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The restrictions of functions holomorphic in a domain to curves lying on its boundary, and discrete $\mathrm{SL}_2(\mathbb{R})$-spectra

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Abstract. We consider the operator of restriction of functions holomorphic in a ball or a polydisc to curves lying on the Shilov boundary. It turns out that any function with polynomial growth near the boundary has such a restriction if the position of the curve satisfies a certain condition: if the domain is a ball, then the curve must be transversal to the standard contact distribution on the sphere, and if the domain is a polydisc, then the curve must be monotonic increasing with respect to all coordinates in the standard coordinatization of the torus. We use assertions of this kind to obtain a simple description of discrete inclusions in spectra (of minimal invariant subspaces) for several problems of $\mathrm{SL}_2(\mathbb{R})$-harmonic analysis.

Introduction

There are many theorems on the restriction of discontinuous functions to submanifolds. The best-known are numerous theorems on the existence of boundary limits of holomorphic functions ([2], [12]; [27], IX.3, IX.8, [29], [30], [33], [34]) and the trace theorem in Sobolev spaces (where the operator of restriction of functions to submanifolds of sufficiently large dimension is well defined; since the functions are discontinuous, they have no values at individual points: see [7], [10], § 2, [27], IX.9). We should also note the Stein–Fefferman theorem [5], § 4 on restrictions of Fourier transforms of functions $f \in L^p(\mathbb{R}^n)$ and the Salem–Zygmund theorem on the capacity of the divergence sets of the Fourier series of functions in Sobolev spaces. The last theorem can be interpreted as a statement on the restriction of Sobolev functions to closed totally disconnected subsets (see [4], [31], [3], V.12, [22], § 2.9).

In this paper we consider the following situation: let B_n be a ball in $\mathbb{C}^n$ (actually, we consider the case when $n = 2$; the same arguments are valid in the general case). We consider the standard (contact) distribution of codimension 1 on the boundary S^{2n-1} of the ball B_n. Let us recall its definition. The tangent space to the sphere S^{2n-1} at the point a consists of the vectors ξ that satisfy the condition $\mathrm{Re}\langle a, \xi\rangle = 0$.

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We consider the subspace of this tangent space that consists of the vectors satisfying the condition $\langle a, \xi\rangle = 0$, that is, the so-called complex tangent space. Let γ be a curve on S^{2n-1} transversal to this distribution. According to the Nagel–Rudin theorem (see [20], [30], Chapter 11.2), any bounded holomorphic function in B_n has a non-tangent limit almost everywhere on γ. We show (§ 2) that any function holomorphic in B_n and with polynomial growth near the boundary has a limit on γ in the sense of distributions.

In a similar way (§ 1), we consider the polydisc U^n: $|z_1| < 1,\ \ |z_2| < 1, \dots,$ $|z_n| < 1$, and a curve γ: $z_1 = e^{ih_1(t)}, \dots, z_n = e^{ih_n(t)}$ in the skeleton of the polydisc U^n. Assume that $h_j'(t) > 0$ for all j. Then any function F holomorphic in the polydisc and with polynomial growth near the boundary has a limit on γ in the sense of distributions.

The following heuristic Nagel–Rudin argument is valid in both cases, although it does not prove the theorem. Assume that the curve is analytic. We continue the map $t \mapsto \gamma(t)$ to a holomorphic map of the strip Π: $|\operatorname{Im}(t)| < \varepsilon$. Then the exact meaning of the conditions of the theorem is that, in a neighbourhood of any point $\gamma(t_0)$, $t_0 \in \mathbb{R}$, $\gamma(\Pi)$ intersects the interior of the domain. We can apply the one-dimensional theorem on the existence of a boundary limit to $\gamma(\Pi)$. On the other hand, our statements are similar to the theorem of the existence of a limit on skeletons in tubular domains (see [27], IX.3, IX.8, [33], [34]), and are based on the same analytical effects. It seems plausible that both kinds of statements are special cases of a more general one.

It was shown in [22] (see also [21], [23]) that theorems on the restriction of discontinuous functions provide a simple explanation for the presence of discrete spectra in many problems of non-commutative harmonic analysis. Moreover, these theorems enable one to produce (in discrete spectra) rather “exotic” unitary representations for certain series of semisimple groups (see [22], [23]), that is, they provide an approach to the problems now usually considered as purely algebraic and requiring sophisticated algebraic techniques. In connection with this, there arises a series of problems on the restriction of functions holomorphic in Cartan domains to various submanifolds in the Shilov boundary (see [22]).

This paper is a sequel to [21]–[23]; its purpose is to find additional means of producing discrete spectra. We consider the simplest non-compact group $\mathrm{SL}_2(\mathbb{R})$ and show that even in this case our methods enable us to obtain new results.

In § 3 we “reformulate” the statements of § 1 in terms of analysis on a one-sheet hyperboloid. In § 5 we discuss several problems on tensor products of unitary representations of $\mathrm{SL}_2(\mathbb{R})$ and show how to use theorems on the restriction of functions to a curve to produce a discrete spectrum in these problems. The methods used in this section are rather general and enable us to produce discrete Harish-Chandra series in various spectra. But even for $\mathrm{SL}_2(\mathbb{R})$, our constructions appear to be new.

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§ 1. The bidisc

1.1. Functions with polynomial growth. Consider the bidisc U^2: $|z_1| < 1$, $|z_2| < 1$ in $\mathbb{C}^2$. A function

$$F(z_1, z_2) = \sum_{k \geqslant 0,\, l \geqslant 0} c_{kl} z_1^k z_2^l$$

holomorphic in the bidisc is called a *function with polynomial growth* if there is an N such that the function

$$F(z_1, z_2)(1 - |z_1|)^N (1 - |z_2|)^N$$

is bounded in U^2. This is equivalent to the following condition on c_{kl}: there are M and A such that

$$|c_{kl}| \leqslant A(k + l)^M. \tag{1.1}$$

Remark. The Hardy spaces $H^p(U^2)$ (see [29], § 3.4) consist of functions with polynomial growth.

1.2. Existence of the restriction operator. Consider a closed C^∞-smooth curve

$$\gamma\colon z_1 = e^{ih_1(t)}, \quad z_2 = e^{ih_2(t)}, \qquad t \in [0, T],$$

in the skeleton $\mathbb{T}^2$: $|z_1| = 1$, $|z_2| = 1$ of U^2. Assume that

$$h_1'(t) > 0, \qquad h_2'(t) > 0$$

for all t.

We claim that under these conditions the restriction of the function $F(z_1, z_2) = \sum c_{kl} z_1^k z_2^l$ with polynomial growth to γ is defined (assertion (i) of Theorem 1) and coincides with the radial limit (assertion (ii)).

Theorem 1. (i) *For any function $F(z_1, z_2) = \sum_{k,l} c_{kl} z_1^k z_2^l$ with polynomial growth in U^2 the series*

$$F|_\gamma = \sum_{k,l} c_{kl} \exp\bigl(ikh_1(t) + ilh_2(t)\bigr) \tag{1.2}$$

converges in the sense of distributions.

(ii) *The family of functions $F(\lambda e^{ih_1(t)}, \lambda e^{ih_2(t)})$ converges in the sense of distributions as $\lambda \to 1 - 0$, and its limit coincides with* (1.2).

Proof. Let φ be an infinitely smooth test function. We have to prove that the series

$$\sum_{k,l} c_{kl} \int_0^T \varphi(t) \exp\bigl(ikh_1(t) + ilh_2(t)\bigr)\, dt \tag{1.3}$$

is absolutely convergent. We estimate the summand

$$I_{k,l} = \int_0^T \varphi(t) \exp\bigl(ikh_1(t) + ilh_2(t)\bigr)\, dt \tag{1.4}$$

in the usual way. Integrating by parts, we obtain

$$\begin{aligned} -I_{k,l} &= \int \left(\frac{\varphi(t)}{ikh_1'(t)+ilh_2'(t)}\right)' \exp\bigl(ikh_1(t)+ilh_2(t)\bigr)\,dt \\ &= \int \frac{\varphi'(t)}{ikh_1'(t)+ilh_2'(t)} \exp\bigl(ikh_1(t)+ilh_2(t)\bigr)\,dt \\ &\quad + \int \frac{\varphi(t)\bigl(-ikh_1''(t)-ilh_2''(t)\bigr)}{\bigl(ikh_1'(t)+ilh_2'(t)\bigr)^2} \exp\bigl(ikh_1(t)+ilh_2(t)\bigr)\,dt. \end{aligned} \tag{1.5}$$

Let

$$\theta = \min_t \bigl(h_1'(t), h_2'(t)\bigr), \qquad C = \max_t \bigl(h_1''(t), h_2''(t)\bigr).$$

Then

$$|I_{k,l}| \leqslant \frac{1}{k+l}\left(\frac{1}{\theta}\int |\varphi'(t)|\,dt + \frac{C}{\theta^2}\int |\varphi(t)|\,dt\right).$$

On the other hand, both summands in (1.5) can be written as (1.4) with another function φ. Repeating this calculation, we obtain the estimate

$$|I_{k,l}| < \frac{1}{(k+l)^N}\left(\sum_{j=0}^{N} A_j^{(N)} \int |\varphi^{(j)}(t)|\,dt\right) \tag{1.6}$$

for any N, and now formula (1.1) obviously implies that (1.3) converges.

(ii) We have to justify the passage to the limit in

$$\sum_{k,l} c_{kl}\lambda^k\lambda^l \int_0^T \varphi(t)\exp\bigl(ikh_1(t)+ilh_2(t)\bigr)\,dt$$

as $\lambda \to 1-0$. This is easy, since (1.6) implies that this series is majorized by a convergent series.

1.3. A view from the boundary. We assume for simplicity that our curve γ is defined by the formula $\varphi_2 = h(\varphi_1)$. Let Δ_ε be the set $h(\varphi_1)-\varepsilon < \varphi_2 < h(\varphi_2)+\varepsilon$ on the torus $\mathbb{T}^2$: $z_1 = e^{i\varphi_1}$, $z_2 = e^{i\varphi_2}$. Let χ_ε be the function on the torus that is equal to 1 on Δ_ε and vanishes outside Δ_ε.

Let

$$F(z_1,z_2) = \sum_{k\geqslant 0,\, l\geqslant 0} c_{kl} z_1^k z_2^l$$

be an element of the Hardy space $H^2(U^2)$, that is, the boundary value of F on $\mathbb{T}^2$ is an element of $L^2(\mathbb{T}^2)$. Let us recall that this is equivalent to the following condition (see [29], § 3.4):

$$\sum_{k,l} |c_{kl}|^2 < \infty.$$

Theorem 2. *For any $F \in H^2(\mathbb{T}^2)$ the restriction of F to γ (that is, expression* (1.2)*) coincides with the limit*

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\chi_\varepsilon(\varphi_1,\varphi_2)F(e^{i\varphi_1},e^{i\varphi_2})$$

in the sense of distributions on the torus. More precisely, the following relation is valid for any C^∞-smooth function $g(\varphi_1,\varphi_2)$ on $\mathbb{T}^2$:

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}&\iint_{\mathbb{T}^2}\chi_\varepsilon(\varphi_1,\varphi_2)F(e^{i\varphi_1},e^{i\varphi_2})g(\varphi_1,\varphi_2)\,d\varphi_1\,d\varphi_2\\ &=\sum_{k,l}c_{kl}\int_0^{2\pi}g(t,h(t))\exp\bigl(ikt+ilh(t)\bigr)\,dt.\end{aligned}$$

Proof. We write $g(\varphi_1,\varphi_2)$ as follows:

$$g(\varphi_1,\varphi_2)=g(\varphi_1,h(\varphi_1))+\bigl[g(\varphi_1,\varphi_2)-g(\varphi_1,h(\varphi_1))\bigr].$$

We denote the bracketed term by $r(\varphi_1,\varphi_2)$. The function r vanishes on $\varphi_2=h(\varphi_1)$. By the Cauchy–Schwarz–Bunyakovskii inequality, the summand

$$\frac{1}{2\varepsilon}\iint_{\mathbb{T}^2}F(e^{i\varphi_1},e^{i\varphi_2})\chi_\varepsilon(\varphi_1,\varphi_2)r(\varphi_1,\varphi_2)\,d\varphi_1\,d\varphi_2 \tag{1.7}$$

is majorized by

$$\biggl(\iint_{\mathbb{T}^2}|F(e^{i\varphi_1},e^{i\varphi_2})|^2\,d\varphi_1\,d\varphi_2\biggr)^{1/2}\biggl(\iint_{\Delta_\varepsilon}\frac{1}{(2\varepsilon)^2}|r(\varphi_1,\varphi_2)|^2\,d\varphi_1\,d\varphi_2\biggr)^{1/2}.$$

The second factor tends to zero as $\varepsilon\to 0$, whence, (1.7) tends to zero. It remains to investigate the behaviour of the integral

$$\frac{1}{2\varepsilon}\iint_{\mathbb{T}^2}F(e^{i\varphi_1},e^{i\varphi_2})\chi_\varepsilon(\varphi_1,\varphi_2)g(\varphi_1,h(\varphi_1))\,d\varphi_1\,d\varphi_2$$

as $\varepsilon\to 0$.

Since the series

$$F(e^{i\varphi_1},e^{i\varphi_2})=\sum_{k,l}c_{kl}\exp(ik\varphi_1+il\varphi_2)$$

converges in $L^2(\mathbb{T}^2)$, we can rewrite the integral under investigation as follows:

$$\sum_{k,l}c_{kl}\biggl(\frac{1}{2\varepsilon}\iint_{\mathbb{T}^2}g(\varphi_1,h(\varphi_1))\chi_\varepsilon(\varphi_1,\varphi_2)\exp(ik\varphi_1+il\varphi_2)\,d\varphi_1\,d\varphi_2\biggr).$$

The change of variables $t = \varphi_1$, $s = \varphi_2 - h(\varphi_1)$ in the integrals reduces the series to the form

$$\sum_{k,l} c_{kl}\left(\frac{1}{2\varepsilon}\int_0^{2\pi} dt \int_{-\varepsilon}^{\varepsilon} g(t,h(t)) \exp\bigl(ikt + il(s+h(t))\bigr)\, ds\right)$$
$$= \sum_{k,l} c_{kl}\left(\int_0^{2\pi} g(t,h(t)) \exp\bigl(ikt + ilh(t)\bigr)\, dt\right)\left[\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} e^{ils}\, ds\right]. \qquad (1.8)$$

We observe that the factors in square brackets are uniformly bounded and tend to 1 for any fixed l as $\varepsilon \to 0$. This enables us to use the theorem on majorized convergence, which completes the proof.

Let us formulate a similar theorem for an arbitrary function F with polynomial growth. It is obvious that for any such function the series

$$F(e^{i\varphi_1}, e^{i\varphi_2}) = \sum_{k,l} c_{kl} e^{ik\varphi_1} e^{il\varphi_2}$$

converges in the sense of distributions on $\mathbb{T}^2$.

Consider a function $\theta(x)$ on $\mathbb{R}$ vanishing outside a small neighbourhood of zero and such that $j\theta(jx) \to \delta(x)$ in measure as $j \to \infty$.

Theorem 3. *For any function F with polynomial growth in U^2 the restriction of F to γ coincides with the limit*

$$\lim_{j\to\infty} jF(e^{i\varphi_1}, e^{i\varphi_2})\theta\bigl(j(\varphi_2 - h(\varphi_1))\bigr)$$

in the sense of distributions on the torus. More precisely, for any function $g(\varphi_1, \varphi_2)$ that is C^∞-smooth on $\mathbb{T}^2$ the limit

$$\lim_{j\to\infty} j \iint_{\mathbb{T}^2} F(e^{i\varphi_1}, e^{i\varphi_2}) g(\varphi_1,\varphi_2)\theta\bigl(j(\varphi_2 - h(\varphi_1))\bigr)\, d\varphi_1\, d\varphi_2$$

coincides with

$$\sum_{k,l} c_{kl}\int_0^{2\pi} g(t,h(t)) \exp\bigl(ikt + ilh(t)\bigr)\, dt.$$

Proof. We write $g(\varphi_1, \varphi_2)$ as follows:

$$g(\varphi_1,\varphi_2) = g(\varphi_1, h(\varphi_1)) + \sum_{m=1}^{M-1} \sin^m\left(\frac{\varphi_2 - h(\varphi_1)}{2}\right) p_m(\varphi_1)$$
$$+ \sin^M\left(\frac{\varphi_2 - h(\varphi_1)}{2}\right) r(\varphi_1,\varphi_2),$$

where p_m and r are smooth functions and M will be chosen later.

We have to investigate the behaviour of the integrals

$$I_0 = j \iint_{\mathbb{T}^2} F(e^{i\varphi_1}, e^{i\varphi_2})\theta\bigl(j(\varphi_2 - h(\varphi_1))\bigr) g(\varphi_1, h(\varphi_1))\, d\varphi_1\, d\varphi_2,$$

$$I_m = j \iint_{\mathbb{T}^2} F(e^{i\varphi_1}, e^{i\varphi_2})\theta\bigl(j(\varphi_2 - h(\varphi_1))\bigr) \sin^m\biggl(\frac{\varphi_2 - h(\varphi_1)}{2}\biggr) p_m(\varphi_1)\, d\varphi_1\, d\varphi_2,$$

$$I_M = j \iint_{\mathbb{T}^2} F(e^{i\varphi_1}, e^{i\varphi_2})\theta\bigl(j(\varphi_2 - h(\varphi_1))\bigr) \sin^M\biggl(\frac{\varphi_2 - h(\varphi_1)}{2}\biggr) r(\varphi_1, \varphi_2)\, d\varphi_1\, d\varphi_2$$

as $j \to \infty$. We apply to I_0 the calculations of the preceding theorem, replacing the bracketed factor in (1.8) by

$$j \int \theta(js) e^{ils}\, ds = \int \theta(s) e^{isl/j}\, ds.$$

These quantities do not exceed 1 and tend to 1 as $j \to \infty$.

A similar calculation for I_m with $1 \leqslant m \leqslant M$ yields

$$\sum_{k,l} c_{kl} \biggl[\int_0^{2\pi} p_m(t) \exp\bigl(ikt + ilh(t)\bigr)\, dt\biggr] \biggl(j \int_0^{2\pi} \sin^m\Bigl(\frac{s}{2}\Bigr) \theta(js) e^{ils}\, ds\biggr).$$

The factors in square brackets do not depend on j and tend to zero as $k, l \to \infty$ more rapidly than any power of $(k+l)^a$. The factors in parentheses tend to zero for a fixed l, and these quantities are uniformly bounded. Hence, $I_m \to 0$ as $j \to \infty$.

Consider I_M. We represent the distribution $F(e^{i\varphi_1}, e^{i\varphi_2})$ as a finite sum:

$$F(e^{i\varphi_1}, e^{i\varphi_2}) = \sum \biggl(\frac{\partial}{\partial \varphi_1}\biggr)^{\alpha} \biggl(\frac{\partial}{\partial \varphi_2}\biggr)^{\beta} G_{\alpha,\beta}(\varphi_1, \varphi_2),$$

where the $G_{\alpha,\beta}$ are functions continuous on $\mathbb{T}^2$ (the pair (α, β) ranges over a finite set). Then integration by parts in I_M yields a sum of expressions of the form

$$\iint G_{\alpha,\beta}(\varphi_1, \varphi_2) \varkappa_{\alpha,\beta,\sigma,\tau}(\varphi_1, \varphi_2) j^{1+\sigma+\tau} \theta^{(\sigma+\tau)}\bigl(j(\varphi_2 - h(\varphi_1))\bigr)$$
$$\times \biggl[\biggl(\frac{\partial}{\partial \varphi_1}\biggr)^{\alpha-\tau-\mu} \biggl(\frac{\partial}{\partial \varphi_2}\biggr)^{\beta-\sigma} \biggl(\sin^M\biggl(\frac{\varphi_2 - h(\varphi_1)}{2}\biggr) r(\varphi_1, \varphi_2)\biggr)\biggr]\, d\varphi_1\, d\varphi_2,$$

where the $\varkappa_{\alpha,\beta,\sigma,\tau}$ are smooth functions. The quantities $G_{\alpha,\beta}$, $\varkappa_{\alpha,\beta,\sigma,\tau}$, and $\theta^{(\sigma+\tau)}$ are majorized by constants, and the term in square brackets is majorized on the support of the function $\theta\bigl(j(\varphi_2 - h(\varphi_1))\bigr)$ by the quantity

$$C\biggl(\frac{1}{j}\biggr)^{M-(\alpha-\sigma)-(\beta-\tau)}.$$

If M is sufficiently large (namely, if M is greater than $\alpha + \beta$ for all pairs (α, β)), this estimate implies that the integral tends to zero.

§ 2. The ball

2.1. Functions with polynomial growth. Let B_2 be the ball $|z_1|^2+|z_2|^2 \leqslant 1$ in $\mathbb{C}^2$. Its boundary is the sphere S^3 defined by the equation $|z_1|^2+|z_2|^2=1$.

A function $F(z_1,z_2)=\sum_{k,l} c_{kl} z_1^k z_2^l$ holomorphic in B_2 is called a *function with polynomial growth* if there are N and A such that

$$F(z_1,z_2) \leqslant \frac{A}{(1-|z_1|^2-|z_2|^2)^N}.$$

Let $\mathcal{H}(B_2)$ be the space of functions with polynomial growth. In terms of Taylor coefficients the condition $f \in \mathcal{H}(B_2)$ is equivalent to the condition

$$c_{kl}\sqrt{\frac{k!l!}{(k+l)!}} = o(k+l)^p \tag{2.1}$$

for some p. This can be rewritten as follows:

$$c_{kl}\sqrt{\frac{k!l!}{(k+l)!}} = o\big((k+1)(l+1)\big)^q \tag{2.2}$$

for some q. The same condition can be written as

$$c_{kl}\left(\frac{k^{k/2} l^{l/2}}{(k+l)^{(k+l)/2}}\right) = o\big((k+1)(l+1)\big)^s \tag{2.3}$$

for some s.

Lemma. *Each of the conditions* (2.1)–(2.3) *on* F *is equivalent to the condition* $F \in \mathcal{H}(B_2)$.

Proof. Let $\mathcal{F}$ be the space of all functions holomorphic in the ball and satisfying any of conditions (2.1)–(2.3). It is obvious that $\mathcal{F}$ is closed relative to the operators $\frac{\partial}{\partial z_1}$, $\frac{\partial}{\partial z_2}$, and

$$J_1 f(z_1,z_2) = \int_0^{z_1} f(u,z_2)\,du, \qquad J_2 f(z_1,z_2) = \int_0^{z_2} f(z_1,u)\,du.$$

Let $F \in \mathcal{H}(B_2)$. Then the function $\widetilde{F} = J_1^a J_2^b F$ is continuous in the closed ball for sufficiently large a and b. Let $\widetilde{F} = \sum_{k,l} a_{kl} z_1^k z_2^l$. It is obvious that $\widetilde{F}$ belongs to the Hardy space $H^2(B_2)$ (see [30]), which is equivalent to the following condition (see [30], § 1.4.9):

$$\sum_{k,l} |a_{kl}|^2 \frac{k!l!}{(k+l+1)!} < \infty.$$

This implies that $\widetilde{F} \in \mathcal{F}$, whence, $F \in \mathcal{F}$.

Conversely, let $F(z_1,z_2) = \sum_{k,l} c_{kl} z_1^k z_2^l \in \mathcal{F}$. Then the Cauchy–Schwarz–Bunyakovskii inequality implies that

$$\begin{aligned}
|F(z_1,z_2)|^2 &\leqslant \biggl(\sum_j \biggl|\sum_{k+l=j} c_{kl} z_1^k z_2^l\biggr|\biggr)^2 \\
&\leqslant \sum_j \biggl(\sum_{k+l=j} |c_{kl}|^2 \frac{k!l!}{(k+l)!}\biggr)\biggl(\sum_{k+l=j} \frac{(k+l)!}{k!l!}|z_1|^{2k}|z_2|^{2l}\biggr) \\
&= \sum_j \biggl(\sum_{k+l=j} |c_{kl}|^2 \frac{k!l!}{(k+l)!}\biggr)\bigl(|z_1|^2+|z_2|^2\bigr)^j \\
&\leqslant \sum_j \biggl(\sum_{k+l=j} (k+l+1)^p\biggr)\bigl(|z_1|^2+|z_2|^2\bigr)^j \\
&= \sum_j (j+1)^{p+1}\bigl(|z_1|^2+|z_2|^2\bigr)^j \\
&\leqslant \sum_j (j+p+1)(j+p)\dots(j+1)\bigl(|z_1|^2+|z_2|^2\bigr)^j \\
&= \bigl(1-|z_1|^2-|z_2|^2\bigr)^{-(p+1)}.
\end{aligned}$$

2.2. The contact distribution on the sphere. Consider the boundary ∂B_2 of B_2, that is, S^3: $|z_1|^2+|z_2|^2=1$. Let $(a,b)\in S^3$. Then the *real tangent space* $T_{(a,b)}$ to S^3 at (a,b) is defined by the equation

$$\operatorname{Re}(a\bar\xi+b\bar\eta)=0, \qquad (\xi,\eta)\in\mathbb{C}^2.$$

We consider the subspace $L_{(a,b)}$ of codimension 1 in $T_{(a,b)}$ that consists of the vectors (ξ,η) satisfying the equation

$$a\bar\xi+b\bar\eta=0.$$

Thus, we obtain a two-dimensional distribution on the three-dimensional sphere (see [30], § 5.4.2). Let us recall that this distribution is not integrable. It is obvious that it is invariant relative to the group $U(2)$ of unitary transformations of $\mathbb{C}^2$.

Consider the action of $U(2)$ on the set of tangent vectors (ξ,η) to S^3 (at all points $(a,b)\in S^3$). It is easy to see that $U(2)$-orbits are distinguished by the following two invariants:

(i) the length of the vector $l=\bigl(|\xi|^2+|\eta|^2\bigr)^{1/2}$;

(ii) the scalar product of (ξ,η) with the unit normal to $L_{(a,b)}$:

$$q=\operatorname{Im}(a\bar\xi+b\bar\eta),$$

where $|q|\leqslant l$.

We shall need the following auxiliary statement.

Lemma. *Any vector* $(\xi,\eta)\in T_{(a,b)}$ *such that* $(\xi,\eta)\notin L_{(a,b)}$ *can be transformed by an element of* $U(2)$ *into a vector* $(\xi',\eta')\in T_{(a',b')}$ *that satisfies the following condition: the numbers* $\operatorname{Im}(\xi'/a')$ *and* $\operatorname{Im}(\eta'/b')$ *are finite, different from zero, and both positive or both negative.*

Proof. It is sufficient to verify that all values of q and l are achieved on vectors satisfying the desired condition.

We fix real a and b. Consider $\xi = iat_1$, $\eta = ibt_2$, where $t_1, t_2 > 0$. Then

$$l = \sqrt{a^2t_1^2 + b^2t_2^2}, \qquad q = a^2t_1 + b^2t_2.$$

If t_1 and t_2 are positive, then q/l can range from a to b, and the statement becomes obvious.

2.3. Boundary values.

Theorem 4. *Let* γ: $z_1 = \alpha(t)e^{ih_1(t)}$, $z_2 = \beta(t)e^{ih_2(t)}$ *be a closed* C^∞*-smooth curve in* $\partial B_2 = S^3$ *transversal to* $L_{(a,b)}$ *at every point. Then the following assertions are valid for any function*

$$F = \sum_{k,l} c_{kl} z_1^k z_2^l \in \mathcal{H}(B_2)$$

with polynomial growth:

(i) *the restriction of* F *to* γ *is defined, that is, the series*

$$\sum_{k,l} c_{kl} \bigl(\alpha(t)e^{ih_1(t)}\bigr)^k \bigl(\beta(t)e^{ih_2(t)}\bigr)^l \tag{2.4}$$

converges in the sense of distributions;

(ii) *the (radial) limit*

$$\lim_{\lambda\to 1-0} F\bigl(\lambda\alpha(t)e^{ih_1(t)}, \lambda\beta(t)e^{ih_2(t)}\bigr)$$

exists in the sense of distributions and coincides with the sum of the series (2.4).

Proof. Let $\varphi(t)$ be a smooth function. We have to prove that the series

$$\sum_{k,l} c_{kl} I_{k,l},$$

where

$$I_{k,l} = \int_0^1 \varphi(t)\bigl(\alpha(t)e^{ih_1(t)}\bigr)^k \bigl(\beta(t)e^{ih_2(t)}\bigr)^l\,dt$$

is absolutely convergent. We can assume without loss of generality that $\varphi(t)$ has a small support. In this case the lemma in § 2.2 allows us to assume that $h_1' > 0$ and $h_2' > 0$. Integration by parts in $I_{k,l}$ yields

$$I_{k,l} = -\int_0^1 \left(\frac{\varphi(t)}{ikh_1'(t) + ilh_2'(t) + k\frac{\alpha'(t)}{\alpha(t)} + l\frac{\beta'(t)}{\beta(t)}}\right)' \times e^{ikh_1(t)}e^{ilh_2(t)}\alpha^k(t)\beta^l(t)\,dt.$$

Applying the Cauchy–Schwarz–Bunyakovskii inequality, we obtain the estimate

$$|I_{k,l}| < C(k+l)^{-1}\biggl(\int_0^1 |\alpha^{2k}(t)\beta^{2l}(t)|\,dt\biggr)^{1/2}.$$

Integrating by parts once again, we obtain the following estimates for the $I_{k,l}$:

$$|I_{k,l}| < C_p(k+l)^{-p}\biggl(\int_0^1 |\alpha^{2k}(t)\beta^{2l}(t)|\,dt\biggr)^{1/2}$$

for any p. To estimate (2.4), we use the Cauchy–Schwarz–Bunyakovskii inequality:

$$\biggl|\sum_{k,l} c_{kl} I_{k,l}\biggr|^2 \leqslant C_p^2 \sum_j \biggl(\sum_{k+l=j} |c_{kl}|^2 \frac{k!\,l!}{(k+l)!}(k+l)^{-2p}\biggr) \times \biggl(\sum_{k+l=j} \frac{(k+l)!}{k!\,l!} \int_0^1 |\alpha(t)|^{2k}|\beta(t)|^{2l}\,dt\biggr).$$

The second factor is equal to 1, whence, our expression is equal to

$$C_p^2 \sum_{k,l} |c_{kl}|^2 \frac{k!\,l!}{(k+l)!}(k+l)^{-2p}.$$

This series converges for sufficiently large p, which completes the proof of (ii).
Assertion (ii) can be proved by the arguments used in the proof of Theorem 1.

§ 3. The space L^2 on the one-sheet hyperboloid

In this section we describe Molchanov's decomposition for L^2 on the one-sheet hyperboloid and discuss its function-theoretic properties.

3.1. The hyperboloid. Consider the group $\mathrm{SO}_0(2,1)$, which consists of the real matrices

$$g = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

satisfying the condition

$$g^t \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} g = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \qquad \det g = +1, \qquad a_{33} > 0.$$

Let us recall that $\mathrm{SO}_0(2,1)$ is isomorphic to the group $\mathrm{PSL}_2(\mathbb{R})$. We consider the one-sheet hyperboloid X defined by the following equation in $\mathbb{R}^3$:

$$x_1^2 + x_2^2 - x_3^2 = 1.$$

It is obvious that X is $\mathrm{SO}_0(2,1)$-invariant (moreover, $X = \mathrm{SO}_0(2,1)/\mathrm{SO}_0(1,1)$).

There is precisely one $\mathrm{SO}_0(2,1)$-invariant measure on X (up to a constant factor). We consider the representation of $\mathrm{SO}_0(2,1) \simeq \mathrm{PSL}_2(\mathbb{R})$ in L^2 relative to this measure. The decomposition of L^2 into irreducible representations is well known (it was actually obtained in [25]); it can be expanded as a double direct integral in the even basic series of unitary representations of $\mathrm{PSL}_2(\mathbb{R})$ plus a single direct sum of all representations of $\mathrm{PSL}_2(\mathbb{R})$ in discrete series (for the series representations of $\mathrm{SL}_2(\mathbb{R})$ see, for example, [17], [25], [26], [28], and § 5 below).

Consider the decomposition

$$L^2(X) = \mathcal{L}_c \oplus \mathcal{L}_d, \tag{3.1}$$

where $\mathcal{L}_c$ is the integral over the basic series and $\mathcal{L}_d$ is the direct sum of the representations in discrete series. The description of this representation was discussed by Molchanov in [17]–[19] (see also [8], [9]). We propose below a convenient explicit description of this decomposition. We also show that Theorems 1 and 2 imply amusing and strange consequences concerning the function-theoretic properties of functions from $\mathcal{L}_c$ and $\mathcal{L}_d$.

3.2. The torus. There are two families of rectilinear generators on the one-sheet hyperboloid. Each of these families is parametrized by the points of a circle. Thus we obtain an embedding of the hyperboloid in a torus (namely, we associate with every point of the hyperboloid the pair of generators that pass through it, and thus with a point of the torus $\mathbb{T}^2$).

We consider coordinates z_1 and z_2 or φ_1 and φ_2 on the two-dimensional torus $\mathbb{T}^2$:

$$z_1 = e^{i\varphi_1}, \qquad z_2 = e^{i\varphi_2}; \qquad \varphi_1, \varphi_2 \in [0, 2\pi]; \qquad |z_1| = |z_2| = 1.$$

Let σ be the Lebesgue measure $d\varphi_1 d\varphi_2$ on $\mathbb{T}^2$. In the new coordinates the group $\mathrm{SO}_0(2,1) \simeq \mathrm{PSL}_2(\mathbb{R})$ becomes the following transformation group of the torus:

$$(z_1, z_2) \mapsto \left(\frac{az_1 + b}{\bar{b}z_1 + \bar{a}}, \frac{az_2 + b}{\bar{b}z_2 + \bar{a}} \right),$$

where $z \mapsto \frac{az+b}{\bar{b}z+\bar{a}}$ is the usual Möbius transformation of the circle $|z| = 1$ (let us recall that $|a|^2 - |b|^2 = 1$). The line Δ: $\varphi_1 = \varphi_2$ (the diagonal) is $\mathrm{PSL}_2(\mathbb{R})$-invariant. The image of X under the map into the torus is the complement of Δ. The invariant measure on the hyperboloid becomes the measure μ on $\mathbb{T}^2$ defined by the formula

$$\sin^{-2}\left(\frac{\varphi_1 - \varphi_2}{2} \right) d\varphi_1 d\varphi_2.$$

The natural isometry

$$J\colon L^2(\mathbb{T}, \mu) \simeq L^2(X) \to L^2(\mathbb{T}, \sigma)$$

is defined by the formula

$$Jf(z_1, z_2) = (z_1 - z_2)^{-1} f(z_1, z_2).$$

The isometric property of J follows from the equality

$$|z_1 - z_2| = 2\left|\sin\left(\frac{\varphi_1 - \varphi_2}{2}\right)\right|.$$

The isometry J transforms the action of $\mathrm{PSL}_2(\mathbb{R})$ in $L^2(X)$ into an action in $L^2(\mathbb{T}, \sigma)$ defined by the formula

$$f(z_1, z_2) \mapsto f\left(\frac{az_1 + b}{\bar{b}z_1 + \bar{a}}, \frac{az_2 + b}{\bar{b}z_2 + \bar{a}}\right)(\bar{b}z_1 + \bar{a})^{-1}(\bar{b}z_2 + \bar{a})^{-1}. \tag{3.2}$$

3.3. The decomposition of $L^2(\mathbb{T}, \sigma)$. It is obvious that the following four subspaces are invariant under operator (3.2): Λ_{++}, Λ_{+-}, Λ_{-+}, Λ_{--}. These subspaces consist of the functions

$$\sum_{k\geqslant 0,\, l\geqslant 0} c_{kl} z_1^k z_2^l, \qquad \sum_{k\geqslant 0,\, l<0} c_{kl} z_1^k z_2^l,$$
$$\sum_{k<0,\, l\geqslant 0} c_{kl} z_1^k z_2^l, \qquad \sum_{k<0,\, l<0} c_{kl} z_1^k z_2^l,$$

respectively. It is obvious that

$$L^2(\mathbb{T}, \sigma) = \Lambda_{++} \oplus \Lambda_{+-} \oplus \Lambda_{-+} \oplus \Lambda_{--}.$$

Λ_{++} coincides with the Hardy space $H^2(U^2)$; the difference between the other three spaces and $H^2(U^2)$ is immaterial.

Using the notation of formula (3.1), we can write

$$\mathcal{L}_c \simeq \Lambda_{+-} \oplus \Lambda_{-+}, \qquad \mathcal{L}_d = \Lambda_{++} \oplus \Lambda_{--}, \tag{3.3}$$

where Λ_{++} is the direct sum of the representations of $\mathrm{PSL}_2(\mathbb{R})$ with highest weight, and Λ_{--} is the direct sum of the representations of $\mathrm{PSL}_2(\mathbb{R})$ with lowest weight.

Remark. The decomposition $\Lambda_{++} \oplus \Lambda_{+-} \oplus \Lambda_{-+} \oplus \Lambda_{--}$ is defined not only for L^2: it can be defined for the space of all distributions on the torus.

3.4. The restriction to curves. Consider a smooth curve γ: $\varphi_1 = h_1(t)$, $\varphi_2 = h_2(t)$ on the torus such that $h_1'(t)$ and $h_2'(t)$ are everywhere different from 0, that is, γ is not tangent to parallels or meridians of the torus. In terms of the hyperboloid this means that γ is not tangent to rectilinear generators. We say that γ is *positive monotonic* if $h_1'(t)$ and $h_2'(t)$ are both positive or both negative. Otherwise, it is said to be *negative monotonic*. If $\gamma(s) = \big(\gamma_1(s), \gamma_2(s), \gamma_3(s)\big) \subset \mathbb{R}^3$ is the corresponding curve on the hyperboloid and

$$\|\gamma'(s)\|^2 := \gamma_1'(s)^2 + \gamma_2'(s)^2 - \gamma_3'(s)^2,$$

then the positive monotonicity means that $\|\gamma'(s)\|^2 < 0$ and the negative monotonicity means that $\|\gamma'(s)\|^2 > 0$.

Theorem 2 and decomposition (3.3) imply the following proposition.

Proposition 5. *The restriction of any $f \in \mathcal{L}_c$ (in the sense of Theorem 2) to any negative monotonic curve is defined, and so is the restriction of any $f \in \mathcal{L}_d$ to any positive monotonic curve.*

A similar statement is valid for the space of distributions on the hyperboloid.

Remark. The diagonal Δ is positive monotonic and $\mathrm{PSL}_2(\mathbb{R})$-invariant. Hence, the restriction of $f \in \mathcal{L}_d$ to Δ is a $\mathrm{PSL}_2(\mathbb{R})$-intertwining operator. Moreover, the restrictions of the derivatives of f of any order to Δ are defined (the derivatives of functions with polynomial growth are functions with polynomial growth). Let S^α be the subspace of functions $f \in \mathcal{L}_d = \Lambda_{++} \oplus \Lambda_{--}$ such that the restrictions of all their partial derivatives of order $\leqslant \alpha$ to Δ are equal to zero. Then we obtain the following filtration in $\mathcal{L}_d$:

$$\mathcal{L}_d \supset S^0 \supset S^1 \supset \cdots,$$

which is invariant under $\mathrm{PSL}_2(\mathbb{R})$. The group $\mathrm{PSL}_2(\mathbb{R})$ acts in the factors S^j/S^{j-1} in a natural way. The corresponding representation is the sum of the representation with the highest weight and the complex-conjugate of the representation with the lowest weight.

The last remark and Proposition 5 are obvious consequences of assertions in § 1. There arise the following two questions concerning analogues of these statements in more sophisticated cases. First, can we explain the discrete spectra in [23] and [14] by analogues of Theorem 2? Second, are there analogues of Proposition 5 for more general hyperboloids, pseudo-Riemannian symmetric spaces, and indefinite Stiefel manifolds (see [19], [6], [32], [13])?

§ 4. The restriction to a curve in a tensor product of Sobolev spaces

4.1. Sobolev spaces on a circle. Any distribution f on the circle S^1 can be represented as the sum of the following series:

$$f = \sum_{k=-\infty}^{\infty} c_k e^{ik\varphi}, \tag{4.1}$$

where $c_n = o(n^p)$ for a sufficiently large p.

Let $\lambda \in \mathbb{R}$. The Sobolev space $W^\lambda = W^\lambda(S^1)$ is the space of distributions on the circle whose Fourier coefficients c_k (see (4.1)) satisfy the condition

$$\|f\|_\lambda^2 = \sum_n (n^2+1)^\lambda |c_k|^2 < \infty.$$

4.2. The restriction to a curve. Consider the tensor product $W^\alpha \otimes W^\beta$. It is natural to interpret this space as the space of distributions

$$F = \sum_{k,l} c_{kl} e^{ik\varphi_1} e^{ik\varphi_2} \tag{4.2}$$

on $\mathbb{T}^2$: $z_1 = e^{i\varphi_1}$, $z_2 = e^{i\varphi_2}$, $\varphi_1, \varphi_2 \in [0, 2\pi]$, that satisfy the condition

$$\|F\|^2 = \sum_{k,l} |c_{kl}|^2 (k^2+1)^\alpha (l^2+1)^\beta < \infty. \tag{4.3}$$

The same formula defines a norm in $W^\alpha \otimes W^\beta$.

Proposition 6. *Let $\alpha + \beta > 1/2$. Let γ be any curve on the torus transversal to the parallels and meridians. Then the operator of restriction of functions $F \in C^\infty(\mathbb{T}^2)$ to γ can be extended to a bounded operator from $W^\alpha \otimes W^\beta$ into the space of distributions on γ.*

Proof. Note that our statement is local. The space $W^\alpha \otimes W^\beta$ is invariant under the changes of variables

$$\tilde{\varphi_1} = q_1(\varphi_1), \qquad \tilde{\varphi_2} = q_2(\varphi_2),$$

where q_1 and q_2 are smooth diffeomorphisms of the circle. Hence, we can assume without loss of generality that γ is defined by the equation $\varphi_1 = \varphi_2$. Now we can write down an explicit expression for the operator of restriction to γ in terms of Fourier coefficients. Namely, we associate with the function (4.2) the series

$$\sum_n \Big(\sum_{k+l=n} c_{kl} \Big) e^{in\theta}. \tag{4.4}$$

By the Cauchy–Schwarz–Bunyakovskii inequality,

$$\begin{aligned} \Big| \sum_{k+l=n} c_{kl} \Big|^2 \leqslant \Big(&\sum_{k+l=n} |c_{kl}|^2 (k^2+1)^\alpha (l^2+1)^\beta \Big) \\ &\times \Big[\sum_k \frac{1}{(k^2+1)^\alpha \big((n-k)^2+1\big)^\beta} \Big]. \end{aligned}$$

The series in the parentheses converges by virtue of (4.3), and the series in square brackets converges for $\alpha+\beta > 1/2$. We have to prove that the sum of the series (4.4) is a distribution, and to do this, it is sufficient to verify that the factor in square brackets has polynomial growth with respect to n. Consider the expression

$$r_k = \frac{1}{(k^2+1)^\alpha \big((n-k)^2+1\big)^\beta}.$$

It is clear that the maximum of r_k over $-n \leqslant k \leqslant 2n$ is majorized by a power of n. If k lies outside this interval, then

$$\begin{aligned} r_k &< \frac{1}{|n-k|^{2(\alpha+\beta)}} \quad &&\text{for} \quad k > 2n, \\ r_k &< \frac{1}{|k|^{2(\alpha+\beta)}} \quad &&\text{for} \quad k < -n. \end{aligned}$$

Now the assertion is obvious.

§ 5. Discrete spectra in tensor products of unitary representations of $\mathrm{SL}_2^{\sim}(\mathbb{R})$

Our purpose in this section is to obtain an explicit construction of irreducible subrepresentations (possibly all of them) in certain tensor products of unitary representations of $\mathrm{SL}_2^{\sim}(\mathbb{R})$, using the theorems on the restriction to curves.

5.1. Representations with the highest weight. We interpret $\mathrm{SL}_2(\mathbb{R})$ as the group of matrices

$$g=\begin{pmatrix} a & b\\ \bar b & \bar a\end{pmatrix},\qquad |a|^2-|b|^2=1.$$

This group acts on the disc $|z|<1$ by the Möbius transformations

$$z\mapsto \frac{az+b}{\bar b z+\bar a}.$$

Let $\mathrm{SL}_2^{\sim}(\mathbb{R})$ be the universal cover of $\mathrm{SL}_2(\mathbb{R})$ (see [26]). For any $s>0$ we define a representation T_s of $\mathrm{SL}_2^{\sim}(\mathbb{R})$ in the space of holomorphic functions on the circle by the formula

$$T_s(g)f(z)=f\left(\frac{az+b}{\bar b z+\bar a}\right)(\bar b z+\bar a)^{-s}.$$

Remark. The parameter s in this formula is real. Therefore, $(\bar b z+\bar a)^{-s}$ is a many-valued function. If s is not an integer, we obtain a representation of the universal cover $\mathrm{SL}_2^{\sim}(\mathbb{R})$ of the group $\mathrm{SL}_2(\mathbb{R})$ rather than of the group itself.

The T_s are called *representations with the highest weight* (see [26]). These representations are unitary in the Hilbert space $\mathcal{H}_s$ of holomorphic functions defined by the reproducing kernel (see, for example, [22], § 1)

$$K_s(z,u)=(1-z\bar u)^{-s}.$$

The norm in $\mathcal{H}_s$ can be written as follows:

$$\left\|\sum c_k z^k\right\|^2=\sum \frac{k!}{s(s+1)\dots(s+k-1)}|c_k|^2. \tag{5.1}$$

In the case when $s>1$ there is a convenient integral representation for the scalar product in $\mathcal{H}_s$:

$$\langle f,g\rangle=\int_{|z|<1} f(z)\overline{g(z)}(1-|z|^2)^{s-2}\,dz\,d\bar z.$$

$\mathcal{H}_1$ coincides with the Hardy space H^2.

Remark. Formula (5.1) implies that the boundary values of $f\in\mathcal{H}_s$ lie in the Sobolev space $W^{-(s-1)/2}$.

The representation $\overline{T}_s$ of $\mathrm{SL}_2^{\sim}(\mathbb{R})$ *with the lowest weight* $(-s)$ is realized in the same space $\mathcal{H}_s$ by the operators

$$\overline{T}_s(g)f(z)=f\left(\frac{\bar a z+\bar b}{bz+a}\right)(bz+a)^{-s}.$$

5.2. The representation $T_s \otimes T_\sigma$. Consider the space $\mathcal{H}_s \otimes \mathcal{H}_\sigma$. It is natural to regard it as the Hilbert space of holomorphic functions in the bidisc U^2 defined by the reproducing kernel

$$K_{s,\sigma}(z_1, z_2; u_1, u_2) = K_s(z_1, \bar{u}_1) K_\sigma(z_2, \bar{u}_2) = (1 - z_1\bar{u}_1)^{-s}(1 - z_2\bar{u}_2)^{-\sigma}.$$

The condition $F = \sum c_{kl} z_1^k z_2^l \in \mathcal{H}_s \otimes \mathcal{H}_\sigma$ is equivalent to the inequality

$$\sum_{k,l} |c_{kl}|^2 (k+1)^{-(s-1)} (l+1)^{-(\sigma-1)} < \infty. \tag{5.2}$$

This implies, in particular, that the space of functions with polynomial growth in the bidisc coincides with

$$\bigcup_{s>0,\,\sigma>0} \mathcal{H}_s \otimes \mathcal{H}_\sigma.$$

The representation $T_s \otimes T_\sigma$ is realized in the space $\mathcal{H}_s \otimes \mathcal{H}_\sigma$ by the operators

$$f(z_1, z_2) \mapsto f\left(\frac{az_1 + b}{\bar{b}z_1 + \bar{a}}, \frac{az_2 + b}{\bar{b}z_2 + \bar{a}}\right)(\bar{b}z_1 + \bar{a})^{-s}(\bar{b}z_2 + \bar{a})^{-\sigma}.$$

Consider the operator of restriction of holomorphic functions $f \in \mathcal{H}_s \otimes \mathcal{H}_\sigma$ to the diagonal Δ: $\varphi_1 = \varphi_2$ of the torus $\mathbb{T}^2$. Consider the subspace Q_0 of the space $\mathcal{H}_s \otimes \mathcal{H}_\sigma$ consisting of functions whose restriction to Δ is zero. Note that the derivatives of a function with polynomial growth are functions with polynomial growth, and so we can apply Theorem 1 to them. For any non-negative integer α we consider the subspace Q_α in $\mathcal{H}_s \otimes \mathcal{H}_\sigma$ that consists of functions whose derivatives of order $\leqslant \alpha$ vanish on Δ. Thus we obtain an $\mathrm{SL}_2^\sim(\mathbb{R})$-invariant filtration

$$\mathcal{H}_s \otimes \mathcal{H}_\sigma = Q_{-1} \supset Q_0 \supset Q_1 \supset \cdots$$

in $\mathcal{H}_s \otimes \mathcal{H}_\sigma$. It can be easily verified that the representation of $\mathrm{SL}_2^\sim(\mathbb{R})$ in Q_{j-1}/Q_j is irreducible and equivalent to $T_{s+\sigma+2j}$. Let R_j be the orthogonal complement of Q_j in Q_{j-1}. We obtain the following decomposition of $\mathcal{H}_s \otimes \mathcal{H}_\sigma$ into a direct sum of minimal invariant subspaces:

$$\mathcal{H}_s \otimes \mathcal{H}_\sigma = \bigoplus_{j=0}^{\infty} R_j.$$

Remark. The phrase "the restriction to Δ is equal to zero" is somewhat unnatural in our case. It would be simpler to say that the holomorphic function f is equal to zero on the diagonal $z_1 = z_2$ of the bidisc (this is an argument from [11]). But see the examples below.

5.3. The representation $T_s \otimes \overline{T}_\sigma$. This representation is realized in $\mathcal{H}_s \otimes \mathcal{H}_\sigma$ by the unitary operators

$$f(z_1, z_2) \mapsto f\left(\frac{az_1 + b}{\bar{b}z_1 + \bar{a}}, \frac{\bar{a}z_2 + \bar{b}}{bz_2 + a}\right)(\bar{b}z_1 + \bar{a})^{-s}(bz_2 + a)^{-\sigma}.$$

This time it is natural to consider the operator of restriction to the curve $\tilde{\Delta}$: $z_1 = e^{it}$, $z_2 = e^{-it}$. This curve, however, does not satisfy the conditions of Theorem 1, which changes the whole situation radically.

According to the remark in Section 5.1, the boundary values of functions $f \in \mathcal{H}_s \otimes \mathcal{H}_\sigma$ belong to the Sobolev space $W^{(1-s)/2}(S^1) \otimes W^{(1-\sigma)/2}(S^1)$.

By virtue of Proposition 6, the operator of restriction to $\tilde{\Delta}$ is well defined in $\mathcal{H}_s \otimes \mathcal{H}_\sigma$ if $s + \sigma < 1$. Thus we obtain a space of functions on $\tilde{\Delta}$ (namely, the restrictions of functions $f \in \mathcal{H}_s \otimes \mathcal{H}_\sigma$) and an action of $\mathrm{SL}_2^\sim(\mathbb{R})$ in this function space. It is easy to verify that one of the series representations of $\mathrm{SL}_2^\sim(\mathbb{R})$ (see [26]) is realized in the functions on $\tilde{\Delta}$.

We can obtain nothing more with the help of $\tilde{\Delta}$. It is well known, however, that there is a discrete spectrum in $T_s \otimes \overline{T}_\sigma$ if $|s - \sigma| > 1$, and our next task is to produce it.

5.4. The dual model of $\overline{T}_s$. Consider the Sobolev space $W^{(s-1)/2}$ (see § 4). Let $\mathcal{L}$ be the subspace consisting of functions that can be continued holomorphically inside the disc $|z| < 1$. Let $\mathring{\mathcal{H}}_s$ be the quotient space $W^{(s-1)/2}/\mathcal{L}$. The group $\mathrm{SL}_2^\sim(\mathbb{R})$ acts in $W^{(s-1)/2}$ by the operators

$$\mathring{T}_s(g)\Phi(z) = \Phi\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)(\bar{b}z+\bar{a})^{s-1}$$

(where $|z| = 1$). It is obvious that the subspace $\mathcal{L}$ is invariant under $\mathring{T}_s$, whence we obtain a representation of $\mathrm{SL}_2^\sim(\mathbb{R})$ in the quotient space $\mathring{\mathcal{H}}_s$. We denote this representation by the same symbol $\mathring{T}_s$.

The scalar product in $\mathring{\mathcal{H}}_s$ is defined by the formula

$$\langle \Phi_1, \Phi_2 \rangle = \lim_{\varepsilon \to 0} \int_{|z|=1} \int_{|u|=1} \frac{\Phi_1(z)\overline{\Phi_2(u)}\, dz\, d\bar{u}}{\bigl(1 - (1-\varepsilon)z\bar{u}\bigr)^s}$$

(if Φ_1 or Φ_2 is holomorphic in the disc, then $\langle \Phi_1, \Phi_2 \rangle = 0$). We write the norm defined by this scalar product as follows:

$$\left\| \sum_{k>0} a_k z^{-k} + \sum_{k \geqslant 0} c_k z^k \right\|^2 = \sum_{k>0} \frac{(s+1)(s+2)\dots(s+k)}{k!} |a_k|^2.$$

Consider the bilinear form $\mathcal{H}_s \times \mathring{\mathcal{H}}_s \to \mathbb{C}$ defined by the formula

$$\{f, \Phi\} = \int_{|z|=1} f(z)\Phi(z)\, dz.$$

It is obvious that

$$\bigl\{T_s(g)f, \mathring{T}_s(g)\Phi\bigr\} = \{f, \Phi\},$$

whence the representations T_s and $\mathring{T}_s$ are mutually dual, that is, $\mathring{T}_s \simeq \overline{T}_s$.

5.5. The remaining discrete spectrum in $\boldsymbol{T_s \otimes \overline{T}_\sigma}$. Let $\sigma > s+1$. Consider the representation $T_s \otimes \overline{T}_\sigma$ in $\mathcal{H}_s \otimes \mathring{\mathcal{H}}_\sigma$. The space $\mathcal{H}_s \otimes \mathring{\mathcal{H}}_\sigma$ is a quotient space of $\mathcal{H}_s \otimes W^{(\sigma-1)/2}$, and $\mathcal{H}_s \otimes W^{(\sigma-1)/2} \subset W^{-(s-1)/2} \otimes W^{(\sigma-1)/2}$. We can now consider the operator of restriction to Δ (see Proposition 6).

Let k be a positive integer such that $s+k < \sigma \leqslant s+k+1$. Let Q_α be the space of all functions $f \in \mathcal{H}_s \otimes \mathring{\mathcal{H}}_\sigma$ the restrictions of whose partial derivatives of order $\leqslant \alpha$ to Δ are equal to zero. Let us recall that differentiation maps W^λ into $W^{\lambda-1}$, whence Proposition 6 enables us to consider spaces Q_α with $\alpha \leqslant k$. We obtain a finite $\mathrm{SL}_2^\sim(\mathbb{R})$-invariant filtration in $\mathcal{H}_s \otimes \mathring{\mathcal{H}}_\sigma$:

$$\mathcal{H}_s \otimes \mathring{\mathcal{H}}_\sigma = Q_{-1} \supset Q_0 \supset Q_1 \supset \cdots \supset Q_k.$$

It is easy to verify that the irreducible representation $\overline{T}_{\sigma-s-2j}$ is realized in the quotient space Q_{j-1}/Q_j.

5.6. An unsuccessful method of producing the discrete spectrum in $\boldsymbol{\overline{T}_s \otimes \overline{T}_\sigma}$. Consider the representation $\mathring{T}_s \otimes \mathring{T}_\sigma$ in $\mathring{\mathcal{H}}_s \otimes \mathring{\mathcal{H}}_\sigma$ (it is very close to $T_s \otimes T_\sigma$). Proposition 6 enables us to consider the operator of restriction to the diagonal in $\mathring{\mathcal{H}}_s \otimes \mathring{\mathcal{H}}_\sigma$, but this yields only finitely many subrepresentations, whereas the set of all such representations is countable (see § 5.2).

5.7. The tensor product of $\boldsymbol{\overline{T}_s}$ and a representation from the basic series. We fix $\tau \in \mathbb{R}$. Consider the representation P_τ of $\mathrm{SL}_2(\mathbb{R})$ in L^2 on the circle $|z| = 1$ defined by the formula

$$P_\tau(g)f(z) = f\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)|\bar{b}z+\bar{a}|^{\frac{1}{2}+i\tau}$$

(these representations are called *representations from the even basic series*).

Consider the tensor product $\mathring{T}_s \otimes P_\tau$. It is natural to realize it in the space $\mathring{\mathcal{H}}_s \otimes L^2$, which is a quotient space of $W^{(s-1)/2} \otimes L^2$. We can apply Proposition 6 to the latter space and consider the operator of restriction to Δ.

This construction provides finitely many irreducible subrepresentations in $\overline{T}_s \otimes P_\tau$, although the discrete spectrum is countable.

5.8. Historical remarks. The problem of calculating the spectrum of the tensor product of two unitary representations of $\mathrm{SL}_2(\mathbb{R})$ was, in principle, solved in the classical paper of Pukanszky [26] (this problem was later discussed in [16], [28]). If we consider $\mathrm{SL}_2^\sim(\mathbb{R})$, then certain additional phenomena appear which are connected with representations that are not weakly contained in the regular representation. The presence of a discrete summand in the situation described in § 5.3 (the tensor product of a representation with a small highest weight and a representation with a small lowest weight) was announced in [21].

5.9. Generalizations. Various generalizations of the situation described in § 5.3 were discussed in [22], §§ 2–4, 7. Among similar problems not considered in [22], let us mention the problem (see [1], [15]) of restricting the harmonic representation (the Friedrichs–Segal–Berezin–Shale–Weyl representation) of $\mathrm{Sp}\bigl(2k(p+q),\mathbb{R}\bigr)$ to $\mathrm{Sp}(2k,\mathbb{R})\times O(p,q)$. It would be interesting to understand whether it is possible to find the spectrum from the papers cited in the case of the restriction to an orbit in the boundary of a Cartan domain of type III. It would be interesting to find out whether the "small" continuous spectra (see [24]) are connected with the operator of restriction to a non-compact orbit.

The construction described in §§ 5.5 and 5.7 works in the following situation. Let G be a semisimple Lie group. Let P and T be two unitary representations of G realized in Hilbert spaces of distributions on a flag manifold M. All that is needed to apply our technique is that one of these representations should act in a Hilbert space of "sufficiently smooth" functions on M. The question of the degree of smoothness of functions in spaces of unitary representations has not been studied. Our technique always works if one of the representations has a "sufficiently large" highest weight. We can show that in this case our construction yields a discrete spectrum that consists of discrete Harish-Chandra series.

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