INTEGRAL OPERATORS WITH GAUSSIAN KERNELS AND SYMMETRIES OF CANONICAL COMMUTATION RELATIONS

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The first chapter of the book "The method of second quantization" by F. Berezin (Moscow, 1965) contains the description of the fundamentals of the analysis in boson and fermion Fock spaces and the basic techniques of dealing with symbol of operators. The greater part of Berezin's book (Chapters II–III) deals with automorphisms of canonical commutation and autocommutation relations. The theory of symbols of operators was afterwards developed by many authors, including Berezin himself (see [2–5]). The problem of the automorphism of commutation relations formulated by Friedrichs in 1953 (see [6]) attracted considerable attention in the 50s and the beginning of the 60s (see [7–10]), and the book by Berezin [1], summarizing the results of that period, contained apparently exhaustive answers to the issues under discussion. It became clear only much later that this is not quite the case. The aim of this paper is to describe both the Berezin construction and (mainly) later constructions that appeared in the course of its development (see [11–15]). We restrict ourselves to the boson case (for the fermion case, see [16]).

§1. Boson Fock space with a finite number of degrees of freedom

1.1. Fock space with n degrees of freedom. Let $n = 0, 1, 2, \dots$. A boson Fock space F_n is a space of holomorphic functions $f(z) = f(z_1, \dots, z_n)$ on $\mathbb{C}^n$ which satisfy the condition

$$\iint_{\mathbb{C}^n} |f(z)|^2 \, d\mu(z) < \infty, \tag{1.1}$$

where $d\mu$ is Gaussian measure on $\mathbb{C}^n$ with density $(1/\pi^n) e^{-|z|^2}$ (so that the measure of the whole $\mathbb{C}^n$ is 1). In F_n the scalar product is defined by

$$\langle f, g\rangle = \iint_{\mathbb{C}^n} f(z)\overline{g(z)} \, d\mu(z). \tag{1.2}$$

It turns out that the space F_n with the scalar product just introduced is a Hilbert space.

Remark. The space F_0 is a one-dimensional space consisting of functions on a singleton $\mathbb{C}^0$.

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1.2. **Standard basis.** It is easy to see by straightforward calculation that the set of functions

$$e_{\alpha_1,\dots,\alpha_n}(z_1,\dots,z_n)=\prod_{j=1}^{n}\frac{z_j^{\alpha_j}}{\sqrt{\alpha_j!}},\tag{1.3}$$

where α_j are nonnegative integers, forms an orthonormal basis in F_n.

1.3. **Overfilled basis.** Let $h=(h_1,\dots,h_n)\in\mathbb{C}^n$. In F_n we consider a function

$$\varphi_h(z)=\exp(z,h)=\exp\Bigl(\sum z_j\overline{h}_j\Bigr).\tag{1.4}$$

A simple calculation shows that

$$\langle\varphi_h(z),\varphi_r(z)\rangle=\exp(r,h)\,.\tag{1.5}$$

Moreover, it is easy to demonstrate that for any function the following remarkable equality (the "reproducing property") holds

$$\langle f(z),\varphi_h(z)\rangle=f(h)\,.\tag{1.6}$$

1.4. **Gaussian vectors.** Let P be a symmetric matrix of size $n\times n$ and $\|P\|<1$ (where $\|\cdot\|$ here and in what follows denotes the *Euclidean norm* of a matrix; for any matrix $\|A\|$ is the greatest of the eigenvalues of the matrix $\sqrt{A^*A}$). Now consider the function

$$b[P]=\exp\Bigl\{\frac12\sum p_{ij}z_iz_j\Bigr\},$$

where p_{ij} are matrix elements of the matrix P. It will be more convenient for us to write this expression in the form

$$b[P]=\exp\Bigl\{\frac12\,zPz^t\Bigr\},\tag{1.7}$$

where z is the row matrix $z=(z_1\ z_2\ \dots\ z_n)$, and the index t denotes the transposition of a matrix.

A simple calculation shows that

$$\langle b[P],b[Q]\rangle=\det[(1-P\overline{Q})^{-1/2}],\tag{1.8}$$

where raising to the power $-1/2$ is defined by the equation

$$(1+x)^{-1/2}=1-\frac12\,x+\frac{(-\frac12)(-\frac12-1)}{2!}\,x^2+\cdots.$$

1.5. **The canonical isomorphism $L^2(\mathbb{R}^n)\leftrightarrow F_n$.** Consider the space $L^2(\mathbb{R}^n)$ with the scalar product

$$\langle f_1,f_2\rangle=\frac{1}{\pi^{n/2}}\int_{\mathbb{R}^n}f(x)\overline{g(x)}\,dx\,.$$

In $L^2(\mathbb{R}^n)$ consider the set of functions

$$\widetilde{\varphi}_h(x) = \exp\Big\{ -\frac{1}{2}\Big(\sum \overline{h}_j^2 - 2\sqrt{2}\sum x_j\overline{h}_j + \sum x_j^2\Big)\Big\}, \tag{1.9}$$

where $h = (h_1, \dots, h_n) \in \mathbb{C}^n$. It is easy to verify that

$$\langle \widetilde{\varphi}_h, \widetilde{\varphi}_r \rangle = \exp(r, h). \tag{1.10}$$

It is more or less evident that linear combinations of the functions φ_h are dense in $L^2(\mathbb{R}^n)$. Therefore (see equations (1.5) and (1.10)), the correspondence

$$\{\widetilde{\varphi}_h \in L^2(\mathbb{R}^n)\} \longleftrightarrow \{\varphi_h \in F_n\}$$

extends to the linear unitary operator $\mathcal{BB}\colon L^2(\mathbb{R}^n) \to F_n$.

It is easy to write out an explicit expression for this operator

$$\begin{aligned}\mathcal{BB}f(z) &= \langle f(x), \widetilde{\varphi}_h(x)\rangle_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}^n} f(x)\exp\Big\{ -\frac{1}{2}\Big(\sum z_j^2 - 2\sqrt{2}\sum z_jx_j + \sum x_j^2\Big)\Big\}\, dx\,.\end{aligned} \tag{1.11}$$

1.6. **Correspondence of the functions.** In $L^2(\mathbb{R}^n)$ consider the set of Hermitian functions

$$H_n(x) = \sigma_n(x)e^{-x^2/2}.$$

Recall that it is determined by the conditions: σ_k is a kth power polynomial, and

$$\langle H_k(x), H_l(x)\rangle_{L^2(\mathbb{R})} = \delta_{k,l}\,.$$

Then the set of functions

$$\tilde{e}_{\alpha_1,\dots,\alpha_n}(x) = H_{\alpha_1}(x_1)\cdot\ldots\cdot H_{\alpha_n}(x_n)$$

forms an orthonormal basis in $L^2(\mathbb{R}^n)$. A simple calculation shows that

$$\mathcal{BB}\tilde{e}_{\alpha_1,\dots,\alpha_n} = e_{\alpha_1,\dots,\alpha_n}\,.$$

Now consider a symmetric matrix of size $n \times n$ with a strictly positive definite real part. Consider the vector

$$\tilde{b}[R] = \exp\Big\{ -\frac{1}{2}\, xRx^t\Big\} \in L^2(\mathbb{R}^n)\,.$$

One can easily see that

$$\langle \tilde{b}[R_1], \tilde{b}[R_2]\rangle = \det(R_1 + R_2)^{-1/2},$$

where $Q^{-1/2}$ is the analytic continuation of the function $A^{-1/2}$ defined on positive definite matrices. The existence of such an analytic continuation is seen from the following formula. Let $Q = A + iB$, where $A = A^t$ is positive definite and real, and $B = B^t$ is real. Then

$$Q = A^{1/2}(E + iA^{1/2}BA^{1/2})\,A^{1/2},$$

and the matrix $E + iA^{-1/2}BA^{-1/2}$ has eigenvalues of the form $1 + is$, where $s \in \mathbb{R}$.

A straightforward calculation shows that

$$\mathcal{BB}\cdot\tilde{b}[R] = \det(1 + R)^{-1/2}b[(R - E)(E + R)^{-1}].$$

1.7. **Symbols of operators (kernels).** Let A be a bounded operator $F_n \to F_m$. With it we associate the function

$$K_A(u,\overline{v}) = \langle A\varphi_v, \varphi_u\rangle_{F_m}, \tag{1.12}$$

which is called the *kernel* or *symbol* of the operator A. It is clear that the function K_A is holomorphic in the variables $u_1,\dots,u_m$ and antiholomorphic in the variables $v_1,\dots,v_n$.

Consider the matrix elements

$$s^{\alpha_1\alpha_2\dots}_{\beta_1\beta_2\dots} = \langle Ae_{\beta_1\beta_2\dots}, e_{\alpha_1\alpha_2\dots}\rangle$$

of the operator A in the standard basis $e^{\dots}_{\dots}$. It is easily verified that

$$K_A(u,\overline{v}) = \sum s^{\alpha_1\dots\alpha_m}_{\beta_1\dots\beta_n} e_{\alpha_1\dots\alpha_m}(u)\, e_{\beta_1\dots\beta_n}(\overline{v}). \tag{1.13}$$

It can be seen from the latter expression that the operator A is reconstructed from its symbol A: the function K_A turns out to be the generating function for matrix elements of the operator A (for this reason in the book by Berezin the symbols K_A are called *generating functionals*).

The operator A can be reconstructed from its symbol K_A by using a very simple and elegant formula

$$Af(z) = \int K(z,\overline{u})\, f(u)\, d\mu(u). \tag{1.14}$$

Thus all bounded operators $F_n \to F_m$ are integral operators in the strict sense of the word.

Note that if the kernel of the operator A is defined by the expression $K(z,\overline{u})$, the kernel of the operator A^* is $K(u,\overline{z})$.

1.8. **Correspondence of the kernels.** We have seen that any operator in F_n has a kernel. In $L^2(\mathbb{R}^n)$ this is certainly not the case. Consider the integral operator

$$Af(x) = \int_{\mathbb{R}^n} \widetilde{K}(x,y) f(y)\, dy$$

in $L^2(\mathbb{R}^n)$. Then the kernel K of the corresponding operator in F_n is obviously given by the expression

$$K(z,\overline{u}) = \langle A\varphi_u, \varphi_z\rangle = \frac{1}{\pi^n}\iint \widetilde{K}(x,y) \exp\Big\{ -\frac{1}{2}\Big(\sum (z_j^2 + \overline{u}_j^2) \\ - 2\sqrt{2}\sum (z_j+\overline{u}_j)x + 2\sum x_j^2\Big)\Big\}\, dx\, dy.$$

§2. Symmetries of commutation relations

2.1. **Creation-annihilation operators.** Consider a linear space $V_{2n} = V_{2n}^+ \oplus V_{2n}^- = \mathbb{C}^n \oplus \mathbb{C}^n$. We write its elements in the form

$$v = (v^+; v^-) = (v_1^+, \dots, v_n^+; v_1^-, \dots, v_n^-).$$

With each $v \in V_{2n}$ we associate the operator $\hat{a}(v)$ in F_n defined by the equation

$$\hat{a}(v)\, f(z) = \Big(\sum v_j^+ z_j + v_j^- \frac{\partial}{\partial z_j}\Big) f(z). \tag{2.1}$$

These operators are usually called *creation-annihilation operators*. It is easy to see that

$$[\hat{a}(v), \hat{a}(w)] = \sum (v_j^- w_j^+ - v_j^+ w_j^-)\, E, \tag{2.2}$$

where $[\,\cdot\,,\cdot\,]$ denotes the commutator of operators. It can also be easily seen that

$$\hat{a}(v_+; v_-)^* = \hat{a}(\overline{v}_-; \overline{v}_+). \tag{2.3}$$

We consider equations (2.3)–(2.4) as formal, not being interested in the domain of definition of the operators.

The equations (2.2) can be represented in a simpler way. We introduce in F_n the *creation operators*

$$\hat{a}_j^+ f(z) = z_j f(z)$$

and the *annihilation operators*

$$\hat{a}_j^- f(z) = \frac{\partial}{\partial z_j} f(z).$$

Then equations (2.2)–(2.3) can be written in the form

$$[\hat{a}_k^+, \hat{a}_m^+] = [\hat{a}_k^-, \hat{a}_m^-] = 0, \qquad [\hat{a}_k^+, \hat{a}_m^-] = -\delta_{km} E, \tag{2.4}$$

$$(\hat{a}_k^+)^* = \hat{a}_k^-. \tag{2.5}$$

The set of formal equalities (2.4)–(2.5) is called the *canonical commutation relations.*

It would be more pleasant for us to speak about creation-annihilation operators $\hat{a}(v)$. So now we introduce on the space V_{2n} simple geometrical structures, which will be important in what follows. First, we define a skew-symmetric bilinear form Λ on V_{2n} by the formula

$$\Lambda(v, w) = \sum (v_j^- w_j^+ - v_j^+ w_j^-). \tag{2.6}$$

Secondly, we fix the antilinear operator C in V_{2n}:

$$C(v_+; v_-) = (\overline{v}_-; \overline{v}_+).$$

It can be easily seen that $C^2 = E$, and the set of fixed vectors of the operator C is a real subspace $V_{2n}^{\mathbb{R}}$ in V_{2n} consisting of vectors of the form $v = (h; \overline{h})$. We can regard the space V_{2n} itself as a complexification of the linear space $V_{2n}^{\mathbb{R}}$.

Finally, in V_{2n} define the indefinite Hermitian form

$$M(v, w) = frac1i\Lambda(v, Cw) = frac1i \sum (v_j^+ \overline{w}_j^+ - v_j^- \overline{w}_j^-) . \tag{2.7}$$

Then equations (2.2)–(2.3) can be written as

$$[\hat{a}(v), \hat{a}(w)] = \Lambda(v, w)E,$$
$$\hat{a}(v)^* = \hat{a}(Cv) .$$

2.2. **Correspondence of the creation-annihilation operators.** Linear combinations of the operators

$$f \mapsto x_j f, \qquad f \mapsto \frac{\partial}{\partial x_j} f$$

in $L^2(\mathbb{R}^n)$ correspond to the creation-annihilation operators in F_n. To be more exact, the correspondence is given by the relations

$$\frac{1}{\sqrt{2}}\left(x_j - \frac{\partial}{\partial x_j}\right) \leftrightarrow z_j, \qquad \frac{1}{\sqrt{2}}\left(x_j + \frac{\partial}{\partial x_j}\right) \leftrightarrow \frac{\partial}{\partial z_j} . \tag{2.8}$$

Now consider the space $V_{2n} = \widetilde{V}_{2n}^+ \oplus \widetilde{V}_{2n}^- = \mathbb{C}^n \oplus \mathbb{C}^n$. For any $v \in \widetilde{V}_{2n}$ define the operator $\tilde{a}(v)$ in $L^2(\mathbb{R}^n)$ by the formula

$$\tilde{a}(v) f(x) = \left(\sum v_k^+ x_k + i \sum v_k^- \frac{\partial}{\partial x_k} \right) f(x) .$$

One can easily see that

$$[\tilde{a}(v), \tilde{a}(w)] = i \sum (v_k^- w_k^+ - v_k^+ w_k^-) E, \tag{2.9}$$
$$\tilde{a}(v^+; v^-)^* = \tilde{a}(\overline{v}^+; \overline{v}^-) . \tag{2.10}$$

Here a simpler language is, of course, possible. Further, we introduce coordinate operators

$$Q_j f(x) = x_j f(x)$$

and "momentum operators"

$$P_j f(x) = i \frac{\partial}{\partial x_j} f(x) .$$

Then

$$[P_k, P_m] = [Q_k, Q_m] = 0, \qquad [Q_k, P_m] = -i\delta_{k,m} E,$$
$$P_k^* = P_k, \qquad Q_k^* = Q_k .$$

In this case, however, we also prefer to deal with the operators $\tilde{a}(v)$ and now introduce additional geometrical structures on $\widetilde{V}_{2n}$.

We define the skew-symmetric bilinear form $\tilde{\Lambda}$ on $\widetilde{V}_{2n}$ by the equation

$$\tilde{\Lambda}(v,w) = i\sum(v_j^- w_j^+ - v_j^+ w_j^-)\,. \tag{2.10}$$

We define the complex conjugation operator $\widetilde{C}$ in the obvious way: $\widetilde{C}v = \overline{v}$. Finally, we introduce the Hermitian indefinite form

$$\widetilde{M}(v,w) = -\Lambda(v, Cw) = -i\sum(v_j^- \overline{w}_j^+ - v_j^+ \overline{w}_j^-)\,. \tag{2.11}$$

Then equations (2.9)–(2.10) can be written in the form

$$[\tilde{a}(v), \tilde{a}(w)] = \Lambda(v,w)E,$$
$$\tilde{a}(v)^* = \tilde{a}(Cv)\,.$$

Observe now that the correspondence relation (2.8) identifies the linear spaces $\widetilde{V}_{2n}$ and V_{2n}. To be more exact, the canonical isomorphism $\widetilde{V}_{2n} \to V_{2n}$ is a linear operator with matrix

$$K = \frac{1}{\sqrt{2}}\begin{pmatrix} E & iE \\ iE & E \end{pmatrix}. \tag{2.12}$$

It is easy to verify that the isomorphism K carries the form $\tilde{\Lambda}$ into the form Λ, the form $\widetilde{M}$ into the form M, and the complex conjugation operator $\widetilde{C}$ into the operator C.

2.3. **Two realizations of the symplectic group.** Consider a group $\mathrm{Sp}\,(2n,\mathbb{R})$ consisting of real matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

which preserve the skew-symmetric bilinear form with matrix $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$.

Thus, real symplectic matrices $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ act in the complex space $\widetilde{V}_{2n}$ preserving the form $\tilde{\Lambda}$. The operator T obviously commutes with the complex conjugation operator $\widetilde{C}$, and, therefore, T also preserves the Hermitian form $\widetilde{M}$. It can be easily deduced that the group $\mathrm{Sp}\,(2n,\mathbb{R})$ consists exactly of all the operators in $\widetilde{V}_{2n}$ which preserve the two forms $\tilde{\Lambda}$ and $\widetilde{M}$.

Since $\widetilde{V}_{2n}$ and V_{2n} can be canonically identified, the group $\mathrm{Sp}\,(2n,\mathbb{R})$ can be defined as a group of matrices in V_{2n}. These matrices have a bloc structure of the type

$$\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} \tag{2.13}$$

and preserve both forms Λ and M.

Remark. If a matrix of type (2.13) preserves one of the forms Λ or M, then it preserves the other one.

2.4. **Symmetry theorem.**

Theorem 2.1 (complex version). (a) *For any symplectic matrix* $Q = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} : V_{2n} \to V_{2n}$ *there exists a unique (up to a factor of proportionality) bounded operator* $W(Q) \colon F_n \to F_n$ *that satisfies the condition*

$$\hat{a}(Qv) = W(Q)\,\hat{a}(v)\,W(Q)^{-1} \tag{2.14}$$

for any $v \in V_{2n}$.

(b) *The operator* $W(Q)$ *is unitary up to multiplication by a factor.*

(c) *For all symplectic matrices* P *and* Q *of the type* (2.13) *the following relation holds*:

$$W(PQ) = \lambda(P,Q)\,W(P)\,W(Q), \tag{2.15}$$

where $\lambda(P,Q) \in \mathbb{C}$.

Theorem 2.1$'$ (real version). (a) *For any matrix* $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2n,\mathbb{R})$ *there exists a unique (up to a factor of proportionality) bounded operator* $\widetilde{W}(S)$ *in* $L^2(\mathbb{R}^n)$ *which satisfies the equation*

$$\tilde{a}(Sv) = \widetilde{W}(S)\,\tilde{a}(v)\,\widetilde{W}(S)^{-1} \tag{2.16}$$

for any $v \in \widetilde{V}_{2n}$.

(b) *The operator* $\widetilde{W}(S)$ *is unitary up to a multiplication by a constant.*

(c) *For all* $S_1, S_2 \in \mathrm{Sp}(2n,\mathbb{R})$ *the following equation is satisfied*:

$$\widetilde{W}(S_1S_2) = \lambda(S_1S_2)\,\widetilde{W}(S_1)\widetilde{W}(S_2), \tag{2.17}$$

where $\lambda(S_1,S_2) \in \mathbb{C}$.

It is clear that both these theorems are equivalent.

2.5. **Proof of the Symmetry Theorem.** We prove the real version of the theorem now.

1. Any bounded operator commuting with all $\hat{a}(v)$ is scalar. This is more or less obvious.

2. Assume that there are two solutions $\widetilde{W}_1(S)$ and $\widetilde{W}_2(S)$ of the equation (2.16). Then

$$\widetilde{W}_1(S)\,\tilde{a}(v)\,W_1(S)^{-1} = \tilde{a}(Sv) = \widetilde{W}_2(S)\,\tilde{a}(v)\,W_2(S)^{-1}.$$

It follows that $W_1(S)^{-1}W_2(S)$ commutes with all $\hat{a}(v)$. Therefore, $\widetilde{W}_1(S)$ and $\widetilde{W}_2(S)$ are proportional to each other.

3. Let the solutions $W(P)$ and $W(Q)$ of equation (2.16) exist for the operators P and Q. Then

$$\tilde{a}(PQv) = \widetilde{W}(P)\,\tilde{a}(Qv)\,\widetilde{W}(P)^{-1} = \widetilde{W}(P)\,\widetilde{W}(Q)\,\tilde{a}(v)\,\widetilde{W}(Q)^{-1}\widetilde{W}(P)^{-1},$$

i.e., the operator $\widetilde{W}(P)\widetilde{W}(Q)$ satisfies equation (2.16) that must be satisfied by the operator $\widetilde{W}(PQ)$. Taking into account the uniqueness of the solution of equation (2.16), we obtain equation (2.17).

4. By virtue of the above statement, it is sufficient to prove the existence of the operators $\widetilde{W}(P)$ for some set of generators of the group $\mathrm{Sp}(2n, \mathbb{R})$.

5. The matrices of the type

$$\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}, \quad \begin{pmatrix} E & T \\ 0 & E \end{pmatrix}, \quad \begin{pmatrix} A & 0 \\ 0 & A^{t-1} \end{pmatrix},$$

where $T = T^t$, generate the whole group $\mathrm{Sp}(2n, \mathbb{R})$.

6. The operator $\widetilde{W}\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ is the Fourier transform

$$\widetilde{W}\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(y)\, e^{-i\sum x_k y_k}\, dy\,. \tag{2.18}$$

Further,

$$\widetilde{W}\begin{pmatrix} E & T \\ 0 & E \end{pmatrix} f(x) = \exp\Big\{ -\frac{i}{2} x T x^t \Big\} f(x), \tag{2.19}$$

$$\widetilde{W}\begin{pmatrix} A & \\ & A^{t-1} \end{pmatrix} f(x) = f(xA)\,. \tag{2.20}$$

7. The operators (2.18) and (2.19) are unitary and the operator (2.20) is unitary up to a multiplication by the constant $\det(A)^{1/2}$.

This ends the proof of the theorem.

Remark. Note the useful formula

$$\widetilde{W}\begin{pmatrix} E & 0 \\ H & E \end{pmatrix} f(x) = \exp\left\{ i \sum h_{kl} \frac{\partial^2}{\partial x_k \partial x_l} \right\} f(x)\,. \tag{2.21}$$

Remark. Almost any symplectic matrix can be represented in the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} E & 0 \\ H & E \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & R^{t-1} \end{pmatrix} \begin{pmatrix} E & T \\ 0 & E \end{pmatrix}.$$

For this reason equations (2.19)–(2.21) provide a more or less explicit expression for the operator $\widetilde{W}(Q)$ for almost all $Q \in \mathrm{Sp}(2n, \mathbb{R})$.

2.6. **Berezin's formula.** In fact, the simplest and most natural way to prove Theorem 2.1 is to produce a formula for the operator $W(Q)$.

Theorem 2.2. *Let* $Q = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}$ *be the symplectic operator* $V_{2n} \to V_{2n}$. *Then*

$$W(Q) f(z) = \int \exp\left\{ \frac{1}{2} (z \overline{u}) \begin{pmatrix} \overline{\Psi}\Phi^{-1} & \Phi^{t-1} \\ \Phi^{-1} & -\Phi^{-1}\Psi \end{pmatrix} \begin{pmatrix} z^t \\ \overline{u}^t \end{pmatrix} \right\} f(u)\, d\mu(u)\,. \tag{2.22}$$

Proof. This can be proved by a straightforward verification.

Proposition 2.1. *The operators*

$$W^0\begin{pmatrix}\Phi & \Psi\\ \overline{\Psi} & \overline{\Phi}\end{pmatrix} = \pm(\det\Phi)^{-1/2}W\begin{pmatrix}\Phi & \Psi\\ \overline{\Psi} & \overline{\Phi}\end{pmatrix} \tag{2.23}$$

are unitary and satisfy the equation

$$W^0(P)W^0(Q) = \pm W^0(PQ).$$

2.7. **The real version of the Berezin formula.** Let $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ be a real symplectic matrix, the matrix C being invertible. Then

$$W\begin{pmatrix}A & B\\ C & D\end{pmatrix} f(x) = \int \exp\left\{ -i\,(xy)\begin{pmatrix}AC^{-1} & -C^{t-1}\\ -C^{-1} & C^{-1}D\end{pmatrix}\begin{pmatrix}x^t\\ y^t\end{pmatrix}\right\} f(y)\,dy.$$

For an accurate definition of this operator one has to do some work. Thus one can define this transformation on $L^1\cap L^2$ and then continuously extend it onto L^2.

2.8. **Gaussian densities and operators $\hat{a}(v)$.** Let T be a symmetric matrix of size $n\times n$ and $\|T\|<1$. Consider the set $L = L_T$ of all $v\in V_{2n}$ such that

$$\hat{a}(v)\,b[T] = 0 \tag{2.25}$$

(cf. equation (2.14)). Equation (2.25) can be written in the form

$$v_+ = Tv_-.$$

The matrix T is symmetric, and, therefore, L is a Lagrangian subspace in V_{2n} (see below, §4.2). The condition $\|T\|<1$ implies that $\|v_+\|^2 < \|v_-\|^2$, and this means that the form M is positive definite on L_T.

Conversely, with each Lagrangian subspace L in V_{2n} on which the form M is negative definite we associate a Gaussian vector $b[T]$ such that $L = L_T$.

The similarity between the operators $W(Q)$ and the Gaussian vectors $b[T]$ cannot but look suspicious (see (1.7) and (2.22), (2.14) and (2.25)). If one carries out the necessary calculations explicitly, the suspicion of similarity is strengthened. Below, in §4 we shall understand that the vectors $W(Q)$ and the operators $b[T]$ are actually objects of equal status.

§3. Integral operators with Gaussian kernels and strange multiplication of matrices

3.1. **Operators with Gaussian kernels.** Consider a symmetric bloc matrix $S = \begin{pmatrix}K & L\\ L^t & M\end{pmatrix}$ of size $(m+n)\times(m+n)$. With each such matrix we associate the operator

$$B[S]\colon F_n \to F_m$$

defined by the expression

$$B\begin{bmatrix}K & L\\ L^t & M\end{bmatrix} f(z) = \int \exp\left\{\frac{1}{2}(z\overline{u})\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}\begin{pmatrix}z^t\\ \overline{u}^t\end{pmatrix}\right\} f(u)\,d\mu(u). \tag{3.1}$$

In this formula z and $\overline{u}$ are row matrices,

$$z = (z_1\, z_2\, \dots\, z_m), \qquad \overline{u} = (\overline{u}_1\, \overline{u}_2\, \dots\, \overline{u}_n).$$

Theorem 3.1. *The operator $B[S]$ is bounded if and only if the matrix*

$$S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$$

satisfies the conditions

1^+. $\|S\| \leqslant 1$;

2^+. $\|K\| < 1$ *and* $\|M\| < 1$.

Example. The operator $B\begin{bmatrix} 0 & L \\ L^t & 0 \end{bmatrix}$ is the change of variable operator $f(z) \mapsto f(zL)$.

3.2. **Comments to Theorem 2.1.** We now prove those statements of Theorem 2.1 that are trivial.

1. For any bounded operator A its kernel $K(u, \overline{v})$ satisfies the condition

$$K(u, \overline{v}) = \langle A\varphi_v, \varphi_u\rangle \leqslant \|A\| \cdot \|\varphi_u\| \cdot \|\varphi_v\| = \|A\| e^{\|u\|^2/2} e^{\|v\|^2/2}.$$

Hence it immediately follows that $\|S\| \leqslant 1$.

2. $B[S] \cdot 1 = b[K]$, $B[S]^* \cdot 1 = b[\overline{M}]$. Hence $\|K\| < 1$ and $\|M\| < 1$.

3. If $\|S\| < 1$, the kernel of the operator $B[S]$ is square-integrable, i.e., $B[S]$ is a Hilbert–Schmidt operator; in particular, it is bounded.

3.3. **Multiplication of the operators $B[S]$.** Let

$$S_1 = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \quad \text{and} \quad S_2 = \begin{pmatrix} P & Q \\ R & T \end{pmatrix}$$

be symmetric bloc matrices of size $(m+n) \times (m+n)$ and $(n+k) \times (n+k)$, respectively, satisfying conditions 1^+ and 2^+ of Theorem 3.1. Then the bounded operators

$$B[S_2] \colon F_k \to F_n, \qquad B[S_1] \colon F_n \to F_m$$

are defined.

Theorem 3.2. $B[S_1]B[S_2] = \lambda(S_1, S_2)\, B[S_1 * S_2]$,
where

$$\lambda(S_1, S_2) = \det[(1 - MP)^{-1/2}], \tag{3.2}$$

$$S_1 * S_2 = \begin{pmatrix} K + LP(1-MP)^{-1}L^t & L(1-PM)^{-1}Q \\ Q^t(1-MP)^{-1}L^t & R + Q^t(1-MP)^{-1}MQ \end{pmatrix}. \tag{3.3}$$

Remark. It is useful to note the following formula:

$$P(1-MP)^{-1} = (1-PM)^{-1}P.$$

Proof. This can be verified by a direct calculation.

3.4. **The category $\mathcal{B}$.** We now introduce a certain category $\mathcal{B}$. Its objects are the numbers $n = 1, 2, \dots$, and the morphisms $n \to m$ are the matrices

$$S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$$

of size $(m+n) \times (m+n)$ which satisfy the conditions:

1^+. S is symmetric;

2^+. $\|S\| \leqslant 1$;

3^+. $\|K\| < 1$, $\|M\| < 1$.

Let the morphisms

$$S_1 = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} : m \to n, \qquad S_2 = \begin{pmatrix} P & Q \\ R & T \end{pmatrix} : k \to n$$

be given. Then their product

$$S_1 * S_2 = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} * \begin{pmatrix} P & Q \\ R & T \end{pmatrix}$$

is, by definition, given by equation (3.3):

$$S_1 * S_2 = \begin{pmatrix} K + LP(1-MP)^{-1}L^t & L(1-PM)^{-1}Q \\ Q^t(1-MP)^{-1}L^t & R + Q^t(1-MP)^{-1}MQ \end{pmatrix}.$$

The definition of the category $\mathcal{B}$ implicitly contains the following theorem.

Theorem 3.3. (a) *If S_1 and S_2 satisfy conditions 1^+–3^+, then $S_1 * S_2$ also satisfies these conditions.*

(b) $S_1 * (S_2 * S_3) = (S_1 * S_2) * S_3$.

Statement (b) can be verified by a direct calculation, although it turns out to be somewhat more complicated than one would expect. On the other hand, the theorem is an obvious logical consequence (corollary) of the difficult Theorem 3.1 and the easy Theorem 3.2. By the way, Theorem 3.3 can also be derived from the easy part of Theorem 3.1 (see 3.2).

3.5. **Category notation.** We shall use the usual category notation: $\mathrm{Ob}(\mathcal{K})$ for the set of objects of the category $\mathcal{K}$, $\mathrm{Mor}_{\mathcal{K}}(V, W)$ for the set of morphisms $V \to W$, $\mathrm{End}_{\mathcal{K}}(V) = \mathrm{Mor}_{\mathcal{K}}(V, V)$ for the semigroup of endomorphisms of the object V, and $\mathrm{Aut}_{\mathcal{K}}(V)$ for the group of automorphisms of the object V.

3.6. **The semigroup $\Gamma_n = \mathrm{End}_{\mathcal{B}}(n)$.** Let us first discuss the semigroup Γ_n of endomorphisms of the object n of the category $\mathcal{B}$ (or, which is almost the same, the semigroup of operators in F_n with Gaussian kernels).

Consider the subset Δ_n consisting of unitary matrices S in Γ_n.

Proposition 3.1. (a) *If $S_1, S_2 \in \Delta_n$, then $S_1 * S_2 \in \Delta_n$.*

(b) *The set Δ_n is a group with respect to the operation $*$. Moreover, Δ_n is isomorphic to the group $\mathrm{Sp}(2n, \mathbb{R})$, and the isomorphism $\tau\colon \mathrm{Sp}(2n,\mathbb{R}) \to \Delta_n$ is given by the formula*

$$\tau \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} = \begin{pmatrix} \overline{\Psi\Phi}^{-1} & \Phi^{t-1} \\ \Phi^{-1} & -\Phi^{-1}\Psi \end{pmatrix}.$$

Remark. Obviously,

$$W \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} = \sigma B \left[\tau \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} \right],$$

where $\sigma \in \mathbb{C}$ (see equation (2.22)).

Further, in Γ_n consider the subset Π_n consisting of matrices $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ such that the matrix L is invertible.

Proposition 3.2. (a) *If $S_1, S_2 \in \Pi_n$, then $S_1 * S_2 \in \Pi_n$.*

(b) *Π_n is isomorphic to a subsemigroup Σ_n in the group $\mathrm{Sp}(2n,\mathbb{C})$.*

We now explicitly describe the semigroup $\Sigma_n \subset \mathrm{Sp}(2n,\mathbb{C})$. This semigroup consists of operators Q in the space V_{2n} (see 2.1) which preserve the skew-symmetric form Λ and "contract" the Hermitian form M. The expression "an operator Q contracts a form M" means that

$$M(Qv, Qv) \leqslant M(v, v)$$

for all $v \in V_n$.

The isomorphism $\Sigma_n \to \Pi_n$ is defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} CA^{-1} & A^{t-1} \\ A^{-1} & -A^{-1}B \end{pmatrix}.$$

Thus, we have the following pattern:

$$\Gamma_n \supset \{\Pi_n \subset \mathrm{Sp}(2n,\mathbb{C})\} \supset \Delta_n = \mathrm{Sp}(2n,\mathbb{R}).$$

The question which naturally arises is whether Γ_n is embedded in $\mathrm{Sp}(2n,\mathbb{C})$. The answer to this question proves to be negative.

Proposition 3.3. *The semigroup Γ_n is not embedded in any other group.*

Proof. $S_1 * O = S_2 * O$ does not imply $S_1 = S_2$.

Thus, we have partially reduced the multiplication $*$ to ordinary matrix multiplication but we have not succeeded in doing that completely.

3.7. **Real version of the same question.** Consider a symmetric matrix $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ of size $(m+n)*(m+n)$. Take the integral operator $\widetilde{B}[S]\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m)$:

$$\widetilde{B}[S]\,f(x) = \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \exp\left\{ -\frac{1}{2}\,(xy) \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \begin{pmatrix} x^t \\ y^t \end{pmatrix} \right\} f(y)\,dy\,. \tag{3.4}$$

Proposition 3.4. *The operator $\widetilde{B}[S]$ is a Hilbert–Schmidt operator if and only if the real part of the matrix is strictly positive definite.*

Proof. Note that only in this case the kernel is square-integrable.

Remark. Of course, in this case the operator $\widetilde{B}[S]$ is smoothing, i.e., it carries $L^2(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^m)$.

Let $S_1 = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ and $S_2 = \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix}$ be matrices of size $(m+n)\times(m+n)$ and $(n+k)\times(n+k)$, respectively, with positive definite real parts. Then the operators $\widetilde{B}[S_1]\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m)$ and $\widetilde{B}[S_2]\colon L^2(\mathbb{R}^k) \to L^2(\mathbb{R}^n)$ are defined.

Theorem 3.3. $\widetilde{B}[S_1]\,\widetilde{B}[S_2] = \lambda(S_1,S_2)\,\widetilde{B}[S_1\circ S_2]$,
where

$$\lambda(S_1,S_2) = \det[(M+P)^{-1/2}]$$

$$S_1\circ S_2 = \begin{pmatrix} K - L(M+P)^{-1}L^t & -L(M+P)^{-1}Q \\ -Q^t(M+P)^{-1}L^t & R - Q^t(M+P^{-1}Q) \end{pmatrix}. \tag{3.5}$$

Note that equation (3.5) gives rise to the same thoughts as equation (3.3).

Consider two examples of the operators $\widetilde{B}[S]$.

Example 1. The solution $\psi(x,t)$ of the heat equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)\psi(x,t) = 0$$

with initial condition $\psi(x,t) = q(x)$ in $L^2(\mathbb{R})$ is given by the operator

$$\psi(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \exp\left\{ -\frac{(x-y)^2}{2t} \right\} q(y)\,dy\,.$$

Example 2. The evolution of the harmonic oscillator, i.e., the solution of the equation

$$\frac{\partial}{\partial t}\,f(x,t) = i\left(-\frac{\partial^2}{\partial x^2} + x^2 \right) f(x,t)$$

with initial condition $f(x,t) = q(x)$ is given by

$$f(x,t) = (i\sin t)^{-1/2}\widetilde{B}\left[i \begin{pmatrix} \cot t & (\sin t)^{-1} \\ -(\sin t)^{-1} & \cot t \end{pmatrix} \right] q(x),$$

where $\operatorname{Im} t \geqslant 0$ and $t \neq \pi k$.

§4. Symplectic category

4.1. **Linear relations.** Let V and W be linear spaces. A linear relation $P\colon V \rightrightarrows W$ is an arbitrary linear subspace P in $V \oplus W$.

Example. A graph $gr(A)$ of any linear operator $A\colon V \to W$ is a linear relation.

If $P\colon V \rightrightarrows W$ and $Q\colon W \rightrightarrows Y$ are linear subspaces, their product $QP\colon V \rightrightarrows Y$ is defined; it consists of all $(v, y) \in V \oplus Y$ such that there exists a $w \in W$ which satisfies the conditions $(v, w) \in P$ and $(w, y) \in Q$.

4.2. **Symplectic category.** Objects of the category Sp are spaces V_{2n} introduced in §2.1 (recall that $n = 0, 1, 2, \dots$). Recall that these spaces are equipped with two forms $\Lambda = \Lambda_{2n}$ and $M = M_{2n}$. On $V_{2n} \oplus V_{2m}$ we introduce the following two forms:

$$\Lambda_{2n,2m}((v, w), (v', w')) = \Lambda_{2n}(v, v') - \Lambda_{2m}(w, w'), \tag{4.1}$$

$$M_{2n,2m}((v, w), (v', w')) = M_{2n}(w, w') - M_{2m}(w, w'). \tag{4.2}$$

We define the set $\mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$ as the set of all linear relations $P\colon V_{2n} \rightrightarrows V_{2m}$ which satisfy the following three conditions:

(a) P is a Lagrangian (with respect to $\Lambda_{2n,2m}$) subspace in $V_{2n} \oplus V_{2m}$;

(b) the form M is taken to be nonnegative on P;

(c) if $(v, 0) \in P$, $v \neq 0$, then $M_{2n}(v, v) > 0$, whereas $(0, w) \in P$, $w = 0$ implies $M_{2m}(w, w) < 0$.

Remark. Recall that the subspace P is said to be *Lagrangian* if $\Lambda(v, w) = 0$ for all $v, w \in P$ andthe dimension of subspace P is the half of dimension of space (in our case it is equal to $m + n$).

Remark. Condition (c) makes (b) only a little bit stronger: (b) would imply that $M_{2n}(v, v) \geqslant 0$, $M_{2m}(w, w) \leqslant 0$.

The morphisms of the category Sp are multiplied as linear relations.

Proposition 4.1. Sp *is really a category.*

The difficulty arise only when verifying that the product of linear relations has the required dimension.

4.3. **Several remarks.** A) Let $A \in \mathrm{Sp}(2n, \mathbb{R})$, i.e., let the operator A in V_{2n} preserve both forms Λ and M. Then its graph $gr(A)$ obviously satisfies conditions (a), (b), and (c), and, consequently, gr_A is a morphism $V_{2n} \to V_{2n}$.

In fact, the condition $\Lambda(Av, Av') = \Lambda(v, v')$ exactly implies that gr_A is Lagrangian, and the condition $M(Av, Av') = M(v, v')$ implies that the form $M_{2n,2n}$ is equal to 0 on gr_A and is, in particular, nonnegative definite.

It can be easily seen that the group of automorphisms of the object V_{2n} of the category Sp coincides exactly with $\mathrm{Sp}(2n, \mathbb{R})$.

B) Let $P \in \mathrm{Mor}(V_{2n}, V_{2n})$ is a graph of some operator A. Then condition (a) implies that

$$\Lambda(Av, Av') - \Lambda(v, v') = 0,$$

i.e., $A \in \mathrm{Sp}(2n, \mathbb{C})$, and condition (b) implies that

$$M(Av, Av) \leqslant M(v, v),$$

i.e., we encounter again the semigroup Σ from 3.6.

C) If $m \neq n$, no $P \in \mathrm{Mor}(V_{2n}, V_{2m})$ can be either a graph of the operator $V_{2n} \to V_{2m}$ or a graph of the operator $V_{2m} \to V_{2n}$.

4.4. **Angular operator.** Our next aim is to choose convenient coordinates on the set $\mathrm{Mor}(V_{2n}, V_{2m})$. But first let us introduce an auxiliary definition.

Let W_1, W_2 be linear subspaces of dimensions k and l, respectively. Let $L \subset W_1 \oplus W_2$ be a subspace of dimension k. Assume that $L \cup W_2 = 0$. Then L is a graph of some linear operator $A\colon W_1 \to W_2$. This operator is called the *angular operator* of the subspace L.

Further, let $\dim W_1 = \dim W_2 = k$. Let a skew-symmetric bilinear form with matrix $\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$ be introduced in $W_1 \oplus W_2$.

Proposition 4.2. *A k-dimensional subspace L in $W_1 \oplus W_2$ such that $L \cap W_2 = 0$ is Lagrangian if and only if its angular operator has a symmetric matrix.*

Proof. Proving the proposition is an easy exercise.

Let $\dim W_1 = k$ and $\dim W_2 = l$. Introduce a Hermitian sign-indefinite form M with matrix $\begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}$ in $W_1 \oplus W_2$.

Proposition 4.3. *Let $L \in W_1 \oplus W_2$ be a k-dimensional subspace. Then the following conditions are equivalent*:

1. *The form M is positive* (*nonnegative*) *definite on L.*
2. *L has an angular operator $A = A(L)$, and $\|A\| < 1$ ($\|A\| \leqslant 1$, respectively).*

Proof. Proving the proposition is an exercise.

4.5. **Coordinates on** $\mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$**.**

Proposition 4.4. (a) *Let $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$. Then P is a graph of the operator*

$$S = S(P) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} : V_{2m}^- \oplus V_{2n}^+ \to V_{2m}^+ \oplus V_{2n}^-,$$

where S satisfies the conditions

1^+. *S is symmetric ($S = S^t$);*

2^+. $\|S\| \leqslant 1$;

3^+. $\|K\| < 1$, $\|N\| < 1$.

(b) *Conversely, a graph of any operator $S\colon V_{2m}^- \oplus V_{2n}^+ \to V_{2m}^+ \oplus V_{2n}^-$ satisfying conditions 1^+–3^+, is a morphism $V_{2n} \to V_{2m}$ of the category* Sp.

Proof. The statement follows from Propositions 4.2–4.3, which should be applied to $W_1 \oplus W_2 = (V_{2m}^- \oplus V_{2n}^+) \oplus (V_{2m}^+ \oplus V_{2n}^-)$.

We call the matrix $S = S(P)$ the *Potapov transformation of the linear relation P*.

Proposition 4.5. *Let $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$ and $Q \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2k}, V_{2n})$. Then*

$$S(PQ) = S(P) * S(Q),$$

where the matrix multiplication $*$ is defined by equation (3.3).

Proof. The above statement is proved by a calculation.

Thus, the category $\mathcal{B}$ of §3 is isomorphic to the category Sp.

4.6. **Basic Theorem.** Suppose that $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$. Let $S(P)$ be its Potapov transformation. Define the operator

$$W(P) = B[S(P)]\colon F_n \to F_m\,. \tag{4.3}$$

Theorem 4.1. (a) *Let $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$ and $Q \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2k}, V_{2n})$. Then*

$$W(P)\,W(Q) = \lambda(P,Q)\,W(PQ),$$

where $\lambda(P,Q) \in \mathbb{C}$.

(b) *The operators $W(P)$ satisfy the equation*

$$\hat{a}(w)\,W(P) = W(P)\,\hat{a}(v) \tag{4.4}$$

for all $v \in V_{2n}$ and $w \in V_{2m}$ such that $(v,w) \in P$.

(c) *Equation* (4.4) *defines the operator $W(P)$ in a unique way up to a factor of proportionality.*

(d) *Let $P \in \mathrm{Sp}(2n,\mathbb{R}) \simeq \mathrm{Aut}(V_{2n})$. Then the operator $W(P)$ up to a factor coincides with the operator $W(P)$ from Theorem* 2.1.

Thus, we have obtained what is called the *projective representation* of a category. With each object V_{2n} of the category Sp a boson Fock space is associated, and with each $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$ a linear operator $W(P)\colon F_n \to F_m$, so that identity (4.3) is satisfied. In this case for any $n = 0,1,2,\dots$ the group $\mathrm{Sp}(2n,\mathbb{R}) \simeq \mathrm{Aut}_{\mathrm{Sp}}(V_{2n})$ acts in F_n, and this operation coincides with the representation of the group $\mathrm{Sp}(2n,\mathbb{R})$ from Theorem 2.1.

4.7. **Proof of Theorem 4.1.** Statement (a) follows from Propositions 4.4–4.5 and Theorem 3.2.

Further, let $K(z,\overline{u})$ be the kernel of the operator $W(P)$. Then the kernel of the operator $\hat{a}(w)\,W(P)$ is

$$\Big(\sum w_j^+ z_j + \sum w_j^- \frac{\partial}{\partial z_j}\Big) K(z,\overline{u}),$$

and the kernel of $W(P)\,\hat{a}(w)$ is

$$\Big(\sum v_j^+ \frac{\partial}{\partial \overline{u}_j} + \sum v_j^- \overline{u}_j\Big) K(z,\overline{u});$$

consequently we obtain the system of equations for the function $K(z,\overline{u})$

$$\Big(\sum w_j^+ z_j + \sum w_j^- \frac{\partial}{\partial z_j} - \sum v_k^+ \frac{\partial}{\partial \overline{u}_k} - \sum v_k^- \overline{u}_k\Big) K(z,\overline{u}) = 0\,. \tag{4.5}$$

It can easily be verified that, on the one hand, the kernel of the operator satisfies the system (4.5) (this yields statement (b)) and, on the other hand, the solution of this set is unique (this yields the statement (c)).

Finally, statement (d) follows from (b).

4.8. **The real version of Theorem 4.1.** Recall that V_{2n} is canonically identified with $\widetilde{V}_{2n}$ (see 2.2).

Theorem 4.1′. (a) *For any $P \in \operatorname{Mor}(\widetilde{V}_{2n}, \widetilde{V}_{2m})$ there exists a unique (up to a factor of proportionality) bounded operator $\widetilde{W}(P)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m)$ such that*

$$\tilde{a}(w)\,\widetilde{W}(P) = \widetilde{W}(P)\,\tilde{a}(v)$$

for all $(v, w) \in P$.

(b) *For all $P \in \operatorname{Mor}(\widetilde{V}_{2n}, \widetilde{V}_{2m})$ and $Q \in \operatorname{Mor}(\widetilde{V}_{2k}, \widetilde{V}_{2n})$ the following equation is satisfied*

$$\widetilde{W}(QP) = \lambda(Q, P)\,\widetilde{W}(Q)\,\widetilde{W}(P).$$

It is clear that this theorem is a corollary of Theorem 4.1. The situation with the expression for the operator $\widetilde{W}(P)$ is somewhat worse than in the complex case.

Consider $P \in \operatorname{Mor}(\widetilde{V}_{2n}, \widetilde{V}_{2m})$. Assume that P is given by equation of the type

$$\begin{pmatrix} w^+ \\ v^+ \end{pmatrix} = \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} w^- \\ v^- \end{pmatrix}.$$

This is not always satisfied, but is valid at least in the case when for any $(v, w) \in P$ the condition $M(v, v) = M(w, w)$ implies $v = 0$ and $w = 0$ (the inequality in condition 2^+ in 4.2 is strict). If this is the case, the operator $\widetilde{W}(P)\colon L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m)$ is given by the expression

$$\widetilde{W}(P) = B\left[i \begin{pmatrix} A & B \\ B^t & -C \end{pmatrix}\right].$$

4.9. **Involution in the category** Sp**.** With each $P \in \operatorname{Mor}(V_{2n}, V_{2m})$ we associate the linear relation $P^{(*)}\colon V_{2m} \rightrightarrows V_{2n}$ which is the orthogonal complement of P in $V_{2n} \oplus V_{2m}$ with respect to $M_{2n,2m}$. It is easily seen that $P^{(*)} \in \operatorname{Mor}(V_{2m}, V_{2n})$. Moreover,

$$P^{(*)(*)} = P, \qquad (PQ)^{(*)} = Q^{(*)}P^{(*)}.$$

Finally,

$$W(P^{(*)}) = W(P)^*.$$

§5. Krein–Shmulyan functor

5.1. **Cartan matrix balls.** Let $n = 0, 1, 2, \dots$. A set of all symmetric complex matrices of size $n \times n$ with norm < 1 will be called a *matrix ball* $\mathcal{Z}_n$.

Example. For $n = 1$ the ball $\mathcal{Z}_n$ is the circle $|z| < 1$ on $\mathbb{C}$, i.e., the Lobachevskiĭ plane.

The group $\mathrm{Sp}(2n, \mathbb{R})$ acts on $\mathcal{Z}_n$ via linear-fractional transformations of the type

$$\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} : T \mapsto (\Phi T + \Psi)(\overline{\Psi} T + \overline{\Phi})^{-1}. \tag{5.1}$$

The stabilizer of the point $0 \in \mathcal{Z}_n$ consists of matrices of the form $\begin{pmatrix} B & 0 \\ 0 & \overline{B} \end{pmatrix}$, where B is a unitary matrix of size $n \times n$. It is easily verified that a matrix ball $\mathcal{Z}_n$ is a $\mathrm{Sp}(2n, \mathbb{R})$-homogeneous space, and thus $\mathcal{Z}_n$ is a homogeneous symmetric space $\mathrm{Sp}(2n, \mathbb{R})/U(n)$.

5.2. **The Siegel upper half-plane.** Let H_n be a space of symmetric matrices of size $n\times n$ with a positive definite imaginary part. The group $\mathrm{Sp}\,(2n,\mathbb{R})$ defined as a group of real symplectic matrices acts on H_n with linear-fractional transformations of the type

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : S \mapsto (AS+B)(CS+D)^{-1}.$$

The stabilizer of the point $S=iE$ consists of matrices of the form $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{Sp}\,(2n,\mathbb{R})$. This stabilizer is isomorphic to the group $U(n)$. It is easily verified that $\mathrm{Sp}\,(2n,\mathbb{R})$ acts on H_n transitively, and, consequently, H_n is the symmetric homogeneous subspace $\mathrm{Sp}\,(2n,\mathbb{R})/U(n)$.

We see that $\mathcal{Z}_n$ and H_n is the same homogeneous space. The isomorphism $\mathcal{Z}_n \to H_n$ is established by the linear-fractional transformation

$$S = (T+iE)(iT+E)^{-1} \tag{5.2}$$

(the "Cayley transformation").

5.3. **Embedding of $\mathcal{Z}_n$ and H_n into the Grassmannian.** In the two previous subsections there were some statements that are quite difficult to prove directly. Everything becomes much more transparent if one turns to a different realization of these regions.

Let $T\in\mathcal{Z}_n$. Consider a linear subspace L_T in V_{2n} with an angular operator $T\colon V_{2n}^- \to V_{2n}^+$, i.e., a set of vectors in V_{2n} of the form $(Tv^-;v^-)\in V_{2n}^+\oplus V_{2n}^-$. As noted above (see 4.4), the space L_T satisfies the following conditions:

(a) L_T is Lagrangian with respect to Λ.

(b) The form M is strictly negative definite on L_T.

Conversely, any subspace L satisfying conditions (a) and (b) has an angular operator $T\in\mathcal{Z}_n$.

Thus, we have established an isomorphism of $\mathcal{Z}_n$ with a certain open region $\mathcal{R}_n$ in a Lagrangian Grassmannian of the space V_{2n}.

Let $T\in\mathcal{Z}_n$ and $g=\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} \in \mathrm{Sp}\,(2n,\mathbb{R})$. Then $L_T\in\mathcal{R}_n$ consists of vectors of the form (Tv_-,v_-). Acting by means of the operators g, we get a vector space of the form

$$\left((\Phi T+\Psi)v_-;(\overline{\Psi}T+\overline{\Phi})v_-\right).$$

Denoting by w_- the vector $(\overline{\Psi}T+\overline{\Phi})v_-$, we see that the space gL_T consists of vectors

$$((\Phi T+\Psi)(\overline{\Psi}T+\overline{\Phi})^{-1}w_-;w_-),$$

i.e., the angular operator of the subspace gL_T is $(\Phi T+\Psi)(\overline{\Psi}T+\overline{\Phi})^{-1}$. Thus, we come to equation (5.1).

The situation is just as simple for the upper half-plane H_n. A subspace with an angular operator $S\colon \widetilde{V}_{2n}^- \to \widetilde{V}_{2n}^+$ is associated with any $S\in H_n$, and this subspace satisfies conditions (a) and (b).

Recalling that V_{2n} and $\widetilde{V}_{2n}$ are the same (see the isomorphism (2.12)), we get the isomorphism $\mathcal{Z}_n\to H_n$ defined by equation (5.2).

Below in this section we shall deal with matrix balls $\mathcal{Z}_n$ only.

5.4. **Krein–Shmulyan functor.** Consider $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_0, V_{2n})$. The morphism P is a subspace in $V_0 \oplus V_{2n} = V_{2n}$, and this subspace satisfies conditions (a) and (b) from the previous subsection. Thus we get the canonical isomorphism

$$\mathcal{Z}_n \simeq \mathrm{Mor}_{\mathrm{Sp}}(V_0, V_{2n}).$$

Now consider the following abstract trivial construction. Let $\mathcal{K}$ be some category and R_0 be one of its objects. Let us construct a functor F from the category $\mathcal{K}$ to the category of sets and mappings using the following rule: for any $R \in \mathrm{Ob}(\mathcal{K})$ the set $F(R)$ is

$$F(R) = \mathrm{Mor}(R_0, R),$$

and for any $\psi \in \mathrm{Mor}(R_1, R_2)$ the mapping

$$F(\psi)\colon F(R_1) \to F(R_2)$$

is defined by the equation

$$F(\psi)\, q = \psi q.$$

Let us apply this unsophisticated construction to the category Sp and to the object V_0. A matrix ball $\mathcal{Z}_n$ is associated with each object V_{2n}. Suppose that $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$, and $S(P) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ is its Potapov transform. Then the transformation

$$\tau(P)T = K + LT(1 - MT)^{-1}L^t \tag{5.3}$$

acting from $\mathcal{Z}_n$ to $\mathcal{Z}_m$ is associated with the morphism P. These transformations are called *generalized linear-fractional Krein–Shmulyan transformations* (it is worth noting that even in the case $m = n$ they cannot generally be written either in the form $(AT + B)(CT + D)^{-1}$ or in the form $(C'T + D')^{-1}(A'T + B')$).

Naturally, the equation

$$\tau(PQ) = \tau(P)\,\tau(Q) \tag{5.4}$$

is satisfied.

5.5. **Equivariant embeddings of matrix balls into Fock spaces.** Let $T \in \mathcal{Z}_n$, $P \in \mathrm{Mor}(V_{2n}, V_{2m})$, and $S(P) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ be the Potapov transform of the morphism P. Then

$$\begin{aligned} W(P)\, b[T] &= B\begin{bmatrix} K & L \\ L^t & M \end{bmatrix} b[T] \\ &= \det[(1 - MT)^{-1/2}]\, b[K + LT(1 - MT)^{-1}L^t]. \end{aligned} \tag{5.5}$$

Now consider a projectivized Fock space $\mathbb{P}F_n$ and the embedding $\sigma\colon \mathcal{Z}_n \to \mathbb{P}F_n$ defined by the expression

$$\sigma(T) = b[T].$$

Then equation (5.5) implies

$$W(P)\sigma(T) = \sigma(\tau(T)). \tag{5.6}$$

Remark. Let $Q \in \mathrm{Mor}_{\mathrm{Sp}}(V_0, V_{2n})$, and $T \in \mathcal{Z}_n$ be its Potapov transformation (i.e., the angular operator). Then the operator $W(Q)\colon F^0 \to F^n$ is defined by the equation

$$s \mapsto s \cdot b[T],$$

where $s \in F_0 \simeq \mathbb{C}$. Therefore equation (5.5) is just a special case of equations (3.2)–(3.3).

5.6. **Fixed points and eigenvectors.**

Proposition 5.1. *Let $P \in \mathrm{End}(V_{2n})$ and $T \in \mathcal{Z}_n$. The following conditions are equivalent*:

(a) *T is a fixed point of the transformation $\tau(P)\colon \mathcal{Z}_n \to \mathcal{Z}_n$.*

(b) *A vector $b[T] \in F_n$ is an eigenvector for the operator $W(P)\colon F_n \to F_n$.*

Proof. See equation (5.5).

Now let $P \in \mathrm{End}(V_{2n})$ satisfy the additional condition

$$(v, w) \in P \implies M(v, v) > M(w, w)$$

(for $v \neq 0$ and $w \neq 0$), i.e., P is interior point of the semigroup $\mathrm{End}(V_{2n})$. Denote by $\mathrm{End}^0(V_{2n})$ the set of all such linear relations. It is easy to see that $P \in \mathrm{End}^0(V_{2n})$ is equivalent to $\|S(P)\| < 1$.

It is easy to verify that in the case $P \in \mathrm{End}^0(V_{2n})$ the mapping $\tau(P)$ takes the matrix ball $\mathcal{Z}_n$ strictly inside $\mathcal{Z}_n$, i.e., there exists a $\varepsilon > 0$ such that $\|\tau(P)T\| \leqslant 1-\varepsilon$ for all $T \in \mathcal{Z}_n$. Since $\mathcal{Z}_n$ is a ball from the topological point of view, we see that $\tau(P)$ necessarily has a fixed point.

5.7. **Estimate for the norm of $B[S]$.** Let the operator $B[S]$ be self-adjoint. This is equivalent to the statement that the matrix $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ satisfies the conditions $K = \overline{M}$ and $L = L^*$.

Let $\|S\| < 1$. Then, as we have seen in the previous subsection, the operator $B[S]$ has an eigenvector of the form $b[T]$. The group $\mathrm{Aut}(V_{2n}) \simeq \mathrm{Sp}(2n, \mathbb{R})$ acts transitively on $\mathcal{Z}_n$, and, consequently, there exists a $P \in \mathrm{Aut}(V_{2n})$ such that $\tau(P)0 = T$. Consider the operator $W(P)B[S]W(P)^{-1}$. This self-adjoint operator has the form $\varkappa \cdot B\begin{bmatrix} K & L \\ L^t & M \end{bmatrix}$, and the vector $b[0] = 1$ is its eigenvector. Hence, $K = 0$ and hence $M = \overline{K} = 0$.

The operator $B\begin{bmatrix} 0 & Q \\ Q^t & 0 \end{bmatrix}$, however, is nothing but the change of variables operator $f(z) \mapsto f(zQ)$. Without loss of generality, one can assume that the matrix Q is diagonal:

$$Q = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

(with $\lambda_j \in \mathbb{R}$ and $|\lambda_j| < 1$). The vectors $e_{\alpha_1 \ldots \alpha_n}$ (see equation (1.3)) are eigenvectors for $B\begin{bmatrix} 0 & Q \\ Q^t & 0 \end{bmatrix}$ with eigenvalues $\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}$. The greatest of these numbers corresponds to $\alpha_1 = \cdots = \alpha_n = 0$, and, consequently, the norm of the operator $B[S]$ is reached on the vector $b[0] = 1$. Therefore, the norm of the operator $b[S]$ is reached on the vector $b[T]$. Thus (see equation (5.5)), the following theorem is proved.

Theorem 5.1. *Let $P \in \operatorname{End}(V_{2n})$ be self-adjoint (i.e., $P^{(*)} = P$). Suppose that $S = S(P) = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ and $T \in \mathcal{Z}_n$ is a fixed point of the transformation $\tau(P)$. Then we have*

$$\|B[S]\| = \det[(1 - MT)^{-1/2}]. \tag{5.7}$$

One can demonstrate (this is not quite obvious, see [56], §3) that if $\|T\| < 1$, then

$$|\det(1 - MT)| \geqslant \det(1 - |M|),$$

where $|M| = \sqrt{M^* M}$. This gives

Corollary. *For a self-adjoint operator $B[S]$ the following inequality holds*:

$$\|B[S]\| \leqslant \det(1 - |M|)^{-1/2}. \tag{5.8}$$

Note that Theorem 3.1 follows from this unequality.

Remark. The above arguments give the following additional information: Let $\|S\| < 1$. Then:

(a) All eigenvectors of the operator $B[S]$ have the form $p(z) \exp\{\frac{1}{2} z T z^t\}$, where $p(z)$ is a polynomial.

(b) A set of eigenvalues of the operator $\|B[S]\|^{-1} B[S]$ has the form $\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}$, where $-1 < \lambda_j < 1$ and $\alpha_1, \dots, \alpha_n$ are nonnegative integers.

§6. Geometry of matrix balls and the Compression Theorem

The subjectmatter of this section stands somewhat apart from the main contents of the paper; the Compression Theorem is used later on, in 7.6.

6.1. **Composite distance.** Let $T_1, T_2 \in \mathcal{Z}_n$. Consider the matrix

$$L(T_1, T_2) = (1 - T_1^* T_1)^{-1/2} (1 - T_1^* T_2)(1 - T_2^* T_2)^{-1/2}. \tag{6.1}$$

Suppose that

$$\lambda_1(T_1, T_2) \geqslant \lambda_2(T_1, T_2) \geqslant \cdots$$

are its *singular values*. Recall that by singular values of a matrix M we mean eigenvalues of the matrix $\|M\| = \sqrt{M^* M}$.

Proposition 6.1. (a) *For all j, we have $\lambda_j \geqslant 1$.*

(b) *Let $T_1, T_2, S_1, S_2 \in \mathcal{Z}_n$. An element $g \in \operatorname{Aut}(V_{2n}) \simeq \operatorname{Sp}(2n, \mathbb{R})$ such that $gT_1 = S_1$ and $gT_2 = S_2$ exists if and only if*

$$\lambda_j(T_1, T_2) = \lambda_j(S_1, S_2)$$

for all j.

The set of numbers
$$\varphi_j = \arccos h\lambda_j(T_1, T_2) \tag{6.2}$$
is called the *composite distance* between T_1 and T_2 in $\mathcal{Z}_n$. It will be more convenient for us to consider the set of numbers
$$\varphi_1(T_1, T_2) \geqslant \varphi_2(T_1, T_2) \geqslant \cdots \tag{6.3}$$
as unbounded, setting $\varphi_j(T_1, T_2) = 0$ for $j > n$.

6.2. **Another description of the values λ_j.** Recall that with each element $T \in \mathcal{Z}_n$ a subspace $P = P_T \subset V_{2n}$, which is a graph of the operator $T\colon V_{2n}^- \to V_{2n}^+$, is associated. Consider the orthogonal (with respect to the form M) projection operator $L\colon P_{T_1} \to P_{T_2}$.

It turns out that λ_j are precisely the singular values of the operator L.

We shall state this in a slightly different manner. The form $(-M)$ is positive definite on P_{T_1} and P_{T_2}. Therefore, P_{T_1} and P_{T_2} are Euclidean spaces. Consider the Hermitian form
$$B(v_1, v_2) = -M(v_1, v_2)\colon P_{T_1} \times P_{T_2} \to \mathbb{C}.$$

Then λ_j are singular values of the form B.

Now we write out the matrix of the form B. Note that the operator
$$Q_T v_- = ((1 - T^*T)^{-1/2} v_-; T(1 - T^*T)^{-1/2} v_-)$$
is a unitary (with respect to the form $(-M)$) operator $V_- \to P_T$, and thus one can regard v_- as a coordinate on P_T; now it is easily seen that
$$-M(Q_{T_1} v_-, Q_{T_2} v'_-) = -M(v_-, L(T_1, T_2) v_-),$$
where $L(T_1, T_2)v_-$ is given by equation (6.1). Thus, we have demonstrated the equivalence of the two definitions of the values λ_j.

6.3. **Minimax description of the values λ_j.** Matrix eigenvalues and singular values have natural minimax descriptions (see, for example, [17],[57]). In our case this gives
$$\lambda_j = \max_{\substack{R \subset P_{T_2} \\ \dim R = j}} \left(\min_{\substack{v \in R \\ M(v,v) = -1}} \left(\max_{\substack{w \in P_{T_1} \\ M(w,w) = -1}} |M(v, w)| \right) \right), \tag{6.4}$$
where R runs over all j-dimensional spaces in P_{T_2}.

Remark. Recall that the *angles* $\psi_1 \geqslant \psi_2 \geqslant \cdots$ between the subspaces P_1 and P_2 in the standard Euclidean space $\mathbb{R}^n$ are given by the expression
$$\cos \Psi_j = \max_{\substack{R \subset P_2 \\ \dim R = j}} \left(\min_{\substack{v \in R \\ \|v\| = 1}} \left(\max_{\substack{w \in P_1 \\ \|w\| = 1}} \langle v, w \rangle \right) \right).$$
Thus, it turns out that the composite distance is the hyperbolic analogue of angles.

6.4. **Infinitesimal metric structure.** Suppose that $T \in \mathcal{Z}_n$ and ∂T is a tangent vector at the point T. We construct a matrix

$$\Sigma(T, \partial T) = (1 - T^*T)^{-1/2} \partial T^* (1 - TT^*)^{-1/2}.$$

Let

$$\sigma_1(T, \partial T) \geqslant \sigma_2(T, \partial T) \geqslant \cdots$$

be its singular values.

Proposition 6.2. *Let $T, S \in \mathcal{Z}_n$, and suppose that ∂T and ∂S are tangent vectors at the points T and S, respectively. The following conditions are equivalent*:

1. *There exists $g \in \mathrm{Aut}_{\mathrm{Sp}}(V_{2n}) \simeq \mathrm{Sp}(2n, \mathbb{R})$ that takes T to S and ∂T to ∂S.*
2. *$\sigma_j(T, \partial T) = \sigma_j(S, \partial S)$ for all j.*

6.5. **Metric in $\mathcal{Z}_n$.** Note that the expression

$$ds^2 = \sum \sigma_j^2(T, \partial T)$$

is a $\mathrm{Sp}(2n, \mathbb{R})$-invariant Riemann metric in $\mathcal{Z}_n$.

Theorem 6.1. *The distance in $\mathcal{Z}_n$ that corresponds to the Riemann metric ds^2 is given by the formula*

$$\rho(T_1, T_2) = \Big[\sum (\varphi_j(T_1, T_2))^2 \Big]^{1/2}.$$

6.6. **Invariant metrics in $\mathcal{Z}_n$.** Let $p(x_1, \dots, x_n)$ be a norm in $\mathbb{R}^n$ invariant under coordinate permutations and under all possible changes of direction of the axes. Consider an invariant Finslerian metric in $\mathcal{Z}_n$,

$$K_p(T, \partial T) = p(\sigma_1(T, \partial T), \dots, \sigma_n(T, \partial T)).$$

Theorem 6.2. *A distance in $\mathcal{Z}_n$ which is defined by the Finslerian metric K_p is given by*

$$\rho_p(T_1, T_2) = p(\varphi_1(T_1, T_2), \dots, \varphi_n(T_1, T_2)).$$

6.7. **The Compression Theorem.**

Theorem 6.3. *Let $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$. Then for all $T_1, T_2 \in \mathcal{Z}_n$ and for all j one has*

$$\varphi_j(\tau(p)T_1, \tau(p)T_2) \leqslant \varphi_j(T_1, T_2). \tag{6.5}$$

Proof. The proof of this fact is rather simple: for $n = m = 1$ it is verified by a straightforward calculation, and the general case is reduced to the one-dimensional one using equality (6.4).

Corollary. *For any metric ρ_p the following inequality holds*:

$$\rho_p(\tau(P)T_1, \tau(P)T_2) \leqslant \rho_p(T_1, T_2). \tag{6.6}$$

Corollary. *Let $\tau(P)$ take the pair $(T, \partial T)$ to $(S, \partial S)$. Then for all j the following inequality holds*:

$$\sigma_j(S, \partial S) \leqslant \sigma_j(T, \partial T)\,.$$

6.8. **Integer distance.** In $\mathcal{Z}_n$ we introduce one more $\mathrm{Sp}\,(2n, \mathbb{R})$-invariant metric

$$n(T_1, T_2) = \mathrm{rk}\,(T_1 - T_2)\,.$$

It is easily seen that $n(T_1, T_2)$ is exactly the number of nonzero numbers in the set (6.3). Obviously, for any $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$ the following inequality holds:

$$n(\tau(p)T_1, \tau(p)T_2) \leqslant n(T_1, T_2)\,. \tag{6.7}$$

6.9. **Inverse theorems.**

Theorem 6.4. *Let $q\colon \mathcal{Z}_n \to \mathcal{Z}_m$ be a holomorphic mapping that satisfies condition* (6.7), *and let there exist $T_1, T_2 \in \mathcal{Z}_n$ such that $n(q(T_1), q(T_2)) \geqslant 2$. Then q is of the form $\tau(P)$ for some $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$.*

Theorem 6.5. *Let $q\colon \mathcal{Z}_n \to \mathcal{Z}_m$ be a holomorphic mapping that satisfies the following conditions*:

1°. *Let the mapping q take the pair (point T; tangent vector ∂T at the point T) to the pair $(S; \partial S)$. Then the condition $\mathrm{rk}\,(\partial T) \leqslant 1$ implies $\mathrm{rk}\,(\partial S) \leqslant 1$.*

2°. *There exist $T_1, T_2 \in \mathcal{Z}_n$ such that $n(q(T_1), q(T_2)) \geqslant 2$.*

3° $\dim q(\mathcal{Z}_n) > 1$

Then q is of the form $\tau(P)$ for some $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$.

Corollary. *Let $q\colon \mathcal{Z}_n \to \mathcal{Z}_m$ be a holomorphic mapping that satisfies condition* (6.5) *and the condition 2° of the Theorem 6.5. Then q is of the form $\tau(P)$ for some $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$.*

Corollary. *Let $q\colon \mathcal{Z}_n \to \mathcal{Z}_m$ be a holomorphic mapping that satisfies condition* (6.6) *and the condition 2° of the Theorem 6.5. Then q is of the form $\tau(P)$ for some $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$.*

§7. The infinite-dimensional case

7.1. **Boson Fock space F_∞ with an infinite number of degrees of freedom.** Note that the space F_n can be canonically isometrically embedded into F_{n+1} according to the relation

$$f(z_1, \dots, z_n) \mapsto f(z_1, \dots, z_n, 0)\,.$$

Therefore, we can identify F_n with a subspace in F_{n+1}. Let $\mathcal{F} = \bigcup_{j=1}^{\infty} F_j$. We denote by F_∞ the completion of $\mathcal{F}$ with respect to the norm defined by the scalar product.

Let $f_j \in F_j$, and let the sequence $f_1, f_2, \dots$ be fundamental in $\mathcal{F}$. It is easily demonstrated that for any vector $u = (u_1, u_2, \dots) \in l_2$ the sequence of numbers $f_j(u_1, u_2, \dots)$ converges. Denote by $f(u)$ the limit of this sequence. It is easily verified that $f(u)$ is a holomorphic function on l_2.

Thus, the space F_∞ is identified with some space of holomorphic functions on l_2.

Further, the canonical embedding $F_n \to F_{n+1}$ identifies the basic vectors $e_{\alpha_1\dots\alpha_n} \in F_n$ and $e_{\alpha_1\dots\alpha_n 0} \in F_{n+1}$. Let $\alpha_1, \alpha_2, \dots$ be an infinite sequence of nonnegative integers, all the α_j's from some place on being equal to 0. Define a function

$$e_{\alpha_1\alpha_2\dots}(z_1, z_2, \dots) = \prod_{j=1}^{\infty} \frac{z_j^{\alpha_j}}{\sqrt{\alpha_j!}}.$$

It is easily seen that this set of functions forms an orthonormal basis in F_∞ (by the way, one can see from this that elements of F_∞ are entire functions on l_2).

For the scalar product in F_∞ we keep the notation

$$\langle f, g\rangle = \int f(z)\overline{g(z)} \exp(-|z|^2)\, dz\, d\overline{z},$$

but we consider this relation as formal (the left-hand side being the definition of the right-hand side). We denote by F_ω the Fock spaces with $\omega = 0, 1, 2, \dots, \infty$ degrees of freedom.

7.2. **Overfilled basis.** Let $a = (a_1, a_2, \dots) \in l_2$. Define the function $\varphi_a(z)$ as before:

$$\varphi_a(z) = \exp(z, a) = \exp\Big(\sum z_j \overline{a}_j\Big).$$

As before,

$$\langle f(z), \varphi_a(z)\rangle = f(a).$$

7.3. **Gaussian vectors.** Let T be a Hilbert–Schmidt operator in l_2; moreover, $T = T^t$, $\|T\| < 1$, and $c = (c_1, c_2, \dots) \in l_2$. Define the vector

$$b[T\,|\,c^t] = \exp\Big\{\frac{1}{2} zTz^t + zc^t\Big\}.$$

Proposition 7.1. *The vectors $b[T\,|\,c^t]$ lie in F_∞, and*

$$\langle b[T\,|\,c^t], b[Q\,|\,d^t]\rangle = \det[(1 - T\overline{Q})^{1/2}] \exp\left\{\frac{1}{2}(c\overline{d}) \begin{pmatrix} -T & E \\ E & -\overline{Q}\end{pmatrix}^{-1} \begin{pmatrix} c^t \\ \overline{d}^t\end{pmatrix}\right\}. \quad (7.1)$$

We denote by F_ω^0 a subspace in F_ω, consisting of finite linear combinations of the Gaussian vectors $b[T\,|\,c^t]$.

7.4. **Operators.** Let A be a bounded operator $F_\omega \to F_\nu$. With it we associate the function $K(z, \overline{u})$ (*the kernel*) defined by the same equation (1.12) as before. Equation (1.13) remains valid.

We write as before

$$Af(z) = \int K(z, \overline{u})\, f(u) \exp(-|u|^2)\, du,$$

but by this we imply the following:

$$Af(z) = \langle f(u), \overline{k_z(\overline{u})} \rangle,$$

where $k_z(\overline{u}) = K(z, \overline{u})$ is the function which is obtained from $K(z, \overline{u})$ for a fixed value of the argument z.

If $K(z, \overline{u})$ is the kernel of the operator $A\colon F_\omega \to F_\nu$, and $L(u, \overline{v})$ is the kernel of the operator $B\colon F_\varkappa \to F_\omega$, then the kernel $M(z, \overline{v})$ of the operator AB can be written in the form

$$M(z, \overline{v}) = \int K(z, \overline{u})\, L(u, \overline{v}) \exp\{-|u|^2\}\, du = \langle l_{\overline{v}}(u), \overline{k_z(\overline{u})} \rangle,$$

where $l_{\overline{v}}(u) = L(u, \overline{v})$ and $k_z(\overline{u}) = K(z, \overline{u})$.

7.5. **Operators $B[S]$.** Let $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ be a matrix of size $(\varkappa + \omega) \times (\varkappa + \omega)$. Define the operator $B[S]\colon F_\omega \to F_\varkappa$ according to equation (3.1).

Theorem 7.1. *If the operator $B[S]$ is bounded, then*
1. $\|S\| \leqslant 1$;
2. $\|K\| < 1$, $\|M\| < 1$;
3. *K and M are Hilbert–Schmidt operators.*

From now on we assume that the matrix S satisfies conditions 1–3.

Theorem 7.2. *The operator $B[S]$ is a well-defined (possibly unbounded) operator that takes F^0_ω to $F^0_\varkappa$.*

Theorem 7.3. *The operators $B[S]$ are multiplied according to the same formula* (3.2), (3.3) *as in the finite-dimensional case.*

Theorem 7.4. *If K and M are the nuclear operators, then the operator $B[S]$ is bounded.*

Theorem 7.5. *If $\|S\| < 1$, the operator $B[S]$ is bounded.*

Theorem 7.6. *If the operator $B[S]$ is not bounded, it still has a unique maximal closed extension.*

7.6. **Sketches of the proofs.** *With regard to Theorem* 7.1. See 3.2.

With regard to Theorem 7.2. See equation (7.1), which makes it possible to calculate $B[S]\, b[T \,|\, c^t]$.

With regard to Theorem 7.4. The estimate (5.8) survives at infinity.

With regard to Theorem 7.5. The infinite-dimensional analogue $\mathcal{Z}_\infty$ of the space $\mathcal{Z}_n$ is the set of symmetric Hilbert–Schmidt matrices. The Compression Theorem remains valid also for $\mathcal{Z}_\infty$, and one can easily deduce from it that the mapping (5.3) is contractive if $\left\| \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \right\| < 1$. Further argument is along the lines similar to that of 5.6.

With regard to Theorem 7.6. Let $0 < \sigma < 1$. Then the operator $B\begin{bmatrix} 0 & \sigma E \\ \sigma E & 0 \end{bmatrix}$ is injective, and the operator

$$H = B\begin{bmatrix} 0 & \sigma E \\ \sigma E & 0 \end{bmatrix} B[S] = B\begin{bmatrix} \sigma^2 K & \sigma L \\ \sigma L^t & M \end{bmatrix}$$

satisfies the assumption of Theorem 7.5. If $B[S]$ had different closed extensions, the bounded operator H would also have different closed extensions, and if $B[S]$ did not have a closed extension, H would also have none.

7.7. **Linear relations.** Consider the Hilbert space $V_{2\infty} = V_{2\infty}^+ \oplus V_{2\infty}^- = l_2 \oplus l_2$ equipped with the two forms Λ and M defined by equations (2.6), (2.7). Just as in 2.1, with each $v \in V_{2\infty}$ we associate an operator $\hat{a}(v)$ according to equation (2.1).

Let $S = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ be a symmetric matrix of size $(\mu+\nu)\times(\mu+\nu)$ satisfying the conditions of Theorem 7.1. We associate with it the linear relation $P = P_S$ consisting of the vectors $(v, w) \in V_{2\nu} \oplus V_{2\mu}$ satisfying the conditions

$$\begin{pmatrix} w_+ \\ v_- \end{pmatrix} = \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \begin{pmatrix} w_- \\ v_+ \end{pmatrix}.$$

It is clear (see Theorem 7.3) that the multiplication of linear relations corresponds to the multiplication of operators $B[S]$.

7.8. Commutation relations. Let $Q = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}$ be a bounded invertible operator in $V_{2\infty}$ preserving the form Λ.

Theorem 7.7. (a) *A bounded operator $W(Q)$ such that the relation*

$$\hat{a}(Qv) = W(Q)\,\hat{a}(v)\,W(Q)^{-1}$$

holds for any $v \in V_{2\infty}$, exists if and only if Ψ is a Hilbert–Schmidt operator. If the operator $W(Q)$ exists, it is unique up to a factor of proportionality.

(b) *The operator $W(Q)$ is given by equation* (2.22).

(c) *The operators*

$$\det[(\Phi^*\Phi)^{-1/4}]\,W(Q)$$

are unitary.

We denote by Sp_∞ the group of all symplectic matrices $\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}$ such that Ψ is a Hilbert–Schmidt operator.

Equation (4.4) also remains valid, but since all the operators appearing in it are, in general, unbounded, the exact formulation of the corresponding theorem is rather long, and so we omit it.

7.9. **Fixed point.** Let $\overline{\mathcal{Z}}_\infty$ be the set of all symmetric ($T = T^t$) operators in l_2 with norm < 1 equipped with the weak operator topology. It follows from the Schauder–Tikhonov theorem that for any matrix S satisfying the conditions of Theorem 7.1 the map

$$T \mapsto K + LT(1 - MT)^{-1}L^t \tag{7.2}$$

has a fixed point $T_0 \in \overline{\mathcal{Z}}_\infty$.

Formula (5.7) for the norm of a self-adjoint operator remains valid (although the proof becomes more complicated).

7.10. **A counter-example.** Let N be a diagonal matrix with eigenvalues $\nu_1, \nu_2, \dots$, where $0 < \nu_j < 1$. Consider the operator $B\begin{bmatrix} N & E-N \\ E-N & N \end{bmatrix}$. The fixed point of the transformation (7.2) is $T_0 = E$, and we see that $\|Q\| = \prod(1-\nu_j)^{-1/2}$. If $\sum \nu_j^2 < \infty$ and $\sum \nu_j = \infty$, then the operator Q turns out to be unbounded.

§8. Lie algebra

Since the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ acts in $L^2(\mathbb{R}^n) \simeq F_n$, the symplectic algebra $\mathrm{sp}(2n,\mathbb{R})$ must also act there.

8.1. **Action in $L^2(\mathbb{R}^n)$.** The Lie algebra of a group of real symplectic matrices consists of real matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \qquad C = C^t, \quad B = B^t.$$

The corresponding operator in $L^2(\mathbb{R}^n)$ is defined by the equation

$$\widetilde{W}\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} = \sum a_{kj} x_k \frac{\partial}{\partial x_j} + \frac{i}{2}\sum b_{kj} x_k x_j - \frac{i}{2}\sum c_{kj}\frac{\partial^2}{\partial x_k\,\partial x_j}.$$

If $Q_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1^t \end{pmatrix}$ and $Q_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2^t \end{pmatrix}$, then

$$[\widetilde{W}(Q_1), \widetilde{W}(Q_2)] = \widetilde{W}([Q_1, Q_2]) + \frac{1}{2}\operatorname{tr}(B_2C_1 - B_1C_2).$$

Hence $\widetilde{W}$ is projective representation of Lie algebra $sp(2n,\mathbb{R})$.The substitution

$$\widetilde{W}'\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} = \widetilde{W}\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} + \frac{1}{2}\operatorname{tr} A$$

yields to the linear representation W' of the algebra $\mathrm{sp}(2n,\mathbb{R})$.

8.2. **Action in F_n.** In this case a symplectic Lie algebra is realized as an algebra of matrices of the type

$$R = i\begin{pmatrix} -P & -Q \\ \overline{Q} & \overline{P} \end{pmatrix}, \qquad P = P^*, \quad Q = Q^t.$$

The corresponding operator in F_n is given by the equation

$$W(R) = i\Big[\sum p_{kj} z_k \frac{\partial}{\partial z_j} + \frac{1}{2}\sum q_{kj} z_k z_j + \frac{1}{2}\sum \overline{q}_{kj}\frac{\partial^2}{\partial z_k\,\partial z_j}\Big].$$

8.3. **Generators of one-parameter semigroups.** In 3.6 and 4.3 we discussed the semigroup $\Xi_n = \mathrm{End}_{\mathrm{Sp}}(V_{2n}) \cap \mathrm{Sp}\,(2n, \mathbb{C})$. It consists of the operators A in V_{2n} satisfying the conditions

$$\Lambda(Av, Av') = \Lambda(v, v'), \qquad M(Av, Av) \leqslant M(v, v).$$

Consider an operator $\mathcal{D}$ in V_{2n} such that

$$\Lambda(\mathcal{D}v, v') + \Lambda(v, \mathcal{D}v') = 0, \tag{8.1}$$
$$M(\mathcal{D}v, v) + M(v, \mathcal{D}v) = 2\,\mathrm{Re}\, M(\mathcal{D}v, v) \leqslant 0. \tag{8.2}$$

The first condition means that $\mathcal{D}$ lies in a complex symplectic algebra, and the second one is called the *M-dissipativity.*

It is easily seen that the following conditions are equivalent:

1. $\mathcal{D}$ satisfies equations (8.1), (8.2);
2. $\exp(t\mathcal{D}) \in \Xi_n$.

Denote by $\mathcal{M}$ the set of all operators $\mathcal{D}$ satisfying equations (8.1)–(8.2). It is easily seen that $\mathcal{M}$ is a convex cone in a complex symplectic algebra, invariant with respect to the associated action of the real symplectic group $\mathrm{Sp}\,(2n, \mathbb{R})$. In addition, it contains the cone C which consists of the operators $\mathcal{D}$ satisfying the conditions

$$M(\mathcal{D}v, w) = M(v, \mathcal{D}w) \qquad (M\text{-self-adjointness}),$$
$$M(\mathcal{D}v, v) \leqslant 0 \qquad (M\text{-dissipativity}).$$

Obviously, $C \subset ig$, $\mathcal{M} = g \oplus iC$.

Theorem 8.1. *If* $\mathcal{D} = \begin{pmatrix} -P & -Q \\ \overline{Q} & \overline{P} \end{pmatrix} \in C$, *then the operator*

$$W'(\mathcal{D}) = \sum p_{kj} z_k \frac{\partial}{\partial z_j} + \frac{1}{2} \sum q_{kj} z_k z_j + \frac{1}{2} \sum \overline{q}_{kj} \frac{\partial^2}{\partial z_k \partial z_j} + \frac{1}{2}\,\mathrm{tr}\, P$$

is negative definite.

In particular, $W'(\mathcal{D})$ generates a contraction semigroup of self-adjoint operators in F_n.

8.4. **One-parameter subgroups in** Sp_∞. Consider a matrix of the form $H = \begin{pmatrix} -P & -Q \\ \overline{Q} & \overline{P} \end{pmatrix}$, where $Q = Q^t$ is a Hilbert–Schmidt operator, and P is, in general, an unbounded self-adjoint operator. Standard statements on one-parameter operator semigroups (see [18, X.8]) demonstrate that for all $t \in \mathbb{R}$ there exists an $\exp(itH)$.

Theorem 8.2. *Let* $\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} = \exp(itH)$. *With the constraints stated above,* $\begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix} \in \mathrm{Sp}_\infty$, *the operator* $(\Phi e^{-iPt} - E)$ *being a nuclear operator.*

This theorem does not describes all one-parameter groups in Sp_∞.

Counter-example. Let matrices P and Q be diagonal with $Q = E$, and the eigenvalues of P increase sufficiently rapidly (e.g. they are equal to j^2). Then $\exp\left(it\begin{pmatrix} -P & -Q \\ \overline{Q} & \overline{P}\end{pmatrix}\right)$ is a one-parameter subgroup in Sp_∞.

8.5. **Self-adjointness of quadratic operators.** The set $F_\infty^k \subset F_\infty$ of all homogeneous functions $f \in F_\infty$ with degree of homogeneity k are naturally identified with the kth symmetric power $S^k l_2$ of the space l_2. For this reason, the Fock space F_∞ is identified with $\bigoplus_{k=0}^\infty S^k l_2$.

Let P be an essentially self-adjoint operator in l_2 with domain of definition L, and Q be the Hilbert–Schmidt operator for which $Q = Q^t$. Denote by $s(Q)$ an operator in F_∞ acting in each F_∞^k as the kth symmetric power of the operator Q. In F_∞ consider the operator

$$\mathcal{H} = \frac{1}{2}\sum q_{kj} z_k z_j + s(Q) + \frac{1}{2}\sum \overline{q}_{kj}\,\frac{\partial^2}{\partial z_k \partial z_j}. \tag{8.3}$$

Theorem 8.3. *The operator $\mathcal{H}$ is essentially self-adjoint on a subspace $\mathcal{L}$ consisting of finite linear combinations of vectors of the type $x_1 \cdot x_2 \cdot \ldots \cdot x_k$, where $x_k \in L$.*

8.6. **Correspondence of one-parameter subgroups.** Suppose that $\begin{pmatrix} -P & -Q \\ \overline{Q} & \overline{P}\end{pmatrix}$ satisfies the conditions of 8.4, and $\mathcal{H}$ is given by equation (8.3).

Theorem 8.3.

$$\exp(it\mathcal{H}) = \det(\Phi e^{iPt})^{1/2} W\left[\exp\left\{it\begin{pmatrix} -P & -Q \\ \overline{Q} & \overline{P}\end{pmatrix}\right\}\right]. \tag{8.4}$$

§9. Canonical forms

We consider here only the finite-dimensional case.

9.1. **Canonical decomposition.** Consider a semigroup $\mathrm{End}_{\mathrm{Sp}}(V_{2n})$ and inside it a subsemigroup Ξ_n consisting of actual operators. Recall that in $\mathrm{End}_{\mathrm{Sp}}(V_{2n})$ the conjugation operation $P \mapsto P_{(*)}$ (see 4.9) is defined, the *unitary elements* $\mathrm{End}(V_{2n})$ (i.e., the elements satisfying the condition $P^{(*)} = P^{-1}$) forming the group $\mathrm{Sp}(2n, \mathbb{R}) \simeq \mathrm{Aut}(V_{2n})$.

Theorem 9.1. *Any element $g \in \Xi_n$ can be uniquely represented in the form of a product $g = UR$, where U is a unitary element, and $R \in \Xi_n$ is self-adjoint ($R^{(*)} = R$).*

Theorem 9.1′. *Any element $P \in \mathrm{End}(V_{2n})$ can be represented in the form of a product $P = UR$, where U is a unitary element, and $R \in \mathrm{End}(V_{2n})$ is self-adjoint.*

9.2. **Theorem 9.2.** *Any self-adjoint element $R \in \Xi_n$ can be uniquely represented in the form $R = \exp(H)$, where H satisfies the conditions*

$$M(Hv, w) = M(v, Hw), \tag{9.1}$$

$$M(Hv, v) \leqslant 0 \tag{9.2}$$

for all $v, w \in V_{2n}$.

Recall that the set of all H satisfying (9.1)–(9.2) forms a convex cone invariant with respect to the associated operation $H \mapsto g^{-1}Hg$ of the group $\mathrm{Sp}(2n, \mathbb{R})$; in 8.3 we have denoted this cone by C.

9.3. **Canonical forms of the elements of the cone C.**

Theorem 9.3. *By the action of the group* $\mathrm{Sp}(2n, \mathbb{R})$ *any element of the cone* C *can be reduced to the canonical form* $K\colon V_{2n}^+ \oplus V_{2n}^- \to V_{2n}^+ \oplus V_{2n}^-$,

$$K = \left(\begin{array}{ccc|ccc|ccc|ccc}
-\lambda_1 & & & & & & & & & & & \\
& \ddots & & & & & & & & & & \\
& & -\lambda_k & & & & & & & & & \\
\hline
& & & -1 & & & & & & 1 & & \\
& & & & \ddots & & & & & & \ddots & \\
& & & & & -1 & & & & & & 1 \\
\hline
& & & & & & -\lambda_1 & & & & & \\
& & & & & & & \ddots & & & & \\
& & & & & & & & -\lambda_k & & & \\
\hline
& & & -1 & & & & & & 1 & & \\
& & & & \ddots & & & & & & \ddots & \\
& & & & & -1 & & & & & & 1
\end{array}\right),$$

where $\lambda_j \geqslant 0$.

Remark. The Jordan form of the matrix K contains only Jordan boxes of size 1×1 and Jordan boxes of size 2×2 with zero eigenvalues. In this case the interior of the cone C (i.e., the set of all H such that $M(Hv, v) < 0$ for $v \neq 0$) consists of operators with nonzero eigenvalues.

Remark. Obviously, this theorem makes it also possible to construct canonical forms for the self-adjoint elements of the semigroup Ξ_n.

9.4. **Canonical forms of quadratic operators.** Theorem 9.3 has the following corollary. Let

$$\mathcal{H} = a + \frac{1}{2}\sum q_{kj} z_k z_j + \sum p_{kj} z_k \frac{\partial}{\partial z_j} + \frac{1}{2}\sum \overline{q}_{kj} \frac{\partial^2}{\partial z_k \partial z_j}$$

be a positive definite quadratic operator (in particular, $\begin{pmatrix} P & Q \\ -\overline{Q} & -\overline{P} \end{pmatrix} \in C$). Then, using the transformation $\mathcal{H} \mapsto W(P)\mathcal{H}W(P)^{-1}$, where $P \in \mathrm{Sp}(2n, \mathbb{R})$, it can be reduced to the form

$$\tau + \sum_{j=1}^{k} \lambda_j z_j \frac{\partial}{\partial z_j} + \frac{1}{2}\sum_{\mu > k} \left(z_\mu + \frac{\partial}{\partial z_\mu} \right)^2,$$

where $\tau \in \mathbb{R}$ and $0 \leqslant \lambda_j$.

Similarly, any positive operator

$$\mathcal{R} = s + i\sum a_{kj} x_k \frac{\partial}{\partial x_j} + \frac{1}{2}\sum b_{kj} x_k x_j + \frac{1}{2}\sum c_{kj} \frac{\partial^2}{\partial x_k \partial x_j}$$

in $L^2(\mathcal{R}_n)$, using the transformation $\mathcal{R} \mapsto \widetilde{W}(P)^{-1}\mathcal{R}\widetilde{W}(P)$, where $P \in \mathrm{Sp}(2n, \mathbb{R})$, can be reduced to the form

$$\sigma + \sum_{j=1}^{k} \mu_j \Big(- \frac{\partial^2}{\partial x_j^2} + x_j^2 \Big) - \sum_{m>k} \frac{\partial^2}{\partial x_m^2},$$

where $\sigma \in \mathbb{R}$ and $\mu_j \geqslant 0$ (i.e., $\mathcal{R}$ is split into the sum of the harmonic oscillator and the Laplace operator).

9.5. **Canonical forms of the elements** $P \in \mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$**.** The group $\mathrm{Sp}(2n, \mathbb{R}) \times \mathrm{Sp}(2m, \mathbb{R})$ acts in an obvious manner on the set $\mathrm{Mor}_{\mathrm{Sp}}(V_{2n}, V_{2m})$. We now describe the orbits of this group.

Consider the following four types of canonical morphisms:

(a) The morphism $\alpha \colon V_0 \to V_2$ consists of elements of the form

$$(0; v^-) \in V_2^+ \oplus V_2^+ = V_2 = V_0 \oplus V_2 .$$

(b) The morphism $\beta \colon V_2 \to V_0$ consists of elements of the form

$$(v^+; 0) \in V_2^+ \oplus V_2^- = V_2 = V_2 \oplus V_0 .$$

(c) The morphisms $\gamma_s \colon V_2 \to V_2$, where $0 < s \leqslant 1$, consist of the elements

$$(x, y; sx, s^{-1}y) \in V_2 \oplus V_2 .$$

(d) The morphisms $\delta_t \colon V_2 \to V_2$, where $0 < t < 1$, consist of the elements

$$(x, y + t(x - y); x + t(y - x), y) \in V_2 \oplus V_2$$

(for different t these morphisms are equivalent).

Further, note that direct sums of the form $V_{2n_1} \oplus \cdots \oplus V_{2n_\mu}$ are obviously identified with $V_{2(n_1+\cdots+n_\mu)}$.

Theorem 9.4. *Any morphism* $P \colon V_{2n} \to V_{2m}$ *is equivalent to a direct sum of morphisms of the form* (a)–(d).

Sketch of the proof. The morphism $P^{(*)}P$ is self-adjoint, and we can make use of Theorem 9.3.

9.6. **Canonical forms of the operators** $B[S]$**.** Let us now state Theorem 9.4 in the language of the operators $B[S]$.

Theorem 9.4′. *The transformation*

$$B[S] \mapsto \lambda W(P) B[S] W(Q)^{-1},$$

where $P \in \mathrm{Sp}\,(2n, \mathbb{R})$, $Q \in \mathrm{Sp}\,(2m, \mathbb{R})$, *and* $\lambda \in \mathbb{C}$, *reduces any operator* $B[S]\colon F_n \to F_m$ *to the form*

$$B\left[\begin{array}{c|c|c|c|c|c}
 & & & \begin{matrix} s_1 & & \\ & \ddots & \\ & & s_k \end{matrix} & & \\ \hline
 & \begin{matrix} t_1 & & \\ & \ddots & \\ & & t_\alpha \end{matrix} & & & \begin{matrix} 1-t_1 & & \\ & \ddots & \\ & & 1-t_\alpha \end{matrix} & \\ \hline
 & & & & & \\ \hline
\begin{matrix} s_1 & & \\ & \ddots & \\ & & s_k \end{matrix} & & & & & \\ \hline
 & \begin{matrix} 1-t_1 & & \\ & \ddots & \\ & & 1-t_\alpha \end{matrix} & & & \begin{matrix} t_1 & & \\ & \ddots & \\ & & t_\alpha \end{matrix} & \\ \hline
 & & & & &
\end{array}\right], \tag{9.3}$$

where $0 < s \leqslant 1$ *and* $0 < t < 1$.

Theorem 9.4$''$. *The transformation*

$$B[S] \mapsto \lambda W(P)\, B[S]\, W(P)^{-1},$$

where $\lambda \in \mathbb{C}$ *and* $P \in \mathrm{Sp}\,(2n, \mathbb{R})$, *reduces any self-adjoint operator* $B[S]\colon F_n \to F_n$ *to the form* (9.3).

In both cases the numbers s_j are invariants of the operator $B[S]$, whereas the set t_μ may be replaced by any other set $t'_1, \dots, t'_\alpha$, where $0 < t'_\nu < 1$.

9.7. **Invariants s_j.** For simplicity, consider the case $n = m$. With each operator $B[S]\colon F_n \to F_n$ we can associate a set of numbers $s_1, \dots, s_n$ consisting of $s_1, \dots, s_\alpha$ (see equation (9.3)) and also of μ ones and $n - \alpha - \mu$ zeros.

Theorem 9.5. *The roots* λ *of the equation*

$$\det\left((1 - S^*S) - \lambda\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - S^*\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S\right)\right) = 0 \tag{9.4}$$

are $\mathrm{Sp}\,(2n, \mathbb{R}) \times \mathrm{Sp}\,(2n, \mathbb{R})$*-invariants of the operator* $B[S]$. *In addition*

$$\lambda = \pm(1 - s_j^2)/(1 + s_j^2)\,. \tag{9.5}$$

Note that the roots of equation (9.4), in their turn, coincide with the eigenvalues of the matrix

$$X(S) = (1 - S^*S)^{1/2}\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - S^*\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S\right)^{-1}(1 - S^*S)^{1/2}. \tag{9.6}$$

Remark. The invariants s_j do not separate the orbits of the group $\mathrm{Sp}\,(2n, \mathbb{R}) \times \mathrm{Sp}\,(2n, \mathbb{R})$ that lie on the boundary of the ball $\|S\| \leqslant 1$, i.e., on the "surface" $\|S\| = 1$.

§10. Formulas for the norm of the operators $B[S]$

We consider the case of the operators $B[S]\colon F_n \to F_n$ only. The general case is reduced to the following one: instead of the operator $B[S]\colon F_n \to F_n$ one can consider the operator $B[\widetilde{S}]\colon F_N \to F_N$, where $N = \max(m, n)$, and $\widetilde{S}$ is a matrix S filled to a required size by zeros.

10.1. **Formulas for the norm of a self-adjoint operator.** Let $\|S\| < 1$, and let $B[S]$ be self-adjoint, where $S = \begin{pmatrix} K & L \\ L^t & \overline{K} \end{pmatrix}$. Let sj be the invariants of the operator $B[S]$.

It is obvious (see 5.7) that the operator $B[S]$ is a nuclear operator. Calculate its trace in two ways. First, the trace is equal to the integral of the kernel of the operator over the diagonal:

$$\begin{aligned} \operatorname{tr} B[S] &= \int \exp\left\{ \frac{1}{2} (z \overline{z}) \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \exp\{-|z|^2\} \right\} dz \\ &= \det\left[\begin{pmatrix} 1-L & -K \\ -M & 1-L^t \end{pmatrix}^{-1/2} \right]. \end{aligned}$$

Secondly, the trace is equal to the sum of the eigenvalues (see 5.7):

$$\operatorname{tr} B[S] = \|B[S]\| \sum s_1^{k_1} \cdots s_n^{k_n} = \|B[S]\| \prod_{j=1}^{n} (1 - s_j)^{-1}.$$

Hence

$$\|B[S]\| = \det\left[\begin{pmatrix} 1-L & -K \\ -M & 1-L^t \end{pmatrix}^{-1/2} \right] \prod_{j=1}^{n} (1 - s_j). \tag{10.1}$$

Further, note that

$$B\begin{bmatrix} 0 & -E \\ -E & 0 \end{bmatrix} f(z) = f(-z),$$

and this operator commutes with any $B[S]$:

$$B\begin{bmatrix} 0 & -E \\ -E & 0 \end{bmatrix} B[S] = B[S] B\begin{bmatrix} 0 & -E \\ -E & 0 \end{bmatrix} = B[JSJ],$$

where $J = \begin{pmatrix} -E & 0 \\ 0 & E \end{pmatrix}$. Calculating $\operatorname{tr}\left(B\begin{bmatrix} 0 & -E \\ -E & 0 \end{bmatrix} B[S] \right)$ in the same two ways, we obtain

$$\|B[S]\| = \det\left[\begin{pmatrix} 1+L & -K \\ -M & 1+L^t \end{pmatrix}^{-1/2} \right] \prod_{j=1}^{n} (1 + s_j). \tag{10.2}$$

10.2. **Norm of an arbitrary operator $B[S]$.** Let $\|S\|<1$. Calculating $\operatorname{tr}(B[S]^*B[S])$ in two ways

$$\operatorname{tr}(B[S]^*B[S])=\int_{\mathbb{C}^n}\int_{\mathbb{C}^n}|\exp\{1/2(z\,\overline{u})S(z\,\overline{u})^t\}|^2d\mu(z)d\mu(u)=\|B[S]\|^2\prod_{j=1}^n(1-s_j^2)$$

, we obtain

$$\|B[S]\|=\det(1-S^*S)^{-1/4}\prod_{j=1}^n(1-s_j^2)^{1/2}. \tag{10.3}$$

Calculating $\operatorname{tr}\left(B[S]^*B\begin{bmatrix}0&-E\\-E&0\end{bmatrix}B[S]\right)$ in two ways, we get

$$\|B[S]\|=\det(J-S^*JSJ)^{-1/4}\prod_{j=1}^n(1+s_j^2)^{1/2}. \tag{10.4}$$

Taking into account the fact that the eigenvalues of the matrix (9.6) are given by (9.5), we can write the latter expression in the form

$$\|B[S]\|=|\det Q|^{-1/4}\det((E+|R^{1/2}Q^{-1}R^{1/2}|)/2)^{-1/2}, \tag{10.5}$$

where

$$Q=J-S^*JS,\qquad R=1-S^*S\,.$$

10.3. **One more formula for the norm of $B[S]$.** Let $S=\begin{pmatrix}K&L\\L^t&M\end{pmatrix}$, the matrix L being invertible, and $\|S\|<1$. By virtue of Theorem 9.4′, any matrix S such that $\|S\|<1$ can be represented in the form

$$S=\begin{pmatrix}P&Q\\Q^t&R\end{pmatrix}*\begin{pmatrix}0&\Sigma\\\Sigma&0\end{pmatrix}*\begin{pmatrix}G&H\\H^t&F\end{pmatrix},$$

where Σ is a diagonal matrix with eigenvalues $s_1,\dots,s_n$, and $\begin{pmatrix}P&Q\\R&T\end{pmatrix}$ and $\begin{pmatrix}G&H\\H^t&F\end{pmatrix}$ are unitary matrices (see 3.6). Then (see Proposition 2.1) the operators

$$|\det(Q)|^{1/2}B\begin{bmatrix}P&Q\\Q^t&R\end{bmatrix},\qquad|\det(H)|^{1/2}B\begin{bmatrix}G&H\\H^t&F\end{bmatrix}$$

are unitary. Taking into account the fact that $\left\|B\begin{bmatrix}0&\Sigma\\\Sigma&0\end{bmatrix}\right\|=1$, we find that the norm of the operator

$$X=|\det(Q)|^{1/2}|\det(H)|^{1/2}B\begin{bmatrix}P&Q\\Q^t&R\end{bmatrix}B\begin{bmatrix}0&\Sigma\\\Sigma&0\end{bmatrix}B\begin{bmatrix}G&H\\H^t&F\end{bmatrix}$$

is equal to 1. Using equations (3.2)–(3.3) to calculate the right-hand side, we obtain

$$X = |\det(Q)|^{1/2} |\det(H)|^{1/2} \det(1 - \Sigma R \Sigma G)^{-1/2} B \begin{bmatrix} K & L \\ L^t & M \end{bmatrix}.$$

However,

$$L = Q\Sigma(1 - \Sigma R \Sigma G)^{-1} H,$$

and, comparing the two last expressions, we obtain

$$\left\| B \begin{bmatrix} K & L \\ L^t & M \end{bmatrix} \right\| = |\det(L)|^{-1/2} |\det(\Sigma)|^{1/2} = |\det(L)|^{-1/2} \prod_{j=1}^{n} |s_j|^{1/2}. \qquad (10.6)$$

10.4. **Remarks.** The formulas just obtained are valid under somewhat more general conditions than those used in the derivation. Thus, equations (10.2), (10.4)–(10.6) remain valid also in the case $\|S\| = 1$.

§11. Remarks

11.1. **Remarks to §1.** *With regard to* 1.1. What we call the "boson Fock space" should be more correctly called the "holomorphic Bargman–Segal–Berezin model" of the boson Fock space.

With regard to 1.3 *and* 1.7. The phenomena described here are related to the Bergman kernel function and are general for Hilbert spaces of holomorphic functions, see, for example, [4]. On the other hand, the sets of functions ϕ_u are special cases of the so-called "overfilled bases" (or "coherent states"), see [5].

With regard to 1.5. The isomorphism $\mathcal{B}B$ is called Bargmann–Segal transform. See [3].

11.2. **Remarks to §2.** *With regard to* 2.5. Theorem 2.1 is a curious example of a theorem that has no author. It appeared in mathematics certainly between 1930 and 1958. On the one hand, it is a trivial consequence of the well-known Stone-von Neumann uniqueness theorem (1930) about a unitary equivalence of the set of self-adjoint operators $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$, such that

$$[P_k, P_l] = [Q_k, Q_l] = 0, \qquad [P_k, Q_l] = i\delta_{kl} E$$

(see, for example, [19, §1.3]). In this case, however, the words "trivial consequence" only imply the triviality of the proof (after the theorem has already been formulated!). Theorem 2.1 itself is usually attributed to I. Segal ([7, 1958]) but this is not quite true since the difficulties connected with the generalization of Theorem 2.1 to the infinite-dimensional case had been discussed earlier in the book by Friedrichs ([6, 1953]).

As soon as the theorem was formulated by Segal in a transparent manner, it evoked a great deal of interest (see A. Weil [20], Cartier [21]). After one of the first responses to this construction the construction itself was then called the "Weil representation" instead of the historically justified but never used term the "Friedrichs–Segal–Berezin representation". It seems that term "harmonic representation" proposed by I.Segal is better.

The representation $W(\,\cdot\,)$ of the group $\mathrm{Sp}\,(2n,\mathbb{R})$ is but one of the representations of only one of the series of semisimple groups, but its actual role in the theory of semisimple group representations is quite important (see, for example, [19, 22, 58, 56]). The role of the "Weil representation" turns out to be still more important in the theory of infinite-dimensional groups, where it plays the part of a sort of "universal object" (see [23, 24, 26]). Finally, the construction is important not only in the representation theory itself, see [1, 19–20, 53, 43].

With regard to 2.6. The remarkable Berezin formula was obtained in [8] (see also [1]) but left unnoticed and was re-discovered much later ([25]).

With regard to 2.7. In all probability, the operators $W(\,\cdot\,)$, which refer to a singular subset in $\mathrm{Sp}\,(2n,\mathbb{R})$, correspond to complex-valued Gaussian measures concentrated on subspaces in $\mathbb{R}^n \oplus \mathbb{R}^n$. I have not seen them demonstrated anywhere else.

11.3. **Remarks to §3.** The basic question formulated in this section was posed many times in many different forms (we limit ourselves with references to [11, 14, 27]) and was solved only in [12, 13, 16].

With regard to 3.1–3.6. The semigroup $\mathrm{End}_{\mathcal{B}}(n)$ appeared in the unpublished paper [11].

With regard to 3.7. See [14].

11.4. **Remarks to §4.** The construction of this section appeared in [12, 13].

With regard to 4.4. See [28, 29].

With regard to 4.8. See the remarks to 2.7.

11.5. **Remarks to §5.** *With regard to* 5.1 and 5.2. See, for example, [30], §11.3.

With regard to 5.3. See, for example, [31].

With regard to 5.4. See [32–34].

With regard to 5.5–5.7. See [13]. Fixed points of generalized linear-fractional mappings were extensively studied in connection with the Krein theorem on the invariant subspace; see, for example, [29].

11.6. **Remarks to §6.** *With regard to* 6.1 and 6.5. See [35, 30]. The problem of the triangle inequality for a composite distance does not seem to be solved and does not seem to have ever been formulated.

With regard to 6.6, 6.8. The description of all invariant metrics on $\mathcal{Z}_n$ seems to be unknown.

With regard to 6.7. The Compression Theorem was announced in [36].

With regard to 6.9. In [36] an inexact assertion on the subject is formulated. Theorems of the type of the "basic theorem of the projective geometry" for symmetric subspaces were extensively discussed in the literature, see, for example, [37–38]. Theorem 6.5 is a strengthening of the theorem from [39].

11.7. **Remarks to §7.** *With regard to* 7.1. See the remarks to 1.1. Literally the construction of the Fock space as $L^2(\mathbb{R}^n)$ collapses when passing to the limit $n \to \infty$. However, if the space L^2 over Gaussian measure, is considered, then this passage can be realized; see [54, 55, 40]. Operators with Gaussian kernels in $L^2(\mathbb{R}^\infty)$ do not seem to have been studied systematically, see [49]

With regard to 7.5–7.10. See [13].

With regard to 7.7. Theorem 7.7 is usually attributed to Shale [9], but Berezin in [1] attributes it to Friedrichs [6]. This theorem is present in the book by Friedrichs, although it is not easy to find it. However, the proof by Friedrichs should be considered as incorrect (although one can correct his proof by using the techniques of §5). Moreover, F. A. Berezin [8] proved Theorem 7.7a at the same time and independently from Shale and also derived the explicit formula for the operators $W(\cdot)$ (Theorem 7.7, (b), (c)). The latter circumstance remained unnoticed in mathematics, and this left a pleasant but not very useful flavor of mysticism on the whole construction (which is basically very simple: one writes a formula and then verifies it). (To a still greater extent this also applies to the Berezin fermion construction.) It is worth noting that the books [1] and [6] are, in fact, written in a way that makes them hard to understand.

11.8. **Remarks to §8.** *With regard to* 8.1, 8.2. These constructions are contemporary to Theorems 2.1 and 7.7.

With regard to 8.3. Invariant cones in Lie algebras were studied by Vinberg, Olshanskiĭ, Paneitz; see [41, 48].

With regard to 8.4–8.6. All the statements are taken from the book [1], they are very important in the applications of the construction to other infinite-dimensional groups. In [1] a statement stronger than Theorem 8.2 is formulated. A good proof of Theorem 8.3 is not known. The existing proof ([1]) is based on a calculation (both sides of equation (8.3) can be calculated explicitly), but the calculation is very laborious, and the justification that it is correct is much longer. It would be interesting to derive statements similar to those of 8.3.

11.9. **Remarks to §9.** *With regard to* 9.1, 9.2. See [28, 41].

With regard to 9.5–9.7. See [11, 15].

11.10. **Remarks to §10.** See [11, 15].For L^p-theory of operators with Gauss kernels see [49–51], more references see [50–51] .

11.11. **Remarks to §11.** *With regard to* 11.3. The problem formulated in §3 arose in connection with the studies of the "hidden structures" related to the infinite-dimensional groups. In fact, formula (3.3) was but one example of "strange multiplications" appearing in the theory of infinite-dimensional groups. The work [11] was done in connection with the following question.

Take the "Weil representation" of the group Sp_∞. Consider the set of all operators of the form $\lambda W(g)$, where $g \in \mathrm{Sp}_\infty$ and $\lambda \in \mathbb{C}$; consider the closure of this set in the weak operator topology. One can understand this closure as a set of limit points of the group Sp_∞. It is easy to see that this weak closure is exactly the semigroup of bounded operators of the type $B[S]$.

In [42] (see also [43]) the complexification of the group of circle diffeomorphisms was constructed, and it has turned out that to understand this construction one needs to analyze the same semigroup of operators of the type $B[S]$.

With regard to 11.4. As soon as the answer to the problem of §3 was obtained, the situation with "hidden structures" was clarified, and by now in most cases the description of "hidden structures" is realized (see [43–47, 12–13, 16]).

The category Sp is but one of the categories of linear relations related to infinite-dimensional classical groups. It appears that the large part of representation theory (including the theory of finite-dimensional representations of the classical groups) can be extended to the representation theory of categories (see [36]).

With regard to 11.5, 11.6. Almost all the statements of §§5 and 6 can be extended to any Riemann symmetric space (see [36]).

With regard to 11.7. For the fermion analog of the operators B, see [16].

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