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Yessenin-Volpin, A. S. [Esenin-Vol′pin, A. S.]
The ultra-intuitionistic criticism and the antitraditional program for foundations of mathematics.
Intuitionism and proof theory (Proc. Conf., Buffalo, N.Y., 1968), pp. 3–45. *North-Holland, Amsterdam*, 1970.

The paper under review (referred to as EV2) is the second published report of the unusual foundational work done by the author during the past 15 years. An important part of this program has been the study of the notion of proof and its relation to validity and the problem of the consistency of set theory, a brief report about which constituted the first paper (referred to as EV1) appearing in *Infinitistic methods* [(Proc. Sympos. Foundations of Math., Warsaw, 1959), pp. 201–223, Pergamon, Oxford, 1961; MR0147389 (26 #4905)]. The underlying assumptions of this program, made much more explicit in EV2, are sharply distinguished from those of traditional foundational studies in that: (1) the notion of natural number series is not categorical; (2) extensive linguistic considerations (from natural language) play a dominant role. An indication of these considerations is sketched below but first it is necessary to discuss two reviews of EV1 [J. Symbolic Logic **32** (1967), 517 and Zbl **134**, 9–10] which unfortunately have had a chilling effect on the discussion of the author's ideas.

These reviews make the following arguments. (α) Nothing in the author's description of "feasible numbers" (used in his constructions of a proof theory for ZF) prevents them from being formalized in non-standard analysis (that is to say, non-standard higher order arithmetic in the sense of A. Robinson) and non-standard analysis in a fairly general formalization has been shown to be a conservative extension of analysis [see G. Kreisel, *Applications of model theory to algebra, analysis, and probability* (Internat. Sympos., Pasadena, Calif., 1967), pp. 93–106, Holt, Rinehart and Winston, New York, 1969; MR0238686 (39 #50)]. (β) In addition to using different length natural number series, the author employs other ideas (such as the construction of a semantical validation procedure for a theorem from the proof of that theorem) all of which have been thought of and applied in other contexts. While these two observations are meant to give insight into the author's work, they are also used to draw the following conclusion: (γ) he is using ideas and principles which are either too weak (by α) or are not unusual enough (by β) to escape from the conclusions of Gödel's work on the consistency of theories.

The problem of escaping from Gödel's arguments is not the unsolvable problem of "formalizing the unformalizable" as some have despairingly felt, but, as explicitly stated by the author (EV2, p. 30), to modify and extend our traditional means of formalization, means which already involve implicit assumptions as to the categorical nature of the natural numbers.

And this leads us to the shortcomings of arguments (α), (β) and (γ). They completely ignore what is the most important element of the author's work, namely, the extensive development and use of linguistic concepts, especially the use of modalities and a theory of denotation. With regard to (α), while certain aspects of the author's proof-theoretical work can be formalized in non-standard analysis (using Kreisel's formula: the feasible numbers are to the standard number as the standard numbers are to the non-standard numbers) it cannot deal with "natural number series" which, while by definition are unending, are nevertheless embeddable in classically finite sets. Of course part of this misunderstanding has been due to the sketchy nature of EV1 [loc. cit.] and EV2 [loc. cit.] does serve the purpose of making the deeper aspects of the author's work more accessible.

The paper begins with a list of assumptions of traditional mathematics (pp. 4, 11). The analysis given shows the circular and interdependent nature of these assumptions, the most outstanding of course being T1, the categoricity of "the" natural number series. The author's position is not so much to say that T1 is wrong as to say that it is inadequate and restricting, that it is very useful to consider discretely proceding processes which are not required by fiat to satisfy induction, which is to say local transitivity is not to necessarily require global transitivity even in "connected" systems. Since justification of the soundness of the axiomatic method (conservation of truth through a deduction) depends on T1, the author must reject the axiomatic method in the deepest questions of foundations.

In the course of a critical analysis of these traditional assumptions, the author gives (p. 10) perhaps the most extensive decomposition of our justification of the principle of mathematical induction, showing that the circularity of this principle is very extensive indeed.

Nevertheless natural number series play an important role in his proof theoretic studies and consequently must be formulated in a way faithful to the above mentioned critique. This is done briefly as follows. A natural number series N is a "discrete process" with an initial event and a successor operation satisfying the usual conditions of being one-to-one and defined on its range. The actual prescription requires: (1) a statement of the rules generating the series; (2) a statement of the deontic modalities in which the rules are to be followed; (3) the specification of a "tactic of attention" which is a collection of rules for "following denotational connections with the sign N" (this is concerned with the assignment of the name N to the objects produced in the course of the discrete process).

Conditions (2) and (3) replace the traditional assumption that mathematical induction must hold along a natural number series and can be used to produce "short" natural number series, e.g., having no more than 3 terms. The heuristic device used in EV1 [loc. cit.] to produce a natural number series with fewer than 10^{12} elements is not necessary.

The combination of two such series, in particular ones of different length, can lead to what the author calls "Zenonian situations" in which the accomplishment of a "finite" (i.e., terminating) process associated with a segment of the "longer" series, apparently requires the completion of the "shorter" (but nevertheless unending) series. Roughly speaking, such a technique is used in the instantiation of the axiom of infinity with a large but "finite" set.

Thus to handle Zenonian situations is an important goal of the author's theories. By way of illustration he includes on p. 8 a short but enlightening discussion of Zeno's runner paradox. This is resolved in much the same way as C. Chihara does in his excellent article [Philosophical Rev. **74** (1965), 74–81].

The major portion of EV2 is devoted to an exposition of the positive part of the "antitraditional" program for the foundations of mathematics including an indication of the considerations required in his analysis of the consistency of ZF. (It should be added here that the author thinks of ZF as a formulation of only a fragment of mathematics. Of course, interest in the consistency of various axiomatic set theories is a natural one in a foundational study.) A considerable amount of terminology including a situational calculus (see p. 15) is introduced in terms of which a number of modal principles are expressed. These are concerned with construction procedures (e.g., the “central ontological hypothesis”, p. 28) as well as principles of inference that take cognizance of the fundamentally new types of obstacles to reasoning recognized by the ultra-intuitionist program, namely those concerned with Zenonian situation (see p. 23).

In addition to the unusual nature of the work, EV2 is very concise and is of a programmatic nature, all of which makes for difficult reading. This problem of conciseness and lack of details is to be rectified with the translation and publication of an extensive report of the author’s work, which was expected to appear in 1972.

One can say that within the author’s foundational framework the nature of the relation between local and global, simple and simplex phenomena as expressed in natural language no longer must assume a paradoxical aspect as it many times does when formalized in terms of traditional mathematical concepts. To the reviewer this is the most exciting feature of the author’s work. In any case, people (meaning mathematicians, logicians, philosophers and linguists) who are interested in foundations should be encouraged to pursue the ideas presented here either by making “classical” sense of them (this approach has already lead to interesting research in proof-theoretic studies about “feasibility” (e.g., the work of R. Parikh at Boston University) and in semantical studies (e.g., the work of E. Engeler and of R. Tereslow at the University of Minnesota and work of A. Ehrenfeucht at the University of Colorado, Boulder, Colorado)) or by a direct approach to his ideas, with the intention of understanding how linguistic considerations may enrich mathematical thought.

The author may, in a certain sense, be placed between L. E. J. Brouwer and G. Mannoury (see *Synthesis*, 1957, for a discussion of Mannoury’s contributions to the foundations of mathematics). Closer to Brouwer in that his antitraditional position permits mathematics to proceed (i.e., mathematical development in the context of his foundations is not impeded by presently unsolved philosophical and psychological problems) and closer to Mannoury in the throwing off of traditional assumptions and the exploitation of the syntactical, semantical and pragmatic nature of natural language.

On the basis of these two reports, EV1 and EV2, which have given us an enticing indication of his ideas and in particular of his consistency proof for ZF, we look forward with great expectation to the appearance of the English translation of the complete report on his foundational studies.

Reviewed by *J. Geiser*