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FRACTIONAL DIFFUSIONS AND QUASI-INVARIANT ACTIONS OF INFINITE-DIMENSIONAL GROUPS

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The purpose of this paper is twofold. First, we want to give a survey of questions which, in one way or another, are concerned with measures that are invariant or quasi-invariant with respect to infinite-dimensional groups. Here we have a contact of two fields of mathematics, i.e., the infinite-dimensional group theory (as usual, this is considered as a pure algebraic science here, although it is not true) and stochastic process theory. During the last 10–15 years, one finds a series of important phenomena on the boundary between these two fields; the understanding of these phenomena is regarded by the fact that experts in algebra and those of probability theory very bad understand each other. The quasi-invariant measures are only a part of this "boundary layer" picture.

The second more particular goal is to discuss actions and representations of the group of diffeomorphisms of a circle and loop groups, which, in one way or another, are related to fractional diffusions. In author's papers [19–21], it was found that the usual constructions of representations of these groups with a leading weight (cf. [32, 33]) admit nontrivial deformations. These deformations representations as well as those with a leading weight are realized in the boson or fermion Fock space. The boson Fock space has a series of models; the most convenient of them is the holomorphic Bargman-Sigal-Berezin model, which was found in the early of 60th. Before that, in the middle of 50th, one considers the space L_2 with respect to a Gaussian measure on an infinite-dimensional space [79] as the Fock space. In [65], there was discovered that the passing to this model of the Fock space allows one to see a series of phenomena which are completely invisible in the holomorphic model.

In Section 1, we present the classical construction of a Gauss measure on a Hilbert space, which is invariant with respect to rotations. As is known, this measure exists but has one (not essential) defiency, that is, this measure is supported outside the Hilbert space itself. Further, we formulate the Feldman-Hajek theorem on a description of linear operators that leave this measure to be quasi-invariant. In the sequel, this theorem is one of our main tools.

In Section 2 (where we present slightly refined results of [65]), we construct a one-parametric family of Gaussian measures on the space of distributions over a circle that are quasi-invariant with respect to twisted changes of variables (this family of measures includes the ordinary Wiener measure, and also stochastic processes, which are called the fractional diffusions). The properties of the constructed actions of the groups of diffeomorphisms Diff are subject to several essential surgeries within the parameter change. Considering invariant sets and quotient systems of these actions, we obtain several constructions of measures, which are quasi-invariant with respect to Diff, in particular, on the group diffeomorphisms of a finite smoothness itself as well as on the space of Cantor subsets of a circle.

In Section 3, we present analogous constructions for loop groups.

In Section 4, we return to the space Cant of Cantor subsets (of a semi-line, but not of a circle). With the help of increasing stable Levy process, we construct a one-parametric family of measures on Cant that are invariant with respect to the group of diffeomorphisms and also we calculate in an explicit form, the Radon-Nykodim derivatives of these measures.

In a natural way, there arises the question on the connection of our constructions with usual constructions of infinite group representation theory (cf. [25]). In Section 2, we show where the usual representations with a leading weight are contained in our picture. In Section 5, we discuss the translation into the language of probability theory (or measure theory) of the problem of constructing the latent structure connected with infinite-dimensional groups.

Section 6 is, in some sense, of a reform character; this is devoted to the enumeration and discussion of various constructions of quasi-invariant measures for different infinite-dimensional groups (symmetric, classical, diffeomorphism, loop groups, etc.).

In Section 7, we discuss the question on the reasonabless of the problem of description of (quasi-) invariant measures and on the form of statement of this problem.

In the described picture, there are more unclear things than clear ones (now only questions connected with the infinite symmetric group and "exchangeability" are relatively well understood). The cases where there are no answers on the simplest questions are fixed by the name "Question". To the author opinion, the most of these "Questions" can be completely solved. The cases where this call strong doubts will be stipulated.

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1 EXTENSION OF A HILBERT SPACE ACCORDING TO I. SIGAL AND HAJEK-FELDMAN THEOREM

The content of this section is classical; see [39], [35], [6] for the proof of theorems, which are formulated below.

1.1. Canonical Extension of a Hilbert space. Let H be a real (separable and infinite-dimensional) Hilbert space. We choose an orthonormal basis $e_1, e_2, ...,$ in H (in the sequel, we shall see that all is independent of this basis). then the Hilbert space H consists of series of the form $v = \sum \alpha_j l_j$, where $\sum \alpha^2 < \infty$. Consider the Hilbert space $\hat{H}$ consisting of series of the form

$$w = \sum x_j e_j.$$

On each copy of $\mathbb{R}$, we intruduce the Gaussian measure with density

$$\frac{1}{\sqrt{2\pi}} e^{\frac{x_j^2}{2}}$$

with respect to the Lebesgue measure (the factor is chosen in such a way that the measure of the whole line $\mathbb{R}$ is equal to 1). Further, we consider the product measure on $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times ...$, which is denoted by μ.

Thus, we have embedded our Hilbert space H into a certain measure space $\hat{H}$. Below, we see that the measure μ replaces successfully the Lebesgue measure, which does not exist on a Hilbert space.

The space $\hat{H}$ will be called the canonical extension of the Hilbert space H.

Example 1.1. The set of all bounded sequences has zero measure on $\hat{H} \cong \mathbb{R}^\infty$. Indeed, the probability of the fact that $|x_j| < C$ is a certain number $\sigma = \sigma(C) < 1$. Therefore the measure of all $x = (x_1, x_2, ...)$ such that $|x_j| < C, ..., |x_n| < C$ is equal to σ^n. Hence the measure of all x such that $|x_j| < C$ for all j is equal to 0.

Corollary 1.2. *The space H has zero measure in $\hat{H}$.*

Proposition 1.3. *Let*

$$\tau_j > 0 \quad \text{and} \quad \sum \tau_j < \infty.$$

Then the series

$$\sum \tau_j x_j^2$$

converges almost everywhere.

Proof. We have the series of positive functions; therefore, it is sufficient to verify the convergence of the series

$$\sum \int \tau_j x_j^2 dx_j = \sum \tau_j < \infty$$

consisting of integrals.

1.2. Example: the Wiener Measure. For a particular functional space H, the space $\hat{H}$ can be precisely and explicitely described (cf., e.g., [53]). For example, let us consider the space of functions f on $[0, \pi]$ such that

$$f(0) = f(\pi) = 0, \quad f'(x) \in L^2[0, \pi]$$

as a space H. Introduce the inner product

$$\langle f, g \rangle = \int f'(t) g'(t) dt$$

in H. Further, we choose the orthonormal basis

$$p_n(t) = \frac{\text{sine}\, nt}{n} \tag{1}$$

in H. Then, by definition, the space $\hat{H}$ consists of "random series"

$$\sum x_n p_n(t) = \sum \frac{x_n}{n} \text{sine}\, nt, \tag{2}$$

where $x = (x_1, \ldots) \in \mathbb{R}^\infty$. Under conditions of Proposition 1.3, we choose the sequence

$$\tau_n = \frac{1}{n^{1+\varepsilon}}.$$

Then in the sum (1) the inequality

$$\sum \frac{x_n}{n^{1+\varepsilon}} < \infty \tag{3}$$

holds with probability 1. In particular, $\sum \left(\frac{x_n}{n}\right)^2 < \infty$, and therefore, series (2) is in $L^2[0, \pi]$.

Thus, we see that $\hat{H}$ (with accuracy up to a set of zero measure) is identified with the set of real functions on $[0, \pi]$; this space consists of series (2) which satisfy inequality (3). Thus, we have constructed the measure on the function space (i.e., the stochastic process); this measure is called the *Wiener* (or *Brownian*) *bridge* (the word "bridge" originates from the boundary conditions $f(0) = f(\pi) = 0$). It is well-known that functions $f \in \hat{H}$ are continuous with probability 1, and moreover, for any $\varepsilon > 0$ and for any $f \in \hat{H}$,

$$|f(t_1) - f(t_2)| \leq C|t_2 - t_1|^{1/2} (\log |t_1 - t_2| + 1)^{1+\varepsilon} \tag{4}$$

for some C and all t_1 and t_2 with probability 1.

Further, let us consider the space of functions $f(t)$ defined on $[0,\infty)$ such that

$$f'(t) \in L^2(0,\infty) \quad \text{and} \quad f(0) = 0$$

as H. In H, we introduce the inner product

$$\langle f, g\rangle = \int_0^\infty f'(t)g'(t)dt.$$

As an orthonormal basis in H, we can choose, for example, primitives of the Laggere functions. The same arguments show us that $\hat{H} \subset L^2[0,\infty)$ (with accuracy to a set of zero measure). In fact, functions $f \in \hat{H}$ are continuous with probability 1 and satisfy the Hölder condition (4).

The just constructed measure is an ordinary Wiener measure. Recall a more direct description of this measure. Let us consider the space $C[0,\infty)$ of continuous functions. Fix certain numbers $t_k > ... > t_2 > t_1 > 0$. Let us assign to each function $f \in C[0,\infty)$ a couple of numbers

$$(u_0, ..., u_k) = (f(0), f(t_1), ..., f(t_k)) \in \mathbb{R}^{k+1}.$$

Thus, we have obtained the mapping $C[0,\infty) \to \mathbb{R}^{k+1}$. Then the image of a Wiener measure on $\mathbb{R}^{k+1}$ is a measure on $\mathbb{R}^k$ with density

$$\left(\frac{1}{2\pi}\right)^{k/2} \prod_{j=1}^{k} \frac{1}{\sqrt{t_k - t_{k-1}}} \exp\left(-\frac{(u_k - u_{k-1})^2}{2(t_k - t_{k-1})}\right)$$

with respect to the Lebesgue measure on $\mathbb{R}^{k+1}$ (we assume $t_0 = 0$).

The standard results on the Wiener measure are [18, 14, 8]; cf. also [6, 39].

1.3. Example: White Noise. The term "white noise" denotes a canonical extension of the space L_2.

For definiteness, consider $L_2[0,\pi]$. Then $\hat{L}_2$ consists of series of the form

$$\sum x_n \operatorname{sine} nt; \tag{5}$$

moreover, we can assume that

$$\sum \frac{x_n^2}{n^{1+\varepsilon}} < \infty. \tag{6}$$

In particular,

$$x_n = o(n^{\frac{1+\varepsilon}{2}}). \tag{7}$$

Of course, series (5) diverge for a fixed t. But, by condition (7), these series converge in the sense of distributions. Thus, we can consider the space of distributions on $[0,\pi]$ satisfying condition (6) as $\hat{L}_2$ (this is one of the Sobolev spaces; see Section 2 below).

It is easy to see that $\hat{L}_2$ consists precisely of derivatives of Wiener trajectories.

1.4. Action of the Orthogonal Group. Denote by $O(H)$ the group of alll orthogonal (i.e., real unitary) operators on a real Hilbert space H. In the case $H = l_2$, we shall denote by $\mathcal{O}(\infty)$ this group.

Fix $A \in O(\infty)$. Let $x = (x_1, x_2, ...) \in \mathbb{R}^\infty = \hat{l}_2$. Apply the matrix A to the vector x; we have

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots \\ a_{21}x_1 + a_{22}x_2 + \cdots \\ \cdots \end{pmatrix}. \tag{8}$$

At first glance, the expression standing on the right-hand side of htis equation has no sense. We can only say that the row $(a_{11}, a_{12}, ...$ is the series

$$\sum a_{1j}^2 < \infty.$$

Therefore $a_{11}x_1 + a_{12}x_2 + ...$ can be arbitrary series, and this series can diverge in principle. In fact, this is not true in general.

Theorem 1.4. (*cf.* [79]) (*a*) *For a given matrix $A \in O(\infty)$ the vector Ax is well defined for almost all $x \in \mathbb{R}^\infty$;*

(*b*) *the mapping $x \mapsto Ax$ preserves the measure μ on $\mathbb{R}^\infty$;*

(*c*) *if $A, B \in O(\infty)$, then*

$$A(Bx) = (AB)x$$

for almost all $x \in \mathbb{R}^\infty$.

In this theorem. the first assertion is surprising; nevertheless, this assertion is an obvious consequence of the following classical Kolmogorov-Khinchin theorem.

Theorem 1.5. (*cf.* [40]). *Let ξ_i be a sequence of independent random variables with zero mean. Then the series $\sum \xi_i$ converges almost everywhere iff the numerical series*

$$\sum D\xi_j = \sum \int \xi_j^2$$

converges.

In the case of Theorem 1.4, we take $\xi_i = a_{ij}x_j$.

1.5. The Canonical Property of the Construction. Assume that two orthonormal bases $e_1, e_2, ...$ and $f_1, f_2, ...$ are chosen in a Hilbert space H. Then the construction of Subsection 1.1 gives us two canonical extensions of the space

H; we denote by $\hat{H}_e$ and $\hat{H}_f$ these extensions. They consist of formal series of the form $\sum x_j e_j$ and $\sum y_j f_j$, respectively. Let A be the transition matrix from the basis e_j to the basis f_j. Then, by Theorem 1.4, the mapping

$$x \mapsto y = Ax$$

is an almost everywhere defined mapping from $\hat{H}_e$ into $\hat{H}_f$, which preserves a measure.

Thus, we have the canonical isomorphism

$$\hat{H}_e \to \hat{H}_f.$$

It should be emphasized that A is an *isomorphism of a measure space* but not a *bijection of sets*, and in particular, A is not *an isomorphism of linear spaces.*

1.6. Remark: Specific and Nonspecific Group Actions. There are two close, but different concepts of the action of a group G on a space M with a measure μ.

Let q be a bijection of a certain subset $M' \subset M$ of a full measure onto a certain subset $M'' \subset M$ of a full measure. One says that q preserves a measure if, for any measurable set $A \subset M$, we have

$$\mu(A) = \mu(qA).$$

An action of a group G on the space (M, μ) is a collection of mappings such that for any $g_1, g_2 \in G$, the equality

$$(g_1 g_2)x = g_1(g_2 x) \tag{9}$$

holds *for almost all* $x \in M$.

If all mappings $x \mapsto gx$ are bijections $M \to M$, and if equation (9) holds *for all* $x \in M$ (i.e., if G *acts on the set* M), then one says that the action is *specific.*

The action of $O(\infty)$ on $\mathbb{R}^\infty = \hat{l}_2$ is a classical example of a nonspecific action (this action could not be made specific by any "corrections" on a set of zero measure).

1.7. Quasi-Invariant Measures. Let M be a space with a measure μ, and let q be a bijection of a certain subset of a full measure in M onto a certain subset of a full measure in M. One says that the measure μ is quasi-invariant with respect to the mapping q if q and q^{-1} transfer subsets of zero measure into these ones.

If a mapping $q : M \to M$ leaves the measure to be quasi-invariant, then there exists the function $q'(x)$ on M (the Radon-Nykodim derivative) defined by the identity

$$\mu(qA) = \int_A q'(x) d\mu,$$

where $A \subset M$ is an arbitrary measurable subset.

We note that in the case where q is a smooth mapping $\mathbb{R}^n \to \mathbb{R}^n$, the Radon-Nykodim derivative is an ordinary Jacobian.

Let a group G acts on M by transformations that leaves the measure μ to be quasi-invariant. Let $s \in \mathbb{R}$. Then we have the unitary representation of the group G in $L^2(M)$, which is defined by the formula

$$T_s(g)f(x) = f(gx)g'(x)^{\frac{1}{2}+is}. \tag{10}$$

1.8. Shift-Invariant Property.

Theorem 1.6. (Cameron-Martin). *Let $a \in H$. Then the mapping $x \to x + a$ leaves the measure μ to be invariant in H; moreover, the Radon-Nykodim derivative is given by the formula*

$$\exp\left(-\frac{1}{2}\sum a_j^2 - \sum a_j x_j\right).$$

This is a very simple assertion. We want to complete the Radon-Nykodim derivative by the formula

$$\frac{\exp\left(-\frac{1}{2}\sum(x_j + a_j)^2\right)\prod\frac{d(x_j+a_j)}{\sqrt{2\pi}}}{\exp\left(-\frac{1}{2}\sum x_j^2\right)\prod\frac{dx_j}{\sqrt{2\pi}}};$$

this questionable-in-the-form computation is easily justified.

1.9. Feldman-Hajek Theorem. Let now A be an arbitrary bounded operator on l_2. Then the matrix elements a_{ij} of this operator satisfy the conditions

$$\sum_j |a_{ij}|^2 < \infty$$

for any i, and therefore, by the Kolmogorov-Khinchin theorem, the mapping

$$x \mapsto Ax$$

is well defined almost everywhere on $\mathbb{R}^\infty$.

Theorem 1.7. *The measure μ is quasi-invariant with respect to the transformation $x \mapsto Ax$ iff the operator A is represented in the form*

$$A = B(1 + S),$$

where $B \in O(\infty)$ and S is a Hilbert-Schmidt operator.

Recall that the operator is called a *Hilbert-Schmidt operator* if the matrix elements s_{ij} of this operator satisfy the condition

$$\sum_{i,j} |s_{ij}|^2 < \infty.$$

In other words, $\mathrm{tr}\,(S^*S) < \infty$.

Recall that an operator S is called nuclear if the operator $|S| = \sqrt{S^*S}$ has the trace in the usual sense.

We want to compute the Radon-Nykodim derivative of a mapping according to the formula

$$\frac{\exp(-\frac{1}{2}\|Ax\|^2)d(Ax)}{\exp(-\frac{1}{2}\|x\|^2)d(x)} = \exp\left(-\frac{1}{2}(\|Ax\|^2 - \|x\|^2)\right)\det(1+S)$$

$$= \exp(-\frac{1}{2}(2\langle Sx, x\rangle + \langle Sx, Sx\rangle))\det(1+S). \tag{11}$$

If the operator S is nuclear, then the answer is, in fact, true. If S is not nuclear, then the situation becomes to be more delicate as the following example shows.

Example 1.8. Let A be a diagonal matrix with eigenvalues $1 + \lambda_j$. Then formula (11) yelds

$$\frac{\prod(1+\lambda_j)}{\exp(\sum \lambda_j x_j^2)\exp(\sum \lambda_j^2 x_j^2)}.$$

Here the numerator and the first factor of the denominator are divergent in general. But their ratio is convergent (this fact is equivalent to the convergence of the series $\sum \lambda_j(x_j^2 - 1)$, which, in turn, is ensured by the Kolmogorov-Khinchin theorem).

Denote by GLO (H) the group of all operators on a Hilbert space H satisfying the conditions of Theorem 1.7.

1.10. The Space $L_2(H)$ as a Boson Fock Space. We do not pass to this aspect in detail (cf. [79, 39]) and make only several remarks.

According to one of the definitions, a boson Fock space $F(H)$ is the sum

$$\bigoplus_{j=0}^{\infty} S^j(H)$$

of symmetric powers S^j of the Hilbert space H. Thus, we have to establish the canonical isomorphism

$$L_2(\hat{H}) = \bigoplus_{j=0}^{\infty} S^j(H).$$

Let $h_k(x)$ be ordinary Hermite polynomials. Recall that they are determined by the following two conditions:

(1) The degree of $h_k(x)$ is equal to k;

(2) $\int\limits_{-\infty}^{\infty} h_k(x)h_l(x)dx = \delta_{kl}$.

Consider the arbitrary sequence of nonnegative integers $J = (j_1, j_2, ...)$ such that $j_s = 0$ for sufficiently large s. Set $|J| = j_1 + j_2 + ...$ Obviously, the functions

$$h_j := h_1(x_1)h_2(x_2)...$$

form an orthonormal basis in $L_2(\mathbb{R}^\infty)$.

Further, the space S^p spanned by functions $h_J(x)$ such that $|J| = p$ is identified with a natural basis of $S^k(l_2)$.

Also we note that the spaces $S^p \subset L_2(\mathbb{R}^\infty)$ are $O(\infty)$-invariant, and the representation of $\mathcal{O}(\infty)$ in S^p is irreducible.

Further, the *birth-annihilation operators* on $L_2(\mathbb{R}^\infty)$ are given by the formulas

$$a_k f = \frac{1}{\sqrt{2}}(-x_k + \frac{\partial}{\partial x})f, \quad a_k^* f = \frac{\partial}{\partial x} f.$$

They satisfy the usual canonical commutation relations

$$[a_k, a_l] = [a_k^*, a_l^*] = 0,$$

$$[a_k, a_l^*] = \delta_{kl}.$$

2 FRACTIONAL DIFFUSIONS AND QUASI-INVARIANT ACTIONS OF THE DIFFEOMORPHISM GROUP OF A CIRCLE

2.1. Sobolev Spaces on a Circle. Let us consider the circle S^1 with coordinate $\varphi \in [0, \pi]$. Real smooth functions on this circle are represented in the form

$$f(\varphi) = \sum_{k=-\infty}^{\infty} c_k e^{ik\varphi}, \tag{12}$$

where $c_k = \bar{c}_{-k}$, and for any N,

$$c_k = o(|k|^{-N}).$$

Distributions on the circle are also represemted in the form (12), but the coefficients c_k satisfy the condition

$$c_k = o(|k|^L)$$

for a certain L (depending on the function f). Let $s \in \mathbb{R}$. Define the Sobolev space H_s as the space of distributions

$$f(\varphi) = \sum_{k=-\infty}^{\infty} c_k e^{ik\varphi}$$

on the circle such that

$$\sum (k^2+1)^s |c_k|^2 < \infty. \tag{13}$$

Introduce the inner product on H_s by the formula

$$(\sum c_k e^{ik\varphi}, \sum b_k e^{ik\varphi}) = \sum (k^2+1)^s c_k b_k. \tag{14}$$

Note that the space H_s is precisely L_2 over the circle. For natural j, the space H_j consists of functions whose j-th derivative belongs to L_2. Respectively, for an arbitrary positive s, the space H_s consists of functions whose fractional derivative of order s belongs to L_2.

2.2. Fractional Diffusion. Consider a canonical extension of the Sobolev space H_s. Let $s > 0$. Under conditions of Proposition 1.3, choosing the sequence $\tau_n = 1/(n^2+1)^{1/2+\varepsilon}$, we obtain that we can take the space

$$\hat{H}_s \simeq H_{s-1/2-\varepsilon}$$

for an arbitrary $\varepsilon > 0$ as the space $\hat{H}_s$. Note that $H_{s-1/2}$ has zero measure in $\hat{H}_s$. The last assertion is equivalent to the fact that the series $\sum x_j^2/\sqrt{j^2+1}$ diverges almost everywhere on $\mathbb{R}^\infty$.

We see that for $s > 1/2$, we can take the space consisting of ordinary functions as $\hat{H}_s$. Moreover, for $\hat{H}_s$, we can take any of the spaces $H_{s-1/2-\varepsilon}$ which are contained in L_2 for a sufficiently small ε.

Remark 2.1. For $s = 1$, our construction differs nonessentially from the Wiener bridge, which was discussed above.

In what follows, for any $\delta > 0$, the inclusion $h_{-\delta} \supset L_1$ holds. Therefore L_1 has zero measure in $\hat{H}_s$ for $s < 1/2$. In fact, for $s \leq 1/2$, the space $\hat{H}_s$ consists of distributions.

2.3. Hölder property. We have just obtained a measure on the function space for each $s \in \mathbb{R}$, i.e., a stochastic process. Differential properties of trajectories of this process are important for us. This question was studied for a long time in probability theory (in particular, for $1/2 < s < 3/2$, these processes are known as fractional abnormal diffusions); we present several results referring to these stochastic processes (cf. [53, 70, 9], 1.7.).

Theorem 2.2. *Let $1/2 < s < 3/2$. Then*

(a) *The function $x(\varphi) \in \hat{H}_s$ is nondifferentiable everywhere with probability 1.*

(*b*) *The function* $x(\varphi) \in \hat{H}_s$ *satisfies the condition*

$$|x(\varphi_1) - x(\varphi_2)| \leq C|\varphi_1 - \varphi_2|^{s-1/2} \times (\log^{1+\varepsilon} |\varphi_1 - \varphi_2| + 1)$$

for a certain C *and all* φ_1, φ_2 *with probability* 1.

Note that the values of degrees in the latter inequality cannot be improved.

Remark 2.3. It is easily seen that the operator $\partial/\partial\varphi$ maps from $\hat{H}_s$ into $\hat{H}_{s-1}$; this allows us to apply Theorem 2.2. to arbitrary spaces $\hat{H}_s$ for $s > 3/2$.

Theorem 2.4. *Let* $1/2 < s < 3/2$. *Then for almost all* $x(\varphi) \in \hat{H}_s$, *the set* A_x *of solutions to the equation* $x(\varphi) = 0$ *satisfies the following conditions:*
(*a*) A_x *is a closed subset of continuum cardinality which has no isolated points.*
(*b*) *The Hausdorff dimension of the set* A_x *is equal to* $3/2 - s$.

Remark 2.5. For $s > 3/2$, a generic function $x(\varphi) \in \hat{H}_s$ is continuously differentiable, and the set A_x is finite with probability 1.

2.4. Changing the Inner Product. We note that formula (14) for the inner product in the Sobolev space seems to be not natural. In fact, this formula is one of the methods (and is not certainly the best one) for making the sense to the formula

$$\langle f, g, \rangle = \int (D^s f)(D^s g) d\varphi,$$

where D^s is the fractional derivation of order s. By the fraction derivation one can mean different operators, and respectively, the inner products on the Sobolev spaces can be written in various forms. Of course, we obtain different measures on canonical extensions of the Sobolev spaces.

Let a Hilbert space H be given with two inner products $\langle \cdot, \cdot \rangle$ and $\{\cdot, \cdot\}$. We say that these inner products are equivalent if there exists a Hilbert-Schmidt operator T such that for all $h_1, h_2 \in H$, the equality

$$\langle h_1, h_2 \rangle = \{(1 + T)h_1, h_2\}$$

holds. It is easy to see that equivalent measure on canonical extensions correspond to equivalent inner products.

Therefore formula (14) can be replaced by any formula of the form

$$\langle \sum c_n e^{in\varphi}, \sum a_n e^{in\varphi} \rangle = \sum \lambda_n a_n c_n,$$

where λ_n is an arbitrary positive sequence satisfying the condition

$$\sum (\lambda_n n^{-2s} - 1)^2 < \infty.$$

There naturally arises the question: what of these inner products is the best one? The answer to this question is not clear, but, in the sequel, we shall see that for $0 < s < 1$, there exists the best inner product. In [65], natural formulas are discussed for other s.

2.5. Basic Construction. Denote by Diff the group of infinitely differentiable diffeomorphisms of the circle which preserve an orientation of the latter one. Denote by Diff^1 the group of C^1-diffeomorphisms.

Fix $s \in \mathbb{R}$. For each s, let us consider a representation T_s of the group in the functional space over a circle given by the formula

$$T_s(q)f(\varphi) = f(q(\varphi))q'(\varphi)^{1/2-s}, \tag{15}$$

where $q \in \mathrm{Diff}$.

Theorem 2.6. *The operators $T_s(q)$ are in the groups* GLO *of the space H.*

This theorem is obtained in [19, 20]; see [22, 65] for the detailed proof of this theorem.

Thus, we obtain the quasi-invariant action of the group Diff on the space $\hat{H}_s$. Moreover, operators (15) are bounded on any space H_t, and in particular, on $H_{s-1/2-\varepsilon}$. Therefore the action of the group Diff on

$$\hat{H}_s \simeq H_{s-1/2-\varepsilon}$$

is specific, i.e., this is *a group action on a set.* We are interested in the properties of this action for various $s \geq 0$.

2.6. Ergodicity. Let a group G acts on a space M with measure μ by transformations leaving μ invariant. Recall that an action is called *ergodic* if any G-invariant subset of M has zero or full measure (i.e., is the complement to a zero measure set).

Theorem 2.7. (*cf.* [80]). *Let H be a real Hilbert space, and let G be a subgroup of the orthogonal group of the space H. Then the following conditions are equivalent:*

(*a*) *The G-action is ergodic on $\hat{H}$;*

(*b*) *The G-action on H has no invariant finite-dimensional subspaces.*

We omit a proof here, but it should be noted that this assertion is very simple: it suffices to understand (cf. Subsection 1.10) when there are invariant vectors in symmetric powers.

Corollary 2.8. *Let A be an orthogonal operator on H. The transformation A of the space $\hat{H}$ is ergodic iff A has no discrete spectrum.*

Question 2.9. Under what conditions imposed on a subgroup $G \subset \mathrm{GLO}$ (and also on the operator $A \in \mathrm{GLO}$) the action of G (resp. A) is ergodic?

Below, we shall see that the difference in ergodic properties of subgroups O and GLO is *considerable.*

2.7. The Case $s = 0$. For $s = 0$, operators (15) are orthogonal in $L_2 = H_0$, i.e., the group Diff is embedded into the group $O(L_2)$. Obviously, the representation of Diff in L_2 is irreducible, and therefore, the action of Diff on $\hat{L}_2$ is ergodic.

This is the well-known action of change of variables on the white noise. This is not interesting from the point of view of representation theory. It is clear that O-invariant subspaces in $L_2(\hat{L}_2)$ are also Diff-invariant, and therefore, the representation of Diff in $L_2(\hat{L}_2)$ is a direct sum of symmetric powers of the representation of Diff in L_2; cf. Subsec 1.10.

2.8. The case $0 < s < 1/2$.

Proposition 2.10. *Let $0 < s < 1/2$. Then*

(a) *The action $T_s(q)$ of the group* Diff *on $L_2(\hat{H}_s)$ is ergodic.*

(b) *Moreover, the action of the group* $\mathrm{PSL}_2(\mathbb{R}) \subset$ Diff *is ergodic.*

(c) *Moreover, the same true for the action of any individual hyperbolic element of* $\mathrm{PSL}_2(\mathbb{R})$ *and also for any individual parabolic element of this group.*

Recall that the group $\mathrm{PSL}_2(\mathbb{R})$ is embedable into Diff. The group $\mathrm{SL}_2(\mathbb{R})$ is isomorphic to the group of matrices of the form $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$, where $|\alpha|^2+|\beta|^2 = 1$. This group acts on the circle $|z| = 1$ by the Möbius transformations

$$z \mapsto (\alpha z + \beta)(\bar{\beta} z + \bar{\alpha}),$$

and in particular, this groups act on the circle $z = e^{i\varphi}$.

Further, it is well-known that for $-1/2 < s < 1/2$, $s \neq 0$, the operators $T_s(q)$, where $q \in \mathrm{PSL}_2(\mathbb{R})$ are unitary with respect to the positive definite inner product

$$\langle f, g\rangle = \int_0^{2\pi} \int_0^{2\pi} \frac{f(\varphi)\overline{g(\psi)}d\varphi d\psi}{\left|\frac{\text{sine }(\varphi-\psi)}{2}\right|^{1+2s}}, \tag{16}$$

which can also be written in the form $\langle \sum c_n e^{in\varphi}, \sum b_n e^{in\varphi}\rangle = \sum \frac{\Gamma(n+\frac{1}{2}+s)}{\Gamma(n+\frac{1}{2}-s)}$. We see from this formula that the inner product (16) is positive definite for $|s| < 1$ (and also we see that there is no positive definiteness for $|s| > 1/2$). Moreover, the latter formula can be considered as the definition of integral (16) in the case $0 < s < 1$ (integral (16) itself is divergent in this case). The representations T_s of the group $\mathrm{SL}_2(\mathbb{R})$ in the functional spaces over the circle with inner product (16) are called the *additional series representations.*

On the other hand, it is easy to see that the inner product (16) is equivalent (in the sense of Subsection 2.4) to the inner product (14) on the Sobolev space H_s. Hence we can choose expression (16) as an inner product on the Sobolev space H_s.

Thus, $\mathrm{PSL}_2(\mathbb{R}) \subset O(H_s)$, and we find ourselves in the domain of applicability of Theorem 2.7. The assertion (b) of Proposition 2.10 follows now from the fact that the representation T_s of the group $\mathrm{PSL}_2(\mathbb{R})$ is irreducible. As for specific operators $T_s(q)$ for $q \in \mathrm{PSL}_2(\mathbb{R})$, we have that their spectra are known and are purely continuous for non-elliptic elements q.

2.6. The Point $s = 1/2$**.** The action of the group Diff at this point is written in the form

$$T_{1/2}f(\varphi) = f(q(\varphi)). \tag{17}$$

It is easy to see that constant functions are invariant with respect to this action, and therefore, it is natural to consider the action of Diff on the quotient space $H_{1/2}/\mathbb{R}$ of the space $H_{1/2}$ by the space of constants. It is appropriate to choose the Sobolev inner product on $H_{1/2}/\mathbb{R}$ in the form

$$\langle f, g, \rangle \frac{1}{2\pi} \int_0^{2\pi} \coth((\varphi - \psi)/2) f(\varphi) g'(\psi) d\varphi d\psi,$$

or, what is the same, in the form

$$\langle \sum c_n e^{in\varphi}, \sum b_n e^{in\psi} \rangle = \sum n c_n \bar{b}_n$$

The group $\mathrm{PSL}_2(\mathbb{R})$ acts on $H_{1/2}/\mathbb{R}$ by orthogonal operators; moreover, the resulting representation is (real) irreducible. Therefore, the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\widehat{H_{1/2}/\mathbb{R}}$ is ergodic.

At this point it appears a series of additional possibilities.

First, we can consider the two-parameter family of affine actions

$$T_{1/2}^{(\alpha,\beta)}(q) f(\varphi) = f(q(\varphi)) + \alpha(q(\varphi) - \varphi) + \beta \log q'(\varphi), \tag{18}$$

together with the linear action (17). By theorem 2.6. and by Martin-Cameron theorem, the measure on $\widehat{H_{1/2}/\mathbb{R}}$ is quasi-invariant with respect to these transformations.

Second, there is the Diff-invariant skew-symmetric bilinear form

$$\{f, g\} = \int_0^{2\pi} f g' d\varphi$$

on $H_{1/2}/\mathbb{R}$. Therefore, in this case, the group Diff is embedded into the so-called infinite-dimensional symplectic group or the authomorphism group of commutation relations (the group $\mathrm{Sp}((2\infty, \mathbb{R}), \mathrm{U}(\infty))$ in the notation of Subsection 6.8.). This group has the well-known representation (the so-called harmonic representation, oscillation representation, Weil representation or the Sigal-Shale-Weil representation). The restriction ρ of this representation to the group Diff is the representation of Diff in $L_2(\widehat{H_{1/2}/\mathbb{R}})$ is isometric to the representation $\rho \otimes \rho^*$; cf. also [22].

2.10. The Case $s > 1/2$. For $s > 1/2$, continuous functions form a set of full measure in $\hat{H}_s$. Consider the following subsets of space of continuous functions:

Ω_+, the space of all positive continuous functions;

Ω_-, the space of all negative continuous functions;

Ω_0, the space of all continuous functions vanishing at least one point.

It is easy to see that all these sets has a nonzero measure in $\hat{H}_s$; obviously, they are also Diff-invariant.

Let us consider the functional

$$I_s(f) = \int_0^{2\pi} \frac{d\varphi}{f(\varphi)^{\frac{1}{s-1/2}}} \tag{19}$$

on Ω_0. Obviously, we have

$$I_s(T_s(q)f) = I_s(f),$$

i.e., the level surfaces of the functional I_s are invariant with respect to Diff.

Let us pass to the set Ω_0. For $1/2 < s < 3/2$, the set of zeros of a generic function $f \in \Omega_0$ is a noncountable closed set of zero measure; we return to this case below. For $s > 3/2$, a generic function $f \in \Omega_0$ has a finite set of isolated zeros. For definiteness, we suppose that $\alpha_1, \alpha_2, ..., \alpha_{2k} = \alpha_1$ are zeros of the function f. Consider a couple of the numbers

$$I_k(f) = \int_{\alpha_j}^{\alpha_{j+1}} \frac{d\varphi}{f(\varphi)^{\frac{1}{s-1/2}}}. \tag{20}$$

It is easy to see that these integrals are almost sure convergent, and the couple of numbers $I_k(f)$ is an invariant of the function f with respect to the action of the transformations $T_s(q)$.

Question 2.11. Is it true that on the level surfaces (19) and (20) there exist conditional measures? Is it true that these measures are quasi-invariant? (The definition of conditional measure see below).

There is no doubt that the answer to both these questions is positive. Nevertheless, it is more important here to give an explicit description of these measures and to compute the Radon-Nykodim derivative than to prove the existence theorem.

Recall the definition of a conditional measure and some other standard definitions of measure theory; cf. [40, 33].

A *Lebesgue space* is a measure space which is equivalent to the union of an interval of the line (which is finite, infinite, or empty) endowed with the ordinary Lebesgue measure and with a family of points (which is finite, infinite or empty) having a nonzero measure.

Let M be a Lebesgue space with a probability measure μ (the measure of the whole M is equal to 1). Let $M = \cup_{\alpha \in A} M_\alpha$ be a partition of M into the pairwise disjoint subsets. A partition is called measurable if there exists a countable system of measurable subsets B_j such that

1. For any M_α and any B_j, the set $M_\alpha \cap B_j$ is either emptyy or coincides with M_α.

2. For any two different M_α and M_β there exists B_j such that one of these sets M_α and M_β is contained in B_j, and this is not the case for the other set.

Let $M = \bigcup_{\alpha\in A} M_\alpha$ be a measurable partition of the space M. Then the quotient space, the set A, is endowed with the canonical measure ν, that is

$$\nu(B) = \mu\left(\bigcup_{\alpha\in B} M_\alpha\right).$$

Further, for almost every (with respect to the measure ν) $\alpha \in A$, a conditional measure μ_α is defined by the following condition: for any measurable $B \in A$, the set $M_\alpha \cap B$ is measurable in M_α for almost all α, and

$$\mu(B) = \int_A \mu_\alpha(M_\alpha \cap B) d\nu.$$

We emphasize the fact that this Rokhlin theorem saysnothing on the existence of conditional measures for a given specific $\alpha \in A$.

2.11. Quotient Actions. Let a group G acts on M leaving the mesure μ to be quasi-invariant. Let the system of sets M_α be the *imprimitive system* for the group G, i.e., for each $g \in G$ and any $M\alpha$, the set gM_α coincides with a certain set M_β. Then we have the action of the group G on the quotient space A, that is, $g\alpha = \beta$ if $gM_\alpha = M_\beta$.

Lemma 2.12. *If a measure μ on M is quasi-invariant with respect to G, then the measure ν on A is also quasi-invariant with respect to G.*

Proof. Let $B \subset A$ be a subset of zero measure. Then

$$\nu(gB) = \mu\left(\bigcup_{\alpha\in B} M_{g\alpha}\right) = \mu\left(g\bigcup_{\alpha\in B} M_\alpha\right);$$

this equals 0, since

$$\mu\left(\bigcup_{\alpha\in B} M_\alpha\right) = \nu(B) = 0,$$

and transformations $g \in G$ send sets of zero measure into sets of zero measure.

2.12. Measures on $\mathbb{T}\backslash\mathrm{Diff}^1$. Denote by $\mathbb{T}$ the rotation group of the circle S^1.

Let $a > 0$. Denote by V_α the surface

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\varphi}{f(\varphi)^{\frac{1}{s-1/2}}} = a$$

in the space $C(S^1)$ of continuous functions of this circle. It is easy to see that the group Diff^1 acts transitively on V_α, and V_α is isometric to $\mathbb{T}\backslash\mathrm{Diff}^1$ as a homogeneous space. Namely, the mapping

$$\mathrm{Diff}^1 \to V_\alpha$$

is given by the formula

$$q(\varphi) \mapsto a^{s-1/2} q'(\varphi)^{1/2-s}.$$

The diffeomorphism q is reconstructed by f according to the formula

$$q(\varphi) = \int \frac{d\varphi}{f(\varphi)^{\frac{1}{s-1/2}}} + C.$$

Therefore, under the condition of a positive answer to Question 2.11, there exists a two-parametric family of measures $\kappa_{s,\alpha}$ on $\mathbb{T}\backslash\mathrm{Diff}^1$ that are quasi-invariant with respect to Diff (where $s > 1/2$, $a > 0$).

Some (other) one-parameter family of Diff-quasi-invariant measures on $\mathbb{T}\backslash\mathrm{Diff}^1$ will be constructed now. Namely, let us consider the action of the group Diff^1 on Ω_+ and note that the set of rays of the form $\lambda f(\varphi)$ (where $f(\varphi) \in \Omega_+$ is fixed and λ runs over all positive numbers) forms an imprimitive system.

The quotient space of the space Ω_+ by this partition is identified with $\mathbb{T}\backslash\mathrm{Diff}^1$ in an obvious way, and thus, we obtain a certain family of measures κ_s on $\mathbb{T}\backslash\mathrm{Diff}^1$ that are quasi-invariant with respect to Diff.

It seems to be true that the measure κ_s is equivalent to the measure $\int\limits_{a>0} \kappa_{s,\alpha} d\alpha$ (of course, if the measures $\kappa_{s,\alpha}$ do exist in reality).

Remark 2.13. The measures κ_s are pairwise singular. This can be seen from the fact that they are supported on diffeomorphisms of a different Hölder smoothness.

Question 2.14. According to [17], the space $\mathbb{T}\backslash\mathrm{Diff}^1$ is identified with the space of univalent functions on the disk $|z| < 1$. Can we describe the measures on $\mathbb{T}\backslash\mathrm{Diff}^1$ in terms of univalent functions? (For example, in terms of their coefficients or in terms of their value distributions at different points.)

This question is not simple, but it should be noted that many of the explicit formulas of representation theory for the group of circle diffeomorphisms can be written precisely via univalent functions (cf. [24]).

2.13. Measures on Diff^1**.** The spaces Diff^1 and $\mathbb{T}\backslash\mathrm{Diff}^1$ are not very different, and if we have a Diff-quasi-invariant measure on one of these spaces, then it is easy to construct measures on the other. Indeed, let us consider a bijection

$$\mathrm{Diff}^1 \to \mathbb{T}\backslash\mathrm{Diff}^1 \times \mathbb{T}, \tag{21}$$

where the mapping $\mathrm{Diff}^1 \to \mathrm{Diff}^1(\mathbb{T}\backslash\mathrm{Diff}^1)$ is a natural projection, and the mapping $\mathrm{Diff}^1 \to \mathbb{T}$ is given by the formula $q \mapsto q^{-1}(0)$. A natural measure on the right-hand side of (21) can be considered as a quasi-invariant measure on the left-hand side of (21), i.e., on Diff^1.

2.14. Measures on the Space of Cantor Subsets. Let $1/2 < s < 3/2$. Denote by Clos the space of all closed subsets of the circle. Consider the mapping

$$(\hat{H}_s \cap \Omega_0) \to \mathrm{Clos}$$

which to each trajectory puts in correspondence the set of its zeros. Denote by β_s the image of the canonical measure on $\hat{H}_s \cap \Omega_0$ under this mapping.

For all $s \in (1/2, 3/2)$, the sets of Hausdorff dimension $(3/2 - s)$ form a set of full measure in Clos.

By Lemma 2.12, the measure β_s is Diff-quasi-invariant. We return to the discussion of questions of such a type in Section 4.

3 FRACTIONAL DIFFUSIONS AND MEASURES INVARIANT WITH RESPECT TO LOOP GROUPS

3.1. Basic Construction. Consider a distribution $f\colon S^1 \to \mathbb{R}^n$. This function can be expanded in the Fourier series of the form

$$f(\varphi) = \sum_{n=1}^{\infty} e^{ik\varphi} v_k, \tag{22}$$

where $v_k = (v_k^1, ..., v_k^n) \in \mathbb{C}^n$, $v_k = \bar{v}_{-k}$.

Denote by H_s^n the space of real functions (22) satisfying the condition

$$\sum (1 + n^2)^s \|v_k\|^2 < \infty.$$

The inner product on H_s^n is defined by the formula

$$\langle \sum e^{ik\varphi} v_k, \sum e^{ik\varphi} w_k \rangle = \sum_{k=1}^{\infty} (1 + n^2)^s \sum_{\alpha=1}^{n} v_k^\alpha \bar{w}_k^\alpha.$$

Denote by $\mathrm{SO}(n) = \mathrm{SO}(n, \mathbb{R})$ the orthogonal group of order n. Denote by $\mathfrak{L}_n$ the group of infinite differentiable functions $S^1 \to \mathrm{SO}(n)$.

The operator

$$Q(R) f(\varphi) = R(\varphi) f(\varphi) \tag{23}$$

on the space H_s^n corresponds to each function $R(\varphi) \in \mathfrak{L}_n$.

Theorem 3.1. (*cf.* [22, 65]).

$$Q(R) \in \mathrm{GLO}(H_s^n).$$

Thus, ve obtain a quasi-invariant action of the group $\mathfrak{L}_n$ on $\hat{H}_s^n$.
the group Diff is also acted on $\hat{H}_s^n$ by the changes of variable

$$T_s(q)f(\varphi) = q'(\varphi)^{1/2-s} f(q(\varphi)).$$

Proposition 3.2. $T_s(q) \in \mathrm{GLO}(H_s^n)$.

(This is a variant of Theorem 2.6).
Thus, the semidirect products of groups

$$\mathrm{Diff} \ltimes \mathfrak{L}_n$$

acts on $\hat{H}_s^n$.

The elements of this semidirect product are the pairs $(q, Q) \in \mathrm{Diff} \times \mathfrak{L}_n$, and the product is given by the formula

$$(q, Q) \cdot (r, R) = (q \circ r, (Q \circ r) \cdot R).$$

3.2. Random Walk on Random Spheres. Let $s > 1/2$. Then continuous functions have a full measure in $\hat{H}_s^n$, and action (23) has an obvious integral: the function $\|f(\varphi)\|^2$ doesn't changes under transformations (23).

Fix a function $r(\varphi) > 0$, and consider further the set S_r of all trajectories $F(\varphi)$ satisfying the condition

$$\|f(\varphi)\| = r(\varphi).$$

The "surface" S_r can be understood as a random walk on a sphere of variable random radius $r(\varphi)$.

Question 3.3. For what functions $r(\varphi)$, the conditional measures are defined and quasi-invariant on the surface S_r?

Remark 3.4. It is known that the Wiener measure on the set of continuous path on a space, which is homogeneous with respect to a compact group, is quasi-invariant with respect to a smooth loop group (cf. [44] and also [82, 83, 61]). Therefore, the answer is positive in the case $s = 1$ and $r = \mathrm{const}$.

3.3. Spherical Part of the Generalized Diffusion. Denote by S^{n-1} the sphere $\|x\| = 1$ in $\mathbb{R}^n$. Assign to each function $f \in \hat{H}_s^n$ the function $S^1 \to S^{n-1}$ by the formula

$$f \mapsto f/\|f\|.$$

Then for each $s > 1/2$, we obtain measures on the space of continuous path $C(S^1, S^{n-1})$. By Theorem 3.1 and Lemma 2.12, these measures are quasi-invariant with respect to the action of the group $\mathfrak{L}_n$. By Proposition 3.2 and Lemma 2.12, these measures are also quasi-invariant with respect to smooth changes of the parameter on the circle.

Remark 3.5. The spherical component of the Brownian motion (cf. [14]) does not coincide with the Brownian motion on the sphere; moreover, this spherical component is not a Markov process.

Remark 3.6. A small variation of this construction also allows one to construct quasi-invariant measures on the continuous loop group itself (cf. [65]).

4 STABLE PROCESSES AND RANDOM CANTOR SETS

4.1. Definition of Measure ξ_α. Denote by Cant the set of closed subsets of the half-line $[0, \infty)$. Now, for any α in the interval $(0, 1)$, we define a certain probability measure ξ_α on Cant.

All these measures have the following property: for any $a > 0$, the measure of the sets $A \in$ Cant which contain a is equal to 0. Therefore, for any $a > 0$ and for almost all $A \in$ Cant, an interval (u, v), which is complement to the set A and which contain the point a, is defined. Thus, for any $a > 0$, the functions

$$u = u_a(A), \quad v = v_a(A)$$

are defined on Cant. Analogously, for any collection of points $0 < a_1 < a_2 < ... < a_k$, there exists the minimal collection

$$(u_1, v_1),\ (u_2, v_2), ..., (u_s, v_s) \tag{24}$$

of complemented intervals to the set A, which cover all points a_j. We set

$$u_1 < v_1 < u_2 < v_2 < ... < v_s.$$

Let us say the same in other words.

We note that the interval (u_i, v_i) can cover several points of the collection a_i.

Fix a natural k. Let

$$I:\ i_1 < i_2 < ... < i_m$$

be a certain subset of the set $\{1, 2, ..., k\}$. Denote by $Q(I) = Q(i_1, i_2, ..., i_m)$ the polyhedron in the space $\mathbb{R}^{2m}$ with coordinates $u_1, v_1, ..., u_m, v_m$ given by the inequalities

$$0 < u_1 < a_1; \quad a_{i_1} < v_1 < u_2 < a_{i_1+1}; ...$$

$$a_{i_l} < v_l < v_{a_l+1}; \ldots a_k < v_m.$$

Let $0 < a_1 < a_2 < \ldots a_k$ be fixed as above.

Then for almost all $B \in \mathrm{Cant}$, a point of the union

$$\mathfrak{Q}(a_1, \ldots, a_k) := \bigcup_{I \subset \{1, \ldots, k\}} Q(I)$$

of polyhedrons $Q(I)$ is defined; namely, couple (24) corresponds to the point B. Thus, Cant is mapped onto $Q(I)$. By definition, we assume that the projection of the measure ξ_α onto $Q(I) \subset \mathfrak{Q}(I)$ is given by the formula

$$\left(\frac{\operatorname{sine} \pi\alpha}{\pi}\right)^m \prod_{j=1}^{m} \frac{du_j dv_j}{(u_j - v_{j-1})^{1-\alpha}(v_j - u_j)^{1+\alpha}} \tag{25}$$

(we set $v_0 = 0$). By this, the measure ξ_α is completely determined.

Let us formulate one more our definition in a somewhat another way. To each point $a > 0$ and to almost every $B \in \mathrm{Cant}$ we put in correspondence the cones of the complemented interval (u, v) which covers a. Project the measure ξ_α onto the subset of $\mathbb{R}^2$ given by the inequalities

$$0 < u < a, \quad v > a.$$

Then, by definition, the image of the measure ξ_α is given by the formula

$$\left(\frac{\operatorname{sine} \pi\alpha}{\pi}\right) \frac{du dv}{u^{1-\alpha}(v - u)^{1+\alpha}}. \tag{26}$$

Again, let the points $0 < a_1 < a_2 < \ldots a_k$. We play the interval (u_1, v_1) covering the point a according to law (26). It can happen that (u_1, v_1) contains some more points a_j. Let a_i be the first of the points a_j which is not contained in (u_1, v_1). Then the interval (u_2, v_2) which contains a_i is plaied according to the law

$$\left(\frac{\operatorname{sine} \pi\alpha}{\pi}\right) \frac{du_2 dv_2}{(u_2 - v_1)^{1-\alpha}(v_2 - u_2)^{1+\alpha}}.$$

In other words, we play (u_2, v_2) according as for (u_1, v_1), but we assume that the half-line start from v_1 but not from zero.

Proposition 4.1. *The set $B \in \mathrm{Cant}$ satisfies the following conditions with probability* 1 *with respect to the measure ξ_α:*

(a) *B is continuum and has no isolated points;*

(b) *The Lebesgue measure of the set B is equal to* 0*;*

(c) $) \in B$.

4.2. Quasi-Invariance of the Measures ξ_α. Denote by Diff^* the group of diffeomorphisms $p(t)$ of the half-line $[0, \infty)$, which satisfy the following conditions:

1^0. $p''(t)$ is continuous;
2^0. There exists a finite $\lim\limits_{t\to} p'(t)$;
3^0. $\int\limits_0^\infty \left|\frac{p''(t)}{p'(t)}\right| dt < \infty$.

Theorem 4.2. (*cf.* [27]). *The measure ξ_α on* Cant *is quasi-invariant with respect to the group* Diff*. *The Radon-Nykodim derivative of the transformation of* Cant *given by the diffeomorphism p is*

$$\frac{p'(0)}{p'(\infty)} \prod_{j=1}^{\infty} \left(\frac{p'(u_j)(v_j - u_j)}{(p(v_j) - p(u_j))} \right)^{1+\alpha}, \tag{27}$$

where the product is taken over all complemented intervals to the set $B \in$ Cant. *Moreover, this product converges almost everywhere on* Cant.

4.3. Local time. On almost every element B of the *measure space* $(\mathrm{Cant}, \xi_\alpha)$, there is a certain measure λ_B, which is canonically defined and is called the *local time*. Let $B \in$ Cant. Let $q > p > 0$. Then, by definition,

$$\lambda_B([p,q] \cap B) = \lim_{\varepsilon\to 0} N(\varepsilon)/\varepsilon^\alpha,$$

where $N(\varepsilon)$ is the number of complemented intervals to B of length $> \varepsilon$ which are in the interval (p,q). Of course, the existence of limit almost everywhere with respect to the measure λ_B is the fact, which requires a proof. It can happen that this limit cannot exist for an arbitrary closed subset of the half-line.

4.4. Stable Processes. Finally, we have to explain the appearance of formulas (25) and (26).

Let us construct, for (almost) every element $B \in$ Cant, a curve in the domain $\{t > 0, x \geq 0\} \subset \mathbb{R}^2$ given by the formula

$$t = \lambda_B([0,x] \cap B).$$

The obtained curve is the graph of the monotonically increasing function $x = x_B(t)$: $[0, \infty \to \mathbb{R}$; this function is a jump function in the sense that

$$x(t) = \sum \left(\lim_{s\to t_j+0} x(s) - \lim_{s\to t_j-0} x(s) \right),$$

where the sum is taken over all points of discontinuity of the monotone function $x(t)$.

Thus we have mapped the space $(\mathrm{Cant}, \xi_\alpha)$ onto the space of increasing jump functions, and therefore, we have obtained a measure on the function space, i.e., a stochastic process. The obtained processes are precisely the *increasing stable processes* (or, what is the same, the *stable subordinators*), cf., e.g., [18, 8, 36, 76]. The jump distribution formula (i.e., formula (26)) for these processes was obtained in [8] (and in [18] in the case $\alpha = 1/2$).

Some Conjectures. Following Paul Lévy, let us consider the following three measures on the space of Cantor subsets of the half-line:

1^*. The measure $\xi_{1/2}$;

2^*. The measure obtained by the Wiener measure (on functions on the half-line) according to the device of Subsection 2.14;

3^*. Consider the Wiener measure on functions on the half-line. To each trajectory $f(t)$ we put in correspondence the set B of all points $a > 0$ such that

$$x(a) \geq x(s) \quad \forall s < a.$$

Thus, we map the set of Wiener trajectories into the set Cant, and we obtain a measure on the set Cant.

Theorem 4.3. [18] *All these three measures are coincided.*

Question 4.4. Is the same true for the arbitrary $\alpha \in (0,1)$ and $s = 3/2 - \alpha$?

Question 4.5. Is there a reasonable explanation of the construction of this section for $\alpha = 0$?

5 STOCHASTIC AND $\mathbb{R}^*$-STOCHASTIC KERNELS

5.1. Problem on the Weak Closure. The constructions described in the previous sections generate a whole series of obvious questions, they are:

Question 5.1. Are the constructed actions ergodic?

Question 5.2. Are there any other imprimitive systems except for those which are presented?

Question 5.1. How the representations of the corresponding groups in L_2 are decomposed?

There are no answer to these questions at all. Some assertions on the representation of the group Diff in $L_2(\hat{H}_s)$ in the case $0 < s < 1/2$ are considered in [22].

In fact, there arises one more question, which is also essential and not obvious. The matter is that as a rule, an infinite dimensional group is a part of some considerably more wide algebraic structure which is rigidly generated by the group itself (see survays [64, 68, 66]), and in many cases, this latent structure but not these group itself plays a key role. For example, the category Shtan of Riemann surfaces of Kontsevich-Sigal (cf. [24, 23]) is connected with representations with leading weight of the group of diffeomorphisms of the circle, the category of bundles over Riemann surfaces (cf. [23]) is connected with representations with leading weight of the loop groups, the operator nodes categories (cf. [68, 66]) are connected with the infinite-dimensional classical groups and the Brauer type categories (cf. [31, 68]) are connected with symmetric groups.

A series of methods for seeking the latent structure is known, these are: the multiplication of double conjugate classes (cf. [68, 66]), holomorphic extensions (cf. [24, 23, 64]), the weak closure (cf. [68]). We now are interested in the last mentioned method, which was suggested by G. I. Ol'shanskii. This method consists in the following.

Let a certain unitary representation in a Hilbert space is known for an infinite-dimensional group G. Consider the set $\rho(G)$ consisting of operators of the form $\rho(g)$, where $g \in G$. Further, let us consider the closure $\overline{\rho(G)}$ of the set $\rho(G)$ in the weak operator topology. It is easy to prove that $\overline{\rho(G)}$ is a compact semigroup (cf. [68]). We see from this that the set $\overline{\rho(G)}$ differs considerably from the group itself. An experimental fact, which is not a priori obvious, consists in the fact that the set $\rho(G)$ can be explicitely described for interesting infinite-dimensional groups, i.e., there cannnnot arise objects of the Bor compactum type.

Note that on a finite-dimensional level, the problem of description of the closure of the image of an equivalent embedding is an analog of the question posed. In various cases, this problem leads one to such ingeneous objects as the Fürstenberg-Satake boundary, Study-Semple-Koncini-Procesi complete symmetric varieties, and the Thurston boundary.

In a natural way there arises the question on the structure of this closure for the representations in L_2, which are connected with the above-constructed quasi-invariant measures. It turns out that in our case, this question can be reformulated in pure probabilistic terms; this reformulation is the goal of the present section.

The next two sections are a certain digression from our goal; we intend to show what sort of semigroups can be obtained from the group Diff.

5.2. Trivial Representation of Diff**.** let Diff act in L_2 on the circle by the formula

$$T(q)f(\varphi) = f(q(\varphi))q'(\varphi)^{1/2}. \tag{28}$$

Proposition 5.4. *The weak closure of the set* $T(\mathrm{Diff})$ *consists of operators of the form*

$$T(q)f(\varphi) = f(q(\varphi))q'(\varphi)^{1/2}\alpha(\varphi),$$

where q *is an arbitrary monotonically increasing mapping of* S^1 *into itself, and*

$$0 < \alpha(\varphi) < 1$$

is a measurable function.

Let us explain the appearance of the additional factor $\alpha(\varphi)$. Consider a diffeomorphism of the form

$$q_n(\varphi) = \varphi + \frac{1}{n}h(n, \varphi),$$

where h is some 2π-periodic function such that $h' > -1$. It is easy to see that $T(q_n(\varphi))$ converges weakly to the multiplication-by a constant operator.

Now Proposition 5.4 becomes to be sufficiently obvious. However, it is interesting fact that the presented arguments do not give an answer to the question on the form of the weak closure $T(\mathrm{Diff})$ in the case of action

$$T(q)f(\varphi) = f(q(\varphi))q'(\varphi)^{1/2+is}.$$

5.3. Tube Semigroup (cf. [66] for more detail). Let $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemannian sphere, D_+ be the disk $|z| \leq 1$, and let D_- be the disk $|z| \geq 1$ on $\bar{\mathbb{C}}$. let $D^0_\pm$ be interiors of these disks.

An element of the *tube semigroup* Γ in the triple $\mathcal{R} = [R, r_+, r_-]$, where R is a Riemann surface. (a one-dimensional complex manifold which is conformally equivalent to the Riemann sphere $\bar{\mathbb{C}}$), and $r_\pm$ are univalent mappings of $D_\pm$ into $\bar{\mathbb{C}}$ which are real analytic on the boundary $|z| = 1$ and such that the following condition holds: $(*)$ the sets $r_-(D_-)$ and $r_+(D_+)$ are disjoint.

Let $\mathcal{R} = [R, r_+, r_-]$ and $\mathcal{Q} = [Q, q_+, q_-]$ be two elements of the semigroup Γ. Then the product of these elements $\mathcal{T} = \mathcal{R}\mathcal{Q}$ is the triple $\mathcal{T} = [T, t_+, t_-]$, where T is the Riemann surface obtained from

$$(R \backslash r_+(D^0_+)) \cap (Q \backslash q_-(Q^0_-)$$

by pasting together the points $r_+(e^{i\varphi})$ and $q_-(e^{i\varphi})$. In addition, we set $t_-(z) = r_-(z)$ and $t_+(z) = q_+(z)$.

Denote by Δ the set of triples $\mathcal{R} = [R, r_+, r_-]$ satisfying the same conditions but with property $(*)$ replaced by the condition
$(**)$ the sets $r_-(D^0_-)$ and $r_+(D^0_+)$ are disjoint, and $r_-(D^0_-) \cup r_+(D^0_+) = \bar{\mathbb{C}}$.

It is easy to see that Δ is a semigroup, and moreover, Δ is a group isomorphic to the group Diff^a of analytic diffeomorphisms of the circle. Namely, to an element $\mathcal{R} = [R, r_+, r_-]$ we put in correspondence the circle diffeomorphism given by the formula

$$e^{i\varphi} \mapsto r_-^{-1} \circ r_+(e^{i\varphi}).$$

Denote by $\bar{\Gamma}$ the set $\Gamma \cup \Delta$.

A natural topology on $\bar{\Gamma}$ is the topology of uniform convergence of univalent functions $p_\pm$ on compact subsets of $D^0_\pm$. As is noted in [64], the group $\Delta = \mathrm{Diff}^a$ is dense in $\bar{\Gamma}$ with respect to this topology. Further, it was shown in [64] that representations of Diff with a leading weight are extended in a unique way to holomorphic representations of the semigroup Γ.

Conjecture 2.5. *Leading weight representations of $\bar{\Gamma}$ are irreducible in the above-mentioned topology.*

If this conjecture is true, then from this conjecture the following collection of assertions is easily deduced.

Corollaries from the conjecture. (a) *The weak closure of the set* $\rho(\mathrm{Diff})$ *for a leading weight representation of* Diff *contains the subgroup* F.

(b) *Let* ρ *and* k *be two irreducible representations of* Diff *with a leading weight and with a following weight respectively. Then the representation* $\rho \otimes k$ *is irreducible.*

(c) *Let* λ *be a representation of the group in* $L_2(H_{1/2}/\mathbb{R})$. *Then the weak closure* $\rho(\mathrm{Diff})$ *contains the semigroup* σ.

Let us make several remarks concerning these corollaries. We have said twicely "contains the semigroup Γ", but not "coincides with the semigroup Γ". In fact, the difference here is not large, and this difference appears due to the fact that the analiticity conditions for the univalent functions $p_{\pm}$ up to the boundary can be weakened (the best possible conditions are unknown).

As for assertion (a), then this assertion follows from the fact that the representation λ is a direct sum of tensor products of leading weight representation by following weight representations.

Question 5.6. What is the structure of the weak closure of the set $\rho(\mathrm{Diff})$ for representations of Diff in $L_2(\hat{H}_s)$?

It is not true that this closure contains Γ; on the other hand, it is not clear at all what is "the deformation of the space of Riemann surfaces".

It remains to comment a degree of likeness of Conjecture 5.6. For leading weight representations of the semigroup Γ, explicit formulas are known (cf. [24]), and it suffices to trace the continuity of these formulas. However, it turns out that this is not simple and is connected with sufficiently delicate properties of coefficients of univalent functions.

Markov Category (cf. [51, 1, 58, 25]). Lebesgue spaces (M, μ) with a probability measure are objects of the category Mar. Morphisms $(M, \mu) \to (N, \nu)$ can be characterized by two equivalent methods, that is as Markov operators or as stochastic kernels.

A Markov operator Q: $L_2(M, \mu) \to L_2(N, \nu)$ is an arbitrary operator satisfying the following conditions:

1^0. If $f \geq 0$ on M, then $Qf \geq 0$ almost everywhere on M;

2^0. $Q \cdot 1 = 1$;

3^0. $Q^* \cdot 1 = 1$.

A *bistochastic kernel* or a *polymorphism* $N \to N$ is an arbitrary measure k on $M \times N$ satisfying the following conditions:

1^*. The projection of k onto M is μ;

2^*. The projection of k onto N is ν.

Let us show that Markov operatoes and bistochastic kernels are, in fact, the same notions.

First, we construct a bistochastic kernel k: $M \to N$ by an arbitrary Markov operator Q: $L_2(M, \mu) \to L_2(N, \nu)$. Let $A \subset M$ and $B \subset N$ be two measurable sets, and let χ_A and χ_B be their characteristic functions. Then

$$k(A \times B) = (Q_{\chi_A}, \chi_B)_{L_2(N)}.$$

Conversely, let a bistochastic kernel k be given. Then, for almost all $n \in N$, the conditional measure $k_n(m)$ is well defined on the set $M \times n$. Now, the operator Q is defined by the formula

$$Qf(n) = \int_M f(m) dk_n(m).$$

The bistochastic kernels (polymorphisms) should be understood as certain "multivalued mappings", only a measure $k_m(n)$ in the image of a point but not a set; thus, a polymorphism gives us a way in which a point $m \in M$ "is spreaded" along the set N and is transformed to the conditional measure $k_m(n)$. If Q: $L_2(M, \mu) \to L_2(N, \nu)$ and R: $L_2(N, \nu) \to L_2(X, \xi)$ are two Markov operators, then, obviously, their product is also a Markov operator.

On the level of polymorphisms, this product is easily interpreted as the "repeated spreading" (cf. [1, 25]). namely, let κ: $(M, \mu) \to (N, \nu)$ and σ: $(N, \nu) \to (X, \xi)$ be two polymorphisms. Let τ: $(M, \mu) \to (X, \xi)$ be the product of these polymorphisms. Consider the corresponding conditional measure

$$\kappa_m(n), \quad \sigma_n(x), \quad \tau_m(x)$$

on the fibers

$$m \times N \subset M \times N, \; n \times X \subset N \times X, \; m \times X \subset M \times X.$$

Then these measures are connected by the equation

$$\tau_m(x) = \int_N \sigma_n(x) d\kappa_m(n);$$

this equation defines the polymorphism τ.

The Markov category is already defined. Denote by Mar this category, by $\mathrm{Mor}(M, N)$ the set of morphisms $M \to N$, and by $\mathrm{End}(M)$ the set of endomorphisms of an object M.

Example 5.7. Consider a measure $\mu \times \nu$ on $M \times N$. This measure corresponds to the "uniform spreading" of points $m \in M$ over the set N. The product of a "uniform spreading" and any polymorphism is also a "uniform spreading".

On the sets $\mathrm{Mor}(M, N)$, we introduce a *topology* by the following condition. Let $\kappa_j, \kappa \in \mathrm{Mor}(M, N)$; then $\kappa_j \to \kappa$ if for any measurable sets $A \subset M$ and $B \subset N$, we have

$$\kappa_j(A \times B) \to \kappa(A \times B).$$

This is an ordinary weak convergence of operators on the level of Markov operators.

Also, we introduce the *involution* $\kappa \to \kappa^*$: $\mathrm{Mor}(M,N) \to \mathrm{Mor}(N,M)$, on the Markov category; this involution is the passing to the adjoint operator on the level of Markov operators, and on the level of polymorphisms, this correspond to the permutation of factors in the product $M \times N$.

5.5. The Group Ams**.** Let (M,μ) be a space with a continuous probability measure (i.e., the measures of all poimts are zero). Denote by ams $= \mathrm{Ams}(M)$ the group transformations of M which preserves the measure μ.

Let $g \in \mathrm{Ams}$. For this element we define a certain bistochastic kernel λ_g: $M \to M$; namely, we set

$$\lambda_g(A \times B) = \mu(gA \cap B).$$

We note that the measure λ_g is supported precisely on the graph of the mapping g: $M \to M$.

The corresponding Markov operators are the ordinary operators of changing the variable $f(m) \mapsto f(gm)$.

Proposition 5.8. *The group* Ams *is dense in the semigroup of all polymorphisms $M \to M$.*

Example 5.9. Let $g \in \mathrm{Ams}$ be mixing. Then the sequence of powers g^n converges in $\mathrm{End}(M)$ to the polymorphism of Example 5.7. Generally speaking, for a given $a \in \mathrm{Ams}$, the set of limit points of the sequence g^n forms a semigroup; this semigroup is an invariant of the automorphism g.

In reality, the categories Ams and Mar as algebraic objects are connected with each other in a very rigid way; we now present several assertions that illustrate this rigidity.

Theorem 5.10. (*cf.* [25]). *Any unitary representation of the group* $\mathrm{Ams}(M)$ *is uniquely extended to a representation of the semigroup* $\mathrm{End}(M)$.

The following form of this theorem is more useful.

Theorem 5.11 *Any unitary representation ρ of the group* $\mathrm{Ams}(M)$ *in a Hilbert space H is uniquely extended to a $*$-representation $\bar\rho$ of category* Mar. *In other words, to each object Q of the category* Mar *we put in correspondence a Hilbert space $H(Q)$ (moreover, $H(M) = H$), and to each morphism $\kappa \in \mathrm{Mor}(Q,R)$ there corresponds a unique operator*

$$\bar\rho(\kappa) : H(Q) \to H(R),$$

so that for any objects Q, R, and T, and for any morphisms $\kappa \in \mathrm{Mor}(Q,R)$ and $\lambda \in \mathrm{Mor}(R,T)$, we have

$$\bar\rho(\lambda\kappa) = \bar\rho(\lambda)\bar\rho(\kappa)$$

(moreover, for $\lambda \in \mathrm{Ams}(M)$, we have $\bar{\rho}(\lambda) = \rho(\lambda)$). Moreover, for any objects Q and R and for any morphism $\lambda \in \mathrm{Mor}(Q, R)$,

$$\rho(\lambda^*) = \rho(\lambda)^*.$$

The following fact is essential here: there are many finite spaces in the category Mar, and the theorem alows one to reduce problems for the group Ams to finite-dimensional ones (in particular, in [25], a description of all unitary representations of Ams is obtained).

Corollary 5.12. *Any invariant action of the group* $\mathrm{Ams}(M)$ *on the measure space* N *is extended to a homomorphism* $\mathrm{End}(M) \to \mathrm{End}(N)$*, which is compatible with the involution.*

Proof. We have a homomorphism γ: $\mathrm{Ams}(M) \to \mathrm{Ams}(N)$, and moreover, we have a unitary representation of the group $\mathrm{Ams}(N)$ in $L_2(N)$, which is denoted by ρ. According to the theorem, $\lambda = \rho \circ \gamma$ is extended to a homomorphism $\bar{\lambda}$ from $\mathrm{End}(M)$ into the semigroup Cont of operators on L_2 with norm ≤ 1. The group $\mathrm{Ams}(M)$ is dense in $\mathrm{End}(M)$, and therefore, the image of $\bar{\lambda}$ is contained in the closure of the image of λ, while the closure of the image of λ is contained in the closure of image of ρ in Cont, i.e., in the semigroup of Markov operators.

5.6. Reformulation of the Question on the Weak Closure. Let a group G act on the measure space M by transfromations that leave the measure to be invariant. Then we have a representation of G in $L_2(M)$. For this representation, the question by Ol'shanskii on the weak closure can be reformulated as follows:

Question 5.13. Describe the closure of the group in the semigroup $\mathrm{End}(M)$.

Example 5.14. Consider the action of the group $O(H)$ onthe space $\hat{H}$. Then $L_2(\hat{H})$ is decomposed into a direct sum of symmetrical povers of the representation of $O(H)$ in H (cf. 1.10). Therefore the representation of $O(H)$ is extended to a representation of the (compact) semigroup $\mathrm{Cont}(H)$ (consisting of all operators on H with norm ≤ 1). Therefore, $\mathrm{Cont}(H)$ should be embedded into the semigroup $\mathrm{End}(O(H))$. this embedding was explicitely described by Nelson in [63].

There are few interesting invariant actions of infinite-dimensional groups, and these naturally arises the question on what replaces the semigroup of polymorphisms in the case of quasi-invariant actions.

5.7. $\mathbb{R}^*$-Polymorphisms (or $\mathbb{R}^*$ Stochastic Kernels. (cf [25]). Denote by $\mathbb{R}^*$ the multiplicative group of positive real numbers. Define the category $\mathbb{R}^*$-Mar. Lebesgue spaces with probability measure are the objects of this category. Let (M, μ) and (N, ν) be two such objects. The morphisms $M \to N$ are probability measures on the direct product

$$M \times N \times \mathbb{R}^*,$$

which satisfy the following two conditions:

1^0. The projection k onto M is μ.

2^0. Denote by x the coordinate on $\mathbb{R}^*$. Let xk be a measure on $M \times N \times \mathbb{R}^*$ with density x with respect to measure k. Then ν is the projection of the measure xk onto N.

We shall call these measures the $\mathbb{R}^*$-stochastic kernels or the $\mathbb{R}^*$-polymorphisms.

Let π: $(M, \mu) \to (N, \nu)$ and λ: $(N, \nu) \to (K, k)$ be two $\mathbb{R}^*$-stochastic kernels. We have to define their product $\sigma = \lambda\pi$: $M \to K$.

First, we do this in the case where all spaces M, N, and K are finite; let these spaces consist of points with measures $m_1, m_2, ..., n_1, n_2, ...,$ and $k_1, ...$ In this case, the morphism π can be understood as a matrix whose matrix elements π_{ij} are measures on $\mathbb{R}^*$, and moreover,

$$\sum_j \pi_{ij}(\mathbb{R}^*) = m_i \quad \text{and} \quad \sum_i (x\pi_{ij})(\mathbb{R}^*) = n_j.$$

Analogously, to a morphism λ there corresponds the matrix λ_{ij}. Then the matrix corresponding to σ is determined by the equation

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots \\ \sigma_{21} & \sigma_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \cdots \\ \lambda_{12} & \lambda_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} n_1^{-1} & 0 & \cdots \\ 0 & n_1^{-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots \\ \pi_{21} & \pi_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

where the product of matrix elements is an ordinary convolution of measures on the group $\mathbb{R}^*$.

Let us pass to the definition of the product in the case of arbitrary spaces M, N, and K, First of all, we note that to any $\mathbb{R}^*$-polymorphism π: $(M, \mu) \to (N, \nu)$ and to any measurable subsets $A \subset M$ and $B \subset N$ there corresponds a measure $\pi_{A,B}$ on $\mathbb{R}^*$, namely, the projection of the measure π on $A \times B \times \mathbb{R}^*$ onto $\mathbb{R}^*$. Now we define the product in terms of systems of measures $\pi_{A,B}$. Let π, λ and k be the same as above. Let $A \subset M$ and $C \subset K$. Consider the projection π^A of the measure k on $A \times N \times \mathbb{R}^*$ onto $N \times \mathbb{R}^*$. Further, let us consider the system π_n^A of conditional measures on $\mathbb{R}^*$ which corresponds to the measure π^A. Further, we construct an analogous system of conditional measures λ_n^C corresponding to the polymorphism λ and to the set $C \subset K$. Now, $\sigma = \lambda\pi$ is determined by the equation

$$\sigma_{A,B} = \int_N \lambda_n^C * \pi_n^A \, d\nu(n)$$

An $\mathbb{R}^*$-polymorphism is not a very visuable object as an ordinary polymorphism. This can be viewed as a sort of "spreading", but we trace not only the place where a point "is spreaded" but the size of "expanding" of this point under "spreading" (if our multivalued "mapping" preserves a measure, then this means that the expansion coefficient is unity, and there no necessity in the factor $\mathbb{R}^*$; if we deal with quasi-invariant measures, then the appearance of the "spreaded" expansion coefficient is inevitable).

Let k_j, k be morphisms $M \to N$. We say that k_j *converges* to k if, for any continuous bounded function φ on $\mathbb{R}^*$ and for any measurable subsets $A \subset M$ and $B \subset N$,

$$\int\int A \times B\varphi(x)dk_j \to \int\int A \times B\varphi(x)dk.$$

(this definition admits several sufficiently obvious modifications).

5.8. The Group Gms**.** Let (M, μ) be a probability space with a continuous measure. Denote by $\mathrm{Gms}(M)$ the group of all transformations of M that leave the measure μ to be quasi-invariant.

Let us describe a canonical embedding of the group $\mathrm{Gms}(M)$ into the semigroup of $\mathbb{R}^*$-polymorphisms of the space M. Let $g \in \mathrm{Gms}(M)$. Embed M into $M \times N \times \mathbb{R}^*$ according to the formula

$$m \mapsto (m, g(m), g'(m)),$$

where g' is the Radon-Nykodim derivative of the transformation g The image of the measure μ on M under this mapping is an $\mathbb{R}^*$-polymorphism.

Proposition 5.15. *The group* $\mathrm{Gms}(M)$ *is dense in the semigroup of* $\mathbb{R}^*$*-polymorphisms of the space* M*.*

Example 5.16. (cf. [25]). Let M be an n-dimensional manifold. If $n > 2$, then the diffeomorphism group of M is dense in the semigroup of $\mathbb{R}^*$-polymorphisms of M.

The category of $\mathbb{R}^*$-polymorphism is rigidly connected with the group Gms as the category of Markov operators with the group Ams. (cf [25]). For example, any unitary representation of the group Gms is uniquely extended to a representation of the category of $\mathbb{R}^*$-polymorphisms.

5.9. The Question on the Weak Closure. So, let a group G acts quasi-invariantly on a measure space (M, μ). Then a series of unitary representation (10) of the group G is connected with this action. The problem on the weak closure for these representation is reduced to the following

Question 5.16. Describe the closure of G in the semigroup of $\mathbb{R}^*$-polymorphisms.

The status of this question is different for various groups G. In many cases, the algebraic structure of the semigroup which is obtained in the closure is *a priori* known. For example, for the action of GLO on $\hat{H}$, this is a certain semigroups of operator nodes (cf. [68, 66]); the answer is analogous for the Pikrelle measures (see Subsec 6.6 below) and for the actions of the symmetric group described below these are semigroups of Brauer type (cf. [31]), and for the group described below in Subsec. 6.15, the answer is contained in [25].

This question is considerably more interesting when the latent structure is not *a priori* known; then it is desirable to use $\mathbb{R}^*$-polymorphisms for computing the latent structure. First of all, this referes to the actions of the group Diff and of the loop group constructed above.

6 A ZOO

Here we discuss various constructions of quasi-invariant measure for different infinite-dimensional groups.

A. Symmetric Group. During the last 30th the infinite symmetric group was more known (or unknown) among the infinite-dimensional groups. A very interesting fact which was recently discovered for the infinite symmetric group is the existence of a substantional harmonic analysis. We describe now the case where this analysis arises.

6.1. Bisymmetric Group. Let S_∞ be the group of all permutations of the set $\mathbb{N}$ of natural numbers. Let S_∞^{fin} be the group of all finite permutations (i.e., permutations that concern with only a finite number of elements).

The *bisymmetric group* SS_∞ is the subgroup of $S_\infty \times S_\infty$ consisting of all pairs of permutations (g_1, g_2) such that the permutation $g_1 g_2^{-1}$ is finite. Note that the group S_∞ is canonically embedded into SS_∞ according to the formula $g \mapsto (g, g)$.

Let G be a group, and let K be a subgroup of this group. Recall that a unitary representation ρ of the group G in a Hilbert space V is called K-*spherical* if there exists avector $v \in V$ which is fixed with respect to K and is cyclic with respect to G (i.e., the set of the vectors $\rho(g)v$ is total in V).

It is known (cf. [16]) that a spherical representation is completely determined by its *spherical function*

$$\varphi(g) = \langle \rho(g)v, v \rangle. \tag{29}$$

Also, we note that the spherical function satisfies the equation

$$\varphi(k_1 g k_2) = \varphi(g) \quad \forall k_1, k_2 \in K. \tag{30}$$

In other words, the spherical function is constant on the double conjugate classes $K \backslash G / k$.

A description of all S_∞-spherical representations of SS_∞ is given by the following theorem, which is a reformulation of the Thom theorem [83]; see [31].

Theorem 6.1. *The set of all irreducibble S_∞-spherical representations of the group SS_∞ is parametrized by couples of numbers*

$$\alpha_1 \geq \alpha_2 \geq \ldots; \quad \beta_1 \geq \beta_2 \geq \ldots$$

such that

$$\sum \alpha_j + \sum \beta_j \leq 1.$$

Moreover, the spherical function is given by the formula

$$\varphi(g_1 g_2) = \prod_{k=2}^{\infty} \left(\sum_{j=1}^{\infty} \alpha_j^k + \sum_{j=1}^{\infty} (-\beta)_j^k \right)^{l_k},$$

where l_k is the number of cycles of order k in the (finite) permutation $g_1 g_2^{-1}$.

See, for example, [31] for explicit constructions of representations.

Note that SS_∞ is a group of type 1 in an appropriate topology, i.e., any representation of this group is uniquely decompose into a direct integral of irreducible representations. A classification of all irreducible representations of SS_∞ was obtained in [31, 28].

6.2. Projective systems of Measures on Groups S_n. Let $\tau \geq 0$. We introduce a probability measure $\tau_s^{(n)}$ on the group S_n of all permutations of n elements in such a way that the measure of a point $g \in S_n$ is equal to

$$\frac{\tau^{[g]}}{\tau(\tau+1)...(\tau+n-1)},$$

where $[g]$ is the number of cycles in the permutation g.

For any $m \geq n$, we define the projection π_n^m: $S_m \to S_n$ as follows. Let $g \in S_m$, and let

$$(a_1 a_2 ... a_p)(b_1 b_2 ... b_q)(c_1 c_2 ... c_r)...$$

be a decomposition of this permutation into independent cycles. Let us delete all elements $a_1, ..., b_1, ..., c_1$, which are greater than n, from this expression. Then we obtain a certain element of the group S_n.

The measures $\tau_s^{(n)}$ are compatible with these projections, i.e., the image of the measure $\tau_s^{(m)}$ under the mapping π_n^m is $\tau_s^{(n)}$. Thus, we have the chain of embeddings

$$... \leftarrow S_n \leftarrow S_{n+1} \leftarrow S_{n=2} \leftarrow,$$

and by the Kolmogorov theorem on projective limits of measures (cf. [40]), we obtain a certain probability measure on the projective limit of this chain.

6.3. Decomposition of the Projective Limit. The following collection of data is called the virtual permutation:

1. A partition of $\mathbb{N}$ into disjoint subsets A_1, A_2, ...
2. A fixed cyclic ordering on each set A_j; these sets are called quasicycles.

We stress the fact that there are no additional requirements to the cyclic ordering on A_j; A_j as a cyclically ordered set can be equivalent to the set of all rational points if the circle.

Let Virt be the set of all virtual permutations. An obvious projection

$$\mathrm{Virt} \to S_n$$

is well defined; namely, we delete all elements greater that N from each cyclically ordered set A_j. Doing this, we obtain a collection of disjoint cycles, i.e., an element of S_n.

We introduce the measure τ_s on Virt by the following condition: the projection of the measure τ_s onto S_n is τ_s^n (for all n).

6.4. Action of SS_∞ on Virt. Generally speaking, it is not possible to multiply two virtual permutations. But the product $k = g\sigma$ of a finite permutation g and a virtual permutation σ is samewhat obvious.

Namely, let a permutation $g \in S_\infty^{\text{fin}}$ deals with only the numbers $1, 2, ..., n$. Let h be the rpojection of an element $\sigma \in$ Virt onto S_n. Compute the product $gh \in S_n$ in a usual way and decompose gh into circle. Now we have to find a certain element $k \in$ Virt which is projected onto gh.

Let $k \leq n$. Take a quasicycle $A_{\alpha(k)}$ of a virtual permutation σ which contains k. Take a segment $\bar{B}_k$ starting from k and terminating by the firs element $m \leq n$ which follows from k in the quasicycle $\bar{B}_k$ (obviously, ($m = hk$) in this quasicycle. Let $B_k = \bar{B}_k \backslash \{k, m\}$ (we stress the fact that B_k is a partially ordered set). Now, we are ready to deskribe the quasicycles of the virtual permutation $g\sigma$.

1. $g\sigma$ contains all cycles A_k of the virtual permutation σ which do not contain an element $k \leq n$;

2. Let $i_1 i_2 ... i_s$ be a cycle of the permutation $g\sigma$. Then $g\sigma$ contains the following quasicycle composed by sets B_j and elements i_k:

$$i_1 B_{i_1} i_2 B_{i_2} ... i_s B_{i_s}.$$

Thus the group S_∞^{fin} acts from the left on Virt. Analogously we define the right action g: $\sigma \mapsto \sigma^{-1}$ of the group S_∞^{fin} on Virt; thus the direct product $S_\infty^{\text{fin}} \times S_\infty^{\text{fin}}$ acts on Virt.

Finally, the action of $S_\infty \subset SS_\infty$ is obvious; this action merely permutates the elements of the set $\mathbb{N}$.

The action of the group SS_∞ on Virt is constructed.

Theorem 6.2. (*cf.* [56]). *The group SS_∞ acts on* Virt *by transformations which leave measures τ_s to be quasi-invariant.*

Note that elements of S_∞ preserve the measure τ_s.

Question 6.3. ([56]). Decompose a representation of SS_∞ in $L_2(\tau_s)$.

Some partial results in this direction were obtained in [56, 15]. We note that for $s \notin \mathbb{N}$, the unit function is a cyclic vector, and therefore, our representation is expanded by Thom representations.

Note that the discussed construction admits an interpretation for $s = \infty$; in this case, this construction gives us a discrete measure supported on S_∞^{fin}; we can consider the measures τ_s as analogs of the Haar measures.

The space Virt is an object appeared in the population genetics and has a series of nonlinear relations; the measures on Virt are well studied; see [57, 52, 74, 75, 15].

B. Infinite-Dimensional Analogs of Classical Groups. Now we describe an analogous construction for classical groups.

6.5. The Group $(U(2\infty),(U(\infty)\times U(\infty))$. Denote by $U(\infty)$ the general unitary group of the complex Hilbert space l_2. Denote by $G=(U(2\infty),(U(\infty)\times U(\infty))$ the group of all unitary transformations of the space $l_2\oplus l_2$ whose matrices are written in the form

$$g=\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where B and C are Hilbert-Schmidt operators.

6.6. Pikkerell Measures on the Grassmannian. (cf. [71]). Denote by Gr_n the set of all n-dimensional subspaces

$$V_+\oplus V_-=\mathbb{C}^n\oplus\mathbb{C}^n.$$

Let Mat_n be the space of complex matrices of order $n\times n$. A generic element of Gr_n is the graph of an operator

$$T: V_+^n\to V_-^n,$$

i.e., we have a mapping $\mathrm{Gr}_n\to\mathrm{Mat}_n$ defined everywhere, except for a set of measure 0.

To the action of the group $U(2n)$ on the Grassmannian there corresponds the action

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}: T\to (A+TC)^{-1}(B+TD) \tag{31}$$

on the space Mat_n.

Let $s\geq -1/2$. Let us introduce a measure μ_s^n with density

$$C(s,n)=\det\,(1+Z^*Z)^{-s-2n}$$

with respect to the Lebesgue measure on Mat_n. We choose the multiplier $C(s,n)$ in such a way that the measure of the whole space Mat_n is equal to 1.

The Radon-Nykodim derivative of transformation (31) is $\det^s(A+TC)$.

Let $m<n$. Consider the mapping

$$\mathbb{T}_m^n:\mathrm{Mat}_n\to\mathrm{Mat}_m$$

which put in correspondence to any matrix of order $n\times n$ the left upper angle of this matrix of order $m\times m$.

Theorem 6.4. *The image of the measure μ_s^n under the mapping π_m^n is π_s^m.*

Thus, we have the chain of mappings

$$\ldots\leftarrow\mathrm{Mat}_n\leftarrow\mathrm{Mat}_{n+1}\leftarrow\ldots,$$

and therefore, we obtain a certain measure μ_s on the projective limit of this chain, i.e. on the space Mat_∞ of all infinite matrices.

By construction, the group $G_0 = \bigcup_{n=1}^{\infty} U(2n)$ acts on the measure space $(\mathrm{Mat}_\infty, \mu_s)$.

Theorem 6.5. *The transformations $g \in \mathrm{Mat}_\infty, \mu_s)$ leave μ_s to be quasi-invariant; moreover, this action is extended to a quasi-invariant action of the group $G = (U(2\infty), (U(\infty) \times U(\infty))$.*

6.7. Two More Measures. Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$. Let $s > 0$. Let

$$\sigma_s(g) = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & s^{-1} \end{pmatrix}. \tag{32}$$

The operator σ_s is not longer a unitary one, but this operator differs from an unitary one by a Hilbert-Schmidt operator. Thus, the mapping $g \mapsto \sigma_s(g)$ is an embedding of the group G into the group GLO of the Hilbert space $l_2 \oplus l_2$ (which is considered as a real Hilbert space).

Thus, the Gauss measure on $\widehat{l_2 \oplus l_2}$ is quasi-invariant with respect to the ation of $\sigma_s(g)$ of the group G.

Let us pass to the second measure. Consider the space $\mathcal{H}$ of Hilbert-Schmidt matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying the condition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Introduce the inner product $\langle g, h \rangle = \mathrm{tr}\,(gh^*)$. Then $\mathcal{H}$ becomes a Hilbert space.

Further, let the group G act on $\mathcal{H}$ by transformations of the form

$$\tau \begin{pmatrix} A & B \\ C & D \end{pmatrix} Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} Q \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}^* + \left[\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}^* - 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \tag{33}$$

It is easy to see that the transformations $\tau(\cdot)$ are affine isometries (i.e., the products of a unitary operator and a shift), and therefore, a Gaussian measure on the Hilbert space $\hat{\mathcal{H}}$ (which is invariant with respect to G).

6.8. (G, K)**-Pairs by Ol'shanskii.** Let $\mathrm{GL}(\infty)$, $U(\infty)$, $O(\infty)$, ..., denote the general linear group, the general unitary group, the general orthogonal group,... of operators on s Hilbert space. We denote by $(G(\infty), K(\infty))$ the group of all operators $A \in G(\infty)$ representable in the form $A = B(1 + T)$, where $B \in K(\infty)$ and T is a Hilbert-Schmidt operator.

Ol'shanskii (cf. [30, 67]) suggests the following list of reasonable groups of the above-described type:

$$
\begin{array}{ll}
(\mathrm{GL}(\infty,\mathbb{R}), O(\infty)) & (U(\infty), O(\infty)) \\
(\mathrm{GL}(\infty,\mathbb{C}), U(\infty)) & (U(\infty)\times U(\infty), U(\infty)) \\
(\mathrm{GL}(\infty,\mathbb{H}), \mathrm{Sp}(\infty)) & (U(2\infty), \mathrm{Sp}(\infty)) \\
(U(\infty,\infty), U(\infty)\times U(\infty)) & (U(2\infty), U(\infty)\times U(\infty)) \\
(O(\infty,\infty), O(\infty)\times O(\infty)) & (O(2\infty), O(\infty)\times O(\infty)) \\
(\mathrm{Sp}(\infty,\infty), \mathrm{Sp}(\infty)\times \mathrm{Sp}(\infty)) & (\mathrm{Sp}(2\infty), \mathrm{Sp}(\infty)\times \mathrm{Sp}(\infty)) \\
(\mathrm{SO}^*(2\infty), U(\infty)) & (O(2\infty), U(\infty)) \\
(O(\infty,\mathbb{C}), O(\infty)) & (O(\infty)\times O(\infty), O(\infty)) \\
(\mathrm{Sp}(\infty,\mathbb{C}), \mathrm{Sp}(\infty)) & (\mathrm{Sp}(\infty)\times \mathrm{Sp}(\infty), \mathrm{Sp}(\infty)) \\
(\mathrm{Sp}(2\infty,\mathbb{R}), U(\infty)) & (\mathrm{Sp}(\infty), U(\infty))
\end{array}
$$

In all the cases, an explicit form of an embedding $K(\infty) \to G(\infty)$ is quite obvious.

All these groups are interpreted as groups of motions of infinite symmetric spaces $G(\infty)/K(\infty)$. This class of groups has a substantional representation theory; see [30, 67, 73]. We have mentioned two groups from these ones; they are the group $\mathrm{GLO} = \mathrm{GL}(\infty,\mathbb{R}), O(\infty))$ and the group $(U(2\infty), U(\infty)\times U(\infty))$. The groups $(\mathrm{Sp}(2\infty,\mathbb{R}), U(\infty))$ and $(O(2\infty), U(\infty))$ are the authomorphism groups of commutation and anticommutation relations; these are the classical objects of mathematical physics.

For all enumerated groups, it is easy to construct embeddings into GLO, which depend on a parameter (according to the model of formula (32)). Also, for groups of the second column it is easy to construct an affine action, which is analogous to (33). Finally, for groups of the second column, there exist Pikrelle measures.

Below (cf. Section 6.9. and 6.14), we shall use this fact.

It is natural to consider the Pikrelle measure connected with the (G, K)-pairs $(K(\infty)\times K(\infty), K(\infty)) = (O(\infty)\times O(\infty), O(\infty))$, $(U(\infty)\times U(\infty), U(\infty))$, $(\mathrm{Sp}(\infty)\times \mathrm{Sp}(\infty), \mathrm{Sp}(\infty))$ the infinite-dimensional analogs of Haar measures (on the finite-dimensional level, the symmetric spaces $K(n)\times K(n)/K(n)$ are the groups $K(n)$ themselves; as usual, in the infinite-dimensional case, a measure is supported on a space which is larger than the group itself).

C. Diffeomorphism Group of a One-Dimensional Manifold.

6.9. Embeddings of Diff **into Infinite-Dimensional Classical Groups.** There are a sufficiently large number of embeddings of the group Diff into the (G, K)-pairs by Ol'shanskii enumerated in Subsection 6.8; see [21, 22] for the list of such constructions. By the constructions of Subsections 6.6–6.8, this gives us several series of quasi-invariant actions of Diff. The construction of Subsection 2.5 is only one of these series.

Let us describe, for example, the embedding Diff into $(U(2\infty), U(\infty)\times U(\infty))$.

Let Diff act in $L_2(S^1)$ according to the formula

$$T_s(q)f(\varphi) = f(q(\varphi)q'(\varphi)^{1/2+s}.$$

We fix decomposition $L_2(S^1) = V_+ \oplus V_-$, where V_+ is spanned by the vectors $e^{ik\varphi}$ with $k \geq 0$, and V_- is spanned by the vectors $e^{ik\varphi}$ with $k < 0$. Then the operators $T_s(q)$ belong to the group $(U(2\infty), U(\infty) \times U(\infty))$ of the space $L_2(S^1) = V_+ \oplus V_-$ (cf. [22]).

6.10. Shavgulidze Measures. See Subsection 6.21 below.

6.11. Quasi-Invariant Measures on the Space of Convergent Sequences. Examples of such measures were constructed in [10]. It is interesting that a slight modification of the construction of Subsections 2.5 and 2.14 also gives us examples of similar measures. Namely, consider the Sobolev space H_s over the closed interval $[0,1]$ and the subspace L of this space consisting of functions $\tilde{f}$ satisfying the condition $f(0) = f'(0) = ... = f^{(k)}(0)$, where k is the entire part of s. Further (analogously to Subsection 2.14), we consider the corresponding measures on the space of closed subsets in $[0,1]$.

It is interesting to understand if there is any connection between these two constructions.

D. Loop Groups. Here we discuss in more detail two constructions of quasi-invariant measures which are close to the construction of Section 3 with $s = 1$ (cf. [13, 42, 43, 44, 50, 61]).

6.12. First Construction. Let H be the space of L_2-functions on S^1 with values in the space of skew-symmetric matrices of order $n \times n$. Let the loop group $\mathfrak{L}_n = C^\infty(S^1, \mathrm{SO}(n))$ acts on H by affine isometric transformations

$$g(\varphi) : f(\varphi) \mapsto g(\varphi)^{-1} f(\varphi) g(\varphi) + g'(\varphi)^{-1} g(\varphi).$$

Then we obtain automatically a quasi-invariant action of $\mathfrak{L}_n$ on $\hat{H}$.

6.13. Second construction. The Wiener measure on the space of continuous paths $S^1 \to \mathrm{SO}(n)$ is quasi-invariant with respect to the left and the right multiplications by elements of $\mathfrak{L}_n$, and therefore, $\mathfrak{L}_n \times \mathfrak{L}_n$ acts in L_2 with the Wiener measure.

There is a simple canonical isomorphism between the spaces L_2 with respect to these quasi-invariant measures (cf. [42, 43, 44]). But the only one copy of the group $\mathfrak{L}_n$ acts visually on $L_2(\hat{H})$.

Question 6.6. Is it possible to give an explicit description of the action of $\mathfrak{L}_n \times \mathfrak{L}_n$ on $L_2(\hat{H})$?

6.14. There are also many embeddings of the group $\mathfrak{L}_n$ in various Ol'shanskii (G, K)-pairs, together with all consequences, which follows from the arguments of Subsections 6.6–6.8. Some examples of such embeddings are given in [22].

E. Poisson Measures.

6.15. (cf. [5] and also [25]). Let M be a Lebesgue space with an infinite continuous measure. Let $\Omega(M)$ be the space of all countable subsets of M. Introduce the ordinary Poisson measure on $\Omega(M)$. By definition, this measure on $\Omega(M)$ is determined according to the following condition. Let $M_1, ..., M_k$ be pairwise disjoint subsets of M of a finite measure. Let l_j be nonnegative integers. Let $S(M_1, l_1; M_2, l_2; ...M_k, l_k)$ be the set of all points $\omega \in \Omega(M)$ such that $\omega \cap M_j$ consists of precisely l_j points. Then, by definition, the measure $S(M_1, l_1; M_2, l_2; ...M_k, l_k)$ is equal to

$$\prod \Big(\frac{\mu(M_j)^{k_j}}{k_j!} e^{-\mu(M_j)} \Big).$$

Let Gms_∞ be the transformation group of M leaving the measure to be quasi-invariant and which satisfy the following condition on the Radon-Nykodim derivative:

$$\int_M |g'(m) - 1| d\mu(m) < \infty.$$

Theorem 6.7. *The measure μ is quasi-invariant with respect to transformations $g \in Gms_\infty$; moreover the Radon-Nykodim derivative is given by the formula*

$$\Bigg(\prod_{m_j \in \omega} g'(m) \exp\Big(\int |g'(m) - 1| d\mu(m) \Big) \Bigg).$$

F. Induced Actions and the Araki Scheme. Often, we have the case where the existence of one G-quasi-invariant measure allow us to construct many other (quasi)-invariant measures and also to construct a larger collection of irreducible representations of the group G, than merely representation (10).

6.16. Group $\mathfrak{B}(M, \Gamma)$. Let Γ be a topological or a discrete group. Let M be a Lebesgue measure space which is equivalent to the close interval $[0, 1]$.

We denote by $\mathfrak{B}(M, \Gamma)$ (resp. $\mathfrak{B}_0(M, \Gamma)$) the group consisting of pairs (q, f), where q is a measure preserving transformation of M and f: $M \to \Gamma$ is a measurable function (resp. a measurable function from M to a certain compact subset of Γ which depends on f). The multiplication on $\mathfrak{B}(M, \Gamma)$ is given by the formula

$$(q, f) \cdot (p, h) = (q \circ p, f \cdot (h \circ p)).$$

In other words, $\mathfrak{B}(M, \Gamma)$ is a semidirect product of the automorphism group Ams of a measure space and the group of measurable functions $M \to \Gamma$.

6.17. Introduced Actions of $\mathfrak{B}(M, \Gamma)$. Let a group Γ act on a space L with a probability measure λ by measure preserving transformations. Then the group $\mathfrak{B}(M,\Gamma)$ acts on the Cartesian product $M \times L$ by the measure preserving transformations

$$(g,f) : (m,l) \mapsto (g(m), f(m)l).$$

6.18. Induced Representation of $\mathfrak{B}(M,\Gamma)$. Let ρ be a unitary representation of a group Γ in a Hilbert space V. Then $\mathfrak{B}(M,\Gamma)$ acts on the space $L_2(M,V)$, which consists of L_2-functions $M \to V$, according to the formula

$$(q,h) : f(m) \mapsto \rho(h(m))f(q(m)).$$

6.19. Actions Induced by Affine Actions of the Group Γ (Araki's Configuration). Let a group Γ acts on a Hilbert space V by transformations of the form

$$v \mapsto \rho(g)v_\gamma(g),$$

where ρ is a unitary operator and $\gamma(g) \in V$. In other words, we assume that Γ acts by affine isometric transformations of the space V.

Then $\mathfrak{B}_0(M,\Gamma)$ acts on $L_2(M,V)$ by affine isometries of the form

$$(q,h) : f(m) \mapsto h(m)f(q(m)) + \gamma(h(m)).$$

Therefore, $\mathfrak{B}_0(M,\Gamma)$ acts on $\widehat{L_2(M,V)}$ by transformations leaving the Gaussian measure to be quasi-invariant.

6.20. Comments. Let now we have succeed in the embedding of the group G into a certain group $\mathfrak{B}(M,\Gamma)$. Then according to Subsection 6.17, we obtain the quasi-invariant action of G from any invariant action of the group Γ, while according to subsection 6.18, we obtain the unitary representation of Γ (in the case where Γ is locally compact and Γ is closed subgroup of G, respectively, the described operation is an ordinary representation induction; see [16].

To apply the construction of Subsection 6.19, it is necessary to have affine isometric actionns of Γ; this situation meets not very often. Among simple Lie groups, these actions have only $\mathrm{SU}(n,1)$, $\mathrm{SO}(n,1)$ (note that both series include $\mathrm{SL}(2,\mathbb{R})$) and also the rank 1 groups over locally compact completely disconnected fields (cf. [60], Theorem 3.5.6; also see, e.g., [3, 5, 84] about the corresponding representations of groups of measurable functions). In addition, interesting affine actions have lattices in these groups; see Subsection 6.24 below.

G. Diffeomorphism Groups of High-Dimensional Manifolds. If all above-mentioned groups have a somewhat reasonable representation theory, then this not the case for diffeomorphism group of high-dimensional manifolds. However, here we have a series of interesting constructions of quasi-invariant measures and unitary representations; we discuss now some of these ones.

6.21. Shavgulidze Measures. We have used many times the symmetries of Gaussian measures, but we have spoke only on the linear symmetries. There exists a series of similar (but different in detail) theorems approximately of the following form (cf. [77, 37, 7]). Let μ be a Gaussian measure on a Hilbert space H (on this space but not on its extension). Let $F\colon H \to H$ be a diffeomorphism of the space H such that the operator $F'(h) - 1)$ is Hilbert-Schmidt, and moreover, the function $h \mapsto F'(h) - 1$ is continuous from H to the space of Hilbert-Schmidt operators. Then the measure μ is quasi-invariant with respect to the diffeomorphism F.

Shavgulidze (cf. [37, 38, 41]) construct a bijective mapping of C^N-diffeomorphism group (where N is sufficiently large) onto a certain Sobolev space H with a Gaussian measure. Under the mapping, the left shifts $p \mapsto q \circ p$ of infinitely smooth diffeomorphisms pass into nonlinear transformations of the above-mentioned form.

This mapping is given by a certain differential operator D whose explicit formula is very cumbersome; at present, this construction, in essence, has the status of existence theorem in all dimensions > 1.

Let us describe briefly what happen in dimension 1. (cf. [38, 23]). Consider the C^k-diffeomorphism group of the closed interval $[0, 1]$, where $k \geq 0$. We consider the expression

$$D(f) = \frac{f^{(k)}(t)}{f'(t)} \tag{34}$$

as the operator D, and take the ordinary Wiener measure for the Gaussian one.

Question 6.8. Is it true that this construction is equivalent to that of Subsection 2.12?

6.22. Corollaries. Let M^n be a smooth manifold of dimension $n > 1$, and let $\mathrm{Diff}^k(M^n)$ be the C^k-diffeomorphism group of M^n endowed with one of the Shavgulidze measures σ. Let $N \subset M^n$ be a smooth submanifold. We construct a measure ν on the set of all submanifolds of M^n which are diffeomorphic to N. To this end, we consider the mapping

$$\mathrm{Diff}^k(M^n) \to P$$

given by the formula $q \mapsto qN$. Obviously, the image of the measure σ is a $\mathrm{Diff}^k(M^n)$-quasi-invariant measures on the set of countable subsets of M^n, on the set of Cantor sets, etc. But now all these have the status of existence theorems.

In certain quasi-invariant measures on the space of convergent sequences of M^n were constructed in [11].

6.23. Ismagilov Embeddings (cf. [12]). Let M^n be a smooth n-dimensional manifold endowed with a volume form ω, let $\bar{\mathcal{D}}$ be the volume preserving diffeomorphism group of M, and let $\mathcal{D}$ be the connected component of the unity in the group $\bar{\mathcal{D}}$. For simplicity, we assume that the fundamental group Γ of the manifold M^n has the trivial center. Denote by $\tilde{M}^n$ the universal covering over

M^n. Then, by the triviality of the center of Γ, any element $q \in \mathcal{D}$ is uniquely lifted to a diffeomorphism $\tilde{q}$ of the covering manifold $\tilde{M}^n$.

Consider any fundamental domain on $\tilde{M}^n$, i.e., a domain with piecewise smooth boundary which is uniquely projected (with accuracy up to partings on the boundary) onto M^n. Then the domains $\gamma_i R$ cover $\tilde{M}^n$ when γ_i runs over Γ.

The embedding

$$\mathcal{D} \to \mathfrak{B}(M^n, \Gamma)$$

is given by the formula

$$p(m) \mapsto (p(m), c(p, m)),$$

where $m \in M^n \to \Gamma$ is determined by the condition

$$c(p, m) = \gamma \Leftrightarrow \tilde{q}m \in \gamma R.$$

The change of a fundamental domain leads us to the adjoint embedding.

6.24. Corollaries. Now, we can apply the arguments of Subsections 6.17–6.20. Consider the case where M^n is the sphere with $g > 1$ handles; denote by C this sphere.

(α) The group $\Gamma = \pi_1(\mathbb{C})$ admits many embeddings into $\mathrm{PSL}_2(\mathbb{R})$ as a co-compact lattice (the set of these embeddings is called the *Teichmüller space*). In what follows, $\mathrm{PSL}_2(\mathbb{R})$ acts on the circle S^1, and therefore, $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ acts on S^1; thus, by Subsection 6.17, we obtain a family of actions of the group $\mathcal{D}$ on the space $C \times S^1$ (moreover, these actions are numbered by points of the Teichmüller space).

(β) Analogously, $\Gamma = \pi_1(C)$ admits many embeddings (as a co-compact lattice) into the groups PSL_2 over locally compact completely disconnected fields $\mathbb{K}$. The group $\mathrm{PSL}_2(\mathbb{K})$ acts on the projective line $\mathbb{PK}^1$; thus, we obtain a family of actions of the group $\mathcal{D}$ on

$$C \times \mathbb{PK}^1.$$

(γ) Consider the embeddings $\Gamma \to \mathrm{PSL}_2$ which were discussed in (α) and (β). Let ρ be a unitary group SL_2. According to the Steger-Bishop-Coweling rigidity theorem (cf. [48, 47]), we have that if ρ is not in the discrete series, then the restriction of ρ to Γ is irreducible. According to the construction of Subsection 6.18, to each representation of the group Γ, there corresponds an induced representation of $\mathcal{D}$. It follows easily from the above-mentioned rigidity theorem that thus constructed unitary representations of the group $\mathcal{D}$ are irreducible (if ρ is not in the discrete series).

(δ) The groups $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{K})$ have nontrivial affine actions (see [3] for $\mathrm{SL}_2(\mathbb{R})$, and see, e.g. [26] for $\mathrm{SL}_2(\mathbb{Q}_p)$). Therefore the embedding of Γ into $\mathrm{PSL}_2(\mathbb{R})$ and $\mathrm{PSL}_2(\mathbb{K})$ allows us to apply the Araki scheme. Now we shall consider a more subtle method of such a type.

(ε) A metric tree is a tree (a connected graph which is free of cycles) whose edge length are fixed (the edges are considered as segments). Further, by deinition, an $\mathbb{R}$-tree is the inductive limit of metric trees embedded into each other (we do not require that under embeddings a vertex is mapped into a vertex; any vertex can be mapped into a point of an edge) (see [81, 62]), and also [46]; hovewer, the latter has no vizual connections with the question discussed now). According to [62, 81], the Thurston boundary of the Teichmüller space is interpreted as the set of all irreducible actions of the group $\Gamma = \pi_1(C)$ on $\mathbb{R}$-trees (an action is called irreducible if there are no invariant subtrees).

Let us assign to each $\mathbb{R}$-tree T a certain Hilbert space $H(T)$. For any ordered pair of points $x, y \in T$, we define a formal vector $e(x,y)$. Then we define the inner product of these formal vectors according to the rule

$$\langle e(x,y), e(u,v)\rangle = \pm l,$$

where l is the length of the intersection of the segments $[x,y]$ and $[u,v]$ on the $\mathbb{R}$-tree T. We take the sign $+$ in the case where when the segment is passed from x to y and from u to v, the segment $[p,q] = [x,y] \cap [u,v]$ is passed in the same direction In the opposite case, we put the sign $-$ (minus).

Further, we assume that formal vectors of the form $e(x,y)$ constitute a total system of vectors in the Hilbert space $H(T)$, and thus, the Hilbert space $H(T)$ is completely determined (we note that if a point a lies on the segment $[x,y]$, then $e(x,a) + e(a,y) = e(x,y)$).

Let now the group Γ act on the tree T. Fix some point u of the tree T. Then the affine action of the group Γ on $H(T)$ is canonically defined by the formula

$$\gamma : \sum \alpha_j e(x_j, y_j) \mapsto \sum \alpha_j e(\gamma x_j, \gamma y_j) + e(u, \gamma u).$$

Now we can apply the construction of Subsection 6.19 and consider an action $\mathcal{D}$ induced from the affine action of the group Γ on $H(T)$.

(ζ) (cf. [12]). Consider the quotient group $\Gamma/[\Gamma,\Gamma]$ of the group Γ by the commutant. As is known,

$$\Gamma/[\Gamma,\Gamma] = \mathbb{Z}^{2g}.$$

Further, we embed $\mathbb{Z}^{2g}$ as a lattice into $\mathbb{C}^g$. Thus, we obtain the affine action of $\mathbb{Z}^{2g}$ by shifts on $\mathbb{C}^g$, and therefore, the action of Γ on $\mathbb{C}^g$.

H. p-Adic Diffusions.

6.25. (cf. [26]) Let $s \in \mathbb{R}$. For simplicity, assume that $|s| \neq 1/2$. Consider the space H_s of real-valued functions on the p-adic projective line $\mathbb{Q}_p\mathbb{P}^1$ with the inner product

$$\langle f_1, f_2\rangle = \int_{\mathbb{Q}_p} |z_1 - z_2|^{-1-2s} f_1(z_1) f_2(z_2)\, dz_1 dz_2 + \int_{\mathbb{Q}_p} f_1(z)dz \cdot \int_{\mathbb{Q}_p} f_2(z)dz,$$

where C is a certain constant choosed in such a way that the inner product is definite. If $|s| < 1/2$, we can set $C = 0$. For $|s| > 1/2$, we have to choose a nonzero C. But under various choices of C, we obtain the inner products, which are equivalent in the sense of Subsection 2.4.

Let now $\mathrm{An}(\mathbb{Q}_p\mathbb{P}^1)$ be the group of analytic diffeomorphisms of $\mathbb{Q}_p\mathbb{P}^1$. Let this group acts on H_s by the formula

$$T_s(q)f(z) = f(q(z))|q'(z)|^{1/2-s}.$$

Then $H_s(q) \in \mathrm{GLO}(H_s)$, and there arises a picture which is analogous to that described in Section 2 (this is not studied at all at present).

7 ON DESCRIPTION OF INVARIANT MEASURES

In the preceedinng 6 Sections, our goal was to construct (quasi)-invariant measures. It is natural to ask are there many (quasi)-invariant actions for a given group. In this section there sre few positive results, but it is of reason to discuss what problem are solved here, and what are reasonable or not.

7.1. Can We Describe all G-Invariant Ergodic Measures? Now we present three silly constructions, which give us many examples of G-invariant measures and ergodic ones.

Example 7.1. Let ρ be aunitary (irreducible in general) representation of the group G in a Hilbert space H. Then G acts on $\hat{H}$ by measure preserving transformations. If ρ has no finite-dimensional representations (this case is typical), then the action is ergodic (see Theorem 2.7 above).

Example 7.2. Consider an irreducible unitary representation of the group $G \times U(n)$ in a Hilbert space H (any such representation is the product of a unitary irreducible representation of the group G and that of the group $U(n)$. Then we have the invariant action of $G \times U(n)$ on the space $\hat{H}$. Further, consider an arbitrary finite subgroup L in $U(n)$. Then G acts on the quotient space $\hat{H}/L$.

Example 7.3. Let the group G be imbedded into a certain group $\mathfrak{B}(M, \Gamma)$. Then (cf. Subsection 6.18) we obtain the quasi-invariant action of the group G from any invariant action of the group Γ.

These constructions show us that an ergodic measure itself is not an interesting object, and the question on a description of all ergodic measures seems to have no reasonable answer even for the simplest infinite-dimensional groups (in contrast to the question on a description of all unitary representations, which is solved in many cases; see [68, 66] and also [28] for references).

7.2. Action of a Group on a Given Space. Let an infinite-dimensional group G acts on a complete metric space M.

Question 7.4. Is it true that any G-(quasi)-invariant measure on M is decomposed into ergodic measures?

Of course, this question is classical, and there is a positive answer for many types of groups (cf. [49, 59]). But, to the author's opinion, these theorems are not applicable to the most of interesting infinite-dimensional groups (although, these theorems are appropriate for certain groups).

Question 7.5. Is it possible to describe all invariant ergodic measures for a given action?

Of course, in the general case, the answer is negative here (precisely as it is not possible to describe orbits of an arbitrary action of any finite group or any Lie group). However, it is important to understand for what groups and for what actions this question has a sense. We pass now to the discussion of this question.

7.3. De Finetti Theorem. Let the group S_∞^{fin} acts on the space $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times ...$ by permutations of coordinates.

Theorem 7.6. (De Finetti theorem). *Any probability measure on* $\mathbb{R}^\infty = \mathbb{R}\times \mathbb{R}\times ...$ *whish is invariant with respect to* S_∞^{fin} *is decomposed into invariant ergodic measures. Any invariant ergodic measure is a product of the same measures on the factors* $\mathbb{R}$.

Denote by L the group of all compactly supported vectors $t = (t_1, t_2, ...)$, $t \in \mathbb{R}$. Denote by G the semidirect product $S_\infty^{\mathrm{fin}} \ltimes L$; this product consists of pairs $(g,t) \in S_\infty^{\mathrm{fin}} \ltimes L$, and the multiplication is given by the formula

$$(g,t)(h,s) = (gh, t + gs).$$

There exists a canonical correspondence between S_∞^{fin}-invariant measures of $\mathbb{R}^\infty$ and S_∞^{fin}-spherical representations (see Subsection 6.1) of the group G. Moreover, irreducivble representations correspond to ergodic measures.

Namely, let ν be a measure on $\mathbb{R}^\infty$ which is invariant with respect to S_∞^{fin}. Then G acts on $L_2(\mathbb{R}^\infty)$ by unitary operators

$$\rho(g,t)f(x) = f(gx)e^{i(\sum t_k x_k)};$$

moreover, it is easy to see that the function $f_0(x) = 1$ is invariant with respect to S_∞^{fin} and is a G-cyclic vector.

Conversely, consider an S_∞^{fin} spherical representation of the group G in a Hilbert space H, and assume that v is a cyclic vector. The Abelian group $L \subset G$ (this is an inductive limit of locally compact groups) acts on the space H, and the vector v is L-cyclic. Decompose this representation of L into a direct integral of characters of the group L. The group of characters of the group L is precisely the additive group of the space $\mathbb{R}^\infty$; we fix the cyclic vector v, and therefore, a spectral measure is fixed on $\mathbb{R}^\infty$, which is defined by ν. It is easy to see that this measure ν is S_∞^{fin}-invariant.

The above-described correspondence between invariant measures and representations of semidirect products is classical and come back to Wiener at least; cf. [16].

By (30), the spherical function of the representation ρ is a function on the set of double conjugat classes $S_\infty^{\mathrm{fin}}\backslash G/S_\infty^{\mathrm{fin}}$, or in other words, this is an S_∞^{fin}-invariant function on the group L, i.e., this function depends only on the non-ordered collection $(t_1, t_2, ...)$. It is easy to see that this function is precisely the characteristic function (the Fourier transform) of the measure ν.

Now, the De Finetti theorem can be reformulated in the following form.

Theorem 7.7. *Any S_∞^{fin}-spherical representation ρ of the group $G = S_\infty^{\mathrm{fin}} \bowtie L$ is decomposed in a direct integral of irreducible spherical representations. Moreover, the spherical function of an irreducible spherical representation has the form*

$$\varphi(g,t) = \prod_{j=1}^{\infty} \psi(t_j). \tag{35}$$

Of course, at present, the De Finetti theorem is a sufficiently obvious assertion from probabilistic (see the discussion in [45]) as well as from group theory points of view.

The assertion on the existence of decomposition of a measure into ergodic components follows here from general theorems of such a type (see discussion in [45]). At the same time, on the representation level, this corresponds to the decomposition of the compact set of spherical functions by extreme points.

Equation (35) is a very simple particular case of Ismagilov-Ol'shanskii multiplicativity (see [66, 68] for the discussion of this phenomenon); in this case, Theorem 7.7 follows from the existence of a multiplication on the set of double conjugate classes

$$S_\infty^{\mathrm{fin}}\backslash G/S_\infty^{\mathrm{fin}}.$$

We make one more remark. Consider the following topology on the general infinite symmetric group. Denote by S_∞^n a subgroup of S_∞ which consists of permutations leaving the elements $1, 2, ..., n$ to be fixed. The topology on S_∞ is defined by the following condition: the subgroups S_∞^n form a fundamental system of neighborhoods of the identity. Consider the group

$$\tilde{G} = S_\infty \bowtie L \supset G = S_\infty^{\mathrm{fin}} \bowtie L.$$

It is easy to show that S_∞^{fin}-spherical representations of $\tilde{G}$ are extended to representations which are continuous on the subgroup S_∞ (see [67, 31] for the discussion of topologies of such a type).

Question 7.8. Describe all unitary representations of the group $\tilde{G}$.

Note that the group $\tilde{G}$ is a simple example of the (G, K)-pairs in the sense of [66], and the problem of classification of representations of this group is look to be simple.

7.4. Exchangeability. There is a wide bibliography devoted to the generalization of the De Finetti Theorem. A remarkable review by Aldous (cf. [45], see also more recent papers [54, 55, 74, 75]) deals with this topic, and we only mention briefly what sort of assertions are known here.

Consider the group $S_\infty^{\mathrm{fin}} \times S_\infty^{\mathrm{fin}}$. Let this group act on the space $\mathrm{Mat}^{\mathrm{fin}}(\infty)$ of infinite compactly supported matrices (i.e., all matrix elements, except for a finite number, are equal to zero), so that the first factor S_∞^{fin} transposes the rows, while the second one transposes the columns. Then the description of invariant ergodic measures is known (cf. [45]). An analogous problem was solved for any finite products $S_\infty^{\mathrm{fin}} \times S_\infty^{\mathrm{fin}} \times \ldots \times S_\infty^{\mathrm{fin}}$ (cf. [54]) and also for the groups $O(\infty) \times \ldots \times O(\infty)$ (cf. [55]).

From the representation theory point of view, first of all, we speak about semidirect products of the form

$$[S_\infty^{\mathrm{fin}} \times S_\infty^{\mathrm{fin}}] \times \mathrm{Mat}^{\mathrm{fin}}(\infty),$$

$$[O^{\mathrm{fin}}(\infty) \times O^{\mathrm{fin}}(\infty)] \bowtie \mathrm{Mat}^{\mathrm{fin}}(\infty), \tag{36}$$

Also, there were discussed representations of the group

$$\mathrm{Ams}([0,\infty)) \bowtie L_1[0,\infty)$$

(see the discussion of the "exchangeable" stochastic processes in [45, 10].

All these groups are (G, K)-pairs in the sense of [66], and we can apply to them the technique of "latent structures", which is standard for infinite-dimensional group representation theory. However, in representaion theory, these groups do not call a special attention. One ezception (which also does not call a great attention) is the motion groups of infinite-dimensional flat matrix spaces ((36) is an example of such a group; these groups are degenerations of Ol'shanskii (G, K)-pairs enumerated in Subsection 6.8; see [72, 69] for their spherical representations).

One more type of groups for which the problem of the "exchangeability" type were discussed are the automorphism groups of thus such that a countable number of edges is emanated from each vertex; cf [45, 13]. It is interesting to understand is there any connection of this work with representation theory results on these groups (a classification of unitary representations of these groups was obtained in [29]).

A description of all measures of the space Virt of virtual permutations which are invariant with respect to S_∞ was obtained in [74, 75] (see also [15]).

Question 7.9. Is it possible to isolate all measures that are quasi-invariant with respect to SS_∞ from all S_∞-invariant measures on Virt? See [15] for the discussion on a large class of such measures.

Finally, we recall 2 questions, which are posed in [2] (it should be noted that this paper contains a series of mistakes), which to the author opinion not yet solved.

Question 7.10. Let Σ_k be the set of all nonordered k-element subsets of $\mathbb{N}$. Let $\mathbb{R}^{\Sigma_k}$ be theset of mappings $\Sigma_k \to \mathbb{R}$. It is required to describe all S_∞^{fin}-invariant measures on $\mathbb{R}^{\Sigma_k}$.

Question 7.11. Let T be the set of tensors of some fixed type over $\mathbb{R}^\infty$. Describe all ergodic measures on T which are invariant with respect to $Q^{\text{fin}}(\infty)$.

7.5. Variations. The following questions concern with "noncommutative analogs" of questions discussed in Subsections 73–74.

Question 7.12. Let Γ be an arbitrary noncommutative discrete or locally compact group (for example $\Gamma = \mathrm{SL}_2(\mathbb{R})$). Let $\mathcal{F}(\mathbb{N}, \Gamma)$ be the group of functions $\mathbb{N} \to \Gamma$ which are equal to 1 everywhere, except for a finite number of points. Is it possible to describe unitary representations of the group $S_\infty \ltimes \mathcal{F}(\mathbb{N}, \Gamma)$?

Question 7.13. Describe all unitary representations of the groups $\mathfrak{B}(M, \Gamma)$ (see Subsection 6.16) and also of the semidirect product of the automorphism group of the measure space $[0, 1]$ and the group of all bounded measurable functions $[0, \infty) \to \Gamma$ which are equal to 1 everywhere, except for a subset of a finite measure.

An attempt to find an approach to the problem of such a type was intended in [25].

Question 7.14. Is it possible (in some sense) to describe all ergodic measures on Diff^1 which are quasi-invariant with respect to the action of Diff by left shifts?

This question is look a sufficiently dangerous. For example, the right shift of a left quasi-invariant measure is a left quasi-invariant measure. Moreover, in the Shavgulidze construction (see Subsection 6.21), we can slightly change the Gaussian measure leaving the differential operator (34) to be the same.

Question 7.15. Fix $\alpha \in \mathbb{C}$. Let Diff act on the space $\mathcal{D}'$ of distributions in the circle by the formula

$$T_\alpha(q)f(x) = f(q(x))q'(x)^\alpha.$$

Is it possible to describe all measures on $\mathcal{D}'$ which are quasi-invariant and ergodic with respect to the action of Diff?

This question contains the preceding one (see construction of Section 2), and therefore, this question is more dangerous. Note that all the constructions of Section 2–4 can be treated as measures on the spaces of distributions.

7.6. Joinings. There is one more type of problems, where the question of description of ergodic measures arises.

Let a group G acts on measure space (M, μ) and (N, μ) and preserves the measures. Then G acts on the product $M \times N$. The *joining* k is a measure on $M \times N$ which is invariant with respect to G and such that the projection of k

onto M is μ and the projection of k onto N is ν. In other words, the joining is a polymorphism commuting with the group action.

As the polymorphisms is naturally considered as morphisms of measure spaces, the joining should naturally be understood as morphisms of dynamical systems.

Joinings were discussed many times in the case where $G = \mathbb{Z}$ or $\mathbb{R}$. (cf. [78, 34]); for example, in the case $M = N$, the semigroup of joinings is an interesting invariant of a dynamical system.

For infinite-dimensional groups, the problem of computing all joinings is not still appeared (to the author knowledge), however, under constructing the harmonic analysis connected with invariant actions this problem should inevitably be appeared.

We note an obvious analogy of joinings to invariant distributions using in Lie groups representation theory (Bruhat's method; see [16]); also, we note that there is a one-to-one correspondence between joinings and Markov interswinging operators.

7.7. Simple Actions. Let (M, μ) be a Lebesgue space with a continuous probability measure. The center of semigroup of polymorphisms of the space (M, μ) into itself consists of polymorphisms of the form $\gamma(s) = s\alpha + (1 - s)\beta$, where α is a "uniform spreading" (see Example 5.7), β is the polymorphism corresponding to the identity mappings, and $0 < s < 1$.

An invariant action of the group G on the measure space is called *simple* if the semigroup of its joinings into itself is trivial (i.e., consists of the polymorphisms $\gamma(s)$). To the author opinion, the simplicity of action for infinite-dimensional groups is a more precise analog of the irreducibility of a representation than the ergodicity.

We can consider the simplicity of action as an analog of the concept of a 2-transitive action of a finite action. Note that all "silly" constructions of the beginning of this section give us actions with a heavy semigroup of joinings into itself (at the same time, in many of these constructions, the action of G on any product $M \times M \times ... \times M$ remains to be ergodic).

Question 7.16. Describe all simple invariant actions of the groups S_∞, O_∞, and Ams.

Consider the following three semigroups.

1. The semigroup PB of all matrices consisting only of zeros and units in which there are not most than one unity stands in each row and each column.

2. The semigroup Cont of all operators on a Hilbert space with norm ≤ 1.

3. The semigroup MAR of all polymorphisms of the segment $[0, 1]$ into itself.

Introduce the weak operator topology and the convolution, an ordinary conjugation of operators on PB and Cont (on MAR, the corresponding structures were introduced in Subsection 5.4).

Not difficult arguments connected with the latent structures (the model of such arguments is the proof of Corollary 5.12) show us that Question 7.16 is reduced to the following one: describe all continuous homomorphisms

$$PB \to \mathrm{MAR}, \quad \mathrm{Cont} \to \mathrm{MAR}, \quad \text{and} \quad \mathrm{MAR} \to \mathrm{MAR}$$

such that the center of the image is trivial in MAR.

References

[1] *Vershik, A.M.* Multivalued Mappings with an Invariant Measure (Polymorphisms) and Markov Processes, *Zap. Nauchn. Seminarov LOMI,*1977, vol. 72. pp. 26–62.

[2] *Vershik, A.M.* On a deskription of Invariant Actions of Certain Infinite-Dimensional Groups, *Dokl. Akad. Nauk SSSR,*1973, vol. 218. pp. 749–752.

[3] *Vershik, A.M., Gelfand, I.M., and Graev, M.I.,* Representations of $\mathrm{SL}(2, R)$, where R is a Function Ring, *Uspekhi Mat. Nauk,*1973, vol. 28, no. 3, pp. 3–41.

[4] *Vershik, A.M., Gelfand, I.M., and Graev, M.I.,* Representation of a Diffeomorphism Group, *Uspekhi Mat. Nauk,*1995, vol. 30, no. 6, pp. 3–50.

[5] *Vershik, A.M., Gelfand, I.M., and Graev, M.I.,* Commutative Model of a Representation of the Current Group $\mathrm{SL}(2, \mathbb{R})^X$ Connected with an Unipotent Subgroup, *Funkts. Anal. Pril.,*1983, vol. 17, no. 2, pp. 70–72.

[6] *Go, H.S., Gaussovy mery v banakhovykh prostranstvakh* (Gaussian Measures on Banach Spaces), Moscow; Mir, 1979.

[7] *Daletskii, Yu. M., and Fomin, S.V., Mery i differentsial'nye uravneniya v beskonechnomernykh prostranstvakh* (Measures and Differential Equations in Infinite-Dimensional Spaces), Moscow: Nauka, 1983.

[8] *Dynkin, E.B.* Several Limit Theorems for Sums of Independent Random Variables with Infinite Expectations, *Izv. Akad. Nauk SSSR, Ser. Mat.,*1955, vol. 19, pp. 247–266.

[9] *Ibragimov, I.A., Rozanov, Yu.A., Gaussovy statsionarnye processy* (Gaussian Stationary Processes), Moscow: Nauka, 1970.

[10] *Ismagilov, R.S.,* Unitary Representations of the Circle Diffeomorphism Group, *Funkts. Anal. Pril.,*1971, vol. 5, no. 3, pp. 45–53.

[11] *Ismagilov, R.S.,* On Unitery Representations of the Diffeomorphism Group of a Compact Manifold, *Izv. Akad. Nauk SSSR, Ser. Mat.,*1972, vol. 36, pp. 180–208.

[12] *Ismagilov, R.S.,* Embedding of Diggeomorphism Group of a Manifold which Preserve the Volume into a Semidirect Product, and its Unitary Representations, *Mat. Sb.,*1980, vol. 113, pp. 81–97.

[13] *Ismagilov, R.S.,* On Unitary Representations of the Groups $C^{\infty}(X, G), G = \mathrm{SU}_2$, *Mat. Sb.,*1976, vol. 100, no. 2, pp. 117–131.

[14] *Ito, K., and McKean, H.P., Diffusionnye processy i ikh traektorii* (Diffusion Processes and Their Trajectories), Moscow: Mir, 1968.

[15] *Kerov, S.V.,* Subordinators and Substitution Actions with a Quasi-Invariant Measure, *Zap. Nauchn. Seminarov LOMI,*1995, vol. 223, pp. 181–218.

[16] *Kirillov, A.A., Elementy teorii predstavlenii* (Elements of Representation Theory), Moscow: Nauka, 1972.

[17] *Kirillov, A.A.,* Koehler Structure on K-Orbits of the Circle Diffeomorphism Group, *Funkts. Anal. Pril.,*1987, vol. 21, no. 2, pp. 42–45.

[18] *Levy, P., Stokhasticheskie protsessy i brounovskoe dvizhenie* (Stochastic Processes and Brownian Motion, Moscow: Mir, 1972.

[19] *Neretin, Yu.A.,* Additional Series of Unitary Representations of the Circle Diffeomorphism Group, *Uspekhi Mat. Nauk,*1982, vol. 224, no. 2, pp. 213–214.

[20] *Neretin, Yu.A.,* Boson Representations of the Circle Diffeomorphism Group, *Dokl. Akad. Nauk SSSR,*1983, vol. 272, no. 3, pp. 528–531.

[21] *Neretin, Yu.A.,* Almost Invariant Structures and Unitary Representation Constructions of the Circle Diffeomorphism Group, *Dokl. Akad. Nauk SSSR,*1985, vol. 294, no. 1, pp. 37–41.

[22] *Neretin, Yu.A.,* Representations of the Virasoro Algebra and Affine Algebras, *Itogi Nauki i Tekhn. VINITI, Ser. Sovr. Probl. Mat. Fund. Napravleniya,* vol. 22, Moscow: VINITI, 1988, pp. 163–224.

[23] *Neretin, Yu.A.,* Spinor Representation of an Infinite-Dimensional Orthogonal Semigroup and the Virasoro Algebra, *Funkts. Anal. Pril.,*1989, vol. 23, no. 3, pp. 32–44.

[24] *Neretin, Yu.A.,* Holomorphic Extensions of the Circle Diffeomorphism Group, *Mat. Sb.,*1989, vol. 180, no. 5, pp. 635–657.

[25] *Neretin, Yu.A.,* Categories of Bistochastic Measures and Representations of Certain Infinite-Dimensional Groups, *Mat. Sb.,*1992, vol. 183, no. 2, pp. 52–76.

[26] *Neretin, Yu.A.,* On Combinatorial Analogs of the Circle Diffeomorphism Group, *Izv. Akad. Nauk SSSR, Ser. Mat.,*1992, vol. 56, no. 5, pp. 1071–1085.

[27] *Neretin, Yu.A.,* Random Cantor Sets and Half-Line Diffeomorphism Group, *Mat. Sb.,*1996, vol. 187, no. 6, pp. 73–84.

[28] *Okun'kov A.Yu.,* Thom Theorem and Representations of an Infinite Bisymmetric Group, *Funkts. Anal. Pril.,*1994, vol. 28, no. 2, pp. 31–40.

[29] *Ol'shanskii, G.I.,* New large Groups of the Type I, *Itogi Nauki i Tekhn. VINITI. Ser. Sovr. Probl. Mat.,* vol. 16, Moscow: VINITI, 1980, pp. 31–52.

[30] *Ol'shanskii, G.I.,* Unitary Representations of (G, K)-Pairs and the Howe Formalism, *Dokl. Akad. Nauk SSSR,*1983, vol. 269, no. 1, pp. 33–36.

[31] *Ol'shanskii, G.I.,* Unitary Representations of (G, K)-Pairs Connected with Ithe Infinite Symmetry Group, *Algebra Analiz*, 1990, vol. 1, no. 4, pp. 178–209.

[32] *Pressley, A., and Sigal, Gr., Gruppy petel'* (Loop Groups), Moskow: Mir, 1990.

[33] *Rokhlin, V.A.,* On Main Concepts of Measure Theory, *Mat. Sb.,*1949, vol. 25, pp. 107–150.

[34] *Ryzhikov, V.,* Joining, Interwiiging Operators, and Mixing Properties of Dynamical Systems, *Izv. Akad. Nauk SSSR, Ser. Mat.,*1993, vol. 57, no. 1, pp. 102–128.

[35] *Skorokhod, A.V., Integrirovanie v gil'bertovom prostranstve* (Integration in a Hilbert Space), Moscow: Nauka, 1975.

[36] *Skorokhod, A.V., Sluchainye processy s nezavisimymi prirachsheniyami* (Stochastic Processes with Independent Increments), Moscow: Nauka, 1986.

[37] *Shavgulidze, E.T.,* Measure which is Quasi-Invariant with Respect to the Diffeomorphism Group of a Finite Dimensional Manifold, *Dokl. Akad. Nauk SSSR,*1988, vol. 303, no. 4, pp. 811–814.

[38] *Shavgulidze, E.T.,* Distributions on Infinite-Dimensional Spaces, *Pyataya Mezhdunarodnaya Konferentsiya po Teorii Veroyatnostei i Matematicheskoi Statistike. Tezisy Soobshchenii,* Vil'nyus, 1989, pp. 359–360.

[39] *Shilov, G.E., and Phan Dyk Tyn, Integral, mera, proizvodnaya v lineinykh prostranstvakh* (Integral, Measure, and Derivative on Linear Spaces), Moscow: Nauka, 1967.

[40] *Shiryaev, A.N., Veroyatnost'* (Probability), Moscow: Nauka, 1979.

[41] *Khafizov, M.U.,* A Certain Quasi-Invariant Measure on the Domain Diffeomorphism Group, Mat. Zametki, 1990, vol. 48, no. 3, pp. 131–141.

[42] *Albeverio, S., Høegh-Krohn, R.,* The energy representations of Sobolev–Lie groups, Compos. Math, 1987, vol. 36, p. 37–52.

[43] *Albeverio, S., Høegh-Krohn, R., Testard, D.,* Irreducibility and reducibility for the energy representations of the group of mappings from Riemann manifolds into a compact Lie group, *J. Funct. Anal.,*1981, vol. 41, pp. 378-396.

[44] *Albeverio, S., Høegh-Krohn, R., Testard, D., Vershik, A.M.,* Factorial representations of path groups. *J. Funct. Anal.,*1983, vol. 51, no. 1, pp. 115–131.

[45] *Aldous, D.,* Exchangeability and related topics, *Lect. Notes Math.,*1985, vol. 1117, pp. 1–198.

[46] *Aldous, D.,* The continuum random tree. I *Ann. Probab.,*1991, vol. 19, pp. 1–28; II Proc. Durham. Symp. Stochastic Analysis 23-70. Cambridge: Univ. Press,1991; III *Ann. Probab.,*1993, vol. 21, pp. 248–289.

[47] *Bishop, C., and Steger, T.,* Representation theoretic rigidity in $\mathrm{PSL}_2(\mathbb{R})$, Acta. Math., 1993, vol. 170, no. 1, pp. 121–149.

[48] *Cowling, M., and Steger, T.,* The irreducibility of restriction of unitary representations of lattices, J. Reine und Angew. Math., 1991, vol. 420, pp. 85–98.

[49] *Dynkin, E.B.,* Sufficient statistics and extreme points, *Ann. Probab.,*1978, vol. 6, pp. 705–730.

[50] *Frenkel, I.B.,* Orbital theory for affine Lie algebras, Invent. Math., 1984, vol. 62, pp. 23–66.

[51] *Hopf, E.,* The general temporally discrete Markov process, J. Rat. Mech. Anal., 1954, vol. 3, pp. 13-45.

[52] *Joyce, P., and Tavare, S.,* Random permutations and neutral evolution models, Stoch.Processes Appl., 1990, vol. 36, pp. 245-261.

[53] *Kahane, J.P.,* Some random series of functions, 2nd ed. Cambridge, 1985.

[54] *Kallenberg, O.,* Symmetries of random arrays and set-indexed processes, *J. Theor. Probab.,*1992, vol. 5, no. 4, pp. 727–765.

[55] *Kallenberg, O.,* Multiple Winer integrals and multivariant version of Schoenberg's theorem, Probab.Theory and Rel. Fields., 1995, vol. 102, pp. 91–143.

[56] *Kerov, S.V., Olshanskii, Gr.I, and Vershik, A.M.,* Harmonic analysis on infinite symmetric group. A deformatiom of regular representation, C. r. Acad. sci. Paris. Ser. Math., 1993, vol. 316, no. 8, pp. 773–778.

[57] *Kingman, J.F.,* Random partitions in population genetics, Proc. Roy. Soc. London A., 1978, vol. 361, pp. 1–20.

[58] *Krengel, U.,* Ergodic theorems. Berlin: W. de Gruyter, 1985.

[59] *Maitra, A.,* Integral representations of invariant measures, *Trans. Amer. Math. Soc.,*1977, vol. 229, pp. 209–225.

[60] *Margulis, G.A.,* Discrete subgroups of semisimple Lie groups. N.Y. etc.: Springer, 1991.

[61] *Malliavin, M., and Malliavin, P.,* Integration on loop groups, *J. Funct. Anal.,*1990, vol. 93, pp. 207–237.

[62] *Morgan, J.W., and Shalen, P.B.,* Valuations, trees, and degenerations of hyperbolic structures, Ann. Math. 2, 1984, vol. 120, no. 3, pp. 401–476.

[63] *Nelson, E.,* The free Markov fields, *J. Funct. Anal.,*1973, vol. 12, pp. 211–227.

[64] *Neretin, Yu.A.,* Infinite-dimensional groups, their mantles, trains and representations //Collection: Topics in representation theory /Ed Kirillov A.A. Providence (R.I.): Amer. Math. Soc., 1991, pp. 103–171. (Adv.Sov.Math.; vol. 2.).

[65] *Neretin, Yu.A.,* Some remarks on quasiinvariant actions of loop groups and the group of diffeomorfisms of the circle, *Communs. Math. Phys.,*1994, vol. 164, no. 3, pp. 599–626.

[66] *Neretin, Yu.A.,* Mantles, trains and representations of infinite-dimensional groups, Proc. First Eur. Math. Congr. Basel etc.: Birkhäuser, 1994, pp. 293–310.

[67] *Olshanskii, G.I.,* Unitary representations of infinite-dimensional pairs (G, K) and the formalism of Howe //Representations of Lie group and related topics. Gordon and Breach, 1990, pp. 269-464.

[68] *Olshanskii, G.I.,* On semigroups related to infinite dimensional groups, //Collection: Topics in representation theory /Ed Kirillov A.A. Providence (R.I.): Amer. Math. Soc., 1991, pp. 67–101. (Adv.Sov.Math.; vol. 2.).

[69] *Olshanskii, G.I., and Vershik, A.M.,* Ergodic unitarily invariant measures on the space of infinite Hermitian matrices: Preprint UTMS 94–61 Dept. Math. Sci. Univ. Tokyo. Oct. 19, 1994. 37 p.; //Amer. Math. Transl. (2). 1996. V. 175. (Contemporary mathematical physics: F.A. Berezin memorial volume).

[70] *Orey, S.,* Gaussian sample functions and Hausdorff dimension of level crossing Zietschr. Wahrscheinlichkeitstheorie. 1970. Bd. 15. S. 249–256.

[71] *Pickrell, D.,* Measures on infinite-dimensional Grassmann manifolds, *J. Funct. Anal.,*1970, vol. 70, no. 2, pp. 323–356.

[72] *Pickrell, D.,* Mackey analysis of infinite-dimensional motion groups, Pacif. J. Math., 1991, vol. 150, no. 1, pp. 139–166.

[73] *Pickrell, D.,* Separable representations for automorphism groups of infinite symmetric spaces, *J. Funct. Anal.,*1990, vol. 90, pp. 1–26.

[74] *Pitman, J.,* Partially exchangeable random partitions: Univ. Calif. Berkeley Techn. Rept., 1992. pp. 343.

[75] *Pitman, J.,* Random discrete distributions invariant under size-bazed permutations: Univ. Calif. Berkeley Techn. Rept., 1992, pp. 344.

[76] *Pitman, J., and Yor, M.,* Arcsine laws and interval partitions derived from stable subordinator, Proc. London Math. Soc., 1992, vol. (3)65, pp. 326-356.

[77] *Ramer, R.,* On nonlinear transformations of Gaussian measures, *J. Funct. Anal.,*1974, vol. 15, no. 2, pp. 166–187.

[78] *Rudolph, D.,* An example of a measure-presering map with minimal self-joinings, and applications, J'd Anal. Math., 1979, vol. 35, pp. 97-122.

[79] *Segal, I.E.,* Tenzor algebras over Hilbert spaces. I, *Trans. Amer. Math. Soc.,*1956, vol. 18, pp. 106–134; II, Ann. Math. Ser. 2. 1956, vol. 63, pp. 160–175.

[80] *Segal, I.E.,* Ergodic subgroups of the ortogonal group of real Hilbert space, Ann. Math. Ser. 2, 1957, vol. 66, pp. 297–303.

[81] *Shalen, P.B.,* Dendrology of groups: An introduction, Collection: Essays in group theory. N.Y. etc.: Springer, 1987, pp. 265-319.

[82] *Revuz, D., and Yor, M.,* Continuous martingals and brownian motion, N.Y. etc.: Springer, 1991.

[83] *Thoma, E.,* Die unzerlegbaren positiv-definiten Klassenfunktionen der abzahlbar undenlichen, symmetrischen Gruppe, Math. Ztschr., 1964, Bd. 85, S. 40–61.

[84] *Vershik, A.M., Gelfand, I.M., and Graev, M.I.,* Models of representations of current groups //Representations of Lie algebras and related topics. Budapest. Akad. Kiado, 1985, pp. 121-180.

[85] *Neretin, Yu.A.,* Categories of symmetries and finite-dimensional groups. Oxford: Oxford Univ. Press, 1996.

[86] *Bogachev, V.I., Gaussovskie mery* (Gaussian Measures), Moscow: Nauka, 1997.

[87] *Khua Lo Ken, Garmonicheskii analis funktsii mnogikh kompleksnykh peremennykh v klassicheskikh oblastyakh* (Garmonical Analysis of Functions of Many Complex Variables in Classical Domains), Moscow, IL, 1959.