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A SEMIGROUP OF OPERATORS IN THE BOSON FOCK SPACE

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A boson Fock space with n degrees of freedom is a space of holomorphic functions on n-dimensional Hilberts space with the scalar product:

$$\langle f, g\rangle = \iint f(z)\,\overline{g(z)}\exp(-(z, z))\,dz\,d\bar{z}.$$

We are interested in operators of the form

$$Bf(z) = \iint \exp\left\{\frac{1}{2}(z\bar{u})\begin{pmatrix}K & L\\ M & N\end{pmatrix}\begin{pmatrix}z^t\\ \bar{u}^t\end{pmatrix}\right\} f(u)\exp(-(u, u))\,du\,d\bar{u}. \tag{0.1}$$

The main problem considered in this article is the problem of the boundedness of these operators.

Unitary operators of the form (0.1) appeared in [1], in such a form Berezin has written down the automorphisms of the canonical commutation relations. In numerous papers of the years 70-80 (we mention only [2-5]) the fundamental role of the automorphisms of canonical commutation and anticommutation relations in the representation theory for infinite dimensional groups has been clarified (this role is the same as for the operators of variables exchange and multiplication by a function in the representation theory of Lie groups). After it had been discovered that a representation of an infinite dimensional group is, in fact, the visible part of a representation of an essentially bigger and invisible with the unaided eye semigroup (see [6]), and, actually, even not a semigroup, but a category, at first a problem of semigroup with the Weil representation has arisen. Ol'shanskii indicated that this semigroup is semigroup BO of all operators of the form (0.1), and then a problem has arisen concerning the algebraic nature of this semigroup, as well as the problem of the boundedness of the operators. It turns out (Ol'shanskii), that for $n < \infty$ the boundedness of the operators (0.1) is equivalent to the pair of conditions: 1) $\left\|\begin{pmatrix}K & L\\ M & N\end{pmatrix}\right\| \leqslant 1$; 2) $\|K\| < 1$, $\|N\| < 1$ (here, as everywhere in this paper, under the norm of a matrix we understand the Euclidean norm). In the joint paper by Ol'shanskii, Nazarov, and the author [9] it has been clarified that the considered semigroup is isomorphic to some semigroup of linear relations.

In Sec. 1 of this paper we introduce an accurate definition of operators of the form (0.1), in Sec. 2 we discuss a realization of the semigroup BO as a semigroup of linear relations, and a semigroup of generalized fraction-linear Krein transformations of an infinite dimensional matrix ball. In Secs. 3 and 4 we formulate and prove theorems on the boundedness of the operators. In Sec. 5 we consider a somewhat more general class of operators.

For applications of the semigroup BO to the representation theory of the Virasoro algebra, and to the conformal quantum field theory (cf. [7; 10]), see the Fermion analog of this paper (cf., [8]).

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1. Operators B[S]

1.1. Boson Fock Space. Let V be a complex Hilbert space of dimension n = 0, 1, 2, ..., ∞ with the scalar product (·, ·). We are interested mainly in the case $n = \infty$. We shall assume simply a noninvariant point of view and suppose that V is the coordinate space C^n or ℓ_2. We shall consider the space F(V) of holomorphic functions on V with the scalar product

$$\langle f, g\rangle = \iint f(z)\overline{g(z)} \exp(-(z, z))\, dz\, d\bar{z}.$$

If $n < \infty$ then this formula should be understood literally [1], for $n = \infty$ the simplest is to consider the inductive limit $\varinjlim F(C^h)$ for $n \to \infty$ take its completion and obtain $F(\ell_2)$.

It is known [1] that any bounded operator in F(V) is an integral operator in the direct meaning of this term. Indeed, let A be a bounded operator. Let us consider the function

$$K(u, \bar{v}) = \langle \exp(z, v);\ A \exp(z, u)\rangle, \tag{1.1}$$

where $e^{(z,v)}$ and $e^{(z,u)}$ are considered as functions of z. Then

$$Af(u) = \iint K(u, \bar{v}) f(v) \exp(-(v, v))\, dv\, d\bar{v}. \tag{1.2}$$

The kernel $K(u, \bar{v})$ is a function which is holomorphic in the variable u and antiholomorphic in the variable v. Taking into account that the norm of the function $\exp(z, v)$ equals $\exp\left(\frac{1}{2}(v, v)\right)$, and also (1.1), we obtain that the kernel $K(u, \bar{v})$ satisfies the estimation

$$|K(u, \bar{v})| \leqslant \|A\| \exp\left(\frac{1}{2}(u, u) + \frac{1}{2}(v, v)\right). \tag{1.3}$$

1.2. Operators B[S]. Operator B[S] in the boson Fock space F(V) is called an operator with a kernel of the form

$$K(z, \bar{u}) = \exp\left\{\frac{1}{2}(z\bar{u})\begin{pmatrix} K & L \\ L^t & M\end{pmatrix}\begin{pmatrix} z^t \\ \bar{u}^t\end{pmatrix}\right\}. \tag{1.4}$$

where z, u are matrix rows, t denotes the matrix transposition, and the matrix $S = \begin{pmatrix} K & L \\ L^t & M\end{pmatrix}$ satisfies the conditions:

0°. $S = S^t$;

1°. $\|S\| \leqslant 1$;

2°. $\|K\| < 1$, $\|M\| < 1$;

3°. K and M are Hilbert–Schmidt operators.

Proposition 1.1 (G. I. Ol'shanskii). If operator A with a kernel of the form (1.4) is bounded, then the conditions 0°-3° are satisfied.

Proof. 1° follows from (1.3).

2°. $A\cdot 1 = \exp\left\{\frac{1}{2} zKz^t\right\}$. This function lies in F(V) if and only if $\|K\| < 1$, where K is a Hilbert–Schmidt operator. To obtain an analogical statement for M $A^*\cdot 1$ should be considered.

Counterexample. Let Λ be a diagonal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots$, where $\Sigma|\lambda_i|^2 < \infty$ and $\Sigma|\lambda_i|^2 = \infty$. Let $S = \begin{pmatrix} \Lambda & 1-\Lambda \\ 1-\Lambda & \Lambda\end{pmatrix}$. Then the operator B[S] is unbounded.

In the sequel we shall see that the necessary conditions of the boundedness 1°-3° are very close to the sufficient conditions, namely, if we assume that $\|S\| < 1$ or if we replace in condition 3° the Hilbert–Schmidt property by the nuclearity, then the operator B[S] will be bounded. In the meantime we should define operators B[S] more accurately.

1.3. Domain of Definition of the Operators B[S]. Let T be a Hilbert–Schmidt operator V with

$$T = T^t, \|T\| < 1. \tag{1.5}$$

We shall define vectors $b[T] = b_z[T] = \exp(1/2\,(zTz^t))$ in the space F(V), the condition $\|T\| < 1$ guarantees that $b[T] \in F(V)$.

Proposition 1.2. Let $K(z, \bar{u})$ be a kernel of the form (1.4), let $\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ satisfy the conditions 0°-3°, and T conditions (1.5). Then

$$\iint K(z, \bar{u})\, b_u[T] \exp(-(u, u))\, du\, d\bar{u} = \det[(1 - MT)^{-1/2}]\, b_z[K + LT(1 - MT)^{-1} L^t],$$

where the integral converges and the RHS of the equality is contained in F(V).

Proof. For the proof we need only the condition $\| K + LT(1 - MT)^{-1} L^t \| < 1$ It follows from the next lemma.

LEMMA 1.1 (Krein and Shmul'yan). Let $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ be a block 2n × 2n-matrix, with $\| X \| \leqslant 1$, $\| X_{22} \| < 1$. Let T be a matrix of dimension n × n. Let $\nu = \| X_{11} + X_{12} T (1 - X_{22} T)^{-1} X_{21} \|$.

a) If $\| T \| \leqslant 1$, then $\nu \leqslant 1$.

b) If $\| T \| < 1$, then $\nu < 1$.

c) If $\| S \| < 1$, $\| T \| \leqslant 1$, then $\nu < 1$.

This is a well-known result (see [11-13]), and a little bit later we shall understand that it has a clear geometrical sense.

Further, we shall introduce the vectors $b[T \mid \alpha] = \exp\left\{\frac{1}{2} z T z^t + \alpha z^t\right\}$, where T satisfies all these conditions and $\alpha \in V$. It is easy to see that $b[T \mid \alpha] \in F(V)$. The set $F_0(V)$ of all finite linear combinations of the vector $b[T|\alpha]$ is dense in F(V).

Proposition 1.3. Let $\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$, T, α be the same as before. Then

$$\iint \exp\left\{\frac{1}{2} (z \bar{u}) \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \begin{pmatrix} z^t \\ \bar{u}^t \end{pmatrix}\right\} \exp\left\{\frac{1}{2} u T u^t + \alpha u^t\right\} \exp(-(u, u))\, du\, d\bar{u} = $$
$$= c b[K + LT(1 - MT)^{-1} L^t \mid L(1 - TM)^{-1} \alpha^t],$$

where c is some constant. The integral is absolutely convergent and the RHS of this relation lies in F(V).

The proposition can be proved by a direct computation.

The explicit form of the RHS is not needed further, we are interested only in the following.

COROLLARY. The set $F_0(V)$ is invariant with respect ot B[S].

Hence all the operators B(S) have a joint dense invariant domain of definition.

1.4. Product of the Operators B[S].

THEOREM 1.1. Let

$$S_1 = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}, \quad S_2 = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}.$$

Then

$$B[S_1]\, B[S_2] = \det((1 - MP)^{-1/2})\, B[S_1 * S_2];$$

where

$$S_1 * S_2 = \begin{pmatrix} K + LP(1 - MP)^{-1} L^t & L(1 - PM) Q \\ Q^t (1 - MP)^{-1} L^t & R + Q^t (1 - MP)^{-1} MQ \end{pmatrix}. \tag{1.6}$$

Proof. The formula can be verified by a direct computation. For the proof we need only the fact that the matrix $S_1 * S_2$ satisfies the conditions 0°-3°. The verification of condition 3° is obvious, condition 2° follows immediately from the Krein–Shmul'yan lemma (b), and to obtain 1° we have to apply the Krein–Shmul'yan lemma to the matrices

$$\begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} K & & L & \\ & 0 & & 1 \\ \hline L^t & & M & \\ & 1 & & 0 \end{array}\right), \quad T = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}.$$

The algebraic structure behind formula (1.6) has been explained in [9]; we shall consider it in the following section.

1.5. Adjoint Operator. Let $S=\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$. We shall put $S^*=\begin{pmatrix} \bar{M} & \bar{L}^t \\ \bar{L} & \bar{K} \end{pmatrix}$.

Proposition 1.4. Let $f_1, f_2 \in F_0(V)$. Then $\langle B[S] f_1, f_2\rangle = \langle f_1, B[S^*] f_2\rangle$.

The proof needs only a direct verification. (Although it would not be cautiously enough to claim that B[S*] = B[S*].)

1.6. Convergence. Proposition 1.5. Let us consider a sequence of bounded operators $B[S_i]$. In order that $B[S_i]$ converge weakly to B[S] it necessary and sufficient that the two conditions hold

1) $B[S_i]$ is uniformly bounded.

2) $S_i \to S$ weakly.

Moreover, if 1) and 2) are satisfied then operator B[S] is bounded.

Proof. We shall show, for instance, the necessity. We shall consider the set R of all vectors from F(V) of the form $f_\alpha(z) = \exp(\alpha z^t)$, $\alpha \in V$. If $B[S_i]$ weakly converges in B[S] then $\langle f_u, B[S_i] f_v\rangle = K_i(u, \bar{v})$ converges for arbitrary u and v to $\langle f_u, B[S_i] f_v\rangle = K(u, \bar{v})$. Hence, the sequence of kernels converges pointwise, that means, also the sequence $(u\bar{v}) S_i \begin{pmatrix} u \\ \bar{v}^t \end{pmatrix}$, should converge pointwise, and this means the weak convergence of the operators S_i.

We notice that usually condition 2) can be easily verified, and, at the contrary, 1) is very difficult to verify.

2. Weil Representation

2.1. Shale–Berezin Symplectic Group. By Sp_n we shall denote the group of all block 2n × 2n-matrices (n = 0, 1, ..., ∞) of the form $Y=\begin{pmatrix} \Phi & \Psi \\ \bar{\Psi} & \bar{\Phi} \end{pmatrix}$, satisfying the conditions:

1) Y preserves the symplectic form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

2) Ψ is a Hilbert–Schmidt operator.

Remark. If n < ∞ then Sp_n is isomorphic to the group $Sp(2n, \mathbf{R})$. To see this, we shall notice that the group Sp_n preserves the n-dimensional subspace consisting of vectors of the form $(a\ \bar{a})$.

THEOREM 2.1 (Berezin [1], see also [14]). The formula

$$\rho(Y) = \det((1-\Psi\Psi^*)^{1/2})\, B\begin{bmatrix} \bar{\Psi}\Phi^{-1} & \Phi^{t-1} \\ \Phi^{-1} & -\Phi^{-1}\Psi \end{bmatrix}$$

defines a projective unitary representation of the group Sp_n in F(V) (the Weil representation).

Remark. Matrix $\begin{pmatrix} \bar{\Psi}\Phi^{-1} & \Phi^{t-1} \\ \Phi^{-1} & -\Phi^{-1}\Psi \end{pmatrix}$ satisfies the conditions 0°-3°. Moreover, as it has been noticed by G. I. Ol'shanskii, this matrix is unitary.

2.2. Symplectic Subgroup. Let us consider two copies V_+ and V_- of the space V, where one of them consists of sequences of the form $(v_1^+, v_2^+, \ldots)$, and the other of sequences of the form $(v_1^-, v_2^-, \ldots)$. We shall introduce in $V_+ \oplus V_-$ a skew-symmetric form $\{\cdot, \cdot\}$ with the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and a Hermitian form $\Lambda(\cdot, \cdot)$ with the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Let $W_+ \oplus W_-$ be a second copy of the space $V_+ \oplus V_-$.

Let matrix $S=\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ satisfy the conditions 0°-2°. We shall construct for it a subspace $\ell(S)$ in $(V_+ \oplus V_-) \oplus (W_+ \oplus W_-)$. This subspace consists of all vectors of the form (v_+, v_-, w_+, w_-) satisfying the condition

$$\begin{pmatrix} v_+ \\ w_- \end{pmatrix} = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \begin{pmatrix} v_- \\ w_+ \end{pmatrix}.$$

The resulting subspace, generally speaking, is not a graph of an operator $V_+ \oplus V_- \to W_+ \oplus W_-$. We shall consider $\ell(S)$ as a linear relation. We recall that linear relation can be multiplied, namely, if R_1, R_2, R_3 are linear spaces, and $L_1 \subset R_1 \oplus R_2$, $L_2 \subset R_2 \oplus R_3$ are linear relations, then, by definition, their product $L_2L_1 = L_3 \subset R_1 \oplus R_3$, consists of all pairs $(r_1, r_3) \in R_1 \oplus R_3$, for which there exists an $r_2 \in R_2$ such that $(r_1, r_2) \in L_1$, $(r_2, r_3) \in L_2$. Linear relations can be naturally considered as graphs of not everywhere defined and multivalued linear operators.

Linear relation $\ell = \ell(S)$ satisfies the conditions: 0^+. ℓ preserves the form $\{\cdot, \cdot\}$, i.e., if and, moreover, ℓ is maximal among spaces satisfying this condition, then $\{p_1, q_1\} = \{p_2, q_2\}$ (this is a consequence of condition 0°).

1^+. ℓ "stretches" the form $\Lambda(\cdot, \cdot)$, i.e., if $(p, q) \in l$, then $\Lambda(p, p) \leqslant \Lambda(q, q)$ (a consequence of condition 2°).

2^+. If $(p, 0) \in l$, then $\Lambda(p, p) \leqslant \varepsilon \| p \|^2$, if $(0, q) \in l$, then $\Lambda(q, q) \geqslant \varepsilon \| q \|^2$, where ε does not depend on p, q (this strengthening of condition 1^+ follows from 2°).

<u>Remark.</u> We shall introduce in $V_+ \oplus V_- \oplus W_+ \oplus W_-$ the forms

$$\{(v_1, w_1), (v_2, w_2)\}' = \{v_1, v_2\} - \{w_1, w_2\},$$
$$\Lambda'((v_1, w_1), (v_2, w_2)) = -\Lambda(v_1, v_2) + \Lambda(w_1, w_2),$$

where $v_i \in V_+ \oplus V_-$, $w_i \in W_+ \oplus W_-$. Then condition 0^+ means that ℓ is a Lagrangian subspace with respect to the form $\{\cdot, \cdot\}'$, and condition 1^+ means that the form Λ' is positive definite on ℓ.

Conversely, one can show that an arbitrary linear relation ℓ satisfying the conditions 0^+-2^+ has the form $\ell = \ell(S)$, where S satisfies the conditions 0°-2°.

<u>Proposition 2.1.</u> $l(S_1)\, l(S_2) = l(S_1 * S_2)$, where the multiplication $*$ is defined by formula (1.6).

The proposition can be verified by a computation.

We have not used yet condition 3°. It determines in the semigroup of all linear relations satisfying conditions 0°-2° a subsemigroup which we shall call a symplectic semigroup ΓSp.

Then Theorem 1.1 can be reformulated in the following way.

<u>THEOREM 2.2.</u> Let $l \in \Gamma\mathrm{Sp}$. Let $S = S(\ell)$ be the corresponding matrix. Then $\ell \to B[S(\ell)]$ is a projective representation of semigroup ΓSp.

This representation generates the Weil representation of the Shale-Berezin group, and thus it can be also called the Weil representation.

<u>2.2. Infinite Dimensional Cartan Domain $\mathscr{L}$.</u> By $\mathscr{L} = \mathscr{L}_n$ we shall denote the set of complete symmetric Hilbert–Schmidt matrices of dimension $n \times n$ $(n \leqslant \infty)$ with the norm <1. If $n < \infty$ then $\mathscr{L}_n$ is one of realizations $\mathscr{L}_n$ of Hermitian symmetric space $\mathrm{Sp}(2n, \mathbf{R}) / U(n)$.

Let $\mathbf{P}F(V)$ be a projective Fock space. We shall consider the mapping $\varkappa\colon \mathscr{L}_n \to \mathbf{P}F(V)$, given by the formula $\varkappa(T) = b[T]$.

Semigroup ΓSp, acting in the Fock space, maps vectors of the form b[T] into vectors of the same form (Proposition 1.1). Thus, we obtain the action of the semigroup ΓSp on $\mathscr{L}$. It is given by the formula

$$\mu\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} T = K + LT(1 - MT)^{-1} L^t.$$

(the Krein–Shmul'yan lemma from 1.3 asserts that the mappings $\mu(S)$ map domain $\mathscr{L}$ into itself).

Mapping $\mu(S)$ is invertible if and only if the matrix S is unitary. The group of all invertible mappings of the form $\mu(S)$ is the Shale–Berezin group, and in the finite dimensional case it is simply $\mathrm{Sp}(2n, \mathbf{R})$.

Further we shall mention that with each $T \in \mathscr{L}$ there is connected a subspace $\lambda(T)$ in $V_+ \oplus V_-$. It consists of all vectors of the form (v_+, Tv_+). By the symmetricity of matrix T the subspace $\lambda(T)$ is Lagrangian with respect to the form $\{\cdot, \cdot\}$. By the condition $\|T\| < 1$ form $\lambda(\cdot, \cdot)$ is positive definite on $\lambda(T)$.

<u>Proposition 2.2.</u> $\lambda(\mu(S)\, T) = l(S)\, \lambda(T)$.

<u>Proof.</u> This is based on a simple verification.

Hence, in such terms, statement a) of the Krein–Shmul'yan lemma becomes obvious.

3. Boundedness Theorems

3.1. Formulation of the Theorems. Let S satisfy the conditions 0°-3° from 1.2.

THEOREM 3.1. If $\|S\| < 1$ then B[S] is bounded.

THEOREM 3.2. If K and M are nuclear operators then S* is bounded.

3.2. Reduction to the Self-Adjoint Case. LEMMA 3.1. a) If S satisfies the assumptions of Theorem 3.1, then S* satisfies the assumption of Theorem 3.1.

b) If S_1, S_2 satisfy the assumption of Theorem 3.1, then $S_1 * S_2$ satisfies the assumption of Theorem 3.1.

LEMMA 3.2. a) If S satisfies the assumption of Theorem 3.2, then S* satisfies the assumption of Theorem 3.2.

b) If S_1, S_2 satisfy the assumption of Theorem 3.2, then $S_1 * S_2$ satisfies the assumption of Theorem 3.2.

Proof. All the statements are obvious, except 3.1b), which can be verified with the help of the Krein–Shmul'yan lemma in the spirit of the proof of Theorem 1.1.

Now we shall recall that to prove the boundedness of an operator A it is sufficient to verify the boundedness of the self-adjoint operator A*A. Hence, by the Lemmas 3.1 and 3.2 we can confine ourselves to the case S = S*.

3.3. Eigenvectors. Proposition 3.1. Let S = S* and for some $T \in \mathcal{X}_n$ hold B[S]b[T] = λb[T]. Then $\|B[T]\| = \lambda$.

Proof. We are to prove that the norm of the operator is assumed on vector b[T]. It is easy to verify that the Shale–Berezin group Sp is acting transitively on the domain $\mathcal{X}_n$, and therefore without loss of generality we can assume that T = 0. Thus b[T] = b[0] = 1. But then matrix S has the form $\begin{pmatrix} 0 & L \\ L^t & 0 \end{pmatrix}$, and consequently, operator B[S] is simply the operator of variable exchange B[S]f(z) = B(Lz).

Now we notice that F(V) decomposes into a direct sum of Hilbert spaces $\mathcal{S}^k V$, where $\mathcal{S}^k V$ is the set of all homogeneous functions of degree k. Operator B[S] is acting in every $\mathcal{S}^k V$ as the k-symmetric poser of L. But $\|L\| \leqslant 1$, therefore $\|B(S)\| = 1$, and thus the norm of B[S] is assumed on the vacuum vector b[0].

COROLLARY (to Proposition 3.1 and 1.2). Let the assumptions of the theorem be satisfied and $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$. Then

$$\| B[S] \| = \det ((1 - MT)^{-1/2}).$$

3.4. Fixed Points. Proposition 3.2. The following conditions are equivalent:

1) b[T] is an eigenvector of the operator B[S];

2) T is a fixed point of the mapping μ(S);

3) the subspace T is invariant with respect to the linear relation μ(T).

This proposition is a tautology. Of course, we shall use the implication 2 ⇒ 1. The implication 2 ⇒ 3 (in a sitaution similar to ours) has been mentioned by Krein [15] (Krein's theorem on an invariant subspace) and consequently it has been often used. An extensive bibliography on this subject has been cited in [16] (see also [18], close to Secs. 4.1 and 4.2 of this paper).

3.5. Finite-Dimensional Case. Here we shall consider the case when n = dim V < ∞. Let $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$.

Proposition 3.3. Let S = S* then $\| B[S] \| \leqslant \det (1 - |M|)^{-1/2}$, where $|M| = \sqrt{M^*M}$.

Proof. First of all everything can be reduced to the case when $\|S\| < 1$. To cope with this case it is enough to consider the sequence $S_n = (1 - 1/n)S$ and apply Proposition 1.5.

Let $\overline{\mathcal{Z}}_n$ be the closure of $\mathcal{Z}_n$ in the space of matrices. By the Brouwer theorem the mapping $\mu(S)$ of the domain $\overline{\mathcal{Z}}_n$ should possess a fixed point. Since $\|S\| < 1$ then by the Krein–Shmul'yan lemma the mapping $\mu(S)$ transforms $\overline{\mathcal{Z}}_n$ into the interion of $\mathcal{Z}_n$, and therefore the fixed point lies inside $\mathcal{Z}_n$. By Proposition 3.2 the operator $B[S]$ has an eigenvector of the form $b[T]$, and hence, by Proposition 3.1 we have $\|B[S]\| = \det((1 - MT)^{-1/2})$. It remains to prove the following lemma.

LEMMA 3.3. Let X and P be square matrices $\|X\| < 1$, where X is positive definite. Let $\|P\| \leqslant 1$. Then

$$|\det(1 - PX)| \geqslant \det(1 - X).$$

Proof. The lemma can easily be derived from the von Neumann–Horn inequality and the Weyl inequality for singular and eigenvalues of a linear operator (see [17]).

In such a way we have obtained a new proof of the theorem of G. I. Ol'shanskii on the boundedness of the operators $B[S]$ in the finite-dimensional case (the original proof of Ol'shanskii gives an exact formula for the norm of the operator $B[S]$, see [9]).

3.6. Proof of Theorem 3.2. We shall denote by P_n the projection from $F(V)$ onto the set of functions depending only on the variables $z_1, \ldots, z_n$. It is easy to see that

$$P_n = B\left[\begin{array}{cc|cc} 0 & & 1_{,} & \\ & 0 & & 0 \\ \hline 1_n & & 0 & \\ & 0 & & 0 \end{array}\right] \stackrel{\text{def}}{=} B[\pi_n],$$

where by 1_n we denote the unit matrix of the rank n. The image of the projection P_n can be naturally identified with $F(\mathbb{C}^n)$. We shall consider operator $B[S_n]$, where $S_n = \pi_n * S * \pi_n$. Let us write S as a block matrix of dimension $(n + \infty) \times (n + \infty)$, and let

$$S = \left(\begin{array}{cc|cc} K_{11} & K_{12} & L_{11} & L_{12} \\ K_{21} & K_{22} & L_{21} & \\ \hline L_{11}^t & L_{21}^t & M_{11} & M_{12} \\ L_{12}^t & L_{22}^t & M_{21} & M_{22} \end{array}\right), \quad \text{then} \quad S_n = \left(\begin{array}{cc|c} K_{11} & 0 & L \\ 0 & 0 & 0 \\ \hline L_{11}^t & 0 & M \\ 0 & 0 & 0 \end{array}\right.$$

Now we prove the theorem. Let $S = S^*$ satisfy the assumptions of the theorem. Then:

1) $S_n \to S$ weakly;

2) $\|B[S_n]\| \leqslant \det(1_n - |K_{11}|)^{-1/2} \leqslant \det(1 - |K|)^{-1/2}$.

Hence, the assumptions of Proposition 1.5 are satisfied, and, therefore, $B[S]$ is bounded.

4. Geometry of Domain $\mathcal{Z}_\infty$, and Proof of Theorem 3.1

4.1. The Geometry of Finite Dimensional Domains $\mathcal{Z}_n$. Let $n < \infty$. We shall introduce in the domain $\mathcal{Z}_n$ a Riemann metric invariant with respect to the group $\mathrm{Sp}(2n, \mathbf{R})$:

$$ds^2 = \operatorname{tr}(1 - Z^*Z)^{-1} dZ^* (1 - ZZ^*)^{-1} dZ.$$

Then the distance between points $Z_1, Z_2 \in \mathcal{Z}$ can be calculated according to the formula

$$\rho^2(Z_1, Z_2) = \frac{1}{4} \sum_k \ln^2 \frac{1 + r_k^{1/2}}{1 - r_k^{1/2}}; \quad r_k = \frac{\lambda_k - 1}{\lambda_k}$$

where $\lambda_1, \lambda_2, \ldots$ is called "compound distance," i.e., the collection of eigenvalues of the matrix

$$(1 - Z_1^*Z_1)^{-1}(1 - Z_1^*Z_2)(1 - Z_2^*Z_2)^{-1}(1 - Z_2^*Z_1).$$

The family of numbers $\lambda_1, \lambda_2, \ldots$ is also invariant with respect to $\mathrm{Sp}(2n, \mathbf{R})$.

We shall introduce also an auxiliary Riemann metric

$$(ds^2)_E = \operatorname{tr} dZ^* dZ$$

and the corresponding distance

$$\rho_E^2(Z_1, Z_2) = \operatorname{tr}(Z_1 - Z_2)^*(Z_1 - Z_2).$$

Proposition 4.1. $\rho(\mu(S)Z_1, \mu(S)Z_2) \leqslant \rho(Z_1, Z_2)$.

Proof. Thanks to the continuity, without loss of generality we can assume that $\|S\| < 1$ and that $\mu(S)$ is injective. Any such S (see [20]) can be represented in the form $S = UL$, where $U \in \mathrm{Sp}(2n, \mathbf{R})$, and $L = L^*$. Since U preserves the metric, without loss of generality we can assume that the fixed point of mapping $\mu(S)$ is the point 0, and thus, $\mu(S)Z = LZL^t$. Now it remains to verify that $\mu(S)$ does not increase the Riemann metric ds^2.

4.2. Geometry of the Domain $\mathscr{L}_\infty$. We shall introduce on $\mathscr{L}_\infty$ a compound distance, metric ρ [it is easy to see that series (4.1) is convergent if $Z_1, Z_2 \in \mathscr{L}_\infty$], and also a metric ρ_E. Metric ρ_E determines on $\mathscr{L}_\infty$ the usual topology in the space of Hilbert–Schmidt operators.

Proposition 4.2. Domain $\mathscr{L}_\infty$ is complete with respect to the metric ρ.

Proof. We shall consider a ball B_C consisting of points $Z \in \mathscr{L}_\infty$, satisfying the condition $\rho(0, Z) \leqslant C$. One can show that on this ball the metrics ρ and ρ_E are equivalent, i.e., $\rho_E(Z_1, Z_2) \leqslant \rho(Z_1, Z_2) \leqslant k\rho_E(Z_1, Z_2)$ for some k depending only on C. The simplest thing to do is to verify the case of domains $\mathscr{L}_n$ for Riemann metrics ds^2 and $(ds^2)_E$, and then to take the limit for $n \to \infty$. But the space of all Hilbert–Schmidt operators is complete with respect to the metric ρ_E, and the ball B_C is closed in the topology given by the metric ρ_E [since the function $\rho(Z_1, Z_2)$ is continuous in the metric ρ_E].

Proposition 4.3. $\rho(\mu(S)Z_1, \mu(S)Z_2) \leqslant \rho(Z_1, Z_2)$ for any $Z_1, Z_2 \in \mathscr{L}_\infty$.

Proof. By Proposition 4.1 holds

$$\rho(\mu(\pi_n * S * \pi_n) Z_1, \ \mu(\pi_n * S * \pi_n)Z_2) \leqslant \rho(Z_1, Z_2)$$

[element π_n is introduced by equality (3.1)]. Further we take the limit for $n \to \infty$.

4.3. Proof of Theorem 3.2. By Sec. 3.4 it is sufficient to show that the mapping $\mu(S)$ for $\|S\| < 1$ has a fixed point in $\mathscr{L}_\infty$. For this end it is enough to verify that the mapping $\mu(S)$ is contractive in metric ρ.

We shall represent S in the form $S = S_1 * T_\varepsilon$, where $\|S_1\| \leqslant 1$, and $T_\varepsilon = \begin{pmatrix} 0 & 1-\varepsilon \\ 1-\varepsilon & 0 \end{pmatrix}$.

LEMMA. $\rho(\mu(T_\varepsilon)Z_1, \mu(T_\varepsilon)Z_2) \leqslant (1-\varepsilon)\rho(Z_1, Z_2)$.

Proof. This inequality can be directly checked in the finite-dimensional case on the level of the Riemann metric ds^2. Next we have to take the limit for $n \to \infty$.

Now $\mu(S) = \mu(S_1)\mu(T_\varepsilon)$. The mapping $\mu(S_1)$ does not increase distances and the mapping $\mu(T_\varepsilon)$ is contractive. Theorem 3.1 has been proved.

5. Generalizations and Remarks

5.1. Nonhomogeneous Operators. The Shale–Berezin groups have an important and frequently used extension. It consists of all affine transformations of the Hilbert space, such that their linear part is contained in the Shale–Berezin group. We shall consider an analogical extension for semigroup $\Gamma\,\mathrm{Sp}$.

For this end we shall introduce the operators

$$B[S \mid v^t] f(z) = B\left[\begin{matrix} K & L \\ L^t & M \end{matrix} \middle| \begin{matrix} \lambda^t \\ \mu^t \end{matrix}\right] f(z) = \iint \exp\left\{\frac{1}{2}(z\bar{u})\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}\begin{pmatrix} z^t \\ \bar{u}^t \end{pmatrix} + z\lambda^t + \bar{u}\mu^t\right\} \exp(-(u,u)) f(u)\, du\, d\bar{u},$$

where $\lambda, \mu \in V$.

To define precisely operators $B[S|v^t]$ we have to repeat all words said in Sec. 1. We omit them and present only the formula for the multiplications of operators

$$B\left[\begin{matrix} K & L \\ L^t & M \end{matrix} \middle| \begin{matrix} \lambda^t \\ \mu^t \end{matrix}\right] B\left[\begin{matrix} P & Q \\ Q^t & R \end{matrix} \middle| \begin{matrix} \pi^t \\ \varkappa^t \end{matrix}\right] = cB\left[\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} * \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix} \middle| \begin{matrix} \lambda + L(1-PM)^{-1}(\pi + P\mu) \\ \varkappa + Q^t(1-MP)^{-1}(M\pi + \mu) \end{matrix}\right]$$

THEOREM 5.1. If $\|S\| < 1$ then the operator $B[S|v^t]$ is bounded.

To prove this theorem we notice at first that in the case $\|S\| < 1$ the operator $B[S|v^t]$ can be represented in the form

$$B\left[\begin{matrix} K & L \\ L^t & M \end{matrix} \middle| \begin{matrix} \lambda^t \\ \mu^t \end{matrix}\right] = B\left[\begin{matrix} 0 & 1-\varepsilon \\ 1-\varepsilon & 0 \end{matrix} \middle| \begin{matrix} \lambda^t \\ 0 \end{matrix}\right] * B\left[\frac{1}{(1-\varepsilon)^2}\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \middle| \begin{matrix} 0 \\ 0 \end{matrix}\right] * B\left[\begin{matrix} 0 & 1-\varepsilon \\ 1-\varepsilon & 0 \end{matrix} \middle| \begin{matrix} 0 \\ \mu^t \end{matrix}\right]$$

By Theorem 3.1 the middle factor in the RHS is bounded for a sufficiently small $\varepsilon > 0$. In such a way, the problem reduces to the boundedness of operators of the form

$$B\begin{bmatrix} 0 & 1-\varepsilon & \lambda^t \\ 1-\varepsilon & 0 & \mu^t \end{bmatrix}$$

As before, without loss of generality, these operators can be considered self-adjoint, i.e., we can assume that $\lambda = \bar{\mu}$.

Given by them mappings in F(V) can be rewritten in the form

$$Af(z) = B\begin{bmatrix} 0 & 1-\varepsilon & b \\ 1-\varepsilon & 0 & b \end{bmatrix} f(z) = f((1-\varepsilon)z + b)\, e^{(z,\, b)}. \tag{5.1}$$

Now we notice that the space F(V) can be decomposed into a tensor product $F(\mathbb{C}^1)^{\otimes n}$ (where $n = \dim V$) with distinguished vectors $f_i(z) = 1$ (cf., [21]), and operator (5.1) decomposes into a tensor product of the operators

$$A_i f(z_i) = f((1-\varepsilon)z_i + b_i)e^{z_i \bar{b}_i}.$$

To verify the boundedness of the operator A it is enough to show that

$$\|A\| = \prod_{i=1}^{n} \|A_i\| < \infty, \quad n = 0, 1, \ldots, \infty. \tag{5.2}$$

The eigenfunctions of the operator A_i are the functions

$$g_m(z_i) = (-\varepsilon z_i + b_i)^m \exp\left(\frac{1}{\varepsilon}\bar{b}_i z_i\right),$$

where m = 0, 1, 2,... . The corresponding eigenvalues are

$$\sigma_m = (1-\varepsilon)^m \exp\left(\frac{1}{\varepsilon}\bar{b}_i z_i\right).$$

Hence, $\|A_i\| = \sigma_0$, that is, the product (5.2) converges, and thus the boundedness of the operator A has been shown. In this way also Theorem 5.1 has been proved.

5.2. Affine Symplectic Semigroup. We have already seen that the operators B[S] correspond to linear symplectic relations. Exactly the same, also operators $B[S|v^t]$ form a representation of affine symplectic relations (for details see [7]).

5.3. Symplectic Category. Operators of the type B[S] acting from one Fock space $F(V_1)$ into another $F(V_2)$ are not less important than operators acting from space F(V) into itself. They determine a representation of certain category of linear relations (see [7, 8]).

5.4. For the connection of operators B[S] with the canonical commutation relations see [7].

5.5. Paper [9] intersects with [22]

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COMPUTATIONAL COMPLEXITY OF IMMANENTS AND REPRESENTATIONS OF THE FULL LINEAR GROUP

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1. Statement of the Problem. Let $A = \| a_{ij} \| \in \mathrm{Mat}(\mathbf{C}, n)$ be a matrix of dimension n × n over the field of complex numbers $\mathbf{C}_n$ and χ be a character of the complex irreducible representation of the symmetric group S_n. The expression

$$d_\chi(A) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

is called the immanent of the matrix A corresponding to χ. The question of the computational complexity of immanents was put by Strassen (see [1]). In this paper, an upper bound on the complexity is given of computing an immanent, which in the case if χ is an alternating character, $\chi(\sigma) = \operatorname{sgn}\sigma$, or a unit character, $\forall\sigma\ \chi(\sigma) = 1$, coincides according to logarithmic order with the known bounds, respectively, for the determinant and permanent and is better in comparison with the ones known for some representations corresponding to intermediate Young diagrams. The author does not know any such nontrivial upper bounds on complexity for Young diagrams of general form. In the paper, use of the theory of representations of the full linear group GL (see [2, pp. 319-347]) is essential.

2. Notation. Let V be an n-dimensional vector space over C with scalar product < , > and an orthonormal basis $e_1, \ldots, e_n$. In the space $V^{\otimes n}$ let us, in the standard manner, introduce the scalar product (denote also < , >), the actions of S_n, permuting factors, and of the full linear group GL(V) [the n-th tensor power $\pi^{\otimes n}$ of the natural representation π of GL(V) in V]. Let us denote by π_χ the symmetrization of π by a representation of S_n with a character χ. For the dimension $\dim \pi_\chi$, an explicit effective formula [2, p. 326] is known.

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