MR0147389 (26 #4905) 02.20

Ésénine-Volpine, A. S. [Esenin-Vol′pin, A. S.]

Le programme ultra-intuitionniste des fondements des mathématiques. (French) 1961

Infinitistic Methods (Proc. Sympos. Foundations of Math., Warsaw, 1959) pp. 201–223 *Pergamon, Oxford*; *Państwowe Wydawnictwo Naukowe, Warsaw*

The author explains his most original and unusual ideas on the foundations of mathematics and sketches a proof for the consistency of the Zermelo-Fränkel set theory (ZF) based on his conception. He rejects the notion of an infinite sequence of natural numbers; instead, he admits that of a "natural sequence" K starting with 0, in which after every number a there is an immediate successor a', while $a' = b'$ implies $a = b$ and no number has 0 as its immediate successor, but such that K is shorter than, say, 10^{12}. As an example he cites the sequence of his heartpulsations during his childhood. From the usual point-of-view the supposition of such a natural sequence is contradictory, but the contradiction cannot be deduced in his system, because the length of a demonstration is also limited to some natural sequence. For instance, if too many steps would be necessary to verify the identity of the two expressions denoted by A in $A\,\&\neg A$, then the latter formula cannot be recognized as a contradiction. Only a rough sketch of the consistency proof can be given here; the reviewer was not able to reconstruct its details from the author's indications. First of all, the consistency of ZF is reduced to that of $ZF_i{}^-$, which results from ZF by omitting the axiom of extensionality, by replacing classical logic by intuitionistic logic, and by adding the axioms $\neg\neg(x = y) \to x = y$ and $\neg\neg(x \in y) \to x \in y$. The system S results from $ZF_i{}^-$ by the adjunction of the axiom (A): There exists a set U which is not equivalent to a natural number and which is not equivalent to one of its proper subsets. Suppose in S a contradiction can be derived by a proof consisting of l_0 formulas. Put $l = 2^{l_0+k}(k \le 50)$ and consider the natural sequence K_l which results from K by replacing every member of K by l new members. A formal process D_l starts with the unit sets formed by the numbers of K_l and forms new sets by two operations: (1) Forming $\{x\}$, where x is a previously constructed set; (2) forming $x + \{y\}$, where x and y are previously constructed sets. The theory $T(D_l, K_l)$ has as axioms the formulas which are true if the symbols are interpreted as indicated in the definitions of K_l and D_l, and the axioms of intuitionistic logic; moreover, the rule $A(t_0), \cdots, A(t_i), \cdots, \vdash A(x)$, where $t_0, \cdots, t_i, \cdots$ are the terms which can be substituted for x. In $T(D_l, K_l)$ every axiom of S can be proved. This remains true if $T(D_l, K_l)$ is replaced by a weaker system $F(D_l, K_l)$ which is a purely formal system. As K_l and D_l form a model for $F(D_l, K_l)$, the latter system is consistent.

Reviewed by *A. Heyting*