A. S. Ésénine-Volpine. *Le programme ultra-intuitionniste des fondements des mathématiques.* Ibid., pp. 201–223.

The general aim of this paper is to establish the consistency of the usual set theories such as the system of Zermelo-Fraenkel, by constructing "pseudo-models" of formal derivations, roughly as follows. Terms, which in the abstract universe of all sets, define infinite sets, would be interpreted by large (hereditarily) finite sets built up from the empty set by means of the operation $(x, y) \to x \cup \{y\}$. Since the pseudo-model of any particular derivation d is required to "satisfy" only the formulas actually occurring in d, and not all their logical consequences, one would be free (as the author points out) to satisfy in the pseudo-model, e.g. (i) A and $A \to B$ but not B if B does not occur in d, or (ii) $t \neq s$, for terms t, s, even if for all terms t' occurring in d, $t' \in t$ and $t' \in s$ are simultaneously satisfied. The author's sketchy construction of such pseudo-models is not particularly convincing; but the general programme is both natural and important for anybody who seriously believes that we derive the properties of arbitrary sets by analogy with the notion of finite set. From this point of view the author's programme is closely related to Hilbert's ε-substitution method which also developed from a dogma of the "finiteness of our thinking."

Put precisely, suppose we formulate the axioms and deductions of set theory in Hilbert's ε-calculus described in Hilbert-Bernays, vol. 2 (V 16). All formulas are propositional combinations of atomic formulas $t = t'$ or $t \in t'$ where t and t' are ε-terms. Then a weak (non-constructive) form of the author's programme is this: given a derivation d, is there a mapping of the terms t in d into hereditarily finite sets $\bar{t}$, such that all formulas in d are true when the atomic formulas are replaced by their truth-values under this mapping? (Note that, for this interpretation, if A and $A \to B$ are satisfied, so is B; no use is made of the possibility (i) in the preceding paragraph.) A positive solution is obtained immediately from the observation, pointed out to us by F. Ville: If A is a quantifier-free formula built up from $=$ and $\in$, and $\exists x_1 \cdots \exists x_n A$ is true in any model of elementary set theory without $\in$-cycles, then there are hereditarily finite sets $\bar{x}_1, \cdots, \bar{x}_n$ which satisfy A. For, in the given derivation d, we need only replace distinct terms $t_1, \cdots, t_n$ by distinct variables $x_1, \cdots, x_n$, denote by A_d the conjunction of the formulas occurring in d, and observe that $\exists x_1 \cdots \exists x_n A_d$ is true in the universe of all sets. Evidently there is a recursive method (by trial and error) of finding $(\bar{x}_1, \cdots, \bar{x}_n)$. This formulation in terms of Hilbert's ε-calculus differs from the author's own scheme, which apparently (p. 206) needs as an auxiliary the fragment of predicate logic built up from $\neg x = y$, $\neg x \in y$ by means of negation, conjunction, and universal quantification; the reviewers have not been able to reconstruct such a scheme. Also on page 206 the author seems to imply a specific conjecture on the size of $(\bar{x}_1, \cdots, \bar{x}_n)$ above as a function of the number l of formulas occurring in d, namely 2^{l+50}, which does not seem plausible.

What is problematic, just as in the case of the ε-substitution method, is not the existence (and, in the present case of set theory, not even a bound on the size) of such finite "pseudo-models," but the principles needed to establish them. There is no explicit discussion of this crucial point in the paper under review. The author uses informally the notion of feasible number (*nombre réalisable* or *exécutable*, p. 203) implying on page 202 that properties of this notion might be used in the kind of consistency proof he envisages. However, he makes no specific proposals, and all the axioms for this notion that the reviewers have been able to find, turned out to give conservative extensions of first-order arithmetic; hence at least these properties cannot provide a consistency proof for the set theories under discussion.

It should be pointed out that the word "ultra-intuitionniste" in the title is completely misplaced. Brouwer was not preoccupied with finiteness, in fact he was one of the first to stress that an intuitive proof is an infinite object (when "fully analyzed," cf. e.g. *Mathematische Annalen*, vol. 97 (1927), pp. 60–75, footnote 8).

At the present time it would seem more fruitful to turn the author's "ultra-finitist" programme upside down, and ask: can we find problematic properties of the intuitive notion of feasible numbers, i.e., axioms satisfied by this notion, which (a) *do* allow a proof of consistency for analysis or Zermelo's set theory, and (b) more important, can in turn be proved to be consistent in other conventional set theories?

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Ibid., pp. 193–200.
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