $\mathrm{End}\,(T)$-valued kernel on Ω satisfying (a)–(c), one can construct a Hilbert space of holomorphic functions on Ω which will be denoted by $H_K(\Omega)$.

The assertion of Sec. 1.6 concerning distributions can be carried over directly to the case of a space of vector-valued functions. In §4 we consider analogs of the invariant kernels of Secs. 1.7–1.8 and give meaningful examples of Hilbert spaces of vector-valued holomorphic functions. Now we present only two simple examples which show how vector-valued functions arise when dealing with scalar functions.

Example 1. Let Jet_m be the space of m-jets of holomorphic functions on $\mathbb{C}^N$. Let $H_K(\Omega)$ be the space of holomorphic functions on Ω. With every $f \in H_K(\Omega)$ we associate a Jet_m-valued function $L^{(m)}f$ such that $L^{(m)}f(z)$ is the m-jet of f at z. Thus, we obtain a Hilbert space of Jet_m-valued holomorphic functions on Ω.

Example 2. Let L be a subspace of $\mathbb{C}^N$, and let $\Sigma = \Omega \cap L$. Let $H_K(\Omega)$ be the Hilbert space of holomorphic functions on Ω, and let $\mathcal{F}_j$ be the space of all $f \in H_K(\Omega)$ which have a zero of order $\geq j$ on Σ. Thus we obtain a filtration on $H_K(\Omega)$ by closed subspaces $\mathcal{F}_j$,

$$H_K(\Omega) = \mathcal{F} \supset \mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots .$$

An element of the Hilbert space $\mathcal{H}_j = \mathcal{F}_j/\mathcal{F}_{j+1}$ can be indentified with a holomorphic function on Σ with values in the space of homogeneous polynomials of degree j on $\mathbb{C}^N/L$. We also have the isomorphism $H_K(\Omega) \simeq \bigoplus\limits_{j=0}^{\infty} \mathcal{H}_j$.

§2. Boundary Hilbert spaces and boundary representations

2.1. Boundary operators. Let Ω and K be the same as in §1. Let μ be a measure on $\partial\Omega$, and let M be the support of μ (as an example, we can imagine $H_s(B_q)$ as $H_K(\Omega)$ and take the real equator of the sphere S^{2q-1} as M, i.e., $M = \partial B_q \cap \mathbb{R}^q$).

Let μ satisfy the conditions

1^*. Almost everywhere on $M \times M$ in measure $\mu \times \mu$ there exists

$$K^*(z, \bar{u}) = \lim_{c \to 1-0} K(cz, c\bar{u}). \tag{2.1}$$

2^*. This limit is dominated, i.e., there exists a function $\varkappa(z, \bar{u}) \in L^1(M \times M)$ such that for all $c \in (0, 1)$ the following inequality holds:

$$|K(cz, c\bar{u})| \leq \varkappa(z, \bar{u}).$$

Denote by $L^1(\mu)$ and $L^\infty(\mu)$ the spaces of functions on M of class L^1 and L^∞, respectively.

For every $\varphi \in L^\infty(\mu)$, we define the function

$$I\varphi = \int\limits_M K(z, \bar{u})\varphi(u)\, d\mu(u).$$

Note that $\varphi\mu$ can be regarded as a distribution on $\mathbb{C}^N$, and then (see (1.7))

$$I\varphi = \Xi(\varphi\mu).$$

Theorem 2.1. *Under conditions 1^* and 2^*, the operator I maps $L^\infty(\mu)$ into $H_K(\Omega)$. This operator is continuous in the following sense: if a sequence $\varphi_j \in L^\infty(\mu)$ is bounded and converges to φ almost everywhere, then $I\varphi_j \to I\varphi$ in the metric of $H_K(\Omega)$.*

Proof. Conditions 1^* and 2^* imply the assumptions of Proposition 1.5. Further,

$$\|I\varphi_j - I\varphi\|^2_{H_K(\Omega)} = \lim_{c\to 1-0} \int\int \overline{K(cz, c\bar{u})} \\ \times (\varphi_j(z)\overline{\varphi_j(u)} - \varphi_j(z)\overline{\varphi(u)} - \varphi(z)\overline{\varphi_j(u)} + \varphi(z)\overline{\varphi(u)})\, d\mu(z)\, d\mu(u).$$

By the Lebesgue theorem on dominated convergence, this expression tends to 0. $\square$

Theorem 2.2. *Assume that conditions 1^* and 2^* are fulfilled. Then for any $f \in H_K(\Omega)$ the limit*

$$Jf(z) = \lim_{c\to 1-0} f(cz) \tag{2.2}$$

exists in the metric $L^1(\mu)$. The operator J is bounded as an operator $H_K(\Omega) \to L^1(\mu)$. Moreover, J is adjoint to I.

Proof. Let $J : H_K(\Omega) \to L^\infty(\mu)^*$ be the operator adjoint to I. The space $L^\infty(\mu)^*$ dual to $L^\infty(\mu)$ is immense, but the linear functional $l = Jf$ is continuous on $L^\infty(\mu)$ in the sense of Theorem 2.1, i.e., if $\varphi_j \to \varphi$ almost everywhere and φ_j is bounded in L^∞, then $l(\varphi_j) \to l(\varphi)$. Therefore, l is uniquely determined by the values it takes on the space of continuous functions on M. Hence l has the form

$$l(\varphi) = \int_M \varphi\, d\sigma,$$

where σ is a finite complex-valued measure on M, and φ is a Borel function. If $\varphi = 0$ in the sense of L^∞ (i.e., $\varphi = 0$ almost everywhere), then $l(\varphi) = 0$. Therefore, σ is absolutely continuous with respect to μ. Hence $\sigma = \rho \cdot \mu$ with $\rho \in L^1(\mu)$.

Thus, J maps $H_K(\Omega)$ into $L^1(\mu)$. Expanding f in homogeneous polynomials, we see that the family of functions $F_c(z) = f(cz)$ depends continuously on c. Now the radial limit (2.2) exists since J is continuous.

2.2. Boundary Hilbert spaces $E^\circ(\mu)$ and $E(\mu)$. The operator I maps $L^\infty(\mu)$ into $H_K(\Omega)$. Thus, on $L^\infty(\mu)$ we have the scalar product

$$\langle \varphi_1, \varphi_2 \rangle = \langle I\varphi_1, I\varphi_2 \rangle_{H_K(\Omega)} = \int\int \overline{K^*(z, \bar{u})}\, \varphi(z)\overline{\varphi(u)}\, d\mu(z)\, d\mu(u). \tag{2.3}$$

This scalar product might be degenerate. We consider the quotient space of $L^\infty(\mu)$ by the kernel of the scalar product (2.3) and take the completion of the resulting pre-Hilbert space. We thus obtain a Hilbert space which will be denoted by $E^\circ(\mu)$. Now we can regard I as an isometric embedding operator,

$$I : E^\circ(\mu) \to H_K(\Omega).$$

We denote by $E(\mu)$ and $\operatorname{Ker} J$ the image and the kernel of the operator J, respectively. Then $H_K(\Omega)/\operatorname{Ker} J$ is a Hilbert space, and J is a bijection from this space onto $E(\mu)$. Identifying $H_K(\Omega)/\operatorname{Ker} J$ with $E(\mu)$, we obtain a Hilbert space structure on $E(\mu)$.

By construction, the spaces $E^\circ(\mu)$ and $E(\mu)$ are dual to each other. The canonical pairing

$$B : E^\circ(\mu) \times E(\mu) \to \mathbb{C}$$

is given by the formula

$$B(\varphi, f) = \int\limits_M \varphi(z)\overline{f(z)}\, d\mu(z) \tag{2.4}$$

which has the literal sense if $\varphi \in L^\infty$ and $f \in E(\mu)$. Moreover, the spaces $E^\circ(\mu)$ and $E(\mu)$ are canonically isomorphic, namely, the operator

$$JI : E^\circ(\mu) \to E(\mu)$$

is unitary. Indeed, $J = I^*$. Therefore, $\operatorname{Im}(I)$ is the orthogonal complement to $\operatorname{Ker} J$. Further, I maps isometrically $E^\circ(\mu)$ onto $\operatorname{Im}(I)$, and J maps isometrically $\operatorname{Im}(I)$ onto $E(\mu)$). Note that we have an explicit formula for JI,

$$(JI\varphi)(z) = \int K^*(z, u)\varphi(u)\, d\mu(u). \tag{2.5}$$

2.3. Boundary representations. Now let a group G act on Ω by biholomorphic transformations $z \mapsto z^g$ (we are interested in the case where G does not coincide with the group of all transformations of Ω). Let $j(g, z)$ be the complex Jacobian of the map $z \mapsto z^g$. Consider the situation described in Sec. 1.7, i.e., let G act on $H_K(\Omega)$ by unitary operators of the form

$$\rho(g)f(z) = j(g, z)^s f(z^g). \tag{2.6}$$

Let M be a compact G-invariant submanifold of $\partial\Omega$, and let μ be the surface Lebesgue measure on M. Let the kernel $K(z, \bar{u})$ and the measure μ satisfy conditions 1^* and 2^*. Then we obviously have unitary representations of G in $E(\mu)$ and $E^\circ(\mu)$. Namely, on $E(\mu)$ the group G acts by the formula

$$\nu(g)f(x) = j(g, x)^s\, f(x^g), \tag{2.7}$$

which coincides with (2.6), and on $E^\circ(\mu)$ it acts by the formula

$$\nu^\circ(g)f(x) = l(g, x)j(g, x)^{-s}\, f(x^g), \tag{2.8}$$

where $l(g, x)$ is the Jacobian of the map $M \to M$ given by $x \mapsto x^g$.

The representations ν° and ν are dual to each other with respect to the form B given by (2.4), i.e.,

$$B(\nu^\circ(g)\varphi, \nu(g)f) = B(\varphi, f). \tag{2.9}$$

On the other hand, ν° and ν are equivalent, with the intertwining operator given by (2.5).

2.4. Example: restriction of holomorphic functions in a ball to the real equator. Let $\Omega = B^q$, and let $M = S^{q-1}$ be the sphere in $\mathbb{R}^q \subset \mathbb{C}^q$ given by the equation

$$\sum_{k=1}^{q} x_k^2 = 1; \quad x_j \in \mathbb{R}.$$

We consider the subgroup $SO(1,q)$ of $SU(1,q)$ consisting of all real matrices $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & \\ & 1_q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} -1 & \\ & 1_q \end{pmatrix}; \quad \det(g) = 1.$$

In other words, $SO(1,q)$ is the subgroup of $SU(1,q)$ consisting of real matrices.

The group $SU(1,q)$ acts on B^q by the transformations (1.18). It is easily shown that S^{q-1} is invariant under the transformations $x \mapsto x^g$ with $g \in SO(1,q)$, and, moreover, that these transformations are conformal transformations of S^{q-1}. For a transformation $x \mapsto x^g$ given by the matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, the dilatation coefficient at the point x is equal to $(a + xc)^{-1}$. Therefore, the Jacobian $l(q,x)$ of the transformation $x \mapsto x^g$ of the sphere into itself is given by the formula

$$l(g,x) = (a + xc)^{-(q-1)}.$$

Let $s > 0$, and let $K_s(z, \bar{u})$ and H_s be as in Sec. 1.5. Consider the action ρ_s of $SU(1,q)$ on H_s by the unitary operators of the form

$$\rho_s(g) f(z) = (a + zc)^{-s} f(z^g)$$

(see Sec. 1.8). Let

$$0 < s < (q-1)/2.$$

Then it is easy to see that, for $M = S^{q-1}$ and the natural measure μ on M, conditions 1^* and $2^*$are fulfilled, and

$$K^*(x,y) = (1 - \sum x_j y_j)^{-s} = \|x - y\|^{-2s}.$$

Here $x, y \in S^{q-1}$, and $\|\cdot\|$ denotes the natural Euclidean norm on $\mathbb{R}^q$.

We shall study the spaces

$$E(\mu) = E_s(S^{q-1}); \quad E^\circ(\mu) = E_s^\circ(S^{q-1})$$

and the unitary representations ν_s and ν_s° of $SO(1,q)$ in these spaces.

The scalar product (2.3) in $L^\infty(\mu) = L^\infty(S^{q-1})$ is given by the formula

$$\langle\varphi_1, \varphi_2\rangle = \int\limits_{S^{q-1}} \int\limits_{S^{q-1}} \frac{\varphi_1(x_1)\overline{\varphi_2(x_2)}dx_1 dx_2}{\|x_1 - x_2\|^{2s}}. \tag{2.10}$$

The space $E_s^\circ(S^{q-1})$ is the completion of $L^\infty(S^{q-1})$ in the norm given by this scalar product. According to (2.8), the group $SO(1,q)$ acts on $E_s^\circ(S^{q-1})$ by the unitary operators

$$\nu_s^\circ(g)f(x) = (a + xc)^{-(q-1)+s} \cdot f(x^g). \tag{2.11}$$

On $E_s(S^{q-1})$ the group $SO(1,q)$ acts by the unitary operators

$$\nu_s(g)f(x) = (a + xc)^{-s} f(x^g). \tag{2.12}$$

The space $E_s(S^{q-1})$ itself will be described in Sec. 2.6, where also a more detailed description of $E_s^\circ(S^{q-1})$ will be given.

Remark. The representations $\nu_s \simeq \nu_s^\circ$ of $SO(1,q)$ are called representations of *spherical complementary series*. Our construction shows that these representations occur in the restrictions $SO(1,q)$ of the "spherical" representations $SU(1,q)$ with highest weight.

2.5. A version of the construction: the Hilbert spaces $F^\circ(M)$. Let $H_K(\Omega)$ be a Hilbert space of holomorphic functions. Let M be a submanifold of $\partial\Omega$, and let μ be the surface Lebesgue measure on M. Denote by $\mathcal{D}^\circ(M)$ the space of distributions on M. Every element $\chi \in \mathcal{D}^\circ(M)$ can be regarded as a distribution on $\mathbb{C}^N$. We denote by $\mathcal{H}(\Omega)$ the space of all holomorphic functions on Ω with the topology of uniform convergence on compact sets. The operator Ξ given by (1.7) is a continuous operator from $\mathcal{D}^\circ(M)$ to $\mathcal{H}(\Omega)$. Let $\mathcal{F}(M)$ be the set of all $\chi \in \mathcal{D}^\circ(M)$ such that $\Xi\chi \in H_K(\Omega)$. Then we obtain the following scalar product on $\mathcal{F}(M)$:

$$\langle\chi, \widetilde{\chi}\rangle = \langle\Xi\chi, \Xi\widetilde{\chi}\rangle_{H_k(\Omega)}.$$

Consider the quotient space of $\mathcal{F}(M)$ modulo the kernel of the above scalar product and denote by $F^0(M)$ the completion of the resulting pre-Hilbert space. The following questions naturally arize.

(A) Is it true that conditions 1^* and 2^* are necessary for $F^\circ(M)$ to be nontrivial?

(B) Is it true that under conditions 1^* and 2^* one has $F^\circ(M) = E^\circ(\mu)$?

(C) Is it true that every element of $F^\circ(M)$ is a distribution? More precisely, is it true that the mapping $\mathcal{F}(M) \to F^\circ(M)$ is surjecture?

The answer to (A) is negative even in the simplest cases, see Sec. 2.8 below.

The answer to (C) is, in general, also negative (e.g., it may happen that M lies in the holomorphically convex hull of Ω, and then distributions are evidently not all the elements of $F^\circ(M)$). However, in all cases considered in the present paper, the answer to (C) is positive. We do not know any general assertions on this subject.

Now we show that the answer to (B) is positive in a case which is very specific but suffucient for our purposes.

Proposition 2.1. *Let a compact group L act on Ω by linear transformations. Let $M \subset \partial\Omega$ be an orbit of L. Then under conditions 1^* and 2^*, we have $F^\circ(M) = E^\circ(M)$.*

Proof. Let $\theta^{(1)}$, $\theta^{(2)}, \ldots$ are all irreducible characters of L. Every $\chi \in \mathcal{D}^\circ(M)$ can be expanded into the series $\chi = \sum \chi^{(j)}$ convergent in $\mathcal{D}^\circ(M)$. Here

$$\chi^{(j)}(x) = \int\limits_L \theta^{(j)}(g)\chi(x^g)dg,$$

and $\chi^{(j)}$ are smooth since $\theta^j(g)$ are smooth. Accordingly, Ξ may be expanded into a series convergent in $\mathcal{H}(\Omega)$,

$$\Xi\chi = \sum \Xi(\chi^{(j)}) = \sum (\Xi\chi)^{(j)},$$

with

$$(\Xi\chi)^{(j)}(g) = \int\limits_L \theta^{(j)}(g)(\Xi\chi)(z^g)dg.$$

Let $\chi \in \mathcal{F}(M)$. Then $\Xi\chi \in H_K(\Omega)$, and, therefore, the series $\sum(\Xi\chi)^{(j)}$ converges to $\Xi\chi$ in the sence of $H_K(\Omega)$. By definition, this means that $\sum \chi^{(j)}$ converges to χ in $F^\circ(M)$. At the same time, $\chi^{(j)} \in E^\circ(\mu)$. $\square$

2.6. Example: the structure of $E_s^\circ(S^{q-1})$ and $E_s(S^{q-1})$. Let Harm_j be the space of homogeneous harmonic polynomials of degree j on $\mathbb{R}^q$ restricted to S^{q-1} (for more details, see [6]). It is known that

$$L^2(S^{q-1}) = \bigoplus_{j=0}^{\infty} \mathrm{Harm}_j,$$

and that the representation of $SO(q)$ in Harm_j is irreducible. Any distribution f on S^{q-1} can be expanded into a series $f = \sum f^{(j)}$, where $f^{(j)} \in \mathrm{Harm}_j$. The series converges in the sense of distributions and

$$\exists n : \|f^{(j)}\|_{L^2(S^{q-1})} = O(j^n), \quad j \to \infty.$$

Moreover,

$$f \in C^\infty(S^{q-1}) \Longleftrightarrow \forall n : \|f^{(j)}\|_{L^2(S^{q-1})} = O(j^{-n}), \quad j \to \infty.$$

Proposition 2.2. *Let $f_1, f_2 \in C^\infty(S^{q-1})$. Then*

$$\langle f_1, f_2\rangle_{E_s^\circ} = C(s,q) \sum \frac{\Gamma(j+s)}{\Gamma(j+q-s-1)} \langle f_1^{(j)}, f_2^{(j)}\rangle_{L^2(S^{q-1})}, \tag{2.13}$$

where $C(q,s)$ is a constant independent of j.

For the proof see [6], X.2.5 and [33].

Corollary. *Let f be a distribution on S^{q-1}. Then*

$$f \in E_s^\circ(S^{q-1}) \Longleftrightarrow \sum j^{-(q-2s-1)} \, \|f^{(j)}\|^2_{L^2(S^{q-1})} < \infty. \tag{2.14}$$

Proof. Formula (2.13) implies the condition

$$f \in E_s^\circ(S^{q-1}) \Longleftrightarrow \sum \frac{\Gamma(j+s)}{\Gamma(j+q-s-1)} \, \|f^{(j)}\|^2_{L^2(S^{q-1})} < \infty$$

which is equivalent to the required condition due to the Stirling formula. □

The condition (2.14) is hard to verify, but there are two convenient tests for checking the inclusion $f \in E_s^\circ(S^{q-1})$.

Proposition 2.3. *Let f be a distribution on S^{q-1} and $0 < c < 1$. Then the function*

$$\gamma(c) = \int\limits_{S^{q-1}} \int\limits_{S^{q-1}} \frac{f(x)\overline{f(y)}\, dx\, dy}{(1 - c^2 \sum x_k y_k)^s} \tag{2.15}$$

increases. A function f lies in $E_s^\circ(S^{q-1})$ if and only if the limit $\lim\limits_{c\to 1-0} \gamma(c)$ exists. This limit coincides with $\|f\|^2_{E^\circ(S^{q-1})}$.

Proof. This is a special case of Proposition 1.5. □

Proposition 2.4. *The space $E_s^\circ(S^{q-1})$ is the Sobolev space $W^{-((q-1)/2-s)}$ on S^{q-1}.*

For the definition of Sobolev spaces see, e.g., [8] and [26].

Proof. Denote by Δ the Laplace operator on the sphere. On each space Harm_j, it acts as the multiplication by $-j(j+q-1)$ (see [6], IX.5.1). Let $\alpha \in \mathbb{R}$. Then $(-1-\Delta)^\alpha$ is a pseudodifferential operator of order 2α and, hence, for each $\sigma \in \mathbb{R}$ it defines a continuous map $W^\sigma(S^{q-1}) \to W^{\sigma-2\alpha}(S^{q-1})$ (see [26], 2.6), which is an isomorphism (the inverse operator is $(-1-\Delta)^{-\alpha}$). Now we take the space $W^\circ = L^2$ as W^σ and apply (2.14). □

The space $E_s(S^{q-1})$ can be characterized as the dual space for $E_s^\circ(S^{q-1})$. Namely, as a topological vector space, $E_s(S^{q-1})$ is the Sobolev space $W^{(q-1)/2-s}(S^{q-1})$. Moreover,

$$f \in E_s(S^{q-1}) \Longleftrightarrow \sum j^{q-2s-1} \|f_j\|^2_{L^2(S^{q-1})} < \infty,$$

and the scalar product in $E_s(S^{q-1})$ is given by the formula

$$\langle f_1, f_2 \rangle = \sum \frac{\Gamma(j+q-s-1)}{\Gamma(j+s)} \, \langle f_1^{(j)}, f_2^{(j)} \rangle_{L^2(S^{q-1})}.$$

This formula is usually expressed in the form

$$\langle f_1, f_2 \rangle = C(q,s) \int\limits_{S^{q-1}} \int\limits_{S^{q-1}} \frac{f_1(x)\overline{f_2(y)}\, dx\, dy}{\|x-y\|^{2(q-1-s)}}.$$

The integral in the right-hand side of the above formula diverges. The kernel $\|x-y\|^{-\lambda}$ should be regarded as a distribution which is analytic (in λ) continuation of $\|x-y\|^{-\mu}$; the latter is integrable if $\mu < q-1$ (see [7, 33]).

2.7. Another version of the construction. Let M be a submanifold of $\partial\Omega$, and let $\mathcal{D}'(M)$ be the space of distributions on $\mathbb{C}^N$ with support in M. Consider the set $R(M)$ of all $\chi \in \mathcal{D}'(M)$ such that $\Xi\chi \in H_K(\Omega)$. Note that $R(M)$ is, generally, larger than the space $\mathcal{F}(M)$ from Sec. 2.5 since $R(M)$ may contain partial derivatives of functions $\chi \in \mathcal{D}^\circ(M)$ along directions normal to M. Denote by $Q^\circ(M)$ the completion of $R(M)$ with respect to the scalar product $\langle \chi, \widetilde{\chi}\rangle = \langle \Xi\chi, \Xi\widetilde{\chi}\rangle$.

Example. Under the conditions of Sec. 2.4, let k be an integer such that $k < (q-1)/2-s$. Then $Q^\circ(M)$ contains the normal partial derivatives of order $\le k$ for smooth δ-functions supported by $S^{q-1} \subset S^{2q-1} = \partial B_q$.

2.8. Example: two-dimensional polydisk. In the two-dimensional polydisk $\mathcal{D}^2 : |z_1| < 1,\ |z_2| < 1$, we consider the positive definite kernel

$$K_{s_1,s_2}(z_1, z_2; \bar{u}_1, \bar{u}_2) = (1 - z_1\bar{u}_1)^{-s_1}(1 - z_2\bar{u}_2)^{-s_2}$$

with $s_1 > 0$ and $s_2 > 0$. Let H_{s_1,s_2} be the corresponding Hilbert space of holomorphic functions. Let $\mathbb{T}^2$ be the torus $|z_1| = |z_2| = 1$, and let $M \subset \mathbb{T}^2$ be a smooth curve. It turns out that the properties of the operators of restriction to M depend heavily (for detailed discussion see [13*]) on the location of M. We discuss briefly some examples.

Example 1. Let M be the curve $z_1 = e^{i\varphi}$, $z_2 = 1$. Then the operator of restriction of $f \in H_{s_1,s_2}$ to M is not defined in any sense.

Example 2. Let M be transverse to the parallels and meridians of $\mathbb{T}^2$. Let $s_1 + s_2 < 1$. Then conditions 1^* and 2^* from Sec. 2.1 are fulfilled. Therefore, the restriction of $f \in H_{s_1,s_2}$ to M (in the sense of the radial L^1-limit) is defined. The resulting function on M is a Sobolev function of the class $1 - s_1 - s_2$. If M is given by $z_1 = e^{i\varphi}$, $z_2 = e^{-i\varphi}$, then every Sobolev function of the class $1 - s_1 - s_2$ is the restriction of a function $f \in H_{s_1,s_2}$ to M. For other M this is not true, in general.

Example 3. Let $\Delta \subset \mathcal{D}^2$ be the diagonal, i.e., the subset defined by $z_1 = z_2$. Let M be the boundary of Δ, and let δ_M be the delta-function of M (i.e., the Lebesgue measure on M regarded as a distribution on $\mathbb{C}^2$). Denote by $\delta_M^{(j)}$ the partial derivative of δ_M of order j in the direction tangent to $\mathbb{T}^2$ and normal to M. Then the space $R(M)$ from Sec. 2.7 contains all distributions of the form $f \cdot \delta_M^{(k)}$, with $f \in C^\infty(M)$ and $k \ge 0$.

2.9. Example: restriction to the Cantor discontinuum. Let M be the standard Cantor discontinuum on the circle $|z| = 1$ (we divide the circle into four equal parts and delete two oposite parts; then we divide each of the remaining parts into three equal parts and delete middle parts, etc.). The operator of restriction of a holomorphic function to M is defined on $H_s(B_1)$ if $s < \ln 2/\ln 3$ (namely, conditions 1^* and 2^* are fulfilled for the natural measure μ on M).

2.10. Remarks. There is a vast literature dedicated to the boundary theory of holomorphic functions (we refer only to Rudin's monograph [25] and to Aleksandrov's survey [1]). Our problems differ tangibly from those usually considered in analysis. Restrictions of holomorphic functions to "small" subsets of the boundary are considered in analysis for a number of function spaces. We refer to the Nagel–Rudin theorem (see [25]) on the restriction of functions from $H^\infty(B_q)$ to curves suitably located in S^{2q-1}. Note that in the case where M is an interpolation set, the space $H^\infty(\Omega)$ contains functions which have no radial limit everywhere on M (see [25], 11.2.2); for interpolation sets see, e.g., [25], Chap. 10, [66], §V.5, and [1], Chap. 5 (note that the spaces under consideration, in general, do not contain H^∞). Example 2.9 belongs to the theory of exceptional sets and is actually known (see [65], V, Theorem 3, and the survey [10]).

On the other hand, our constructions are related to the well-known theorem on the trace in Sobolev spaces, see [8], §3.2, and [26], §1.3.

2.11. Boundary values of vector-valued functions. All the above can easily be generalized to Hilbert spaces of vector-valued functions. Let Ω, K, and T be the same as in Sec. 1.9, let M be a compact subset of $\partial\Omega$, and let μ be a finite posite measure on M. Suppose that $\mathrm{End}\,(T)$-valued kernel K satisfies the conditions

1^{**}. On $M \times M$ the limit

$$K^*(z, \bar{u}) = \lim_{c \to 1-0} K(cz, \overline{cu})$$

exists almost everywhere with respect to the measure $\mu \times \mu$.

2^{**}. The above limit is dominated, i.e., for all $c \in (0, 1)$, the inequality

$$\|K(cz, c\bar{u})\| \le \varkappa(z, \bar{u})$$

holds with some function $\varkappa$ on $M \times M$ integrable with respect to the measure $\mu \times \mu$.

Then Theorems 2.1 and 2.2 and their proofs remain valid. In this case, by L^1 and L^∞ we mean the spaces L^1 and L^∞ of functions on M with values in the finite-dimensional space T.

Exactly as in Sec. 2.2, one can define the Hilbert spaces $E^\circ(\mu)$ and $E(\mu)$. In $E^\circ(\mu)$, the scalar product is given the formula

$$\langle f_1, f_2 \rangle_{E^\circ(\mu)} = \int\limits_M \int\limits_M \langle f_1(z), K^*(z, \bar{u}) f_2(u) \rangle_T \, d\mu(z)\, d\mu(u). \tag{2.16}$$

The canonical pairing $E^\circ(\mu) \times E(\mu) \to \mathbb{C}$ is given by the relation

$$B(f, h) = \int\limits_M \langle f(z), h(z) \rangle_T \, d\mu(z), \tag{2.17}$$

and the canonical isomorphism $E^\circ(\mu) \to E(\mu)$ has the form (2.5).

Assume that a Lie group G acts on Ω by biholomorphic transformations $z \mapsto z^g$. Suppose that we are given a unitary representation of G in $H_K(\Omega)$ defined by the operators of the form

$$\rho(g) f(z) = R(g, z)\, f(z^g), \tag{2.18}$$

where R is a function on $G \times \Omega \mapsto GL(T)$ holomorphic in z (for a detailed discussion of this situation see Sec. 4.0 below).

Let $M \subset \partial\Omega$ be a manifold invariant under G. Then, under conditions 1^{**} and 2^{**}, we have unitary representations ν° and ν of G in the Hilbert spaces $E^\circ(\mu)$ and $E(\mu)$, respectively. The representation ν in $E(\mu)$ is given by the formula

$$\nu(g)f(x) = R(g,x)f(x^g); \quad g \in G, \quad x \in M \tag{2.19}$$

(this formula essentially coincides with (2.18)). The representation ν° in $E^\circ(\mu)$ is given by the formula

$$\nu^\circ(g)f(x) = l(g,x)[R(g,x)^*]^{-1}f(x^g) \tag{2.20}$$

where $l(g,x)$ is the Jacobian of the map $M \to M$ defined by $x \mapsto x^g$.

§3. Scalar functions in matrix balls

In Secs. 3.1–3.4, we present the necessary information on the geometry of the matrix balls

$$B_{p,q} = U(p,q)/(U(p) \times U(q)).$$

In what follows, we describe the natural Hilbert spaces of holomorphic functions in matrix balls which were introduced by Berezin in [4] (Sec. 3.5) and construct the operator of restriction to a certain specific submanifold in the Shilov boundary of $B_{p,q}$ invariant under the group $O(p,q)$.

3.1. The group $U(p,q)$. Let $p \leq q$. We denote by $\mathbb{C}^{p,q}$ the space $\mathbb{C}^{p+q}$ equipped with the indefinite Hermitian form

$$s((x_1, \ldots, x_{p+q}); (y_1, \ldots, y_{p+q})) = -\sum_{i=1}^{p} x_i \bar{y}_i + \sum_{j=p+1}^{p+q} x_j \bar{y}_i \tag{3.1}$$

and by $U(p,q)$ the group of transformations of $\mathbb{C}^{p,q}$ preserving the form $s(\cdot,\cdot)$. We write the elements of $U(p,q)$ in the form of block matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of order $(p+q) \times (p+q)$. It can easily be seen that $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $U(p,q)$ if and only if the following relation holds:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1_p & \\ & 1_q \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} -1_p & \\ & 1_q \end{pmatrix}. \tag{3.2}$$

We denote by $SU(p,q)$ the subgroup of $U(p,q)$ consisting of the matrices with determinant 1 and by $O(p,q)$ the subgroup of $U(p,q)$ consisting of real matrices. The group $O(p,q)$ has four connected components. Denote by $SO_0(p,q)$ the connected component of the identity. An element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is contained in $O(p,q)$ if the following conditions hold:

$$\det(g) = +1; \quad \det(a) > 0; \quad \det(d) > 0$$

(any two of these conditions imply the remaining one.)

The Lie algebra $u(p,q)$ of $U(p,q)$ consists of all matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}; \quad \alpha + \alpha^* = 0, \quad \delta + \delta^* = 0, \quad \beta = \gamma^*.$$

The Lie algebra $o(p,q)$ of $O(p,q)$ is the subalgebra of $u(p,q)$ consisting of real matrices.

3.2. The matrix balls $B_{p,q}$. Denote by $B_{p,q}$ (resp., $\overline{B_{p,q}}$) the set of all complex matrices z of size $p \times q$ such that $\|z\| < 1$ (resp., $\|z\| \leq 1$). Let $\|\cdot\|$ denote the usual norm of a matrix regarded as an operator from $\mathbb{C}^p$ to $\mathbb{C}^q$.

The group $U(p,q)$ acts on $B_{p,q}$ by the biholomorphic transformations

$$z \mapsto z^g = (a + zc)^{-1}(b + zd). \tag{3.3}$$

We note that a scalar matrix induces the identity transformation $z \mapsto z^g$, so, without loss of generality, we may assume that $g \in SU(p,q)$. The differential of the mapping $z \mapsto z^g$ is given by the formula

$$[\delta z] \mapsto -(a + zc)^{-1}[\delta z](-cz^g + d). \tag{3.4}$$

If $g \in SU(p,q)$, then the complex Jacobian of (3.3) equals to

$$j(g,z) = \det{}^{-(p+q)}(a + zc). \tag{3.5}$$

Remark 1. The fact that $z \in B_{p,q}$ implies $z^g \in B_{p,q}$ follows from the construction of Sec. 3.4 below. Further, due to (3.2), we have $a^*a = 1 + c^*c$. Hence $|a| > |c|$ and, therefore, $a + zc$ is invertible for all $z \in B_{p,q}$, $g \in U(p,q)$. This means that $z \mapsto z^g$ is continuous on $\overline{B_{p,q}}$.

Remark 2. If $p \neq q$, any biholomorphic transformation of $B_{p,q}$ into itself has the form (3.3). If $p = q$, we also have the transposition $z \mapsto z^t$, (see [34] and [42]).

Remark 3. The action of $U(p,q)$ on $B_{p,q}$ is transitive. The stabilizer of $0 \in B_{p,q}$ consists of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ with $a \in U(p)$ and $d \in U(q)$. Therefore, we have an isomorphism of homogeneous spaces

$$B_{p,q} \simeq U(p,q)/(U(p) \times U(q)).$$

Remark 4. The domains $B_{p,q}$ form one of the four series of classical homogeneous Cartan domains (or, what is the same, noncompact symmetric Hermitian spaces, see [34, 24, 27], and [28]).

3.3. The Shilov boundary. The boundary $\partial B_{p,q}$ of $B_{p,q} \subset \mathbb{C}^{pq}$ consists of all matrices such that $\|z\| = 1$. The set $\partial B_{p,q}$ consists of the orbits $N_0, \dots, N_{p-1}$ of $U(p,q)$, where N_j is determined by the condition rank $(1 - zz^*) = j$ (see [24]). Obviously, the closure of N_j contains N_{j-1}. Our interest is only with the smallest orbit N_0. It consists of z such that $zz^* = 1$. The set N_0 is the Shilov boundary of $B_{p,q}$ in the usual sense.

The condition $zz^* = 1$ means that the map $\mathbb{C}^p \to \mathbb{C}^q$ given by $x \mapsto xz$ is an isometric embedding. The image of this map is a p-dimensional subspace in $\mathbb{C}^q$ with the distinguished orthonormal basis, namely, the image of the standard basis of $\mathbb{C}^q$. Thus N_0 is identified with the manifold of all orthonormal p-frames in $\mathbb{C}^q$, i.e., with the *complex Stiefel manifold.*

In our case, the *real Stiefel manifold* $M = M_{p,q}$ is an analog of the "real equator" $S^{q-1} \subset \partial B_q$ from §2. This manifold consists of all real matrices satisfying the condition $zz^* = 1$ or, what is the same, of isometric embeddings $\mathbb{R}^p \to \mathbb{R}^q$.

3.4. The embedding of $B_{p,q}$ into a Grassmanian. Denote by $\mathcal{L}_-$ the set of all p-dimensional subspaces of $\mathbb{C}^{p,q}$ on which the form (3.1) is strictly negative definite.

Let $z \in B_{p,q}$. Denote by $\Gamma_z \subset \mathbb{C}^{p,q}$ the graph of the operator $z : \mathbb{C}^p \to \mathbb{C}^q$. It can easily be seen that $\Gamma_z \in \mathcal{L}_-$ and, moreover, that the map $z \mapsto \Gamma_z$ is a bijection from $B_{p,q}$ to $\mathcal{L}_-$.

The group $U(p,q)$ acts naturally on $\mathcal{L}_-$ and, thus, also on $B_{p,q}$. This action is given by (3.3).

Now we describe the other objects of Sec. 3.2–3.3 in these terms.

(a) $z \in \overline{B_{p,q}}$ means that the form (3.1) is nonpositive definite on Γ_z.

(b) $z \in N_j$ means that the form (3.1) is degenerate on Γ_z and that on Γ_z its rank is equal to j.

(c) $z \in N_0$ is equivalent to the isotropy of Γ_z (i.e., on Γ_z the form (3.1) is equal to 0).

(d) The points $z \in M$ correspond to real isotropic subspaces of $\mathbb{R}^{p+q} \subset \mathbb{C}^{p,q}$.

3.5. The spaces H_s. Suppose that the group $U(p,q)$ (more precisely, the universal covering of $U(p,q)$) acts on the space of holomorphic functions on $B_{p,q}$ by the formula

$$\rho_s(g)f(z) = \det{}^{-s}(a + zc)f(z^g) \tag{3.6}$$

(i.e., we are under the conditions of Sec. 1.7). A simple argument similar to that from Sec. 1.8 (we present a more general argument in Sec. 4.3 below) shows that only scalar products given by kernels of the form

$$K_s(z,\bar{u}) = \det{}^{-s}(1 - zu^*) \tag{3.7}$$

are invariant under operators $\rho_s(g)$. Denote by Σ_+ the interval $s > p-1$ on $\mathbb{R}$, and denote by Σ the union of Σ_+ and the set of points $0, 1, \dots, p-1$.

Berezin's Theorem (see [4])**.** *The kernel $K_s(z,\bar{u})$ is positive definite on $B_{p,q}$ if and only if $s \in \Sigma$. In this case the representation is unitary in $H_s = H_{K_s}(B_{p,q})$.*

We denote by $H_s = H_s(B_{p,q})$ the Hilbert space of holomorphic functions on $B_{p,q}$ defined by the kernel $K_s(z,\bar{u})$.

Remark 1. If $s \in \Sigma_+$, then H_s contains the space Pol of all polynomials. If $s \in \Sigma \setminus \Sigma_+$, this is not the case any more. In the latter case, the description of $H_s \cap \mathrm{Pol}$ can be found in Berezin's paper [4]. This paper also contains a description of the scalar product on H_s in terms of $U(p) \times U(q)$-harmonics on $\mathbb{C}^{pq}$. If $s = 0$, the space H_s is one-dimensional.

Remark 2. If $s > p + q - 1$, then the scalar product on H_s can be written in the form

$$\langle f_1, f_2 \rangle_{H_s} = c(s) \int\limits_{B_{p,q}} f_1(z)\overline{f_2(z)} \det{}^{s-p-q}(1 - zz^*)dz\, d\bar{z},$$

where $c(s)$ is a normalizing factor. The point $s = q - 1$ corresponds to an analog of the Hardy space H^2,

$$\langle f_1, f_2 \rangle_{H_{q-1}} = \int_{N_0} f_1(z)\overline{f_2(z)} dz\, d\bar{z}.$$

3.6. The Stiefel manifold. We describe the real Stiefel manifold $M \subset \mathbb{C}^{pq}$ (see Sec. 2.3) in more detail. The group $SO_0(p,q)$ acts on M by transformations (3.3). The subgroup $SO(p) \times SO(q) \subset SO_0(p,q)$ acts on M by transformations of the form

$$z \mapsto a^{-1} z d,$$

with $a \in SO(p)$ and $d \in SO(q)$. Clearly, the action of $SO(q)$ on M is transitive, the stabilizer of the point

$$z_0 = (1_p 0) \in M \tag{3.8}$$

being $SO(q-p)$ (this is quite clear if we pass to the model of Sec. 3.4).

Consequently, the group $SO_0(p,q)$ acts transitively on M as well. Now we describe the stabilizer P of z_0 in $SO_0(p,q)$. Let $l_1, \dots, l_{p+q}$ be the standard basis in $\mathbb{R}^{p+q}$.

In $\mathbb{R}^{p+q}$ we consider the following basis $f_1, \dots, f_{p+q}$:

$$\begin{aligned} f_i &= \frac{1}{\sqrt{2}}(l_i + l_{i+p}) \quad \text{if} \quad i \le p, \\ f_j &= l_{j+p} \quad \text{if} \quad p < j \le q, \\ f_{k+q} &= \frac{1}{\sqrt{2}}(l_k - l_{k+p}) \quad \text{if} \quad k \le p. \end{aligned}$$

The subspace Γ_{z_0} (the graph of the operator z_0 given by (3.8), see Sec. 3.4) is spanned by $f_1, \dots, f_p$. The orthogonal complement to Γ_{z_0} with respect to the form (3.1) is spanned by $f_1, \dots, f_q$. By definition, an element $g \in P$ stabilizes Γ_{z_0} and, therefore, g also stabilizes the orthogonal complement to Γ_{z_0}. Thus, in the basis f_j, $g \in P$ has the form

$$g = \begin{array}{c} \begin{pmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & 0 \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \end{array} \begin{array}{l} \}p \\ \}q-p; \\ \}p \end{array} \qquad g \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} g^t = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}. \tag{3.9}$$

The subgroup P is the product of a reductive subgroup R and a maximal nilpotent subgroup $\mathcal{N}$. The reductive part $R \subset P$ consists of all matrices of the form

$$g = \begin{pmatrix} g_{11} & & \\ & g_{22} & \\ & & g_{11}^{t-1} \end{pmatrix}; \quad g_{11} \in GL_+(p, \mathbb{R}), \quad g_{22} \in SO(q-p), \tag{3.10}$$

where $GL_+(p, \mathbb{R})$ is the group of all matrices with positive determinant. The nilpotent group $\mathcal{N}$ consists of all matrices of the form

$$g = \begin{pmatrix} 1 & 0 & 0 \\ g_{21} & 1 & 0 \\ g_{31} & -g_{21}^t & 1 \end{pmatrix}; \quad g_{31} + g_{31}^t = -g_{21}^t g_{21}. \tag{3.11}$$

We note that

$$R = GL_+(p, \mathbb{R}) \times SO(q-p).$$

It will be necessary to write explicit formulas in terms of the old basis $l_1, \dots, l_{p+q}$ of $\mathbb{R}^{p+q}$. In this case, the elements $g \in P \subset SO_0(p,q)$ have the form

$$g = \begin{pmatrix} a & b_1 & b_2 \\ c_1 & d_{11} & -b_2 \\ c_2 & c_2 & d_{22} \end{pmatrix} \begin{matrix} \}p \\ \}p \\ \}q-p \end{matrix} \quad ; \quad a + c_1 = b_1 + d_{11} \tag{3.12}$$

The semisimple part $R \subset P$ consists of matrices of the form

$$g = \begin{pmatrix} a & b_1 & \\ b_1 & a & \\ & & d_{22} \end{pmatrix} ; \quad aa^t - b_1 b_1^t = 1, \quad d_{22} \in SO(q-p). \tag{3.13}$$

The blocks of (3.10) can be expressed in terms of a, b_1, and d_{22} by the formula

$$g_{11} = a + b_1^t, \quad g_{11}^{t-1} = a - b_1^t, \quad g_{22} = d_{22}.$$

Let μ be the surface Lebesgue measure on $M \subset \mathbb{C}^{pq}$. It is clear that μ is $SO(p) \times SO(q)$-invariant.

Proposition 3.1. *Let $g \in SO_0(p,q)$. Then the Jacobian $l(g,z)$ of the map $z \mapsto z^g$ of the manifold M into itself is given by the formula*

$$l(g,z) = \det{}^{-(q-1)}(a + cz). \tag{3.14}$$

Proof. Both sides of (3.14) are invariant under the transformation $(g,z) \mapsto (gh^{-1}, z^h)$, where $h \in SO(q)$. Therefore, without loss of generality, we may assume that $z = z_0$. Further, both sides of (3.14) are invariant under the transformations $(g,z) \mapsto (\tau g, z)$, where $\tau \in SO(p)$. Thus, without loss of generality, we may assume that $z_0^g = z_0$, i.e., that $g \in P$. To calculate $l(g, z_0)$ for $g \in P$ it is sufficient to calculate the determinant Det_g of the differential of the mapping $z \mapsto z^g$ on the tangent space T at z_0.

The space T consists of all block matrices $(\xi \eta)$ of size $p \times (p + (q-p))$ such that $\xi + \xi^t = 0$. If $g \in P$ belongs to the maximal nilpotent subgroup, then $\mathrm{Det}_g = 1$ (since a nilpotent subgroup acts by unipotent operators). Therefore, we may restrict ourselves to the case where $g \in R = GL_+(p, \mathbb{R}) \times SO(q)$, i.e., g has the form (3.10). It can easily be seen that here the differential of the map $z \mapsto z^g$ at z_0 is given by the formula

$$(\xi\, \eta) \mapsto \begin{pmatrix} g_{11}^{-1} \xi\, g_{11}^{t-1} & g_{11}^{-1} \eta\, g_{22} \end{pmatrix},$$

and now the assertion is obvious. $\square$

3.7. $SO(q)$**-finite functions.** We recall some definitions concerning Fourier series connected with compact groups (see [40, 9]). Let τ be a representation of a compact group K in a complete topological vector space V. Let $\lambda_1, \lambda_2, \dots$ be a complete set of irreducible representations of K, and let $\theta_1, \theta_2, \ldots$ be their characters. We define the operators $\pi_i : V \to V$ by the formula

$$\pi_i v = \dim \lambda_i \int\limits_K \overline{\theta_i(g)} \tau(g) v \, dg. \tag{3.15}$$

Then $\pi_i^2 = \pi_i$. Let $V_i = \pi_i V$. Then V_i is the isotypic component of V corresponding to λ_i (i.e., V_i is the sum of all subrepresentations in V equivalent to λ_i). A smooth vector $v \in V$ can be expanded into the series $v = \sum \pi_i v$ which converges in V. A vector $v \in V$ is called K-finite if only a finite number of $\pi_i v$ is different from 0 (or, what is the same, the linear span of the vectors $\tau(g)v$ is finite-dimensional).

Let $SO(q)$ act in the space $\mathcal{D}^0(M)$ of distributions on M. Since this action is transitive, the spaces $\pi_i \mathcal{D}^0(M)$ are finite-dimensional. It is seen from (3.15) that these spaces consist of C^∞-smooth functions. We denote by Fin the space of all $SO(q)$-finite functions on M.

Proposition 3.2. *The space* Fin *is exactly the set of the restrictions to M of holomorphic polynomials in $\mathbb{C}^{pq}$.*

Proof. Let $\sigma : \mathrm{Pol} \to \mathrm{Fin}$ be the restriction operator. Then, by the Stone–Weierstrass theorem, the set $\sigma(\mathrm{Pol})$ it is dense in $C(M)$. Consider the sequence of subspaces $\sigma(\pi_k \mathrm{Pol}_j) \subset \pi_k(\mathrm{Fin})$ increasing with j. The union of these subspaces coincides with $\sigma(\pi_k \mathrm{Pol})$ and, thus, is dense in $\pi_k(\mathrm{Fin})$. However, $\pi_k(\mathrm{Fin})$ is finite-dimensional, and, therefore, for some j we have $\sigma(\pi_k \mathrm{Pol}_j) = \pi_k(\mathrm{Fin})$. The proposition is proved. $\square$

3.8. The existence of boundary operators.

Theorem 3.1. *Let*

$$s \in \Sigma \quad \text{and} \quad s < (q - 2p + 1)/2. \tag{3.16}$$

Then for M, μ and K_s conditions 1^* *and* 2^* *of Sec.* 2.1 *hold.*

Theorem 3.2[1]**.** *Under conditions* (3.16), *the function*

$$h(z) = \det{}^{-s}(1 - z\, z_0^*) \tag{3.17}$$

is integrable on M.

Remark. If z is written in the form of a block $p \times (p + (q-p))$-matrix $z = (z^{(1)} z^{(2)})$, then $z z_0^* = z^{(1)}$.

Now we deduce Theorem 3.1 from Theorem 3.2.

Proof of Theorem 3.1. It is obvious that almost everywhere on $M \times M$ the following relation holds:

$$\lim_{c \to 1-0} \det{}^{-s}(1 - c^2 z u^*) = \det{}^{-s}(1 - z u^*).$$

[1]In [15*] integral of (3.17) over Stiefel manifold is explicitly evaluated. This gives sufficient condition $s < (q - p + 1)/2$ for existence of the restriction operator. Plancherel formula obtained in [16*] show that this condition is nessessary

Further, it is obvious that

$$\det{}^{-s}(1 - c^2 zu^*) \le 2^{ps} \det{}^{-s}(1 - zu^*).$$

Indeed, let $\sigma_1, \dots, \sigma_p$ be the eigenvalues of zu^*. Then $|\sigma_j| \le 1$, and the above inequality reduces to the obvious inequality

$$2(1 - c^2\sigma_j) \ge 1 - \sigma_j.$$

Further, from $SO(q)$-invariance considerations and from Theorem 3.2 it follows that the function $h_u(z) = \det{}^{-s}(1 - zu^*)$ is integrable on M for each $u \in M$. Hence the function $\det{}^{-s}(1 - zu^*)$ is integrable on $M \times M$. $\square$

3.9. An estimate of the integral (3.17).

Lemma 3.1. *Let A be a square matrix with $\|A\| < 1$. Then*

$$|\det(1 - A)| \ge \det(1 - |A|). \tag{3.18}$$

Here $|A| = (A^*A)^{1/2}$.

Proof. Let λ_j be the eigenvalues of A, and let α_i be the singular numbers (i.e., the eigenvalues of $|A|$). We must check the inequality

$$\prod_j |1 - \lambda_j| \ge \prod_j (1 - \alpha_j).$$

To this end, it suffices to verify the inequality

$$\prod_j (1 - |\lambda_j|) \ge \prod_j (1 - \alpha_j),$$

which is equivalent to

$$\sum_j \ln(1 - |\lambda_j|) \ge \sum_j \ln(1 - \alpha_j).$$

We write this inequality in the form

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_j |\lambda_j|^n \right) \le \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_j |\alpha_j|^n \right).$$

The latter follows from the inequality

$$\sum_j |\lambda_j|^n \le \sum_j \alpha_j^n,$$

which is a special case of the G. Weyl inequalities for eigenvalues and singular numbers, see [52, 2].

Proof of Theorem 3.2. Let $z = (z^{(1)} z^{(2)}) \in M$. Due to the lemma, we have

$$\det{}^{-s}(1 - zz_0^*) = \det{}^{-s}(1 - z^{(1)}) \leq \det{}^{-s}(1 - |z^{(1)}|).$$

This expression is invariant under the action of $SO(p) \times SO(p) \times SO(q-p) \subset SO_0(p,q)$, i.e., under the transformations of the form

$$(z^{(1)} z^{(2)}) \mapsto x^{-1} (z^{(1)} z^{(2)}) \begin{pmatrix} y & 0 \\ 0 & h \end{pmatrix}, \tag{3.19}$$

where $x \in SO(p)$, $y \in SO(p)$, and $h \in SO(q-p)$. It can easily be seen that if $q > 2p$ (this inequality follows from (3.16)), then by means of the transformations (3.19), we can bring any $z \in M$ to the form

$$\begin{pmatrix} r_1 & & & \sqrt{1-r_1^2} & & & \\ & \ddots & & & \ddots & & 0 \\ & & r_p & & & \sqrt{1-r_p^2} & \end{pmatrix},$$

where $1 \geq r_1 \geq \ldots \geq r_p \geq 0$. We denote by Δ the collection of all such sets of r_j. The function $\det{}^{-s}(1 - |z^{(1)}|)$ can be regarded as a function on Δ:

$$\det{}^{-s}(1 - |z^{(1)}|) = \prod (1 - r_j)^{-s}.$$

The projection of the measure μ onto the simplex Δ is given by the formula

$$C \prod_{1 \leq i < j \leq p} (r_i^2 - r_j^2) \prod_{i=1}^{p} (1 - r_i^2)^{(q-2p-1)/2} \prod_{i=1}^{p} dr_i,$$

where C is a constant. This relation is equivalent to the formula for the radial part of the invariant measure on the symmetric space $SO(q)/(SO(p) \times SO(q-p))$, see [27], X.I. or [28]. Now the assertion is obvious. $\square$

3.10. Boundary representations of $SO_0(p,q)$. Let $s \in \Sigma$ satisfy the condition

$$s < (q - 2p + 1)/2.$$

Then we can apply Secs. 2.1–2.2. In particular, on $L^\infty(M)$ we have the scalar product

$$\langle f_1, f_2 \rangle = \int_M \int_M \det{}^{-s}(1 - zu^*) f_1(z) \overline{f_2(u)} \, d\mu(z) \, d\mu(u), \tag{3.20}$$

and this scalar product is nonnegative definite (which is by no means obvious from the form of (3.20)). In the Hilbert space $E_s^\circ(M) := E^\circ(\mu)$ equipped with the scalar product (3.20), we have a unitary representation of $SO_0(p,q)$ given by the formula

$$\nu_s^\circ(g) f(z) = \det{}^{-(q-1)+s}(a + cz) f(z^g) \tag{3.21}$$

(see (2.8) and (3.14)). On the space $E_s(M) = E(\mu)$ dual to $E_s^\circ(M)$, the group $SO_0(p,q)$ acts by the unitary operators

$$\nu_s(g)f(z) = (a+zc)^{-s} f(z^g)$$

(see Sec. 2.7). We note that ν_s° and ν_s are not equivalent.

Remark 1. From all the above it follows that ν_s is a subrepresentation of the restriction to $SO_0(p,q)$ of the unitary representation ρ_s of $SU(p,q)$.

Remark 2. The representations ν_s° and ν_s are similar in form to representations of the degenerate complementary series and, possibly, coincide with these representations when $s \in \Sigma_+$. Here the degenerate principal series corresponds to $s = (q-1)/2 + i\sigma$ with $\sigma \in \mathbb{R}$. We do not know whether the representations ν_s are unitary for $s \in [(q-2p+1)/2, (q-1)/2)$.

3.11. Remarks on E_s° and E_s. First let $s \in \Sigma_+$. Then $H_s(B_{p,q}) \supset \mathrm{Pol}$, and, by Prop. 3.2, the space $E_s(M)$ contains Fin (and, therefore, $E_s(M)$ is a completion of Fin). From the duality considerations, the natural projection Fin $\to E_s^\circ(M)$ is injective and has dense image (the latter follows from the fact that Fin is dense in $C(M)$). Therefore, $E_s^\circ(M)$ is the completion of Fin with respect to the scalar product 3.2.1.

We cannot describe $E_s^\circ(M)$ and $E_s(M)$ directly in the spirit of Sec. 2.6. We only note that $E_s^\circ(M)$ and $E_s(M)$ are not Sobolev spaces (for $p > 1$) and they are not anisotropic Sobolev spaces.

If $s \in \Sigma \setminus \Sigma_+$, then $E_s(M) \cap \mathrm{Fin}$ is a proper subspace of Fin. In this case, the projection Fin $\to E_s^\circ(M)$ has a nontrivial kernel.

In the case in question, Proposition 2.1 implies

Proposition 3.3. *Let f be a distribution on M. Then $f \in E^\circ(M)$ if and only if the function*

$$\gamma(c) = \int\limits_M \int\limits_M \det{}^{-s}(1 - c^2 z u^*) f(z) \overline{f(u)}\, dz\, du$$

is bounded for $0 < c < 1$. Moreover,

$$\|f\|_{E_s^\circ} = \lim_{c \to 1-0} \gamma(c).$$

Assume that S is a submanifold of M, σ is the surface Lebesgue measure on S, δ_S is the measure σ regarded as a distribution on M.

Theorem 3.3. *Let the integral*

$$\int\limits_S \int\limits_S \det{}^{-s}(1 - z u^*) d\sigma(z)\, d\sigma(u) \tag{3.21}$$

converge. Then for any $\varphi \in L^\infty(S)$, the distribution $\varphi \cdot \delta_S$ is contained in $E_s^\circ(M)$.

Proof. This assertion is a special case of Prop. 3.3. $\square$

Remark 1. For the normal derivatives of δ_S, the same statement is also easy to check.

Remark 2. Convergence of (3.21) depends both on the dimension of S and on its location. It would be interesting to obtain the convergence conditions in terms of the tangent spaces to S (it is possible that such conditions do not exist).

§4. Hilbert spaces of vector-valued functions in matrix balls

4.0. Multiplicator and induced representations. Let a group G act on a space X by transformations $x \mapsto x^g$. Let V be a linear space, and let $GL(V)$ be the group of all invertible operators on V. Assume that G acts in the space $\mathcal{F}$ of functions $X \to V$ by the formula

$$T(g)f(x) = R(g,x)f(x^g), \tag{4.1}$$

where $g \in G$, $x \in X$, and $R(g,x)$ is a fixed function $G \times X \to GL(V)$. It is easy to see that the relation $T(g_1g_2) = T(g_1)T(g_2)$ is equivalent to the equation

$$R(g_1g_2, x) = R(g_2, x^{g_1})R(g_1, x). \tag{4.2}$$

The operator-valued functions $R(g,x)$ satisfying (4.2) are called *multiplicators*.

Fix a function $A : X \to GL(V)$ and consider the transformation $f(x) \mapsto A(x)f(x)$ of $\mathcal{F}$. It takes operators $T(g)$ from (4.1) to the operators of the same form but with the new multiplicator

$$\widetilde{R}(g,x) = A(x^g)^{-1}R(g,x)A(x). \tag{4.3}$$

It is natural to regard the multiplicators R and $\widetilde{R}$, linked by (4.3), as equivalent.

In the case where X is a G-homogeneous space, the description of all measurable multiplicators up to equivalence is well-known (see [9], 13.2). Namely, let $x_0 \in X$, and let H be the stabilizer of x_0. Then $\pi(g) = R(g, x_0)$ is a representation of H, i.e., we have a map from the set of multiplicators (defined up to equivalence) into the set of representations of H (the case of disconnected multiplicators, which is not pathologic, needs more accuracy). This map turns out to be bijective; for description of the inverse map see, e.g., [9], 13.2. In particular, with any representation of H one associates the multiplicator representation of G of the form (4.1) in the space of "all" functions on X. This association is called *inducing*.

We recall one of the ways of constructing the induced representation (this will be important for us only in §5 and Sec. 7.7). Let V be the space of a representation π of the group H. Consider the product W of G and V over H, i.e., the quotient space of $G \times V$ modulo the equivalence relation

$$(hg, \pi(h)\delta) \sim (g, \delta); \quad g \in G, \quad \delta \in V, \quad h \in H.$$

There is a natural projection from W to $X = G/H$ (namely, to $(g,v) \in G \times H$, we assign $Hg \in G/H$; this map factors through the above equivalence relation). The pre-image of any $x \in X$ is identified with the space V, i.e., W is a vector bundle over X.

The group G acts on $G \times V$ by the transformations $p : (g, \delta) \mapsto (gp, \delta)$, with $p \in G$. These transformations send equivalence classes into equivalence classes. Therefore, we obtain an action of G on W. Hence G acts on the sections of W over $X = G/H$ (indeed, the graph of a section $X \to W$ is a subset of W, and G acts on the space of sections by interchanging their graphs).

The action of G on the sections of W is the required induced representation. Now we consider a measurable map $W \to X \times V$ such that the fiber over $x \in X$ maps linearly to

$x \times V \subset X \times V$ (we identify $x \times V$ with V). The sections of W correspond to the functions $X \to V$. The corresponding action on the space of functions $X \to V$ has the form (4.1).

In general, the properties of the representation depend heavily on the choice of the topology in the space of functions on X. In an important special case where G is a semisimple Lie group and X is a compact set, the choice of the topology is not significant, and the construction of the induced representations can be considered as purely algebraic.

4.1. Multiplicator representations of $U(p,q)$. We define the functions

$$\gamma_1 : U(p,q) \times B_{p,q} \to GL(p,\mathbb{C}),$$
$$\gamma_2 : U(p,q) \times B_{p,q} \to GL(q,\mathbb{C})$$

by the relations

$$\begin{aligned}\gamma_1(g,z) &= (a+zc)^{-1},\\ \gamma_2(g,z) &= -d+cz^g.\end{aligned} \tag{4.4}$$

A direct calculation shows that

$$\gamma_i(g_1g_2,z) = \gamma_i(g_2,z^{g_1})\gamma_i(g_1,z),$$

i.e., γ_1 and γ_2 are multiplicators in the sense of Sec. 4.0, see also (3.4).

Let π be a holomorphic representation of the group $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ or its universal covering group (the term "holomorphic representation" means that the matrix-valued function π is holomorphic). Then the function

$$R(g,z) = \pi(\gamma_1(g,z),\gamma_2(g,z)) \tag{4.5}$$

is obviously a multiplicator.

This example is universal in the sense that any multiplicator representation of $U(p,q)$ (or its universal covering group) in the space of holomorphic vector-valued functions is given by a multiplicator (4.5).

4.2. Holomorphic representations of $GL(n,\mathbb{C})$ (preliminaries). We begin with the description of the so-called fundamental representations. Let $\tau(g) = g$ be the tautological n-dimensional representation of $GL(n,\mathbb{C})$. Let $\Lambda^j\mathbb{C}^n$ be the space of all j-polyvectors on $\mathbb{C}^n$, i.e., the space of all expressions of the form

$$\sum a_{\alpha_1,\dots,\alpha_j} e_{\alpha_1} \wedge \dots \wedge e_{\alpha_j},$$

where $\wedge$ denotes the exterior product and $\{e_k\}$ is the standard basis in $\mathbb{C}^n$. Let τ_j be the natural representation of $GL(n,\mathbb{C})$ in $\Lambda^j(\mathbb{C}^n)$, $j \le n$, and let v_j denote the vector

$$v_j = e_1 \wedge e_2 \wedge \dots \wedge e_j \in \Lambda^j\mathbb{C}^n.$$

We denote by $\widetilde{GL}(n,\mathbb{C})$ the universal covering group of $GL(n,\mathbb{C})$. All irreducible holomorphic representations of $\widetilde{GL}(n,\mathbb{C})$ are numbered by the signatures

$$\Lambda = (\lambda_1,\dots,\lambda_n),$$

where $\lambda_j \in \mathbb{C}$, and $\lambda_j - \lambda_{j+1}$ are nonnegative integers. Now we describe the representation π_Λ corresponding to Λ, and the representation space V_Λ.

Consider the tensor product

$$W = \bigotimes_{j=1}^{n-1} (\Lambda^j \mathbb{C}^n)^{\otimes(\lambda_j - \lambda_{j+1})}. \tag{4.6}$$

In W, we consider the distinguished vector

$$v_\Lambda = \bigotimes_{j=1}^{n-1} v_j^{\otimes(\lambda_j - \lambda_{j+1})}. \tag{4.7}$$

The group $\widetilde{GL}(n, \mathbb{C})$ acts on W by the formula

$$\widehat{\pi}_\Lambda(g) = \det{}^{\lambda_n}(g) \bigotimes_{j=1}^{n-1} \tau_j(g)^{\otimes(\lambda_j - \lambda_{j+1})}. \tag{4.8}$$

We define V_Λ to be the linear span of the vectors $\widehat{\pi}_\Lambda(g)v_\Lambda$, where g runs through $\widetilde{GL}(n, \mathbb{C})$. We define π_Λ to be a subrepresentation of $\widehat{\pi}_\Lambda$ which acts on the space V_λ.

Remark. The vector v_Λ is a vector of the highest weight in the usual sense, i.e., for any matrix of the form

$$g = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & r_{nn} \end{pmatrix}$$

the following relation holds:

$$\pi_\Lambda(g)v = r_{11}^{\lambda_1} \cdot r_{22}^{\lambda_2} \cdot \ldots \cdot r_{nn}^{\lambda_n} v.$$

Now let $\lambda_j \in \mathbb{R}$ (our concern is only with this case). Then, in the space V_Λ, we have a unique (up to a scalar factor) positive definite scalar product $\langle \cdot, \cdot \rangle_{V_\Lambda}$ such that

$$\langle \pi_\Lambda(g)\omega, (\pi_\Lambda(g)^*)^{-1}\omega' \rangle_{V_\Lambda} = \langle \omega, \omega' \rangle_{V_\Lambda}. \tag{4.9}$$

This scalar product can easily be defined. Namely, the standard scalar product on $\mathbb{C}^n$ induces a scalar product in $\Lambda^j\mathbb{C}^n$ and, therefore, also in the space (4.6), hence in V_Λ.

Example. The signature $(1, \dots, 1, 0, \dots, 0)$ with 1 at the first j places corresponds to the representation τ_j. The one-dimensional representation $g \mapsto \det{}^\lambda(g)$ has the signature $(\lambda, \dots, \lambda)$.

4.3. Invariant kernels. Consider a signature

$$\Lambda = (\lambda_1, \dots, \lambda_p; \lambda_{p+1}, \dots, \lambda_{p+q}), \tag{4.10}$$

where $\lambda_j \in \mathbb{R}$, and $\lambda_j - \lambda_{j+1}$ is a nonnegative integer whenever $j \neq p$. Consider the signatures

$$\Lambda' = (\lambda_1, \dots, \lambda_p), \quad \Lambda'' = (\lambda_{p+1}, \dots, \lambda_{p+q}) \tag{4.11}$$

for the groups $\widetilde{GL}(p,\mathbb{C})$ and $\widetilde{GL}(q,\mathbb{C})$, respectively.

Consider the representation

$$\pi_\Lambda := \pi_{\Lambda'} \otimes \pi_{\Lambda''} \tag{4.12}$$

of the group $\widetilde{GL}(p,\mathbb{C}) \times \widetilde{GL}(q,\mathbb{C})$ in the space

$$V_\Lambda = V_{\Lambda'} \otimes V_{\Lambda''}. \tag{4.13}$$

Let $\mathcal{H}_\Lambda$ be the space of all holomorphic V_Λ-valued functions on $B_{p,q}$. Consider the representation ρ_Λ of the universal covering group of $U(p,q)$ in $\mathcal{H}_\Lambda$ connected with the multiplicator

$$R_\Lambda(g,z) = \pi_{\Lambda'}(a+zc) \otimes \pi_{\Lambda''}(d-cz^g). \tag{4.14}$$

This representation is defined by the formula

$$\rho_\Lambda(g)f(z) = R_\Lambda(g,z)f(z^g). \tag{4.15}$$

Remark. Consider the action of the Lie algebra $gl(p+q)$ (the complexification of $u(p,q)$) on $\mathcal{H}_\Lambda$ corresponding to th

e action (4.15) (one can easily obtain the formulas). Then the constant function

$$v_\Lambda = v_{\Lambda'} \otimes v_{\Lambda''} \tag{4.16}$$

$B_{p,q} \to V_\Lambda$ is a vector of the highest weight in ρ_Λ, i.e., a vector annihilated by the upper triangular matrices from $gl(p+q)$.

Theorem 4.1. *Assume that there exists a Hilbert space of V_Λ-valued holomorphic functions on Ω in which the operators $\rho_\Lambda(g)$ are unitary. Then this Hilbert space is defined by the kernel*

$$K = K_\Lambda(z,\bar{u}) = \pi_{\Lambda'}(1-zu^*) \otimes \pi_{\Lambda''}(1-u^*z)^{-1}. \tag{4.17}$$

Conversely, if the kernel given by (4.17) is positive definite, then the representation ρ_λ is unitary in the Hilbert space defined by this kernel.

Lemma 4.1. *The operators $\rho_\Lambda(g)$ are unitary in $H_K(\Omega)$ if and only if the following relation holds:*

$$K(z,\bar{u}) = R_\Lambda(g,z)K(z^g,\overline{u^g})R_\Lambda(g,u)^*. \tag{4.18}$$

Proof of Lemma 4.1. Let the operators $A_z : H_K(B_{p,q}) \to V_\Lambda$ be the same as in Sec. 1.9, i.e., $A_z f = f(z)$. Then

$$A_u\rho_\Lambda(g)f = A_u[R_\Lambda(g,z)f(z^g)] = R_\Lambda(g,u)f(u^g) = R_\Lambda(g,u)A_u f.$$

Let the operators $\rho_\Lambda(g)$ be unitary. Then

$$K(z,\bar{u}) = A_z A_u^* = [A_z\rho_\Lambda(g)][A_u\rho_\Lambda(g)]^* = R_\Lambda(g,z)K(z^g,\overline{u^g})R_\Lambda(g,u)^*.$$

Conversely, by the above calculation, relation (4.18) implies the equation

$$A_z A_u^* = A_z(\rho_\Lambda(g)\rho_\Lambda(g)^*)A_u^*.$$

Let $\xi \in V_\Lambda$. Consider the functions $\psi_{u,\xi} = A_u^*\xi \in H_K(B_{p,q})$. The latter equation is equivalent to the relation

$$\langle \psi_{z,\eta}, \psi_{u,\xi}\rangle = \langle \psi_{z,\eta}, \rho_\Lambda(g)\rho_\Lambda(g)^*\psi_{u,\xi}\rangle,$$

and now the identity $\rho_\Lambda(g)\rho_\Lambda(g)^* = 1$ is obvious. □

Proof of Theorem 4.1. Identity (4.18) for K_Λ follows from the relations

$$1_p - zu^* = (a+zc)(1_p-(z^g)(u^g)^*)(a+uc)^*, \quad (d-u^gc)^*(1_q-u^*z)(d-z^gc) = 1_q-(u^g)^*z^g$$

that are easy to check using (3.2).

Now we prove the uniqueness. The kernel $K(z,\bar{u})$ is uniquely determined by its values on the diagonal, i.e., by the function $K(z,\bar{z})$. Further, due to (4.18), the function $K(z,\bar{z})$ is uniquely determined by $K(0,0)$ since $B_{p,q}$ is $U(p,q)$-homogeneous. Let $h \in U(p)\times U(q)$. We have $0^h = 0$. Then (4.18) implies the relation

$$K(0,0) = \pi_\Lambda(h)K(0,0)\pi_\Lambda(h)^*.$$

Since the operators $\pi_\Lambda(h)$ are unitary (see (4.9)), we obtain that $K(0,0) \in GL(V_\Lambda)$ commutes with the action of $U(p)\times U(q) \subset GL(p,\mathbb{C})\times GL(q,\mathbb{C})$. Therefore, since π_Λ is irreducible, the operator $K(0,0)$ is scalar.

4.4. Positive definiteness conditions. By the well-known construction of Harish–Chandra [37], the unitary multiplicator representations of a semisimple Lie group G in spaces of vector-valued holomorphic functions are exactly the representations of G with the highest weight. Recall that such representations exist only for the automorphism groups of Cartan domains (i.e., for the groups $U(p,q)$, $\mathrm{Sp}\,(2n,\mathbb{R})$, $SO^*(2n)$, $SO_0(n,2)$, and two real forms of E_6 and E_7). In different ways, the description of all unitary multiplicator representations in the spaces of scalar-valued functions was obtained independently by Berezin [4] and Gindikin [56], Wallach [51], and Rossi and Vergne [47]; see also [55], XIII for the exposition of these results. For the single-valued representations of $U(p,q)$ in spaces of vector-valued functions, the problem was solved by Kashiwara and Vergne [39] and Jakobsen [57], for the representations of universal covering of $U(p,q)$ the answer was given by one of the authors [20]. Finally, for the universal covering groups of all groups G the result was obtained by Enright, Howe, and Wallach [35], and later by Enright and Joseph [67, 68] in a different way.

We cite the result in the case $G = U(p,q)$ of our interest. Let

$$\Lambda = (\lambda_1,\dots,\lambda_p;\, \lambda_{p+1},\dots,\lambda_{p+q}) \tag{4.19}$$

be a signature of the form (4.10), i.e., $\lambda_j - \lambda_{j+1}$ are nonnegative integers for all $j \neq p$, and let $\lambda_j \in \mathbb{R}$. We denote by a a number of the coordinates λ_i such that $i \leq p$ and $\lambda_i \neq \lambda_1$.

Similarly, let b be the number of the coordinates λ_j such that $j \geq p+1$ and $\lambda_j \neq \lambda_{p+q}$. Let

$$c = \min(p+b, q+a) - 1.$$

We denote by Σ the set of all signatures Λ of the form (4.19) such that $t = \lambda_{p+q} - \lambda_1$ belongs to the set

$$\{a+b; a+b+1; \dots ; c\} \cup \{t > c\}.$$

Let Σ_+ be the subset of Σ defined by the condition $t > c$.

Theorem 4.2. *A kernel K_Λ given by the formula* (4.17) *is positive definite if and only if $\Lambda \in \Sigma$.*

Proof. See [20, 35, 67, 68]. □

Denote by $H_\Lambda = H_\Lambda(B_{p,q})$ the Hilbert space of V_Λ-valued holomorphic functions associated with the kernel $K_\Lambda(z, \bar{u})$. Thus, for $\Lambda \in \Sigma$, we have a unitary representation ρ_Λ of the universal covering group of $U(p,q)$ in H_Λ.

We denote by $\operatorname{Pol}_\Lambda$ the space of V_Λ-valued polynomial functions on $B_{p,q}$. The set $H_\Lambda \cap \operatorname{Pol}_\Lambda$ is dense in H_Λ, see Sec. 1.3. It is shown in [20] that

$$\Lambda \in \Sigma_+ \Longleftrightarrow H_\Lambda \supset \operatorname{Pol}_\Lambda. \tag{4.20}$$

4.5. The existence of boundary operators.

Theorem 4.3[2]**.** *Let*

$$q \geq 2p; \quad \Lambda \in \Sigma; \quad \lambda_{p+1} - \lambda_p < (q - 2p + 1)/2. \tag{4.21}$$

Then, for a Stiefel manifold M and a measure μ on M, conditions 1^{**} and 2^{**} from Sec. 2.11 *are fulfilled.*

Proof. Let $z, u \in M$, $0 < c < 1$. Then

$$\|K_\Lambda(cz, c\bar{u})\| = \|\pi_{\Lambda'}(1 - c^2 z u^*)\| \cdot \|\pi_{\Lambda''}(1 - c^2 u^* z)^{-1}\|.$$

Further,

$$\pi_{\Lambda'}(h) = \det{}^{\lambda_p}(h) \cdot \pi_{\widetilde{\Lambda'}}(h),$$

where $\widetilde{\Lambda'} = (\lambda_1 - \lambda_p, \dots, \lambda_{p-1} - \lambda_p, 0)$. The representation $\pi_{\widetilde{\Lambda'}}$ of the group $GL(p, \mathbb{C})$ is polynomial (i.e., the matrix elements of $\pi_{\widetilde{\lambda'}}$ are polynomials in the matrix elements h (see the explicit construction from Sec. 4.2). Therefore, the expression $\|\pi_{\widetilde{\Lambda'}}(1 - c^2 z u^*)\|$ is bounded on $M \times M \times [0,1]$.

Similarly,

$$\pi_{\Lambda''}(h) = \det{}^{-\lambda_{p+1}} \cdot \pi_{\widetilde{\Lambda''}}(h)$$

[2]The condition $q \geq p$; $\Lambda \in \Sigma$, $\lambda_{p+1} - \lambda_p < (q - p + 1)/2$ is sufficient for existence of the boundary operator, see [15*].

where $\widetilde{\Lambda}'' = (0, \lambda_{p+2} - \lambda_{p+1}, \dots, \lambda_{p+q} - \lambda_{p+1})$. The expression $\pi_{\Lambda''}(h)^{-1}$ is polynomial, and, therefore, $\sigma(z, u, c) = \|\pi_{\widetilde{\Lambda}''}(1 - c^2 u^* z)^{-1}\|$ is bounded on $M \times M \times [0, 1]$. Thus,

$$\|K(z, \bar{u})\| \leq C \cdot \det{}^{\lambda_p - \lambda_{p+1}}(1 - c^2 z u^*),$$

and now we can use Theorem 3.1.

4.6. Boundary representations of $SO_0(p, q)$. By the construction from Sec. 2.1–2.3, 2.11, we have two Hilbert spaces

$$E_\Lambda^\circ(M) := E^\circ(\mu), \quad E_\Lambda(M) := E(\mu)$$

and representations ν_Λ° and ν_Λ of $SO_0(p, q)$ in these spaces whenever conditions (4.21) are fulfilled.

The space $E_\Lambda^\circ(M)$ is the space of functions $M \to V_\Lambda$ with the scalar product

$$\langle f_1, f_2 \rangle_{E_\Lambda^\circ(M)} = \int\limits_M \int\limits_M \langle f_1(z), K_\Lambda(z, \bar{u}) f_2(u) \rangle_{V_\Lambda} d\mu(z)\, d\mu(u). \tag{4.22}$$

More exactly, consider the space L^∞ of functions $M \to V_\Lambda$. Then the scalar product (4.22) is nonnegative definite. We consider the quotient space L^∞ modulo the kernel of the scalar product and complete the resulting space.

The unitary representation ν_Λ° of $SO_0(p, q)$ in $E_\Lambda^\circ(M)$ is given by the formula

$$\nu_\Lambda^\circ(g) f(z) = \det{}^{-(q-1)}(a + zc) \pi_{\Lambda'}((a + zc)^t)^{-1} \otimes \pi_{\Lambda''}(((d - z^g c)^t)^{-1}) f(z^g). \tag{4.23}$$

The space $E_\Lambda(M)$ is a subspace of the space of integrable V_Λ-valued functions on M consisting of the boundary values of $f \in H_\Lambda(B_{p,q})$. The unitary representation ν_Λ of $SO_0(p, q)$ in $E_\Lambda(M)$ is given by the formula

$$\nu_\Lambda(g) f(z) = (\pi_{\Lambda'}(a + zc) \otimes \pi_{\Lambda''}(d - z^g c)) f(z^g). \tag{4.24}$$

By definition, this formula coincides with (4.15).

The unitary representations ν_Λ° and ν_Λ are equivalent, with the intertwining operator $E_\Lambda^\circ(M) \to E_\Lambda(M)$ given by

$$Tf(z) = \int\limits_M K_\Lambda(z, \bar{u}) f(u) d\mu(u).$$

We denote by Fin_Λ the space of all $SO(q)$-finite V_Λ-valued functions on M. As in Sec. 3.7, the space Fin_Λ consists of the restrictions of the functions $f \in \mathrm{Pol}_\Lambda$ to M.

Therefore, if $\Lambda \in \Sigma_+$, then $E_\Lambda(M)$ contains Fin_Λ (since $H_\Lambda \supset \mathrm{Pol}_\Lambda$ for $\Lambda \in \Sigma_+$), and the canonical map of $\mathrm{Fin}_\Lambda \subset L^\infty$ to $E^\circ(M)$ is injective. Therefore, both spaces $E_\Lambda^\circ(M)$ and $E(M)$ can be regarded as completions of Fin_Λ.

If $\Lambda \in \Sigma \setminus \Sigma_+$, then the space $\mathrm{Fin}_\Lambda \cap H_\Lambda$ is a proper subspace of Fin_Λ, and the canonical projection $\mathrm{Fin}_\Lambda \to E^\circ_\Lambda(M)$ has a nontrivial kernel. Thus, $E_\Lambda(M)$ turns out to be the completion of a certain subspace of Fin_Λ (consisting of the boundary values of the functions $f \in \mathrm{Pol}_\Lambda \cap H_\Lambda$), and $E^\circ(M)$ is the completion of a certain quotient space of Fin_Λ.

4.7. The representations ν°_Λ and ν_Λ as induced representations. Let z_0 and P be as in Sec. 3.6, i.e., $z_0 = (1_p 0)$, and $P \subset SO_0(p,q)$ is the stabilizer of z_0. The actions ν°_Λ and ν_Λ of $SO_0(p,q)$ have the form (4.1). In particular, for ν_Λ, the multiplicator is defined by the formula

$$R_\Lambda(g,z) = \pi_{\Lambda'}(a+zc) \otimes \pi_{\Lambda''}(d - z^g c).$$

Consider the representation $\tau_\Lambda(h) = R_\lambda(h, z_0)$ of P. This representation has the form

$$\tau_\Lambda(h) = \pi_{\Lambda'}(\sigma_1(h)) \otimes \pi_{\Lambda''}(\sigma_2(h)) \tag{4.25}$$

where the homomorphisms

$$\sigma_1 : P \to GL_+(p,\mathbb{R}), \quad \sigma_2 : P \to GL_+(q,\mathbb{R})$$

can be expressed in terms of the matrix elements of $h \in P$ by the formulas

$$\begin{aligned} \sigma_1(h) &= a + b_1, \\ \sigma_2(h) &= \begin{pmatrix} d_{11} - b_1 & -b_2 \\ 0 & d_{22} \end{pmatrix} \end{aligned} \tag{4.26}$$

(see (3.12)).

The mapping σ_1 is the natural projection of the group P on its quotient group $GL_+(p,\mathbb{R})$. The representation σ_2 is more complicated. On the reductive part $GL_+(p,\mathbb{R}) \times SO(q-p)$ of P, the mapping σ_2 is the natural embedding $GL_+(p,\mathbb{R}) \times SO(q-p) \to GL_+(q,\mathbb{R})$ (note that the p-dimensional representations $(a+b_1)$ and $(d_{11}-b_1)$ of $GL_+(p,\mathbb{R})$ are connected by the relation $((a+b_1)^t)^{-1} = d_{11} - b_1$). It is significant that the q-dimensional representation σ_2 of P is nontrivial (i.e., different from the identity) on the nilpotent subgroup N of P. Therefore, the representation τ_Λ of P, generally, is not completely reducible.

Thus, we see that, for $\Lambda \in \Sigma_+$, the representation ν_λ is induced by the representation τ_Λ of P. In this case ν°_Λ is also an induced representation.

If $\Lambda \in \Sigma \setminus \Sigma_+$, then ν_Λ is a subrepresentation of an induced representation, and ν°_Λ is a quotient representation of an induced representation.

§5. Singular unitary representations of $SO_0(p,q)$

5.0. Let $q > 2p$. Assume that P is the parabolic subgroup of $SO_0(p,q)$ defined in Sec. 3.6, and N is the maximal nilpotent subgroup of P. Recall that $P/N = GL_+(p,\mathbb{R}) \times SO(q-p)$.

Suppose that $\varkappa$ is a finite-dimensional representation of $GL_+(p,\mathbb{R})$, and σ is an irreducible representation of $SO(q-p)$. Consider the representation θ of P which is trivial on N and coincides with

$$\varkappa \otimes \sigma = \varkappa(a+b_1) \otimes \sigma(d_{22})$$

(see (3.12)) on $GL_+(p,\mathbb{R}) \otimes SO(q-p)$. We denote by $\mathrm{Ind}\,(\varkappa, \sigma)$ the representation of $SO_0(p,q)$ induced (in the nonunitary sense) by the representation θ of P (see Sec. 4.0).

We show that a great number of $\mathrm{Ind}\,(\varkappa,\sigma)$ are unitarizable, and, in some cases, $\mathrm{Ind}\,(\varkappa,\sigma)$ has unitarizable subrepresentations and quotient representations. Note that if $\varkappa$ has the form $\varkappa(r) = \det{}^{(q-1)/2+is}(r)$, then $\mathrm{Ind}\,(\varkappa,\sigma)$ are representations of degenerate principal series, and these representations are unitary by construction. If $\varkappa(r) = \det{}^{(q-1)/2+s}(r)$ with $s \in \mathbb{R}$, then, for sufficiently small $|s|$, some $\mathrm{Ind}\,(\varkappa,\sigma)$ are unitarizable as small deformations of unitarizable representations of the degenerate principal series. In this case, $\mathrm{Ind}\,(\varkappa,\sigma)$ are representations of degenerate complementary series.

If $\varkappa$ is not one-dimensional, then a priori we have no visible reasons for $\mathrm{Ind}\,(\varkappa,\sigma)$ to be unitarizable. However, we shall see that ν_Λ° and ν_Λ from Sec. 4.6 can be decomposed into finite direct sums of representations of the form $\mathrm{Ind}\,(\varkappa,\sigma)$. Thus, the direct summands must be unitary.

5.1. Filtrations in V_Λ. Let $\Lambda \in \Sigma$, and let $\pi_\Lambda = \pi_{\Lambda'} \otimes \pi_{\Lambda''}$ be a representation of $GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$ in $V_\Lambda = V_{\Lambda'} \otimes V_{\Lambda''}$ (see Secs. 4.4–4.5).

Consider the representation τ_Λ of P given by (4.25)–(4.26). Denote by δ the homomorphism $P \to GL(p,\mathbb{R}) \times GL_+(q,\mathbb{R})$ defined by the formula

$$\delta \begin{pmatrix} a & b_1 & b_2 \\ c_1 & d_{11} & -b_2 \\ c_2 & c_2 & d_{22} \end{pmatrix} = \left(a + b_1;\ \begin{pmatrix} d_{11} - b_1 & -b_2 \\ 0 & d_{22} \end{pmatrix} \right) \tag{5.1}$$

(see (3.12)). Then $\tau_\Lambda = \pi_\Lambda \circ \delta$. The image of δ is the subgroup $Q \subset GL_+(p,\mathbb{R}) \times GL_+(q,\mathbb{R})$ consisting of all matrices of the form

$$\left(g;\ \begin{pmatrix} g^{t-1} & \xi \\ 0 & \eta \end{pmatrix} \right), \tag{5.2}$$

where $g \in GL_+(p,\mathbb{R})$, $\eta \in SO(q-p)$, and ξ is real. If $\dim V_{\Lambda''} \neq 1$, then the representation τ_Λ is not completely reducible. In V_Λ, we consider P-invariant filtration which has no refinement:

$$\{0\} = W_0 \subset W_1 \subset \ldots \subset W_N = V_\Lambda. \tag{5.3}$$

In each subquotient W_{j+1}/W_j, we have an irreducible representation of P. It is obvious (see [58], §8.7) that this representation is trivial on N, and, on the reductive part, it has the form $\varkappa_j \otimes \sigma_j$, $\varkappa_j$ and σ_j being irreducible representations of $GL_+(p,\mathbb{R})$ and $SO(q-p)$, respectively. Let Γ_Λ denote the set of all pairs $(\varkappa_j, \sigma_j)$ that can be obtained in this way. We do not know a complete description of Γ_Λ. We note only that $(\varkappa_j, \sigma_j) \in \Gamma_\Lambda$ if and only if $\varkappa_j \otimes \sigma_j$ enters the restriction of π_Λ to the subgroup $GL_+(p,\mathbb{R}) \times SO(q-p) \subset P$. This problem is a combination of the following three problems, which were studied intensively:

(1) The restriction of a finite-diminsional representation from $GL(q)$ to $GL(p) \times GL(q-p)$.

(2) The restriction of an irreducible representation of $GL(q-p)$ to $SO(q-p)$.

(3) Tensor products of representations of $GL(p)$.

5.2. Filtrations in $C^\infty(M, V_\Lambda)$. Let C^∞_Λ denote the space of all infinitely smooth V_Λ-valued functions on M. Consider the representation $\bar\nu$ of $SO_0(p,q)$ in C^∞_Λ given by the formula

$$\bar\nu_\Lambda(g)f(z) = (\pi_{\Lambda'}(a+zc)\otimes\pi_{\Lambda''}(-d+z^g c))f(z^g).$$

This formula coincides with (4.24) for ν, but if $\Lambda \notin \Sigma_+$, then ν is only a subrepresentation of $\bar\nu_\Lambda$. The restriction of $\bar\nu_\Lambda$ to $SO(q)$ has a spectrum of finite multiplicity. Therefore, when dealing with $SO_0(p,q)$-invariant subspaces, we can pay no attention to the topology (see [40]). In particular, the $SO_0(p,q)$-invariant subspaces of C^∞_Λ are in one-to-one correspondence with the subspaces of Fin_Λ that are invariant under the Lie algebra $so(p,q)$).

Let $W \subset V_\Lambda$ be a P-invariant subspace. For any $u \in M$, we consider a $g \in SO_0(p,q)$ such that $u = z_0^g$. Further, we define the subspace

$$W^{(u)} = W\cdot g \subset V_\Lambda.$$

It is clear that $W^{(u)}$ is independent of the choice of g. Thus, we have a fiber bundle on M such that the fiber over $u \in M$ is $W^{(u)} \subset V_\Lambda$.

We denote by $\widetilde W$ the subspace of C^∞_Λ consisting of all f such that

$$f(u) \in W^{(u)} \quad \text{for all} \quad u \in M.$$

In particular, to each P-invariant filtration (5.3) there corresponds $SO_0(p,q)$-invariant filtration

$$\{0\} = \widetilde W_0 \subset \widetilde W_1 \subset \ldots \subset \widetilde W_N = C^\infty_\Lambda \tag{5.4}$$

of C^∞_Λ. The representation of $SO_0(p,q)$ in $\widetilde W_{j+1}/\widetilde W_j$ is, obviously, $\mathrm{Ind}\,(\varkappa_j, \sigma_j)$.

Theorem 5.1. *Assume that $\Lambda \in \Sigma_+$ satisfies (4.21) and $(\varkappa_j,\sigma_j) \in \Gamma_\Lambda$. Then the representation $\mathrm{Ind}\,(\varkappa_j,\sigma_j)$ of $SO_0(p,q)$ is unitarizable.*

Proof. Indeed, the above representation is a subquotient in the unitarizable representation $\bar\nu_\Lambda = \nu_\Lambda$. $\square$

It is obvious that if $\Lambda \in \Sigma_+$, then

$$\nu_\Lambda = \bigoplus_j \mathrm{Ind}\,(\varkappa_j,\sigma_j).$$

In the case $\Lambda \in \Sigma\setminus\Sigma_+$, an additional difficulty arises. In this case $\bar\nu_\Lambda$ is unitarizable in the proper subspace $C^\infty_\Lambda \cap E_\Lambda(M)$ of C^∞_Λ. Then the filtration

$$\{0\} \subset (\widetilde W_0 \cap E_\Lambda(M)) \subset (\widetilde W_1 \cap E_\Lambda(M)) \subset \ldots \subset (\widetilde W_N \cap E_\Lambda(M)) = E_\Lambda(M) \tag{5.5}$$

corresponds to (5.3).

It is clear that the representation of $SO_0(p,q)$ in

$$(\widetilde W_{j+1} \cap E_\Lambda(M))/(\widetilde W_j \cap E_\Lambda(M)) \tag{5.6}$$

is unitarizable. However, this representation can be zero-dimensional for some j.

Proposition 5.1. *Let $\Lambda \in \Sigma$. Then the highest subquotient of* (5.5) *is nontrivial, i.e.,*

$$\widetilde{W}_{N-1} \cap E_\Lambda(M) \neq E_\Lambda(M).$$

Proof. The space H_Λ contains all constant functions, and thus $E_\Lambda(M)$ also contains all constant functions. If $f \in \widetilde{W}_{N-1}$, then $f(z_0) \in W_{N-1} \neq V_\Lambda$, i.e., there exist constants which do not belong to W_{N-1}.

The following question naturally arises: which representations $\varkappa_j \otimes \sigma_j$ can occur in the highest terms of a filtration of the form (5.3)?

A partial answer to this question is given in the next section.

5.3. Additional remarks about filtrations in V_Λ. We consider the subgroup R in $GL(q,\mathbb{C})$ consisting of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, \quad a \in GL(p,\mathbb{C}), \quad c \in GL(q-p,\mathbb{C}) \tag{5.7}$$

(cf. (5.1)). In $V_{\Lambda''}$ we consider an R-invariant filtration which has no refinement,

$$\{0\} = Y_0 \subset Y_1 \subset \ldots \subset Y_L = V_{\Lambda''}. \tag{5.8}$$

Then we obtain an irreducible representation of R in Y_{j+1}/Y_j which is trivial on a nilpotent subgroup, i.e., actually it is the group $GL(p,\mathbb{C}) \times GL(q-p,\mathbb{C})$ that acts in Y_{j+1}/Y_j. The representation has the form $\psi_j \otimes \varphi_j$, where ψ_j and φ_j are irreducible holomorphic representations of $GL(p,\mathbb{C})$ and $GL(q-p,\mathbb{C})$, respectively.

Let

$$Z_j = V_{\Lambda'} \otimes Y_j \subset V_\Lambda.$$

Then the filtration

$$\{0\} = Z_0 \subset Z_1 \subset \ldots \subset Z_L = V_\Lambda \tag{5.9}$$

is clearly P-invariant (see (5.7) and (5.1)). Further, R and $Q = \delta(P)$ have the same nilpotent subgroup. Therefore, the action of the nilpotent subgroup $N \subset P$ on Z_{j+1}/Z_j is trivial, i.e., in fact, it is the reductive part $GL_+(p,\mathbb{R}) \times SO(q-p)$ of P that acts in Z_{j+1}/Z_j. Hence the representation of P in Z_{j+1}/Z_j is completely reducible.

Now we describe the highest subquotient Y_L/Y_{L-1} of the filtration (5.7). Let ω be a vector of the lowest weight in $V_{\Lambda''}$. Let U denote its $GL(p,\mathbb{C}) \times GL(q-p,\mathbb{C})$-cyclic span. The space U is the tensor product of the representation of $GL(p,\mathbb{C})$ with signature

$$B = (-\lambda_{p+q}, \ldots, -\lambda_{q+1})$$

and the representation of $GL(q,\mathbb{C})$ with signature

$$\Delta = (-\lambda_q, \ldots, -\lambda_{p+1}).$$

Let C be a $GL(p,\mathbb{C}) \times GL(q-p,\mathbb{C})$-invariant complement with respect to U. Clearly, C is R-invariant.

Thus, we can take C in the role of Y_{L-1} in (5.8). Then we obtain the representation $\pi_B \otimes \pi_\Delta$ of $GL(p,\mathbb{C}) \times GL(q-p,\mathbb{C})$ in Y_L/Y_{L-1}.

We decompose Z_L/Z_{L-1} into the sum

$$Z_L/Z_{L-1} = \oplus(\varkappa_\alpha \otimes \sigma_\alpha)$$

of irreducible representations of $GL_+(p,\mathbb{R}) \times SO(q-p)$.

Let Γ_Λ^0 be the set of all pairs $(\varkappa_\alpha, \sigma_\alpha)$ that occur in this expression. It is clear that these pairs are characterized by the following conditions:

1) $\varkappa_\alpha$ occurs in the decomposition of $\pi_{\Lambda'} \otimes \pi_B$.

2) σ_α occurs in the restriction of π_Δ to $SO(q-p)$.

Thus, each $(\varkappa_\alpha, \sigma_\alpha) \in \Gamma_\Lambda^0$ can occur as the upper term of a filtration (5.3).

5.4. The existence of a unitary subrepresentation in $\operatorname{Ind}(\varkappa, \sigma)$**.**

Theorem 5.2. *Assume that* $\Lambda \in \Sigma$ *satisfies* (4.21) *and* $(\varkappa_\alpha, \sigma_\alpha) \in \Gamma_\Lambda^0$. *Then* $\operatorname{Ind}(\varkappa_\alpha, \sigma_\alpha)$ *has a nonzero unitary subrepresentation.*

Proof. The theorem follows from Prop. 5.1 and the construction from the preceding section. $\square$

Let $(\varkappa_\alpha, \sigma_\alpha) \in \Gamma_\Lambda^0$. Consider a filtration W_j of the form (5.3) such that $\varkappa_\alpha \otimes \sigma_\alpha$ is realized in its highest term W_N/W_{N-1}. Denote by $I(\varkappa_\alpha, \sigma_\alpha)$ the representation of $SO_0(p,q)$ in $(E_\Lambda(M) \cap C_\Lambda^\infty)/(E_\Lambda(M) \cap \widetilde{W}_{N-1})$. We note that $\Lambda \in \Sigma_+$ implies $I(\varkappa_\alpha, \sigma_\alpha) = \operatorname{Ind}(\varkappa_\alpha, \sigma_\alpha)$.

5.5. Irreducibility.

Theorem 5.3. *Assume that* $\Lambda \in \Sigma$ *satisfies the condition*

$$\lambda_{p+1} - \lambda_p < (q-2p+1)/4, \tag{5.10}$$

and $(\varkappa_\alpha, \sigma_\alpha) \in \Gamma_\Lambda^0$. *Then* $I(\varkappa_\alpha, \sigma_\alpha)$ *is irreducible.*

Proof. We follow the Bruhat method (see [9], 13.5) based on searching invariant distributions, which is quite complicated in general. However, in the case under consideration we have a simplifying reasoning.

Thus, $I(\varkappa_\alpha, \sigma_\alpha)$ is realised in the highest term of a $SO_0(p,q)$-invariant filtration in $E_\Lambda(M)$. Dually, the same representation is realized in the lowest term of a $SO_0(p,q)$-invariant filtration in $E_\Lambda^\circ(M)$. Namely, we have the following vector bundle on M. For each $u \in M$, we choose a subspace $X^{(u)} \subset V_\Lambda$ which depends smoothly on u. The group $SO_0(p,q)$ acts in sections of the vector bundle by formula (4.24), and each $\nu_\Lambda^\circ(g)$ takes sections into sections. Let $\mathcal{F}$ be the space of sections of this vector bundle. In $\mathcal{F}$ we have an invariant (possibility, degenerate) scalar product, namely,

$$\langle f_1, f_2 \rangle = \int\limits_M \int\limits_M \langle f_1(z), K_\Lambda(z,\bar{u}) f_2(u) \rangle_{V_\Lambda} d\mu(z)\, d\mu(u). \tag{5.11}$$

We note that in this formula it is the function

$$K(z,u;\xi,\eta) := \langle \xi, K_\Lambda(z,\bar u)\eta\rangle_{V_\Lambda}; \quad z,u\in M, \quad \xi\in X^{(z)}, \quad \eta\in X^{(u)}$$

that is significant, but not the function $K_\Lambda : M\times M \to GL(V_\Lambda)$.

Condition (4.21) means that $\|K(\cdot,\cdot)\| \in L^1(M\times M)$. Condition (5.10) is stronger. This condition means that $\|K(\cdot,\cdot)\| \in L^2(M\times M)$; in particular, the scalar product (5.11) is continuous in the L^2-topology on $\mathcal{F}$.

Assume that $I(\varkappa_\alpha,\sigma_\alpha)$ is reducible. Then an $SO_0(p,q)$-invariant projector P exists in $I(\varkappa_\alpha,\sigma_\alpha)$. Therefore, in $\mathcal{F}$, we have another $SO_0(p,q)$-invariant scalar product

$$\{f_1,f_2\} := \langle \mathcal{P}f_1, \mathcal{P}f_2\rangle_{E^\circ(M)}.$$

Obviously,

$$\{f,f\} \le \langle f,f\rangle_{E^\circ(M)}. \tag{5.12}$$

It follows from (5.12) that $\{\cdot,\cdot\}$ is continuous in the L^2-topology, and thus $\{\cdot,\cdot\}$ has the form

$$\{f_1,f_2\} = \langle f_1, Bf_2\rangle_{L^2},$$

where B is a bounded operator in L^2. However, the scalar product (5.11) also has the form $\langle f_1,f_2\rangle = \langle f_1, Af_2\rangle_{L^2}$, where A is a Hilbert–Schmidt operator in L^2. Now inequality (5.12) can be written in the form

$$\langle f, Bf\rangle_{L^2} \le \langle f, Af\rangle_{L^2}.$$

Hence $B \le A$, and B is also a Hilbert-Schmidt operator. (Indeed, let $s_1 \ge s_2 \ge \dots$ and $t_1 \ge t_2 \ge \dots$ are the eigenvalues of B and A, respectively. Then the minimax characterization of eigenvalues (see [2]) implies inequalities $s_j \le t_j$.)

Thus, $\{\cdot,\cdot\}$ has the form

$$\{f_1,f_2\} = \int\limits_M\int\limits_M \langle f_1(z), S(z,u)f_2(u)\rangle d\mu(z)d\mu(u),$$

where $\|S(\cdot,\cdot)\| \in L^2(M\times M)$. The scalar product $\{\cdot,\cdot\}$ is invariant if and only if the following relation holds

$$S(z,u) = \det{}^{q-1}(a+cz)R_\Lambda(g,z)\,S(z^g,u^g)R_\Lambda(g,\bar u)^* \tag{5.13}$$

(cf. (4.18)). We show that this relation uniquely determines S and, thus, there are no invariant scalar products different from (5.11).

The group $SO_0(p,q)$ has an open orbit O on $M\times M$. This orbit contains the point

$$(z_0;-z_0) = ((1_p0);(-1_p0)) \in M\times M,$$

and the stabilizer St of this point is isomorphic to $GL_+(p,\mathbb{R})\times SO(q-p)$. Due to (5.13), the function $S(z,u)$ is uniquely determined by the matrix $K(z_0,-z_0)\in GL(V_\Lambda)$. Choosing $g\in St$ in (5.13), we see that

$$(\xi,\eta) = \langle \xi, K(0,0)\eta\rangle_{V_\Lambda}; \quad \eta\in X^{(z_0)},\ \xi\in X^{(-z_0)},$$

is St-invariant. Since the representation of St in $X^{(z_0)}$ is irreducible (it is equivalent to $\varkappa_\alpha \otimes \sigma_\alpha$), such form is unique. $\square$

5.6. Class $\mathcal{B}$. Thus, any representation of $SO_0(p,q)$ of the form ν_Λ is decomposable into a finite direct sum of unitary representations of the form $\mathrm{Ind}\,(\varkappa_\alpha, \sigma_\alpha)$ or subrepresentations of $\mathrm{Ind}\,(\varkappa_\alpha, \sigma_\alpha)$. Let $\mathcal{B}$ denote the set of all unitary representations of $SO_0(p,q)$ which can be obtained in this way. In §§6–7 we show that the representations from $\mathcal{B}$ have a lot of interesting and unusual properties.

§6. Passage to the limit as $q \to \infty$

6.0. The groups $U(p,\infty)$ and $O(p,\infty)$. Consider the coordinate Hilbert space l_2. The elements of l_2 have the form $x = (x_1, \dots, x_p; x_{p+1}, \dots)$. We equip l_2 with the indefinite Hermitian form

$$\{x, y\} = -\sum_{j\le p} x_j \bar{y}_j + \sum_{j>p} x_j \bar{y}_j.$$

Let $U(p,\infty)$ be the group of all invertible operators in l_2 which preserve the form $\{\cdot,\cdot\}$. The elements of $U(p,\infty)$ will be written in the form of block matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of size $(p+\infty)\times(p+\infty)$. Let $O(p,\infty)$ be the subgroup of $U(p,\infty)$ consisting of all real matrices.

These two groups equipped with the weak operator topology are topological groups.

The group $O(p,\infty)$ has two connected components. The component containing the identity is determined by the condition $\det(a) > 0$; we denote this component by $O_0(p,\infty)$.

The group $U(p,\infty)$ is not simply connected. It can be contracted to the subgroup $U(p) \times U(\infty)$ in the standard way (in the same way as $U(p,q)$ is contractable to $U(p) \times U(q)$). The group $U(\infty)$ is simply connected; therefore, the fundamental group of $U(p,\infty)$ is isomorphic to $\mathbb{Z}$. Denote by $U(p,\infty)^\sim$ the universal covering group of $U(p,\infty)$.

6.1. Holomorphic representations of $GL(\infty,\mathbb{C})$. Denote by $GL(\infty,\mathbb{C})$ the group of all bounded operators in l_2 which have bounded inverse operators.

Consider a signature of the form

$$M = M(\infty) = (\mu_1, \mu_2, \dots), \tag{6.1}$$

where $\mu_1 \ge \mu_2 \ge \dots$ are nonnegative integers, and $\mu_j = 0$ for sufficiently large j. Repeating word for word the construction of Sec. 4.2, we obtain a Hilbert space V_M and a representation π_M of $GL(\infty,\mathbb{C})$ in V_M (note that all tensor products in the formulas of Sec. 4.2 are finite since $\mu_j = 0$ for sufficiently large j. For the same reason, the factor $\det^{\lambda_n}(g)$ is absent in (4.8)).

Remark. The representations π_M are continuous in the weak topology on $GL(\infty,\mathbb{C})$, but $GL(\infty,\mathbb{C})$, equipped with the weak topology, is not a topological group.

Assume that we are given a signature (6.1). Then, for each $n \in \mathbb{N}$, we consider the signature $M(n) = (\mu_1, \dots, \mu_n)$. For any m and n such that $m < n \le \infty$, we obviously obtain canonical embeddings

$$L^{(M)}_{m,n} : V_{M(m)} \to V_{M(n)} \tag{6.2}$$

satisfying the condition

$$L^{(M)}_{m,n}\pi_{\Lambda(m)}(g) = \pi_{\Lambda(n)}(g)L^{(M)}_{m,n}; \quad g \in GL(m, \mathbb{C}).$$

Thus, we have a chain of embedded spaces

$$V_{M(1)} \to V_{M(2)} \to \dots .$$

Obviously, the union of these spaces is dense in $V_{M(\infty)}$.

6.2. Holomorphic representations of $U(p,\infty)^\sim$. Let Σ_∞ be the set of all signatures

$$\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_p; \lambda_{p+1}, \dots) \tag{6.3}$$

satisfying the following conditions:

(a) for all $j \neq p$, $\lambda_{j+1} - \lambda_j$ is a nonnegative integer.

(b) $\lambda_{p+j} = 0$ for sufficiently large j.

(c) Denote by a the number of $\lambda_1, \dots, \lambda_p$ different from λ_1 and by b the number of $\lambda_{p+1}, \dots$ different from 0. Then λ_1 belongs to the set

$$\{a+b, a+b+1, \dots, p+b-1\} \cup \{\lambda_1 > p+b-1\}.$$

We define the matrix ball $B_{p,\infty}$ as the set of complex matrices z of size $p \times \infty$ such that $\|z\| < 1$. The group $U(p,\infty)$ acts on $B_{p,\infty}$ by the standard formula (3.3).

Let Λ be a signature of the form (6.3). As before, let

$$\Lambda' = (\lambda_1, \dots, \lambda_p), \quad \Lambda'' = (\lambda_{p+1} \dots),$$

$$V_\Lambda = V_{\Lambda'} \otimes V_{\Lambda''}.$$

For any $\Lambda \in \Sigma_\infty$, we define an operator-valued kernel

$$K_\Lambda : B_{p,\infty} \times B_{p,\infty} \to GL(V_\Lambda)$$

by the formula

$$K_\Lambda(z, \bar{u}) = \pi_{\Lambda'}(1 - zu^*) \otimes \pi_2(1 - u^*z)^{-1}. \tag{6.4}$$

As in the finite-dimensional space, for K_Λ we construct the Hilbert space $H_\Lambda(B_{p,\infty})$ consisting of the V_Λ-valued holomorphic functions on $B_{p,\infty}$. The group $U(p,\infty)^\sim$ acts on $H_\Lambda(B_{p,\infty})$ by the formula

$$\rho_\Lambda(g)f(z) = [\pi_{\Lambda'}(a + zc) \otimes \pi_{\Lambda''}(d - z^g c)]f(z^g). \tag{6.5}$$

Moreover, ρ_Λ is a unitary representation. The construction of the next section shows that the present constructions are well defined.

6.3. Limit construction for ρ_Λ. We may consider the matrix ball $B_{p,n}$ as a subset of $B_{p,\infty}$. Indeed, to $z \in B_{p,n}$ we can assign the matrix $(z\, 0)$ of size $p \times (n + \infty)$. Consider a signature of the form (6.3) satisfying conditions (a)–(b), and for each positive integer n, we define the signature

$$\Lambda(n) = (\lambda_1, \dots, \lambda_p; \lambda_{p+1}, \dots, \lambda_{p+n}).$$

It should be noted that this signature satisfies the conditions of Sec. 4.4. Consider the corresponding representation $\rho_\Lambda(n)$ of $U(p,n)$ in the space $H_{\Lambda(n)}(B_{p,n})$.

Let $m < n$. We shall define the operator of isometric embedding

$$A_{m,n} : H_{\Lambda(m)}(B_{p,m}) \to H_{\Lambda(n)}(B_{p,n}).$$

To this end, we consider the system of functions

$$\varphi^{(l)}_{u,\xi}(z) = K_{\Lambda(l)}(z, \bar{u})\xi \in H_{\Lambda(l)}(B_{p,l}); \quad u \in B_{p,l}, \quad \xi \in V_{\Lambda(l)},$$

and put

$$A_{m,n}\varphi^{(m)}_{u,\xi} = \varphi^{(n)}_{u,\eta}; \quad \eta = \big(1_{V_{\Lambda'}} \otimes L^{\Lambda''}_{m,n}\big)\xi$$

(where $L_{m,n}$ is defined in 6.1). It is easy to see that

$$\langle A_{m,n}\varphi^{(m)}_{z,\eta}, A_{m,n}\varphi^{(m)}_{u,\xi}\rangle_{H_{\Lambda(n)}(B_{p,n})} = \langle \varphi^{(m)}_{z,\eta}, \varphi^{(m)}_{u,\xi}\rangle.$$

Therefore, $A_{m,n}$ is actually an isometric embedding, moreover,

$$A_{m,n}\rho_{\Lambda(m)}(g)f = \rho_{\Lambda(n)}(g)A_{m,n}f; \quad g \in U(p,m) \tag{6.6}$$

(this identity is obvious if $f = \psi_{u,\xi}$; then we extend by linearity.)

Remark. A distribution χ on $\bar{B}_{p,m}$ with values in $V_{\Lambda(m)}$ can be regarded as a $V_{\Lambda(n)}$-distribution on $\bar{B}_{p,n}$. It is easy to see that, in notation of Sec. 1.6, we have

$$A_{m,n}\Xi\chi = \Xi\big(1_{V_{\Lambda'}} \otimes L^{(\Lambda'')}_{m,n}\chi\big). \tag{6.7}$$

Thus, we obtain a chain of embedded Hilbert spaces

$$\dots \to H_{\Lambda(k)}(B_{p,k}) \overset{A_{k,k+1}}{\to} H_{\Lambda(k+1)}(B_{p,k+1}) \to \dots .$$

We denote by $H_\Lambda(B_{p,\infty})$ the corresponding inductive limit, i.e., the completion of the union of these spaces. By construction, the group

$$U^0(p,\infty) := \bigcup_{n=1}^{\infty} U(p,n)$$

acts in $H_\Lambda(B_{p,\infty})$.

Proposition 6.1. *The representation of the group $U^0(p,\infty)^\sim$ in $H_\Lambda(B_{p,\infty})$ can be extended to a weakly continuous representation of $U(p,\infty)^\sim$.*

Proof. Consider the map

$$\left(\bigcup_{n=1}^{\infty} B_{p,n}\right) \times \left(\bigcup_{n=1}^{\infty} V_{\Lambda(n)}\right) = \bigcup_{n=1}^{\infty} (B_{p,n} \times V_{\Lambda(n)}) \to H_\Lambda(B_{p,\infty})$$

defined by the formula

$$\alpha(u,\xi) = \psi_{u,\xi}.$$

Since

$$\langle \psi_{z,\eta}, \psi_{u,\xi}\rangle = \langle \eta, K_\Lambda(z,\bar{u})\xi\rangle,$$

we obtain that α can be extended to a continuous map

$$B_{p,\infty} \times V_\Lambda \to H_\Lambda(B_{p,\infty})$$

($B_{p,\infty}$ is equipped with the uniform topology and V_Λ with the weak topology). Let $g \in U(p,\infty)$, and let $R_\Lambda(g,z)$ be the multiplicator from (6.5) (see (4.14)). We define unitary operators $\rho_\Lambda(g)$ by the formula

$$\rho_\Lambda(g)\psi_{u,\xi} = \psi_{u^{g^{-1}},\eta}; \quad \eta = R_\Lambda(g,u)^{*-1}\xi.$$

It can easily be seen that the quantities $\langle \rho_\Lambda(g)\psi_{z,\zeta}, \psi_{u,\xi}\rangle$ depend continuously on $g \in U(p,\infty)$. $\square$

6.4. Dual construction. The limit construction of the preceding section can be described in a slightly different way (cf. [60]). Namely, consider the map

$$A^*_{m,n} : H_{\Lambda(n)}(B_{p,n}) \to H_{\Lambda(m)}(B_{p,m})$$

which sends a function $f \in H_{\Lambda(n)}(B_{p,n})$ to the restriction of the function $\left(1_{V_{\Lambda'}} \otimes (L^{(\Lambda'')}_{m,n})^*\right)$ $f(z)$ to $B_{p,m} \subset B_{p,n}$.

Thus, we obtain a chain of Hilbert spaces

$$\ldots \longleftarrow H_{\Lambda(k)}(B_{p,k}) \overset{A^*_{k,k+1}}{\longleftarrow} H_{\Lambda(k+1)}(B_{p,k+1}) \leftarrow \ldots .$$

We define $H_\Lambda(B_{p,\infty})$ as the inverse limit of this chain. Obviously,

$$A^*_{m,n}\rho_{\Lambda(n)}(g) = \rho_{\Lambda(m)}(g)A^*_{m,n}; \quad g \in U(p,m).$$

Therefore, the group $U^0(p,\infty)$ acts on $H_\Lambda(B_{p,\infty})$. Then we extend the representation by continuity.

6.5. The restriction of ρ_Λ to $O(p,\infty)$.

Theorem 6.1 ([19])**.** *The restriction of the representation ρ_Λ to $O_0(p,\infty)$ is irreducible.*

Proof. Consider the restriction of ρ_Λ to $U(\infty)$. For $g \in U(\infty)$, formula (6.5) can be written in the form

$$f(z) \mapsto \big(1_{V_{\Lambda'}} \otimes \pi_{\Lambda''}(g)\big) f(zg).$$

It is clear that this representation of $U(\infty)$ is decomposable into a direct sum of the representations π_M from Sec. 6.1 (regarded as representations of $U(\infty) \subset GL(\infty,\mathbb{C})$). It can be shown (see [21], see also [8*]) that the restrictions of these π_M to $O(\infty)$ are irreducible and pairwise nonequivalent.

Therefore, any $O_0(p,\infty)$-invariant subspace $Q \subset H_\Lambda(B_{p,\infty})$ is also $U(\infty)$-invariant (since it is $O(\infty)$-invariant). In particular, this subspace is invariant under the groups $SO_0(p,n) \subset U(p,n)$ and $U(n) \subset U(p,n)$. But the group $U(p,n)$ is generated by its subgroups $SO_0(p,n)$ and $U(n)$. Hence Q is $U(p,n)$-invariant and, therefore, $U^0(p,\infty)$-invariant. $\square$

Theorem 6.2 ([21])**.** *Any irreducible unitary representation of $O_0(p,\infty)$ is a restriction of some representation ρ_Λ of $U(p,\infty)$. If $\Lambda \neq \Lambda'$, then the restrictions of ρ_Λ and $\rho_{\Lambda'}$ to $O_0(p,\infty)$ are distinct.*

Denote by γ_Λ the restriction of ρ_Λ to $O_0(p,\infty)$.

6.6. Some remarks on inductive limits. Consider a unitary representation γ_Λ of $O_0(p,\infty)$ in $H_\Lambda(B_{p,\infty})$. In the space $H_\Lambda(B_{p,\infty})$ we consider a chain of subspaces

$$T_1 \subset T_2 \subset T_3 \subset \dots$$

such that T_n is invariant under $SO_0(p,n)$. It is clear that there are a lot of such chains (see, e.g., the discussion of inductive limits of groups in [59]). For example, it is possible to take $H_{\Lambda(n)}(B_{p,n})$ as T_n. This example is uninteresting since the representation of $SO_0(p,n)$ in $H_{\Lambda(n)}(B_{p,n})$ is strongly reducible. In Sec. 6.9 below we present a chain of T_n such that the representation of $SO_0(p,n)$ in T_n is irreducible. Note that while the existence of such chains for the representations ρ_Λ of $U(p,\infty)$ is obvious since the representations have the highest weight, there are no simple visible reasons for the existence of such chains of irreducible representations in the case of the group $O(p,\infty)$.

Remark. Assume that we are given an abstract chain of Hilbert spaces $T_1 \subset T_2 \subset T_3 \subset \dots$ with a representation σ_n of $O(p,n)$ on T_n such that

$$\sigma_n(g)v = \sigma_{n-1}(g)v \quad \forall v \in T_{n-1}, \quad g \in SO_0(p,n-1).$$

Then the group $\bigcup\limits_n SO_0(p,n)$ acts on the inductive limit $\widetilde{T}$ of the spaces T_j, but this representation is not weakly continuous, in general. The following question arises: is it true that if all σ_n are irreducible, then the limit representation is weakly continuous? The same question can be put somewhat differently. Suppose we are given a chain of irreducible representations σ_n of $SO_0(p,n)$ such that σ_{n-1} embeds into the restriction of σ_n to $SO_0(p,n-1)$. Is it true that, starting from some index, the chain σ_n coincides with one of the chains constructed below?

6.7. The representation γ_Λ as the inductive limit of the representations $\nu^\circ_{\Lambda(n)}$. Let a signature Λ satisfy the conditions of Sec. 6.2. Then, for sufficiently large n, $\Lambda(n)$ satisfies (4.21), and, thus, the representations $\nu^\circ_{\Lambda(n)}$ and the spaces $E^\circ_{\Lambda(n)}(M_{p,n})$ are defined. Here $M_{p,n}$ is the Stiefel manifold of p-frames in $\mathbb{R}^n$. Let $m < n$. We construct the canonical embedding

$$C_{m,n} : E^\circ_{\Lambda(m)}(M_{p,m}) \to E^\circ_{\Lambda(n)}(M_{p,n}).$$

Let φ be a bounded $V_{\Lambda(m)}$-valued function on $M_{p,m}$. Consider the $V_{\Lambda(n)}$-valued function $(1 \otimes L_{m,n})\varphi(z)$ on $M_{p,n}$ (here the operator $L_{m,n}$ is the same as in Sec. 6.1). The function $L_{m,n}\varphi$ can be regarded as a $V_{\Lambda(n)}$-valued distribution on $M_{p,n}$. The corresponding linear functional on C^∞ is given by the formula

$$\psi \mapsto \int\limits_{M_{p,m}} \langle (1 \otimes L_{m,n})\varphi(z), \overline{\psi(z)} \rangle_{V_{\Lambda(n)}} d\mu_{p,m}(z).$$

The required embedding $C_{m,n}$ is obtained (it is significant that $C_{m,n}\varphi$ is actually contained in
$E^\circ_{\Lambda(n)}(M_{p,n})$ (see Prop. 2.1). Obviously,

$$C_{m,n}\nu^\circ_{\Lambda(m)}(g) = \nu^\circ_{\Lambda(n)}(g)C_{m,n}; \quad g \in SO_0(p,m).$$

Thus, we have a chain of embeddings

$$\ldots \to E^\circ_{\Lambda(k)}(M_{p,k}) \xrightarrow{C_{k,k+1}} E^\circ_{\Lambda(k+1)}(M_{p,k+1}) \to \ldots .$$

We denote by E^* the inductive limit of this chain in the category of Hilbert spaces (i.e., the completion of the union $\cup E^\circ_{\Lambda(k)}(M_{p,k})$). By construction, E^* is $\cup SO_0(p,k)$-invariant subspace of $H_\lambda(B_{p,\infty})$. Therefore, $E^* \simeq H_\Lambda(B_{p,\infty})$, and the representation of $O_0(p,\infty)$ in E^* coincides with γ_Λ.

6.8. The representation γ_Λ as a limit of irreducible representations of $SO(p,n)$. Now, in each $E^\circ_{\Lambda(k)}(M_{p,k})$ we shall find an $SO(p,k)$-invariant subspace $I^\circ_{\Lambda(k)}$ such that the operator $C_{m,n}$ sends $I^\circ_{\Lambda(m)}$ to $I^\circ_{\Lambda(n)}$. This space is a space of generalized sections (distributions) of some $SO(p,k)$-invariant subbundle $F_{\Lambda(k)}$ of a trivial vector bundle $M_{p,k} \times V_{\Lambda(k)}$ over $M_{p,k}$.

Let $R^\circ_\Lambda(g,z)$ be the multiplicator of the representation $\nu^\circ_{\Lambda(k)}$ (see (4.23)). Let P be the stabilizer of $z_0 = (1_p\, 0) \in M_{p,k}$. Consider the representation

$$\tau(h) = R^\circ_{\Lambda(k)}(h, z_0)$$

of the group P associated to $R^\circ_{\Lambda(k)}$. Then $SO_0(p,k)$-invariant subbundles of $M_{p,k} \times V_{\Lambda(k)}$ are in one-to-one correspondence with the subspaces of $V_{\Lambda(k)}$ invariant under the representation τ of P. In $V_{\Lambda(k)}$, we consider the subspace $W_{\Lambda(k)}$ generated by all vectors of the form

$\tau(h)\delta_{\Lambda(k)}$, where $\delta_{\Lambda(k)}$ is the vector of the highest weight in $V_{\Lambda(k)}$. It is easy to see that $W_{\Lambda(k)}$ is a minimal P-invariant subspace of $V_{\Lambda(k)}$.

Let $F_{\Lambda(k)}$ be a subbundle of $M_{p,k} \times V_{\Lambda(k)}$, invariant under the group $SO_0(p,k)$ and corresponding to the subspace $W_{\Lambda(k)} \subset V_{\Lambda(k)}$. It can easily be seen that $C_{m,n}$ sends the generalized sections of $F_{\Lambda(m)}$ to those of $F_{\Lambda(n)}$. Denote by $I^\circ_{\Lambda(k)}$ the subspace of $E^\circ_{\Lambda(k)}(M_{p,k})$ consisting of generalized sections of $F_{\Lambda(k)}$. By Theorem 5.3, the representation of $SO_0(p,k)$ in $I^\circ_{\Lambda(k)}$ is irreducible. The required chain of irreducible representations is constructed.

Remark. "Tracing" the above construction shows that, for sufficiently large k, the representation of $SO_0(p,k)$ in $I^\circ_{\Lambda(k)}$ is a quotient representation of an induced representation of $SO_0(p,k)$. This latter is induced by the representation of P which is trivial on the nilradical, and has the form $\varkappa \otimes \sigma$ on the reductive part $GL(p,\mathbb{R}) \otimes SO(q-p)$ of P. Moreover, the representation $\varkappa$ of $GL(p,\mathbb{R})$ is given by the signature

$$(-(q-1)-\lambda_p, \dots, -(q-1)-\lambda_1),$$

and the representation σ of $SO(k-p)$ is given by the signature

$$(\lambda_{p+1}, \lambda_{p+2}, \dots, \lambda_k).$$

6.9. The representation γ_Λ as the inverse limit of $\nu_{\Lambda(n)}$. Let $m < n$, and let m be sufficiently large. Consider the map

$$C^*_{m,n} : E_{\Lambda(n)}(M_{p,n}) \to E_{\Lambda(m)}(M_{p,m})$$

adjoint to $C_{m,n}$. If $f \in E_{\Lambda(n)}(M_{p,n})$, then $C_{m,n}f$ is the restriction of the function $\left(1_{V_{\Lambda'}} \otimes (L^{\Lambda''}_{m,n})^*\right) f(z)$ to $M_{p,m} \subset M_{p,n}$ (cf. [60], see also the next section). Now we have a chain of projections

$$\dots \leftarrow E_{\Lambda(k)}(M_{p,k}) \xleftarrow{C^*_{k,k+1}} E_{\Lambda(k+1)}(M_{p,k+1}) \leftarrow \dots .$$

We can consider the representation of $\cup SO_0(p,k)$ in the inverse limit of this chain. It can easily be seen that this representation coincides with γ_Λ.

§7. Applications of the trace theorems and some problems in noncommutative harmonic analysis

A. Fourier analysis problems for complementary series of unitary rerpresentations of the group $SO_0(1,q)$.

7.1. The principal spherical series of representations of $SO_0(1,q)$. This series is defined as follows. Assume that $SO_0(1,q)$ acts on the sphere S^{q-1} as described in Sec. 2.4. For every $s \in \mathbb{R}$, we define the unitary representation of *principal series* $\sigma_s = \sigma_s^{(q)}$ of $SO_0(1,q)$ in $L^2(S^{q-1})$ by the formula

$$\sigma_s(g)f(x) = (a+xc)^{-(q-1)/2+is} f(x^g).$$

The representations σ_s and σ_{-s} are equivalent. The corresponding intertwining operator is given by the formula

$$K_s f(x) = \int\limits_{S^{q-1}} \frac{f(y)dy}{\|x-y\|^{q-1+2is}}.$$

In $SO_0(1,q)$, we consider the subgroup $SO(1,q-1)$ consisting of the matrices which leave fixed the last base vector in $\mathbb{R}^{q+1}$. Denote by ξ the restriction of $\sigma^{(q)}$ to $SO_0(1,q-1)$. On the sphere S^{q-1} (defined by the equation $x_1^2+\ldots+x_q^2=1$) the group $SO_0(1,q-1)$ has three orbits, namely, the equator Eq defined by $x_q=0$, and the hemispheres $\mathcal{D}_+$ and $\mathcal{D}_-$ defined by the inequalities $x_q>0$ and $x_q<0$, respectively. It is clear that

$$L^2(S^{q-1}) = L^2(\mathcal{D}_+) \oplus L^2(\mathcal{D}_-).$$

Further, the homogeneous spaces

$$\mathcal{D}_\pm = SO_0(1,q-1)/SO(q-1)$$

are $(q-1)$-dimensional Lobachevskii spaces. The restriction of ξ to $L^2(\mathcal{D}_\pm)$ has the form (4.1) and, thus, it is the representation induced by a one-dimensional representation of the isotropy subgroup $SO(q-1)$. But this subgroup has a unique one-dimensional representation, namely, the trivial one. Thus, we have proved the following statement.

Proposition 7.1. *The restriction of $\sigma_s^{(q)}$ to $SO_0(1,q-1)$ is independent of s. This representation is the sum of two copies of representations of $SO_0(1,q-1)$ in $L^2(SO_0(1,q-1)/SO(q-1))$.*

Thus, we have reduced the problem to the well-studied one (see [11; 6], §X.4; [14]). Recall that a representation in L^2on the Lobachevskii space is decomposable into a direct integral of multiplicity one of σ_s over all $s \geq 0$.

Now we consider the decomposition problem for the tensor product of the representations σ_s and σ_τ of $SO_0(1,q)$.

Proposition 7.2. *The representation $\sigma_s \otimes \sigma_\tau$ is independent of s and τ and it is equivalent to the representation of $SO_0(1,q)$ in*

$$L^2(SO_0(1,q)/(SO_0(1,1)\times SO(q-1))).$$

Proof. The group $SO_0(1,q)$ acts in $L^2(S^{q-1}\times S^{q-1})$ by the formula

$$T_{s,\tau}(g)f(x,y) = (a+xc)^{-\frac{q-1}{2}+is}(a+yc)^{-\frac{q-1}{2}+i\tau} f(x^g,y^g).$$

The unitary operator

$$A_{i\alpha} f(x,y) = \|x-y\|^{i\alpha} f(x,y) \tag{7.1}$$

intertwines the representations $T_{s,\tau}$ and $T_{s+\alpha,\tau+\alpha}$. On the other hand, σ_s is equivalent to σ_{-s}, hence , $T_{s,\tau} \simeq T_{-s,\tau}$. From these two remarks, it follows that $T_{s,\tau}$ is independent of s and τ. Further, $SO_0(1,q)$ has two orbits in $S^{q-1}\times S^{q-1}$, namely, the diagonal

$\Delta = S^{q-1} \subset S^{q-1} \times S^{q-1}$ and the complement C with respect to the diagonal. It is clear that $L^2(S^{q-1} \times S^{q-1}) = L^2(C)$ and, for $s = t = 0$, the representation under consideration is the natural representation of $SO_0(1, q)$ in

$$L^2(C) = L^2(SO_0(1, q)/(SO_0(1, 1) \times SO(q - 1)).$$

Thus, this problem is also reduced to a standard one, namely, to the decomposition of L^2 on a symmetric pseudo-Riemannian space. □

Proposition 7.3. *The tensor product of the representation σ_s of the principal spherical series and the representation ν_τ of the complementary spherical series is independent of s and τ and is equivalent to the tensor product of two representations of the principal spherical series.*

Lemma 7.1. *Let $P \subset SO_0(1, q)$ be the stabilizer of a point of S^{q-1}. Then the restrictions of σ_s and ν_τ to P are irreducible and equivalent.*

Proof of Lemma 7.1. We apply the stereographic projection $S^{q-1} \to \mathbb{R}^{q-1}$. Then P can be regarded as the group of conformal transformations of $\mathbb{R}^{q-1}$ (i.e., P is generated by the rotation group $SO(q)$, the group $\mathbb{R}^{q-1}$ of translations, and the group $\mathbb{R}^*$ of similarity transformations). The group P acts on the space of functions on $\mathbb{R}^{q-1}$ by the operators

$$f(x) \mapsto f(x^g)\lambda(g)^\alpha, \tag{7.2}$$

where λ is a similarity ratio, $\alpha = (q - 1)/2 + is$ for σ_s and $\alpha = (q - 1)/2 - \tau$ for ν_τ.

We apply the Fourier transform on $\mathbb{R}^{q-1}$. Then the shift operators transform into operators of multiplication by functions of the form $\exp(\sum a_j x_j)$. Therefore, if an operator Q commutes with the given representation, then Q must commute with the multiplications by $\exp(\sum a_j x_j)$. Hence, Q is an operator of multiplication by a function which is invariant under rotations and homotheties. Hence Q is a scalar operator.

Thus, the representations in question are irreducible. The intertvining operator for two representations of the form (7.2) is a convolution with a function of the form $|x|^\lambda$ for an appropriate λ. □

Proof of Proposition 7.3. Let H be a subgroup of G. Denote by $\mathrm{IND}_H^G(\rho)$ the representation of G induced (in the sense of unitary representations) by a representation ρ of H (see [9], §13). Denote by $\mathrm{RES}_H^G(\pi)$ the restriction to H of a representation π of G. Recall the following obvious relation

$$\mathrm{IND}_H^G(\rho) \otimes \pi = \mathrm{IND}_H^G(\rho \otimes \mathrm{RES}_H^G(\pi)), \tag{7.3}$$

where ρ is a unutary representation of H and π is a unitary representation of G. Now the statement of Prop. 7.3 is obvious since $\sigma_s \otimes \sigma_{s'}$ and $\sigma_s \otimes \nu_\tau$ are induced by the same representation of the parabolic subgroup P. □

7.2. The restriction problem for representations of the complementary series. We preserve the notation of Sec. 2.4, 2.6 for the spaces $E_s(S^{q-1})$ and $E_s^\circ(S^{q-1})$ and the representations ν_s and ν_s°.

Consider the representation ν_s° of the complementary spherical series of $SO_0(1-q)$. Let k be the largest integer such that

$$s + k < q/2 - 1.$$

Theorem 7.1. *The restriction of* ν_s° *to* $SO_0(1,q-1)$ *is the direct sum of the representation*

$$\nu_s^\circ \oplus \nu_{s+1}^\circ \oplus \ldots \oplus \nu_{s+k}^\circ \tag{7.4}$$

of $SO_0(1,q-1)$ *and the restriction to* $SO_0(1,q)$ *of a representation of the principal spherical series* $SO_0(1,q)$.

Remark. If $s > q/2-1$, then the sum (7.4) is empty.

First, we present discrete components of (7.4). Let δ_{Eq} denote the delta-function of the equator $Eq \subset S^{q-1}$, i.e., the distribution on S^{q-1} defined by the formula

$$\int_{S^{q-1}} \psi(x)\delta_{Eq}(x)dx = \int_{Eq} \psi(y)dy; \quad \psi \in C^\infty(S^{q-1}).$$

We denote by $\mathcal{F}_0$ the space of distributions on S^{q-1} of the form $\varphi(x)\delta_{Eq}$, where $\varphi(x) \in C^\infty$. Let $\vec{n}$ be the field of unit normals to the equator (for definiteness, directed toward the upper hemisphere $\mathcal{D}_+$). On S^{q-1} we consider the space $\mathcal{F}_j$ of distributions of the form

$$\varphi_0(x)\delta_{Eq} + \varphi_1(x)\frac{\partial}{\partial\vec{n}}\,\delta_{Eq} + \ldots + \varphi_j(x)\frac{\partial^j}{\partial\vec{n}^j}\,\delta_{Eq}; \quad \varphi_\alpha \in C^\infty. \tag{7.5}$$

We also can say that $\mathcal{F}_j$ is the space of derivatives of order $\le j$ along the smooth vector fields of functions from $\mathcal{F}_0$.

Proposition 7.4. *The space* $E_s^\circ(S^{q-1})$ *contains* $\mathcal{F}_k$.

Proof. This follows from Prop. 2.3. Moreover, this statament is a special case of the trace theorem for Sobolev spaces (see [8], §3.2 and [26], §1.3). □

It is obvious that the spaces $\mathcal{F}_j$ are $SO_0(1,q-1)$-invariant. Let $\overline{\mathcal{F}_j}$ be the closure of $\mathcal{F}_j$ in $E_s^\circ(S^{q-1})$, which means that the coefficients of φ_j in (7.5) are functions from $E_{s+j}^\circ(S^{q-2})$).

It is obvious that the representation of $SO_0(1,q-1)$ in $\mathcal{F}_0$ is exactly ν_s°. Now we consider the following representation of $SO_0(1,q-1)$ in $\mathcal{F}_j/\mathcal{F}_{j-1}$:

$$\nu_s^\circ(g)\left(\psi(x)\frac{\partial^j}{\partial\vec{n}^j}\,\delta_{Eq}\right) = (a+xc)^{-(q-1)+s}\psi(x^g)\left(\left(\frac{\partial}{\partial\vec{n}}\right)^j \delta_{Eq}\right)(x^g).$$

Since the transformations $x \mapsto x^g$ are conformal and the coefficient of dilatation is $(a+xc)^{-1}$, we can transform the right-hand side of the last equation to the form

$$(a+xc)^{-(q-1)+s}\psi(x^g)(a+xc)^j\left(\frac{\partial}{\partial\vec{n}}\right)^j \delta_{Eq}(x) + \ldots,$$

where dots stand for the terms from $\mathcal{F}_{j-1}$. Thus, we see that the representation of $SO_0(1,q-1)$ in $\mathcal{F}_j/\mathcal{F}_{j-1}$ is equivalent to ν_{s+j}^o.

7.3. Dual description of the discrete spectrum. Instead of ν_s°, we consider the representation ν_s equivalent to ν_s°. Let k be the same as in Sec. 7.2. It is easy to show that, for $f \in E_s(S^{q-1})$, the restrictions of $\left(\frac{\partial}{\partial \vec{n}}\right)^j f$ to E_q are well defined if $j \le k$.

Let $j \le k$. Denote by Q_j the space of all $f \in E_s(S^{q-1})$ such that

$$f(y) = \frac{\partial}{\partial \vec{n}} f(y) = \ldots = \left(\frac{\partial}{\partial \vec{n}}\right)^j f(y) = 0$$

identically on E_q.

Then, in $E_s(S^{q-1})$, we have the filtration

$$E_s(S^{q-1}) \supset Q_0 \supset Q_1 \supset \ldots \supset Q_k.$$

Now the representation ν_{s+j} of $SO_0(1, q-1)$ is realized in Q_j/Q_{j-1}.

7.5. Separation of the discrete spectrum. We must complete the proof of Theorem 7.1. The main point of the proof is very simple. The operator

$$B_s f(x) = |x_q|^{(q-1)/2-s} f(x)$$

intertwines the restriction of ν_s to $SO_0(1, q-1)$ with the restriction of $\sigma_0 \simeq \nu_{(q-1)/2}$ to $SO_0(1, q-1)$. The kernel of this operator consists exactly of the distributions supported by Eq.

However, B_s is the operator of multiplication by a nonsmooth function in the space of distributions. This operator is unbounded. Hence the above argument needs clarification.

In the Hilbert space $E_s^\circ(S^{q-1}) \oplus L^2(S^{q-1})$, we consider the subspace K which consists of the pairs $(f,$
$|x_q|^{(q-1)/2-s} f)$ with $f \in C^\infty(S^{q-1})$. Let $\bar{K}$ be the closure of K. Consider the projection operators $Q : \bar{K} \to E_s^\circ(S^{q-1})$ and $R : \bar{K} \to L^2(S^{q-1})$.

The following lemma implies Theorem 7.1.

Lemma 7.2. (a) *The operator Q is injective and has a dense image.*

(b) *The operator R has a dense image and the kernel of R consists of the pairs $(\psi, 0) \in \bar{K}$ such that ψ is a distribution supported by Eq.*

Proof. Both statements concerning the image are obvious. We prove (a). Let $\varphi_i \in C^\infty(S^{q-1})$. Let $\varphi_i \to 0$ in $E_s^\circ(S^{q-1})$ (hence $\varphi_i \to 0$ in the sense of distributions), and let $|x_q|^{(q-1)/2-s}\varphi_i \to \psi$ in L^2. Consider a function $\theta \in C^\infty(S^{q-1})$ which vanishes in the neighborhood of Eq. Then $\theta\varphi_i \to 0$ in the sense of distributions, and $|x_q|^{(q-1)/2-s}\varphi_i \to 0$ in L^2. Hence $\theta\psi = 0$ in L^2 for all θ, and $\psi = 0$ in L^2.

Now we prove (b). Let $\varphi_i \to \psi$ in E_s° and $|x_q|^{(q-1)/2-s}\varphi \to 0$ in L^2. Let θ be as above. Then $\varphi_i\theta \to \psi\theta$ in the sense of distributions, and $|x_q|^{q-1}\varphi_i\theta \to 0$ in L^2. Therefore, $\varphi_i\theta \to 0$ in L^2, hence $\theta\psi = 0$. Thus, the support of ψ is a subset of Eq. $\square$

7.6. The limit of the representations ν_s° of $SO_0(1, n)$ as $n \to \infty$. For the representations ν_s° of the complementary spherical series of $SO_0(1, n)$, the construction of Sec. 6.7 is especially transparent.

We have a chain of canonical embeddings

$$\ldots \to E_s^\circ(S^{n-1}) \to E_s^\circ(S^n) \to \ldots . \tag{7.6}$$

Namely, to any function f on S^{n-1} we assign the distribution $f \cdot \delta_{Eq(n)}$ on S^n (here $Eq(n)$ denotes the equator of S^n). Let E_s^* be the limit of the chain (7.6) (i.e., the completion of the union of $E_s^\circ(S^n)$). By construction, the group $\cup SO_0(1,q)$ acts on E_s^*. We denote by ν_s^* the corresponding limit representation.

Theorem 7.2. *The representation ν_s^* can be extended to a representation of $O_0(1,\infty)$, continuous in the weak topology. Here $O_0(1,\infty)$ denotes the subgroup of $O(1,\infty)$ consisting of the matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a > 0$.*

Proof. It is sufficient to prove that the given representation is weakly continuous on the subgroup $G_0 := \cup SO(n)$. Let G_n denote the subgroup of G_0 consisting of the matrices leaving fixed the first n basis vectors. In [21], the following criterion of weak continuity of unitary representations of G_0 is obtained.

Proposition 7.5. *Let ρ be a unitary representation of the group $\bigcup_{n=1}^{\infty} SO(n)$ in a space H. Let $H^{(n)}$ be the set of vectors $h \in H$ invariant under G_n. The following stataments are equivalent:*

(a) *ρ is continuous in the weak topology on $\bigcup_{n=1}^{\infty} SO(n)$,*

(b) *$\bigcup_{n=1}^{\infty} H^{(n)}$ is dense in H.*

(For the proof, see [21], Sec. 1.4, 3.2).

Let $H = E_s^*$. It can easily be seen that $H^{(n)}$ is invariant under $SO_0(1,n)$. Therefore, $\cup H^{(n)}$ is invariant under $\cup SO_0(1,n)$. Clearly, ν_s^* is irreducible. Therefore, either $\cup H^{(n)}$ reduces to 0, or $\cup H^{(n)}$ is dense in H. Thus, it is sufficient to find at least one vector in $\cup H^{(n)}$.

Let $\xi_n \in E_s^\circ(S^{n-1})$ be a function of the form $\lambda_n \cdot 1$, where $\lambda_n > 0$, and let $\|\xi_n\| = 1$. It follows from Lemma 7.3 below that the sequence ξ_n converges in $H = E_s^*$. The limit σ of this sequence is obviously $\cup SO(n)$-invariant. Hence $\cup H_n$ is nonempty. $\square$

Lemma 7.3. *Let $k > n$. Then*

$$\langle \xi_n, \xi_k \rangle = \left(\frac{c_k}{c_n} \right)^{1/2},$$

where

$$c_m = \frac{\Gamma(m/2)\Gamma((m-1)/2 - s)}{\Gamma((m-s)/2)\Gamma((m-1-s)/2)}.$$

Proof. It suffices to calculate the following simple integral

$$I_{k,n} = \int\limits_{S^{k-1}} \int\limits_{S^{n-1} \subset S^{k-1}} \frac{dx\, dy}{\|x-y\|^{2s}} = \sigma_k \sigma_n c_k,$$

where σ_m is the area of S^{m-1}. We have

$$\langle \xi_n, \xi_k \rangle = I_{k,n}/(I_{k,k} \cdot I_{n,n})^{1/2}. \quad \square$$

7.7. A general remark. We want to formalize the method that we used when presenting the discrete spectrum in Secs. 7.2, 7.3.

Assume that a Lie group G acts on a manifold B by transformations $x \mapsto x^g$, $\mathcal{L}$ is a vector bundle over B, and $\pi : \mathcal{L} \to B$ is the canonical projection (the preimages of points are linear spaces). Suppose that G acts on $\mathcal{L}$ by transformations $l \mapsto l^g$ in such a way that fibers go to fibers, i.e.,

$$\pi l^g = (\pi l)^g; \quad l \in \mathcal{L}, \quad g \in G,$$

and the map $l \mapsto l^g$ from the fiber over x into that over x^g is linear.

Let $s : B \to \mathcal{L}$ be a section of the vector bundle (i.e., a function s such that $\pi(s(x)) = x$). Define the section $\rho(g)s$ by the formula

$$(\rho(g)s)x = [s(x)]^g.$$

Thus, we have a representation of G in the sections of the vector bundle.

Remark. All the above is simply a different language for describing the multiplicator representations (see Sec. 4.0). If we regard sections of a vector bundle as functions on B, then we obtain a representation given by a formula of the form (4.1). For more details, see [9], §13.

Assume that M is a compact G-invariant submanifold of B, $\mathcal{K}$ is the restriction of the bundle $\mathcal{L}$ to M, and $\mathcal{N}$ is the conormal vector bundle on M (i.e., to any $x \in M$ the quotient space of the tangent space to B at x modulo the tangent space to M at x is assigned). The group G acts on $\mathcal{K}$ and $\mathcal{N}$ in a natural way. Therefore, G acts in any linear bundle obtained from $\mathcal{K}$ and $\mathcal{N}$ by tensor operations.

Let $C^\infty(\mathcal{L})$ be the space of all C^∞-sections of $\mathcal{L}$. Denote by Q_j the space of all C^∞-sections of $\mathcal{L}$ having a zero of order $\geq j$ at all points of M. In $C^\infty(\mathcal{L})$, we obtain a filtration

$$C^\infty(\mathcal{L}) = Q_0 \supset Q_1 \supset \dots$$

which is invariant under G. Obviously, we have an isomorphism

$$Q_j/Q_{j+1} \simeq \{\text{the space of} \quad C^\infty\text{-sections of} \quad \mathcal{K} \otimes S^j \mathcal{N}\},$$

where $S^j\mathcal{N}$ is the j-th symmetric power of $\mathcal{N}$. It is easily seen that the representation of G in Q_j/Q_{j+1} coincides with the representation of G in sections of $\mathcal{K} \otimes S^j\mathcal{N}$.

Assume that M is a G-homogeneous space (this is the case of our interest). Let $x_0 \in M$ and let $P \subset G$ be the stabilizer of x_0. Let $\varkappa$ be the natural representation of P in the fiber of $\mathcal{K}$ over x_0 (this fiber is P-invariant), and let ν be the representation of P in the fiber of $\mathcal{N}$ over x_0. It is not difficult to see that the representation of G in Q_j/Q_{j+1} is simply the representation induced from the representation $\varkappa \otimes S^j\nu$ (here $S^j\nu$ denotes the jth symmetric power of ν).

Now we describe the dual construction. Each smooth section of $\mathcal{K}$ over M can be regarded as a generalized delta-like section of $\mathcal{L}$ over B. Let $\mathcal{F}_0$ be the space of all generalized sections of $\mathcal{L}$ with support in M which can be obtained in this way. Let $\mathcal{F}_j$ be the set of partial derivatives of order $\leq j$ of functions from $\mathcal{F}_0$. We come to a G-invariant filtration

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots .$$

We obviously have a homomorphism

$$\mathcal{F}_j/\mathcal{F}_{j-1} \simeq \{\text{the space of } C^\infty\text{-sections in } \mathcal{K}' \otimes S^j \mathcal{N}'\},$$

where $\mathcal{K}'$ and $\mathcal{N}'$ denote the dual vector bundles for $\mathcal{K}$ and $\mathcal{N}$, respectively.

If M is homogeneous, then the representation of G in $\mathcal{F}_j/\mathcal{F}_{j-1}$ is induced by the representation of P given by the formula

$$J \otimes \varkappa' \otimes S^j \nu'.$$

Here, $\varkappa'$ and ν' denote the representations of P dual to $\varkappa$ and ν, respectively, and J is the Jacobian of the map $x \mapsto x^g$ $(g \in P)$ at the point x_0.

The above remarks are obvious. They become interesting only in the case where the trace theorem is valid. In a special case, this method was applied in the paper by Jakobsen and Vergne [60] (namely, they studied some problems concerning decompositions of tensor products of representations of $U(p,q)$ with highest weight, which corresponds to the situation described in Example 2, Sec. 1.9 above).

7.8. Tensor products of representations of the complementary series. Consider the representation $\nu_s^\circ \otimes \nu_\tau^\circ$ of $SO(1,q)$ in $E_s^\circ \otimes E_\tau^\circ$. The space $E_s^\circ \otimes E_\tau^\circ$ is a space of functions on $S^{q-1} \times S^{q-1}$ with the scalar product

$$\langle f_1, f_2 \rangle = \iiiint\limits_{(S^{q-1})^4} \frac{f_1(x_1, y_1)\overline{f_2(x_2, y_2)}\, dx_1\, dx_2\, dy_1\, dy_2}{\|x_1 - x_2\|^{2s} \cdot \|y_1 - y_2\|^{2\tau}}.$$

The group $SO_0(1,q)$ acts on $E_s^\circ \otimes E_\tau^\circ$ by the formula

$$(\nu_s^\circ \otimes \nu_\tau^\circ)(g) f(x,y) = (a + xc)^{-s}(a + yc)^{-\tau} f(x^g, y^g).$$

As it is shown in Sec. 2.6, the elements of the spaces E_s° and E_τ° can be regarded as distributions, and the spaces E_s° and E_τ° themselves are Sobolev spaces. Therefore, the elements of $E_s^\circ \otimes E_\tau^\circ$ can be regarded as distributions on $S^{q-1} \times S^{q-1}$. (We emphasize that $E_s^\circ \otimes E_\tau^\circ$ is not a Sobolev space.)

Proposition 7.6. *Let ψ be a distribution on $S^{q-1} \times S^{q-1}$. Then the function*

$$\gamma(c) = 2^{s+\tau} \iiiint\limits_{(S^{q-1})^4} \frac{\psi(x_1, y_1)\overline{\psi(x_2, y_2)}\, dx_1\, dx_2\, dy_1\, dy_2}{(1 - c\sum x_1^{(j)} x_2^{(j)})^{2s}(1 - c\sum y_1^{(j)} y_2^{(j)})^{2\tau}}$$

increases if $0 < c < 1$. *We have* $\psi \in E_s^\circ \otimes E_\tau^\circ$ *if and only if there exists a finite limit* $\lim_{c\to 1-0} \gamma(c)$. *Moreover,*

$$\|\psi\|^2_{E_s^\circ \otimes E_\tau^\circ} = \lim_{c\to 1-0} \gamma(c).$$

Proof. Consider the Hilbert space of holomorphic functions on $B_q \times B_q$ defined by the positive definite kernel

$$K_{s,\tau}(z_1, z_2; \bar{u}_1, \bar{u}_2) = (1 - \sum z_1^{(j)} \overline{u_1^{(j)}})^{-s} (1 - \sum z_2^{(j)} \overline{u_2^{(j)}})^{-\tau}.$$

Consider the submanifold $S^{q-1} \times S^{q-1}$ of the boundary of $B_q \times B_q$. The boundary Hilbert space E° associated with $S^{q-1} \times S^{q-1}$ (see Secs. 2.1, 2.2) is $E_s^\circ \otimes E_\tau^\circ$. Now the monotonicity of $\gamma(c)$ is a special case of Prop. 1.5. The second statement of Prop. 7.6 follows from Prop. 2.1. □

We denote by $\mathcal{F}_0$ the space of distributions on $S^{q-1} \times S^{q-1}$ of the from $\varphi \cdot \delta_\Delta$, where δ_Δ is the delta function of the diagonal $\Lambda \subset S^{q-1} \times S^{q-1}$, and $\varphi \in C^\infty$. Let $\mathcal{F}_j$ be the space of partial derivatives of order $\leq j$ of functions from $\mathcal{F}_0$.

Let $s + \tau < (q-1)/2$, and let k be the largest integer such that $s + \tau + k < (q-1)/2$.

Proposition 7.7. *The space* $E_s^\circ \otimes E_\tau^\circ$ *contains* $\mathcal{F}_k$.

Proof. This follows from Prop. 7.6. □

The representation of $SO_0(1,q)$ in $\mathcal{F}_k$ can easily be described by means of Remarks from Sec. 7.7 (this representation is a sum of a finite number of representations of $SO_0(1,q)$ from the complementary series (not only spherical ones)).

Theorem 7.3. *The representation* $\nu_s^\circ \otimes \nu_\tau^\circ$ *is a direct sum of two representations, namely, the representation of* $SO_0(1,q)$ *in* $\mathcal{F}_k$ *(this summand is present only if* $s + \tau < (q-1)/2$*) and a representation equivalent to the tensor product of representations of the principal series.*

Proof. Let $s \geq \tau$. Consider the operator

$$A_{\frac{q-1}{2}-s} f(x,y) = \|x - y\|^{(q-1)/2-s} f(x,y)$$

(cf. Sec. 7.1) intertwining $\nu_s^\circ \otimes \nu_\tau^\circ$ with $\sigma_0 \otimes \nu_{\tau+(q-1)/2-s}$. The kernel of this operator is the closure of $\mathcal{F}_k$, and the image is dense in $L^2 \otimes E^\circ_{\tau+(q-1)/2-s}$ (to justify these assertions one should repeat the reasoning from Sec. 7.5). Finally, we apply Prop. 7.3. □

7.9. Historical notes. Theorem 7.3 was discovered by Pukansky [45] in the case $q = 2$ ($SO_0(1,2) \simeq PSL_2(\mathbb{R})$), and by Naimark [15] in the case $q = 3$ ($SO_0(1,3) \simeq PSL_2(\mathbb{C})$). Theorem 7.1 was obtained by Boyer [33]. The simple interpretation of the discrete series was not observed in the above papers. The statements of the present section were announced in [15].

B. Singular unitary representations of $SO_0(p,q)$.

7.10. The restriction to the subgroup $SO_0(p, q-1)$, and tensor products. Note that the arguments which enabled us to present the discrete spectra in Sec. 7.2, 7.3 remain valid for the representations of class $\mathcal{B}$ (see Sec. 5.6). All we need is the trace theorem. For $E^{\circ}(M)$ this theorem is stated in Sec. 3.11, and for $E^{\circ}_{\Lambda}(M) \otimes E^{\circ}_{\widetilde{\Lambda}}(M)$ it can be justified exactly as for $E^{\circ}_{s}(S^{q-1}) \otimes E^{\circ}_{\tau}(S^{q-1})$, see Prop. 7.6. The method described in Sec. 7.7 makes it possible to obtain explicit formulas.

On the other hand, we do not know whether the method of separating the discrete spectrum (see Sec. 7.5) can be applied in this case.

7.11. The Gelfand–Tsetlin basis. Let ρ be a representation of class $\mathcal{B}$. It is easy to see that the restriction of ρ to $SO_0(q)$ has a spectrum of finite multiplicity. This fact is rather surprising since, by general considerations, the spectrum of the restriction of a representation of $SO_0(p, q)$ to the maximal compact subgroup $SO(p) \times SO(q)$ is of finite multiplicity. Therefore, for any intermediate subgroup H such that $SO_0(p, q) \supset H \supset SO(q)$, the spectrum of the restriction of ρ to H is also of finite multiplicity and discrete. In fact, the spectrum of the restriction of ρ to $SO_0(p-1, q)$ is simple and consists of representations of the same type. Therefore, for ρ, we have the Gelfand–Tsetlin basis with respect to the following sequence of subgroups

$$SO_0(p, q) \supset SO_0(p-1, q) \supset \ldots \supset SO(q) \supset SO(q-1) \supset \ldots .$$

For details, see [22].

C. Singular representations of $\mathrm{Sp}\,(p, q)$ and $U(p, q)$.

7.12. The groups $\mathrm{Sp}\,(p, q)$. Consider the matrix ball $B_{2p,2q}$ and the representation ρ_{Λ} of $U(2p, 2q)$ with highest weight (see Secs. 4.3–4.4). In the Shilov boundary of $B_{2p,2q}$, we consider the subset $M_{\mathbb{H}}$ which consists of the block matrices

$$h = \begin{pmatrix} h_{11} & \ldots & h_{1q} \\ \ldots & \ldots & \ldots \\ h_{p1} & \ldots & h_{pq} \end{pmatrix}, \quad h^* h = 1_{2p},$$

where each block h_{kl} is a 2×2 matrix of the form

$$h_{kl} = \begin{pmatrix} a_{kl} & b_{kl} \\ -\overline{b_{kl}} & \overline{a_{kl}} \end{pmatrix}.$$

In other words, h_{kl} is a quaternion in the complex form. The manifold $M_{\mathbb{H}}$ (this manifold is nothing but the quaternion complex Stiefel manifold) is invariant under the subgroup $\mathrm{Sp}\,(p, q) \subset U(2p, 2q)$. Now we can simply repeat the construction of §§3–5.

7.13. The groups $U(p, q)$. Consider the representation $\rho_{\Lambda} \otimes \rho_{\widetilde{\Lambda}}$ of $U(p, q) \otimes U(p, q)$. This representation is realized in vector-valued holomorphic functions on $B_{p,q} \times B_{p,q}$. On the Shilov boundary of $B_{p,q} \times B_{p,q}$, we consider the subset $M_{\mathbb{C}}$ consisting of all points of the form

$$(z, \bar{z}) \in \bar{B}_{p,q} \times \bar{B}_{p,q}; \quad z^* z = 1_p.$$

Let $G \simeq U(p,q)$ be the subgroup in $U(p,q)\times U(p,q)$ consisting of the matrices $(g,\bar g)$. It is clear that the complex Stiefel manifold $M_{\mathbb C}$ is G-invariant. Thus, we can repeat the reasoning from §§3–4.

Remark. For the representation $\rho_\Lambda\otimes\rho_{\widetilde\Lambda}$ of the group $U_{p,q}\times U_{p,q}$, the problem of restriction to the subgroup G is equivalent to the problem of decomposing of the tensor product of the representation ρ_Λ of $U(p,q)$ with highest weight and the representation $\bar\rho_{\widetilde\Lambda}$ with lowest weight.

7.14. Restriction to a subgroup. Argument connected with the trace theorem enables us to present also a finite number of discrete components for the following problems on restricting representations of class $\mathcal B$ to a subgroup:

$$U(2p,2q)\to \mathrm{Sp}\,(p,q),\quad U(p,q)\to O(p,q),$$
$$O(2p,2q)\to U(p,q),\quad \mathrm{Sp}\,(p,q)\to U(p,q).$$

§8. Remarks. Relations with other spectral problems.

A great number of papers are devoted to numerous spectral problems like the problem of restriction of a representation to a subgroup, decomposition of a tensor product, decomposition of an induced representation. We shall discuss relationship between the present paper and the papers of other authors on this subject.

8.1. The harmonic representation of $\mathrm{Sp}\,(2n,\mathbb R)$. (For more details, see [3, 39, 18], and [43]; the other names: *"the Weyl representation," "the Segal–Shale–Weyl representation," "the oscillator representation"*). It is convenient to regard $\mathrm{Sp}\,(2n,\mathbb R)$ as the group of complex block matrices of size $(n+n)\times(n+n)$ written in the form

$$g=\begin{pmatrix}\phi&\psi\\ \bar\psi&\bar\phi\end{pmatrix} \tag{8.1}$$

and satisfying the conditions

$$g\begin{pmatrix}0&1\\-1&0\end{pmatrix}g^t=\begin{pmatrix}0&1\\-1&0\end{pmatrix};\quad g\begin{pmatrix}1&\\&-1\end{pmatrix}g^*=\begin{pmatrix}1&\\&-1\end{pmatrix}.$$

In particular, $\mathrm{Sp}\,(2n,\mathbb R)\subset U(n,n)$.

Denote by F_n the boson Fock space with n degrees of freedom, i.e., the Hilbert space of holomorphic functions on $\mathbb C^n$ with the scalar product

$$\langle f,g\rangle=\int_{\mathbb C^n} f(z)\overline{g(z)}\exp(-\sum|z_j|^2)dz\,d\bar z.$$

The corresponding positive definite kernel is equal to

$$K(z,\bar u)=\exp(\sum z_j\bar u_j).$$

To a matrix $g \in \mathrm{Sp}\,(2n, \mathbb{R})$ of the form (81), we assign the integral operator

$$we_{2n}(g)f(z) = \int\limits_{\mathbb{C}^n} A(z, \bar{u}) f(u) \exp(-|u|^2) du d\bar{u}$$

with the kernel

$$A(z, \bar{u}) = \det(\phi)^{-\frac{1}{2}} \exp\left\{ \frac{1}{2}(z, \bar{u}) \begin{pmatrix} \bar{\psi}\phi^{-1} & \phi^{t-1} \\ \phi^{-1} & -\phi^{-1}\psi \end{pmatrix} \begin{pmatrix} z^t \\ \bar{u}^t \end{pmatrix} \right\}.$$

A direct calculation shows that $we_{2n}(g)$ is a unitary projective representation of $\mathrm{Sp}\,(2n, \mathbb{R})$. $\blacksquare$

It is easy to see that the spaces of even and odd functions (we denote them by F_+ and F_-, respectively) are invariant under $\mathrm{Sp}\,(2n, \mathbb{R})$. For definiteness, we consider the representations of $\mathrm{Sp}\,(2n, \mathbb{R})$ in F_+.

First, recall that the Cartan domain of the third type C_n is the space of symmetric complex matrices of size $n \times n$ with norm < 1. The group $\mathrm{Sp}\,(2n, \mathbb{R})$ acts on C_n by the biholomorphic transformations

$$w \mapsto w^g := (\phi + w\bar{\psi})^{-1}(\psi + w\bar{\phi}).$$

To each $w \in C_n$ we assign the function

$$\varphi_w(z) = \exp\left\{ \frac{1}{2} \sum \bar{w}_{kl} z_k z_l \right\} \in F_n.$$

Consider the operator J from F_+ to the space of holomorphic functions on C_n defined by the formula

$$(Jf)(w) = \langle f, \varphi_w \rangle_{F_+}.$$

It can easily be checked that J is a unitary operator from F_+ to the space $H_k(C_n)$ of holomorphic functions on C_n given by the positive definite kernel

$$K(w, \xi) = \det{}^{-1/2}(1 - w\bar{\xi}). \tag{8.2}$$

We note that functions from this space satisfy rigid conditions, namely, the system of partial differential equations

$$\left(\frac{\partial^2}{\partial w_{kl} \partial w_{mj}} - \frac{\partial^2}{\partial w_{kj} \partial w_{lm}} \right) f = 0.$$

The action of the group $\mathrm{Sp}\,(2n, \mathbb{R})$ on $H_k(C_n)$ is defined by the formula

$$g : f(\omega) \mapsto (\phi + w\bar{\psi})^{-1/2} f(w^g).$$

Now we are in a situation very close to that of §3, and the exponent $1/2$ in the formula (8.2) is very small. Therefore, this opens up many possibilities to present discrete spectra in the restriction problems for $\mathrm{Sp}\,(2n, \mathbb{R})$ and various subgroups.

8.2. Restriction to $O(p,q)$. Consider the natural embeddings

$$O(p,q) \to \mathrm{Sp}\,(2n,\mathbb{R}) \times O(p,q) \to \mathrm{Sp}\,(2k(p+q),\mathbb{R}). \tag{8.3}$$

We explain the second embedding. The group $\mathrm{Sp}\,(2n,\mathbb{R})$ acts on $\mathbb{R}^{2k}$ preserving the skew-symmetric bilinear form, and $O(p,q)$ acts on $\mathbb{R}^{p+q}$ preserving the symmetric bilinear form. Therefore, $\mathrm{Sp}\,(2n,\mathbb{R}) \times O(p,q)$ acts on $\mathbb{R}^{2k} \otimes \mathbb{R}^{p+q}$ preserving the tensor product of these forms, i.e., a skew-symmetric bilinear form.

We also consider the embeddings

$$U(p,q) \to U(k) \times U(p,q) \to \mathrm{Sp}\,(2k(p+q),\mathbb{R})). \tag{8.4}$$

It is obvious that the restriction of $we_{2k(p+q)}$ to $U(p,q)$ is a direct sum of representations with the highest weight. It follows from [39, 20, 35, 67, 68] that all unitary representations ρ_Λ of $U(p,q)$ with the highest weight and integral signature Λ can be obtained in this way. Denote by Σ^{int} the set of all signatures $\Lambda \in \Sigma$ of the form (4.19) with integer λ_j.

Further, we note that the composition of the embedding $O(p,q) \to U(p,q)$ with the embedding (8.4) is the embedding (8.3). Thus, the problem of restriction of $we_{2p(k+q)}$ to $O(p,q)$ is closely connected with the problem of restriction of ρ_Λ to $O(p,q)$ for $\Lambda \in \Sigma^{\mathrm{int}}$.

Let π be a unitary representation of a group G. Denote by $\mathrm{Spec}\,\pi$ the set of all irreducible representations of G weakly contained in π. Let, as above, $\mathrm{RES}\,^G_H(\pi)$ denote the restriction of π to H.

In this notation, we have

$$\bigcup_{k=0}^{\infty} \mathrm{Spec}\,(\mathrm{RES}\,^{\mathrm{Sp}\,(2k(p+q),\mathbb{R})}_{\mathcal{O}(p,q)}(we_{2k(p+q)})) = \bigcup_{\Lambda\in\Sigma^{\mathrm{int}}} \mathrm{Spec}\,(\mathrm{RES}\,^{U(p,q)}_{O(p,q)}(\rho_\Lambda)). \tag{8.5}$$

A similar relation holds for discrete spectra. The discrete spectrum in the left-hand side of (8.5) was studied in the paper by Adams [31].

8.3. The hyperboloids $O(m,n)/O(m-1,n)$. Consider the problem of restriction of $we_{2(p+q)}$ to $O(p,q)$ from another point of view. To this end, we consider the sequence of subgroups

$$\mathrm{Sp}\,(2(p+q),\mathbb{R}) \supset SL(p+q,\mathbb{R}) \supset O(p,q)$$

of the symplectic group. The restriction of the harmonic representation $we_{2(p+q)}$ to $SL(p+q,\mathbb{R})$ is the natural representation of $SL(p+q,\mathbb{R})$ in $L^2(\mathbb{R}^{p+q})$ (this becomes clear if we consider the real model of the harmonic representation, see, e.g., [43, 39]). Next, we restrict the representation of $SL(p+q,\mathbb{R})$ to $O(p,q)$. The orbits of $O(p,q)$ in $L^2(\mathbb{R}^{p+q})$ are the hyperboloids $\mathrm{Hyp}\,_\alpha$

$$x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2 = \alpha.$$

We have

$$\mathrm{Hyp}\,_\alpha = \begin{cases} O(p,q)/O(p-1,q); & \alpha > 0 \\ O(p,q)/O(p,q-1); & \alpha < 0 \end{cases}$$

(the cone $\alpha = 0$ has zero measure and it can be neglected).

Thus, $L^2(\mathbb{R}^{p+q})$ is decomposable into the direct integral of $L^2(\mathrm{Hyp}_\alpha)$ and, in particular,

$$\mathrm{Spec}\,(\mathrm{RES}\,^{\mathrm{Sp}\,(2(p+q),\mathbb{R})}_{O(p,q)}(we_{2(p+q)}) = \\ = \mathrm{Spec}\,(L^2(O(p,q)/O(p-1,q)) \cup \mathrm{Spec}\,(L^2(O(p,q)/O(p,q-1)).$$

Discrete spectra on hyperboloids and Plancherel formula for hyperboloids were obtained in [71]. For further information concerning the discrete spectrum in L^2 on hyperboloids, see [13, 49, 61]. The above reasoning is due to Howe [38].

8.4. Indefinite Stiefel manifolds. Consider the embeddings

$$\mathrm{Sp}\,(2n,\mathbb{R}) \to O(k) \times \mathrm{Sp}\,(2n,\mathbb{R}) \to \mathrm{Sp}\,(2kn,\mathbb{R})$$

(this is the special case of (8.2)). It is well known that

$$\mathrm{RES}\,^{\mathrm{Sp}\,(2kn,\mathbb{R})}_{\mathrm{Sp}\,(2n,\mathbb{R})}(we_{2kn}) = we_{2n}^{\otimes k}.$$

We put $n = p+q$ and restrict to $O(p,q)$ both sides of this relation. In the left-hand side, we obtain representations discussed in Sec. 8.2. In the right-hand side, we obtain

$$L^2(\mathbb{R}^{p+q})^{\otimes k} = L^2((\mathbb{R}^{p+q})^k),$$

where the action of $O(p,q)$ on $(\mathbb{R}^{p+q})^k$ is the product of k copies of the actions on $\mathbb{R}^{p+q}$. In other words, $O(p,q)$ acts on k-tuples of vectors from $\mathbb{R}^{p+q}$. It is clear that the stabilizer of a generic point has the form $O(p-\alpha, q-\beta)$, where $\alpha+\beta = k$ (if $k \geq p+q$, the stabilizer is trivial). Thus, $L^2((\mathbb{R}^{p+q})^k)$ is decomposable into a direct integral of spaces of the form

$$L^2(O(p,q)/O(p-\alpha, q-\beta)); \quad \alpha+\beta = k.$$

In particular,

$$\mathrm{Spec}\,(\mathrm{RES}\,^{\mathrm{Sp}\,(2k(p+q),\mathbb{R})}_{O(p,q)}(we_{2k(p+q)})) = \bigcup_{\alpha+\beta=k} \mathrm{Spec}\,(L^2(O(p,q)/O(p-\alpha, q-\beta)).$$

On the other hand, we have the identity (8.5).

On the indefinite Stiefel manifolds

$$O(p,q)/O(p-\alpha,q-\beta);\ U(p,q)/U(p-\alpha,q-\beta);\ \mathrm{Sp}\,(p,q)/\mathrm{Sp}\,(p-\alpha,q-\beta),$$

the problem of decomposition of L^2 has been studied thoroughly, but until now this problem has not found a final solution. See [48, 41]. In [46], Rawnsley, Schmid, and Wolf studied the spaces

$$L^2(U(p,q)/(U(p-\alpha,q-\beta) \times U(\alpha) \times U(\beta)) \subset L^2(U(p,q)/U(p-\alpha,q-\beta)).$$

8.5. Remarks. 1) From the above, it follows that some of singular unitary representations constructed in the present paper were constructed previously by Schlichtkrull, Adams, and Kobayashi. However, our construction contains a continuous parameter while the representations from [30, 48, 41] depend only on a discrete set of parameters. The continuous parameter arises since we consider the representations ρ_Λ with nonintegral signature, and such representations cannot be embedded into the harmonic representation of a larger symplectic group. It seems that our construction has one more advantage, namely, it is relatively simple.

2) After Kostant [69] and Zuckerman [53], tensor products of infinite-dimensional unitary representations of semisimple groups and finite-dimensional representations were studied in a number of papers. It is well known that sometimes such tensor product are decomposable into unitary representations, although we do not see simple reasons for this. In some special cases, this phenomenon can be explained by our method. Namely, the representations in singular distributions from §7 are decomposible into such tensor products.

3) Some singular representations of $SO_0(p,q)$ constructed in this paper coincide with the representations first constructed by Molchanov in [62].

4) Some problems mentioned in Sec. 7.14 were studied by Kobayashi in [63].

5) The tensor product of two representations of the complementary series for SL_2 over the p-adic field can contain a representation of the complementary series, see Tsuchikawa [50]. This also can be explained by the trace theorem.

6) In some papers (beginning with [64]) the problems of harmonic analysis for representations which are unitary with respect to the indefinite scalar product are studied. In such problems, partially discrete spectra arise. These spectra also can be explained by the trace theorem.

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ADDENDUM

The paper was published 6 years ago. Here we add a short description of modern status of the problems discussed in Sections 3-4 of our paper and some bibliography.

A.1. Kernel representations. In Section 3 we discussed the representations of $G = O(p,q)$ obtained by the restriction of the highest weight representation (3.6) of the 'overgroup' $G^* = U(p,q)$.

Let us define the similar representations for an arbitrary classical group G.

Let $\mathbb{K}$ denote $\mathbb{R}$, $\mathbb{C}$, or the quaternions $\mathbb{H}$. We say that a *matrix ball* is

– the set of $p \times q$ matrices over $\mathbb{K}$ with norm < 1

– or the set of $n \times n$ matrices over $\mathbb{K}$ with norm < 1 satisfying some natural symmetry condition.

The *natural symmetry conditions* are

– over $\mathbb{R}$: $z = z^t$ or $z = -z^t$;

– over $\mathbb{C}$: $z = z^t$, $z = -z^t$, $z = z^*$;

– over $\mathbb{R}$: $z = z^*$ or $z = -z^*$.

Fix a matrix ball B. Consider the group G of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ over $\mathbb{K}$ such that the linear fractional map

$$z \mapsto (a + zc)^{-1}(b + zd)$$

is a bijection $B \to B$.

Denote by K the stabilizer of the point $z = 0$ in G. Then B is the homogeneous space G/K, moreover B is a Riemannian noncompact symmetric space, and moreover all the classical Riemannian noncompact symmetric spaces can be obtained in this way (a table of matrix balls is present, for instance, in [11*]). In particular, for $G = O(p,q)$ the matrix ball G/K consists of real $p \times q$ matrices.

Consider a kernel

$$K(z,u) = |\det(1 - zu^*)|^{-s}$$

on B. For sufficiently large values of s, this kernel is positive defined and hence it defines some Hilbert space H_s of functions on G/K. The operators

$$T_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = f\big((a+zc)^{-1}(b+zd)\big) \det(a+zc)^{-s}$$

are unitary in H_s.

We say that T_s is *a scalar kernel representation* of the group G ([12*]).

Other terms for kernel representations are *canonical representation, Berezin transform, Berezin quantization.*

A definition of a vector kernel representation is similar (we must consider an invariant matrix-valued kernel instead of $K(z,u)$, for details see [12*]).

A.2. Another definition of the kernel representations. Consider a Hermitian noncompact symmetric symmetric space G^*/K^* and consider a unitary highest weight representation ρ of G^*.

Consider a symmetric subgroup $G \subset G^*$ and the associated Riemannian noncompact symmetric space G/K. There are two possibilities.

a) G/K is a complex submanifold in G^*/K^*. In this case, the restriction of ρ to the subgroup G is a direct sum of highest weight representations. This case leads to purely combinatorial theory (see [60]).

b) Otherwise, G/K is a totally real submanifold in G^*/K^* (the list of embeddings of this type for classical spaces is contained in [6*], exceptional cases are reduced to [10*], see [18*]). We define *kernel representations* of G as restrictions of unitary highest weight representations of G^* to the subgroup G (see [17*], [12*]).

This construction is slightly more general than previous. It adds to construction of Subsection A.1 two series of representations related to the Hermitian symmetric spaces

$$G^*/K^* = SO_0(p,2)/SO(p) \times SO(2)$$

and also representations related to 6 exceptional spaces G^*/K^*.

In this language, Sections 3–4 of our paper contain the explicit construction of discrete spectra in the kernel representations of $O(p,q)$ (and also $U(p,q)$ and $Sp(p,q)$ in 7.12–7.14)

A.3. Relations with some other problems of harmonic analysis (the Howe dual pairs, L^2 on the Stiefel manifolds, L^2 on pseudo-Riemannian symmetric spaces) are discussed in [12*]. See also the survey [23*] on purely discrete spectra.

A.4. Berezin's formula. In the short paper [1*] published in 1978 Berezin obtained the Plancherel formula for the scalar kernel representations of $U(p,q)$, $Sp(2n,\mathbb{R})$, $SO^*(2n)$, $SO_0(p,2)$ (in these cases, G/K are Hermitian symmetric spaces and $G^* = G \times G$). Berezin died soon after this and the proof of Berezin's formula was published by Upmeier and Unterberger [21*] in 1994.

The results of Sections 3–4 of our paper have empty intersection with Berezin's work, since Berezin considered kernel representations for large values of the parameter s and we considered the discrete part of the spectra for small values of the parameter s.

A.5. General Plancherel formula. In the case of non-Hermitian symmetric spaces the Plancherel formula for scalar kernel representations was obtained in [14*] for large values of s.

For scalar kernel representations, the general Plancherel formula including small s was obtained in [16*] (a partial Plancherel formula for $U(p,q)$ was obtained in [5*]); this also includes description of spectra.

For vector valued kernel representations the Plancherel formula is not known.

A6. Operators of restriction to the boundary and separation of spectra. For small values of the parameter s, the spectrum of a kernel representation is complicated and it contains many different series. A natural decomposition of the kernel representation to subrepresentations with uniform spectra is obtained in [15*] in the following way.

Consider the case $G = O(p,q)$ and the picture described in our Section 3. Consider the subset $B_{p,q}(\mathbb{R})$ of the complex matrix ball $B_{p,q}$ consisting of real matrices. The group $O(p,q)$ has p orbits L_θ on the boundary of $B_{p,q}(\mathbb{R})$, they are enumerated by the number $\theta = rk\,(1 - zz^t) = 0, 1, \dots, p-1$.

There exist the constants s_θ such that for $s < s_\theta$ the operator of restriction of a function $f \in H_s$ to the orbit L_θ is well defined (above in §3 we consider the case $\theta = 0$).

For the groups $O(p,q)$ the constants s_θ are evaluated in [15*].

Hence we obtain a collection of canonical subrepresentations in the kernel representation enumerated by θ.

The same argument is valid for the vector kernel representations ([15*]) but the spectrum in the subrepresentation associated with an orbit L_θ is yet complicated.

Some upper estimates of spectra of vector kernel representations are obtained in [8*].

A.7. Limits as $s \to \pm\infty$. For large values of s the spectrum of the kernel representation is supported by the principal nondegenerated series of representations of G, and a kernel representation is equivalent to the natural representation of G in the space $L^2(G/K)$ (see [1*], [4*], [17*],[20*]).

An explicit construction of a unitary operator intertwining $L^2(G/K)$ and the kernel representation is discussed in [26*] – [27*].

As was observed by Berezin, a natural limit of kernel representations as $s \to +\infty$ is $L^2(G/K)$, see, for instance, [16*].

For a negative integer n, the kernel representation is a unitary representation of the compact form G_c of the group G; the natural limit of the kernel representations as $n \to -\infty$ is L^2 on the corresponding compact symmetric space [16*].

For the spaces $G_c/K = U(2n)/U(n) \times U(n)$ the Plancherel formula is equivalent to the Pickrell's decomposition [24*] of some distinguished K-invariant function on G_c/K.

A8. Singular unitary representations. a) Unitary representations of $U(p,q)$ that admit Gelfand–Tsetlin basis were investigated in [9*].

b) Let G be a semisimple group, let H be a symmetric subgroup. For numerous pairs (G,H) Kobayashi (see [63], [7*], and also [41]) constructed families of unitary representations of G, whose restrictions to H have purely discrete spectra. For $G = O(p,q), U(p,q), Sp(p,q)$ Kobayashi construction has an intersection with our construction. Nevertheless, the explicit correspondence of the parameters of the representations obtained by Kobayashi and us is not known. It is interesting also to construct explicitly the intertwining operators.

A.8. Functional theoretical phenomena. In [13*] one of the authors tried to find new functional theoretical phenomena using known facts about discrete spectra.

A.9. Special functions. Special functions related to the kernel representations are discussed in [25*] – [27*].

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