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HOLOMORPHIC EXTENSIONS OF REPRESENTATIONS OF THE GROUP OF DIFFEOMORPHISMS OF THE CIRCLE

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YU. A. NERETIN

ABSTRACT. This paper gives the construction of a semigroup Γ which could be thought of as the complexification of the group Diff of analytic diffeomorphisms of the circle, and it is shown that any unitary projective representation of Diff with highest weight has a holomorphic extension to Γ. For this, Γ is embedded in the semigroup of "endomorphisms of canonical commutation relations" (this is a certain part of the Lagrange Grassmannian in complex symplectic Hilbert space).

Bibliography: 25 titles.

In this paper we study the semigroup Γ, constructed in [13], which from a geometric point of view is an "infinite-dimensional complex domain", and which has the group Diff of diffeomorphisms of the circle as "skeleton" (the "Shilov boundary"). It is proved that representations of Diff with highest weight (they are usually called "representations of the Virasoro algebra") have holomorphic extensions to Γ. The existence of holomorphic extensions of representations of a Lie group G to some open subsemigroup in the complexification $G_{\mathbf{C}}$ of G is a general feature of representations with highest weight (see [15]). However, there is no group $\mathrm{Diff}_{\mathbf{C}}$ (see §1.10), and therefore even the existence of a complex semigroup containing Diff seemed for a long time to be a rather doubtful conjecture (put forward by Ol′shanskiĭ). It turns out that representations of the semigroup Γ are given by fairly simple explicit formulas, while the existing earlier formula for representations of Diff (see [14]) contained rather inexplicit operations such as the calculation of Fredholm determinants and finding the operator inverse to an integral.

We should like to remark on the connection between the theory of representations of the semigroup Γ and such an isolated domain of mathematics as the geometric theory of functions, and also its connection with a whole series of recent papers on the Virasoro algebra and Riemann surfaces [4]–[7], [23].

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In §1 of the paper we consider the simplest algebraic and geometric properties of the semigroup Γ; §§2 and 3 contain a summary of the necessary information about

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operators in Fock space and about representations of the Virasoro algebra. In §4 we prove the theorem about the holomorphic extension of representations of Γ, and at the end of that section there are the explicit formulae for the representations of Γ, the cocycles on Γ and the characters of Γ that arise from this proof.

§1. The semigroup Γ

1.0. Notation. We let S' denote the circle $|z| = 1$ in the complex plane $\mathbf{C}$; D_+ the circle $|z| \leq 1$; D_- the domain $|z| \geq 1$ in the extended complex plane $\overline{\mathbf{C}}$; and D_-^0 and D_+^- the interiors of these domains. We let Diff be the group of analytic orientation-preserving diffeomorphisms of S^1, Vect the Lie algebra of analytic vector fields on S^1, and $\mathrm{Vect}_{\mathbf{C}}$ its complexification. It is natural to consider Vect to be the Lie algebra of the group Diff. The words "the function f is holomorphic (single-valued) in the closed domain B up to the boundary" means that f extends holomorphically (single-valuedly) to some neighborhood of B.

1.1. The local semigroup $\mathrm{L}\Gamma$. Let $\mathrm{LDiff}_{\mathbf{C}}$ be the set of all analytic maps $\rho : S^1 \to \mathbf{C} \setminus 0$ such that the Jordon curve $\rho(e^{i\varphi})$ surrounds 0 and proceeds anticlockwise and, further, $\rho'(e^{i\varphi}) \neq 0$. The set $\mathrm{LDiff}_{\mathbf{C}}$ is a local group in the following sense. If $\rho, \mu \in \mathrm{LDiff}_{\mathbf{C}}$ and ρ extends analytically to a domain containing the contour $\mu(e^{i\varphi})$, then the composition $\rho \circ \mu$ makes sense.

Let $\mathrm{L}\Gamma$ be the local subsemigroup in $\mathrm{LDiff}_{\mathbf{C}}$ consisting of the maps ρ satisfying the additional condition $|\rho(e^{i\varphi})| < 1$. At first glance it seems obvious that $\mathrm{L}\Gamma$ cannot be extended to a global semigroup. In fact this is not so.

1.2. First construction of the semigroup Γ. An element γ of the semigroup Γ is a triple (K, p, q), where K is a complex Riemann surface with boundary biholomorphically equivalent to an annulus, where $p, q: e^{i\varphi} \to K$ are fixed analytic parametrizations of the components of the boundary of K, and where on going round the contour $p(e^{i\varphi})$ the surface K remains on the right and on going round $q(e^{i\varphi})$ it remains on the left. Elements $\gamma_1 = (K_1, p_1, q_1)$ and $\gamma_2 = (K_2, p_2, q_2)$ of the semigroup are considered to be the same if there exists a conformal map $R: K_1 \to K_2$ such that $p_2 = R \circ p_1$ and $q_2 = R \circ q_1$. Now let $\gamma_1 = (K_1, p_1, q_1)$ and $\gamma_2 = (K_2, p_2, q_2)$; then their product $\gamma_1\gamma_2$ is the triple (K_3, p_3, q_3) where the Riemann surface K_3 is obtained from the disjoint union $K_1 \cup K_2$ by identifying points of the form $q_1(e^{i\varphi})$ with $p_2(e^{i\varphi})$, where $\varphi = [0, 2\psi]$ and $p_3 = p_1$, $q_3 = q_2$.

The local isomorphism A: $\mathrm{L}\Gamma \to \Gamma$ is constructed in the following way. Let $\rho \in \mathrm{L}\Gamma$; then $A(\rho) = (K, p, q)$, where K is the annular domain in $\mathbf{C}$ bounded by the contours $\rho(e^{i\varphi})$ and S^1, while $q(e^{i\varphi}) = e^{i\varphi}$ and $p(e^{i\varphi}) = \rho(e^{i\varphi})$.

The elements of Diff should be considered as "infinitesimally narrow bands" in Γ; in other words, we have a closed curve with two parametrizations p and q, and moreover only their "relation" $p^{-1} \circ q \in \mathrm{Diff}$ is essential.

1.3. Second realization of Γ. An element of the semigroup $\widehat{\Gamma}$ is the triple $\gamma = (S_\gamma, p_\gamma, q_\gamma)$, where S_γ is a Riemann surface conformally equivalent to the sphere $\overline{\mathbf{C}}$, where p_γ (q_γ respectively) is a map $D_- \to S_\gamma$ ($D_+ \to S_\gamma$) that is holomorphic and single-valued up to the boundary; and where the domains $p_\gamma(D_-^0)$ and $q_\gamma(D_+^0)$ do not intersect. Elements $\gamma = (S_\gamma, p_\gamma, q_\gamma)$ and $\delta = (S_\delta, p_\delta, q_\delta)$ are considered to coincide if there exists a conformal map $R: S_\gamma \to S_\delta$ such that $p_\delta = R \circ p_\gamma$ and

$q_\delta = R \circ q_\gamma$. Now let $\gamma = (S_\gamma, p_\gamma, q_\gamma)$, $\mu = (S_\mu, p_\mu, q_\mu) \in \widehat{\Gamma}$. Then their product $\nu = \gamma\mu$ is the triple (S_ν, p_ν, q_ν), where S_ν is obtained from the disjoint union of $S_\gamma \setminus q_\gamma(D_+^0)$ and $S_\mu \setminus p_\mu(D_-^0)$ by identifying points of the form $q_\nu(e^{i\varphi}) \in S_\gamma$ with $p_\mu(e^{i\varphi}) \in S_\mu$, where $\varphi \in [0, 2\pi]$, and $p_\nu = p_\gamma$, $q_\nu = q_\mu$.

We consider the subsemigroup Γ of $\widehat{\Gamma}$ consisting of the elements $\gamma = (S_\gamma, p, q)$ that satisfy the additional condition: $p(D_-)$ does not intersect $q(D_+)$. Clearly this semigroup is isomorphic to the semigroup introduced in the previous subsection (the annulus K_γ is $S_\gamma \setminus (p(D_-^0) \cup q(D_+^-)))$. It is obvious that the semigroup Γ has a complex structure in the new realization.

Later on we consider the subsemigroup Δ in $\widehat{\Gamma}$ consisting of all $\gamma = (S_\gamma, p, q)$ such that $p(D_-) \cup q(D_+) = S_\gamma$. Then the map $r_1 = p^{-1} \circ q$ is an analytic diffeomorphism of S^1, and in addition the map $\gamma \to r_\gamma$ is an isomorphism between Δ and Diff (to construct the inverse map $r \to \gamma(r)$ it is sufficient to "glue" the domains D_+ and D_- together by identifying the points $e^{i\varphi}$ and $r(e^{i\varphi})$). Then we obtain the Riemann sphere S with fixed maps $D_- \to S$ and $D_+ \to S$.

Later on it will be convenient for us to consider as the basic object neither Γ nor $\widehat{\Gamma}$, but the semigroup $\overline{\Gamma} = \Gamma \cup \mathrm{Diff} \subset \widehat{\Gamma}$.

1.4. Third realization of Γ. An element of the semigroup Γ is the formal product

$$p \cdot A(t) \cdot q, \tag{1.1}$$

where $p, q \in \mathrm{Diff}$, $p(1) = 1$, $t > 0$, and $A(t)$ is a map of $\mathbf{C}$ into itself of the form $z \to e^{-t}z$. In order to multiply two elements of the form (1.1) it is sufficient to know how to reduce formal triple products of the form $A(s) \cdot p \cdot A(t)$ to the form (1.1).

A. Let $t = \varepsilon$ be sufficiently small for the diffeomorphism p to extend holomorphically to the annulus $e^{-\varepsilon} \le |z| \le 1$. Then the product $\rho = A(s)pA(\varepsilon)$ makes sense as a product of maps, and it belongs to $\mathrm{L}\Gamma$. Let K be the annular domain contained between S^1 and $\rho(S^1)$. Let Q be the canonical conformal map of K onto an annulus of the form $e^{-t'} \le |z| \le 1$ such that $Q(1) = 1$; see [3], V.1. Then $\rho = p' \cdot A(t) \cdot q'$, where p' is the restriction of Q^{-1} to S^1 and q' is found from the condition

$$A(s)pA(\varepsilon) = p'A(t')q'.$$

B. Let t be arbitrary. Then there exists $\varepsilon = t/n$ so small that the product

$$A(s) \cdot p \cdot A(t) = (((A(s) \cdot p \cdot A(\varepsilon)) \cdot A(\varepsilon)) \cdots) \cdot A(\varepsilon)$$

can be computed (that is, reduced to the form (1.1)) by a sequence of applications of the procedure described in step A. The following lemma shows that this is possible.

LEMMA 1.1. *Let $t, \mu > 0$ and $\alpha \in \mathrm{Diff}$ be extended single-valuedly over the annulus $e^{-\mu} \le |z| \le e^{\mu}$ with image in the interior of the annulus $e^{-t} < |z| < e^{t}$. Let $0 < \nu < \mu$, let $\rho = A(t)\alpha A(\varepsilon)$, and let $\varphi A(t')\psi$ be a representation of ρ in the form* (1.1). *Then ψ extends single-valuedly over the annulus $e^{-\mu-\nu} \le |z| \le e^{\mu+\nu}$ with image in the interior of the annulus $e^{-t'} < |z| < e^{t'}$.*

PROOF. It is easy to see that ρ takes the annulus $0 \le |z| \le e^{\mu+\nu}$ into the disc $|z| < 1$; therefore φ^{-1} is well-defined on the image of this annulus. Consequently,

$\psi = A(-t')\varphi^{-1}\rho$ takes the domain $0 \le |z| \le e^{\mu+\nu}$ to the interior of the annulus $0 < |z| < e^{t'}$. If we apply the Riemann-Schwarz principle, we get an extension of ψ over the annulus $e^{-\mu-\nu} \le |z| \le e^{\mu+\nu}$.

It remains to verify that multiplication does not depend on choice and is associative. The following lemma enables one to reduce this to checking it in the simple case when the values of the parameters t are sufficiently small.

LEMMA 1.2 (on domains depending analytically on a parameter). *Let Λ be a domain in $\mathbf{R}^n$. Let $l(\varphi, \tau)$ and $m(\varphi, \tau)$ be analytic maps from $S^1 \times \Lambda$ into $\mathbf{C}$, where, for fixed τ, the contours $l(\varphi, \tau)$ and $m(\varphi, \tau)$ bound an annular domain Ω_τ. Let $\kappa(\varphi): \Lambda \to S^1$ be an analytic function. Let $f_\tau(z)$ denote the canonical biholomorphic map of the domain Ω_τ onto a domain of the form $e^{-t(\tau)} \le |z| \le 1$ such that $f_\tau(\kappa(\varphi)) = 1$. Then the function $F(z, \tau) = f_\tau(z)$ is analytic.*

PROOF. We just give the proof of the theorem about reducing the annular domain to a canonical form in which the analytic dependence on the parameter can be clearly followed; the missing details of the arguments can be found in [4], V.1 and VI.4, and [10], 21.1.

Let $u(z)$ be a solution of the Dirichlet problem for the equation $\Delta u = 0$ with boundary conditions $u|_l = 0$ and $u|_m = 1$ (it is given as the potential of the dual fiber). Let $v(z)$ be the harmonic function dual to $u(z)$. Then $g(z) = u(z)+iv(z)$ is a multivalued holomorphic function mapping the annular domain Ω onto the strip $0 \le \operatorname{Re} z \le 1$. Let $g^{-1}(0) = g^{-1}(ic)$. Then $f = \exp(2\pi g/c)$ is the required map of Ω onto the annulus $e^{-\alpha} \le |z| \le 1$.

We now show that the new definition of Γ is equivalent to the two preceding ones. For this we make the formal product $\gamma = r \cdot A(t) \cdot h$ correspond to the triple (K, p, q), where K is the annulus $e^{-1} \le |z| \le 1$, $q(e^{i\varphi}) = h(e^{i\varphi})$, and $p(e^{i\varphi}) = e^{-t} r(e^{i\varphi})$.

1.5. The endomorphisms of $\mathrm{Vect}_{\mathbf{C}}$. If $\mathfrak{g}$ is a semisimple Lie algebra, then its Lie group can be realized as the group of automorphisms of $\mathfrak{g}$. We try to apply this scheme to $\mathrm{Vect}_{\mathbf{C}}$.

Let $G\Gamma$ be the set of triples of the form (K, p, q), where K is a Riemann surface with boundary consisting of two circles, where $p, q : S^1 \to K$ are analytic parametrizations of components of the boundary, and where on going round the contour $p(e^{i\varphi})$ the surface remains on the left and on going round $q(e^{i\varphi})$ it is on the right. The equivalence of the triples and their product is determined in the same way as in §1.2. The semigroup $G\Gamma$ clearly consists of a countable number of connected components, and the connected component of the identity is Γ.

Let $\gamma = (K, p, q)$. Let D be a holomorphic vector field on K, let $\alpha(D)$ ($\beta(D)$ respectively) be the preimage of D under the map $e^{i\varphi} \to p(e^{i\varphi})$ ($e^{i\varphi} \to q(e^{i\varphi})$). The set of all fields of the form $\alpha(D)$ ($\beta(D)$) forms a dense subalgebra in $\mathrm{Vect}_{\mathbf{C}}$, which we denote by A (by B). The map $L_\gamma: \alpha(D) \to \beta(D)$ is thus an isomorphism of the subalgebra A onto the subalgebra B. So, for each $\gamma \in G\Gamma$ we have constructed an unbounded densely-defined operator in $\mathrm{Vect}_{\mathbf{C}}$ that preserves the commutativity operation; that is, an unbounded endomorphism of $\mathrm{Vect}_{\mathbf{C}}$.

Here it is convenient to translate this into another language and as usual, to consider the graphs of unbounded operators instead of the unbounded operators themselves. An arbitrary linear subspace in $V \oplus V$ will be called a *linear relation* in the space V. As usual the product PQ of relations P and Q is the subspace $R \subset V \oplus V$ consisting of all pairs $(w_1, w_2) \in V \oplus V$ such that there exists $v \in V$

satisfying $(w_1, v) \in R$ and $(v, w_2) \in P$. We say that a linear relation R in a Lie algebra preserves the commutativity operation if $(v_1, v_2), (w_1, w_2) \in R$ implies $([v_1, w_1], [v_2, w_2]) \in R$.

Now let $\gamma = (K, p, q) \in G\Gamma$. Then the set of pairs of the form

$$(\alpha(D), \beta(D)) \in \mathrm{Vect}_{\mathbf{C}} \oplus \mathrm{Vect}_{\mathbf{C}}$$

forms a linear relation L_γ in $\mathrm{Vect}_{\mathbf{C}}$. It is easy to see that $L_{\gamma_1\gamma_2} = L_{\gamma_1} L_{\gamma_2}$. Thus we have realized the semigroup $G\Gamma$ as a semigroup of linear relations in $\mathrm{Vect}_{\mathbf{C}}$ which preserve the commutative operation.

1.6. Generalized Borel subalgebras in $\mathrm{Vect}_{\mathbf{C}}$ and generalized flag spaces. A point of the flag space $\Omega_{g,n}$ is a collection $(R, c_1, \dots, c_n, q)$, where R is a Riemann surface of genus g with a distinguished, unordered collection of points $c_1, \dots, c_n$, and $q: D_+ \to R$ is a function single-valued up to the boundary, and where $c_1, \dots, c_n \notin q(D_+)$. Two collections $(R, c_1, \dots, c_n, q)$ and $(R', c_1', \dots, c_n', q')$ are considered to be equivalent if there exists a biholomorphic map $H: R \to R'$ such that $q' = H \circ q$ and $c_j' = H(c_j)$ for some ordering of the collection c_j'.

Let us define the action of the semigroup Γ on $\Omega_{g,n}$. Let Γ be realized as in §1.3, let $\gamma = (S_\gamma, p, r) = \overline{\Gamma}$, and let $x = (R, c_1, \dots, c_n, q) \in \Omega_{g,n}$. Then γx is the collection $(Q, d_1, \dots, d_n, t)$, where Q is the Riemann surface obtained from the disjoint union of $R \setminus q(D_+^0)$ and $S_\gamma \setminus p(D_-^0)$ by identifying pairs of points of the form $q(e^{i\varphi})$ and $p(e^{i\varphi})$.

The space $\Omega_{g,n}$ can be realized as a space of generalized Borel subalgebras in $\mathrm{Vect}_{\mathbf{C}}$. Let $x = (R, c_1, \dots, c_n, q) = \Omega_{g,n}$. Then the vector field v belongs to the subalgebra B_x in $\mathrm{Vect}_{\mathbf{C}}$ if the image of v under the map q extends to a holomorphic vector field V on $R \setminus q(D_+^0)$, where V is 0 at the points $c_1, \dots, c_n$. The classic Borel subalgebras in $\mathrm{Vect}_{\mathbf{C}}$ correspond to the points of $\Omega_{0,1}$.

1.7. The Siegel-Kirillov domain $K = \Omega_{0,1}$. Let S be the space of functions that are univalent in D_+ up to the boundary and have a Taylor series of the form $z + c_1 z^2 + \cdots$. Let $f \in S$. Then the triple $X_f = (\overline{\mathbf{C}}, \infty, f)$ is the element $K = \Omega_{0,1}$. It is not difficult to check that the function $f \to X_f$ is a one-to-one map $S \to K$, and in addition the action of Diff on K, considered in [4] and [5], is translated by this map into an action of $\mathrm{Diff} \subset \overline{\Gamma}$ on K. Obviously K is a Diff-homogeneous space, and moreover the stabilizer of a point is isomorphic to the groups of rotations of the circle.

REMARK. The space $\Omega_{0,0}$ is also Diff-homogeneous, and $\Omega_{0,0} = \mathrm{Diff}/\mathrm{PSL}_2(\mathbf{R})$ (see also [4] and [5]). The other spaces $\Omega_{g,n}$ are no longer Diff-homogeneous, and the Diff-orbits are not even invariant under $\overline{\Gamma}$.

1.8. The universal covering $\widetilde{\Gamma}$ over Γ and the central extension of Γ. We define an element of the semigroup $\widetilde{\Gamma}$ to be a quadruple (R, θ, p, q), where R is a Riemann surface which is a biholomorphically equivalent to the disc, θ is a hyperbolic automorphism of R (it is convenient to consider the stationary points of θ as "punctures" in the boundary of R), and p and q are maps of $\mathbf{R}$ into the boundary of R with $p(x + 2\pi) = \theta p(x)$ and $q(x + 2\pi) = \theta q(x)$, and where the surface remains on the left when going round the curve $p(x)$ and on the right when going round $q(x)$.

REMARK. It is convenient to consider R to be the strip $c \le \mathrm{Im}(z) \le 0$, the curves $p(x)$ and $q(x)$ to be the straight lines $\mathrm{Im}(z) = c$ and $\mathrm{Im}(z) = 0$, and θ to be translation by 2π.

Quadruples (R, θ, p, q) and (R', θ', p', q') are equivalent if there exists a biholomorphic map $L: R \to R'$ such that $\theta' \circ L = L \circ \theta$, $p' = L \circ p$, and $q' = L \circ q$. We define the product of two quadruples $\gamma_1 = (R_1, \theta_1, p_1, q_1)$ and $\gamma_2 = (R_2, \theta_2, p_2, q_2)$ to be the quadruple $\gamma_1\gamma_2 = (R_3, \theta_3, p_3, q_3)$, where R_3 is the Riemann surface obtained from the disjoint union of R_1 and R_2 by identifying the pairs of points $q_1(x) \in R_1$ and $q_2(x) \in R_2$ for all $x \in \mathbf{R}$, and the automorphism θ_3 coincides with θ_1 on R_1 and with θ_2 on R_2, and $p_3 = p_1$, $q_3 = q_2$.

Let us define the covering map $\tau: \widetilde{\Gamma} \to \Gamma$. Suppose that $(R, \theta, p, q) \in \widetilde{\Gamma}$; then it determines the triple $(K, \pi, \kappa) \in \Gamma$ (Γ is realized as in §1.2), where $K = R/\theta$ and the diffeomorphisms π and κ are covered by the diffeomorphisms p and q.

Finally, the theorem proved in §4 implies that the semigroup Γ has a nontrivial central extension; in other words, there exists a "holomorphic" (see §1.11) function $c: \Gamma \times \Gamma \to \mathbf{C}$ such that

$$(\gamma_i, u_1)(\gamma_2, u_2) = (\gamma_1\gamma_2, u_1 + u_2 + c(\gamma_1, \gamma_2))$$

is an associative multiplication in $\Gamma + \mathbf{C}$. There are explicit formulas (see §§4.9 and 4.10).

1.9. Involution. Let $\gamma = p \cdot A(t) \cdot q \in \Gamma$ (see §1.4). Then we define γ^* as $q^{-1} \cdot A(t) \cdot p^{-1}$. If the definition of §1.3 is used, then

$$(S, p(z), q(z))^* = (S, q(z^{-1}), p(z^{-1})).$$

It is easy to see that $\gamma \to \gamma^*$ is an antiholomorphic involution: $(\gamma_1\gamma_2)^* = \gamma_2^*\gamma_1^*$.

1.10. Does the group $\mathrm{Diff}_{\mathbf{C}}$ **exist?** No satisfactory definition of an infinite-dimensional Lie group is known (at least, it seems that none of the groups mentioned in this paper satisfies any of the existing definitions). Therefore the question of whether the group $\mathrm{Diff}_{\mathbf{C}}$—a complex envelope of the group Diff—exists is heuristic. There are arguments, given below, "against" its existence which seem convincing to the author.

Let $G_n \subset \mathrm{Diff}$ be a group of all transformations of S^1 of the form

$$z \to \sqrt[n]{\frac{az^n + b}{\overline{b}z^n + \overline{a}}},$$

where $|a|^2 - |b^2| = 1$. Obviously G_n is an n-sheeted covering of the group $\mathrm{PSL}_2(R)$; when $n > 2$ this three-dimensional group has no complexification. Hence, $\mathrm{Diff} \supset G_n$ does not have one either. This inference is still not completely convincing, since G_n may be embedded in the group U of unitary operators on Hilbert space and the group GL of all invertible bounded operators may be considered as the complexification of U by "stretching a point". The "point to be stretched" consists of the following: if elements of the Lie algebra of a Lie group are considered as generators of the one-parameter subgroups, then the "Lie algebra" $\mathfrak{u}$ of the real group U consists of all skew-Hermitian (generally speaking, unbounded) operators, and the Lie algebra $\mathfrak{gl}$ of the complex group GL consists of only the bounded operators; that is, $\mathfrak{gl} \neq \mathfrak{u}_{\mathbf{C}}$. Nevertheless the existence of an embedding of the group G_n into Diff implies that the following facts:

1. $\mathrm{Diff}_{\mathbf{C}}$ does not contain Γ.
2. $\mathrm{Diff}_{\mathbf{C}}$ does not contain $\mathrm{LDiff}_{\mathbf{C}}$.
3. For any $n > 2$ at least one of the vector fields $z^{n+1}\partial/\partial z$ and $z^{-n+1}\partial/\partial z$ is not a generator of a one-parameter subgroup in $\mathrm{Diff}_{\mathbf{C}}$.

This list of requirements appears to be rather pathological.

1.11. Diffeology. A precise meaning can be given to the words "Γ is an infinite-dimensional Lie semigroup" in the following way. We shall say that a family $\gamma(u)$ of elements of Γ *depends holomorphically on* $u \in \Omega$, where Ω is a domain in $\mathbf{C}^n$ if in a small neighborhood of any point $u_0 \in \Omega$ the family $\gamma(u)$ has the form $(\overline{\mathbf{C}}, p_u, q_u)$, where p_u and q_u are functions which depend holomorphically on u.

PROPOSITION. *If* $\gamma_1(u)$ *and* $\gamma_2(u)$ *are families which depend holomorphically on* u, *then* $\gamma_1(u)\gamma_2(u)$ *depends holomorphically on* u.

This obvious assertion means, according to Souriau [22], that a holomorphic diffeology is introduced on Γ. We know which of the functions from $\mathbf{C}^n$ into Γ are holomorphic, and consequently we know which of the functions from Γ into $\mathbf{C}^m$ are holomorphic. Namely, a function $f: \Gamma \to \mathbf{C}^m$ is holomorphic if, for any holomorphic family $\gamma(u) f(\gamma(u))$ is holomorphic.

Diffeology, in the case of the semigroup Γ, turns out to be an extremely flexible tool. We do not use it below, but we note that, if one does use it, Theorem 2 of §4.1 can be proved by the following method: we prove that a representation of Diff can be extended to $\mathrm{L}\Gamma$, where, by holomorphicity arguments, we can show that (4.1) does in fact give the representation. Technically this is, however, considerably more complicated than the proof we give in §4.

1.12. Tangent spaces. The definition of the tangent space T_γ to Γ at the point γ is completely obvious. As for Lie groups, the map $\gamma \to \mu\gamma$ induces a linear operator $A: T_\gamma \to T_{\mu\gamma}$. However, in contrast to Lie groups, the operator A is not bijective (it is injective but not surjective).

1.13. The tangent cone and causal paths. We define the tangent cone $\overline{C}$ (C, respectively) to the semigroup $\overline{\Gamma}$ at the identity to be the set of all vector fields on S^1 of the form $a(\varphi)\partial/\partial\varphi$, where $\operatorname{Im} a(\varphi) \geq 0$ (> 0 respectively). In other words $\overline{C}$ consists of the fields directed "into the disc". We note that every element of the cone C generates a one-parameter real semigroup in Γ.

We say an analytic path $\gamma(t) \in \Gamma$ is *causal* if for every $t > s$ there exists an element $\mu(t, s) \in \Gamma$ such that $\gamma(t) = \mu(t, s)\gamma(s)$. For any causal path the derivative is defined as an element of the cone $\overline{C}$, namely $\gamma'(s) = d\mu(s+\varepsilon, \varepsilon)/d\varepsilon$. We note that by §1.12 the concept of the derivative as an element of $\mathrm{Vect}_{\mathbf{C}}$ does not exist for arbitrary paths in Γ. Conversely, let $\alpha(t)$ be an analytic trajectory in C, $t \in [t_0, t_1]$, and let $\gamma_0 \in \Gamma$. Then there exists a unique causal path $\gamma(t)$ such that $\gamma(t_0) = \gamma_0$ and $\gamma'(t) = \alpha(t)$; see [15] for tangent cones.

1.14. Borel semigroups in Γ. Let $\mathfrak{b} \subset \mathrm{Vect}_{\mathbf{C}}$ be the algebra of vector fields on S^1 that can be extended into D_+. The intersection of $\mathfrak{b}$ with the tangent cone C is nonempty, so there exists a subsemigroup B in Γ corresponding to $\mathfrak{b}$.

We shall say that $\gamma = (S_\gamma, p, q) \in \Gamma$ is an element of B if the map $p \circ q^{-1}$, defined on the contour $q(e^{i\varphi})$, extends to a single-valued map π from $S_\gamma \setminus q(D_-^0)$ into $p(D_+)$. As a semigroup, B is isomorphic to the semigroup of biholomorphic maps of the circle D_+ into itself.

Similarly we define a semigroup B_0, connected with the subalgebra $\mathfrak{b}_0 \subset \mathfrak{b}$, to consist of those fields that are 0 at the point $0 \in D_+$. It is then necessary to impose on π the additional condition $\pi(p(0)) = p(0)$. The semigroup B_0 is isomorphic to the semigroup of biholomorphic maps of D_+ into itself that preserve the point 0.

1.15. The category Shtan. An object of the category is a nonnegative integer. A morphism from m into n is a collection (R, r_i^+, r_j^-), $1 \le i \le m$, $1 \le j \le n$, where R is a compact closed (maybe disconnected) Riemann surface, r_i^+, $D_+ \to R$ and $r_j^- : D_- \to R$ are holomorphic functions which are single-valued up to the boundary, and the domains $r_\alpha^\pm(D_+)$ do not intersect pairwise. Two morphisms (R, r_i^+, r_j^-) and (P, p_i^+, p_j^-) are considered to be the same if there exists a biholomorphic map $\pi: R \to P$ such that $\pi \circ r_i^+ = p_i^+$ and $\pi \circ r_i^- = p_i^-$.

Now let $(P, p_i^+, p_j^-) \in \mathrm{Mor}(m, n)$ and $(Q, q_j^+, q_\alpha^-) \in \mathrm{Mor}(n, k)$. Then their product $(R, r_i^+, r_\alpha^-) \in \mathrm{Mor}(m, k)$ is defined in the following way. The surface R is obtained from the disjoint union of $P \setminus \bigcup p_j^-(D_-^0)$ and $Q \setminus \bigcup q_j^+(D_+^0)$ by identifying the points $p_j^-(e^{i\varphi})$ and $q_j^+(e^{i\varphi})$, where $j = 1, \dots, n$ and $\varphi \in [0, 2\pi]$. Further $r_i^+ = p_i^+$ and $r_\alpha^- = q_\alpha^-$.

The author thanks M. L. Kontsevich for discussing this category.

1.16. The category Shtan$^\sim$. The objects of the category are the same as for Shtan. The morphism from m into n is the collection (P, p_i^-, p_j^-, π), where (P, p_i^+, p_j^-) is a morphism of Shtan and π is a maximal isotropic lattice in the first integral homology of the Riemann surface $P \setminus \bigcup p_\alpha^\pm(D_\pm^0)$ (recall that there is a skew-symmetric bilinear form on the first integral homology of a Riemann surface—the intersection index). The product of $(P, p_i^+, p_j^-, \pi) \in \mathrm{Mor}(m, n)$ and $(Q, q_j^+, q_\alpha^-, \kappa) \in \mathrm{Mor}(n, k)$ is defined as the quadruple $(R, r_i^+, r_\alpha^-, \rho)$, where (R, r_i^+, r_α^-) is the product of (P, p_i^+, p_j^-) and (Q, q_j^+, q_α^-) in the sense of Shtan, and the lattice ρ is the sum of the lattices π and κ.

REMARK. In the paper by Krichever and Novikov [7], which may be interpreted in terms of the semigroup $G\Gamma$, the maximal isotropy lattice in the first homology occurs implicitly as a one-parameter family of contours on the Riemann surface.

§2. The symplectic semigroup

This section contains a summary of the results of Ol'shanskii, Nazarov, and the author about a class of operators in the Fock boson space that are indispensable later on. A large part of the assertions may be checked by direct, although cumbersome calculations. The exception is the conditions for the operators to be bounded; they are useful in understanding what follows but they are not used formally (see also [2] and [15]).

2.1. Motivation (see [15]). The Weyl representation of the group $\mathrm{Sp}(2n, \mathbf{R})$ may be extended by holomorphicity to the representation of a certain open semigroup $G \subset \mathrm{Sp}(2n, \mathbf{C})$. The domain G is biholomorphically equivalent to a domain of the form $\widetilde{G} \setminus A$, where $\widetilde{G}$ is a classical complex symmetric domain of the third type, and A is a complex submanifold of codimension 1 in $\widetilde{G}$. It turns out that, like the multiplication in G, the Weyl representation of the semigroup G extends by continuity onto the whole domain $\widetilde{G}$. Moreover the points of the manifold A correspond not to operators from $\mathbf{C}^{2n}$ into $\mathbf{C}^{2n}$ but to points of the Lagrange Grassmannian in $\mathbf{C}^{2n} \oplus \mathbf{C}^{2n}$; that is (see §1.5), to linear relations. We are interested in the infinite-dimensional analogue of this situation (we note that in the finite-dimensional case the elements of the Lagrange Grassmannian are, as a rule, graphs of operators, but in the infinite-dimensional case this is not so).

2.2. The Ol′shanskiĭ symplectic semigroup $\Gamma\mathrm{Sp}$. Let $V_{\mathbf{R}}$ be a complex Hilbert space with inner product $(\cdot\,,\cdot)$, which we shall consider as a real Hilbert space with a complex structure operator I: $I^2=-1$. Let $V=V_{\mathbf{R}}\oplus iV_{\mathbf{R}}$ be the complexification of the space $V_{\mathbf{R}}$. If we extend the form $\mathrm{Re}(\cdot\,,\cdot)$ to V by sesquilinearity, we get a Hilbert space structure on V. If we extend the form $\mathrm{Im}(\cdot\,,\cdot)$ by bilinearity to a form $\{\cdot\,,\cdot\}$ on V, we get the structure of a symplectic space on V. Finally, if we extend the form $\mathrm{Im}(\cdot\,,\cdot)$ to V by sesquilinearity, we get an indefinite Hermitian form $\Lambda(\cdot\,,\cdot)$ on V. Further, let $V_{\pm}=\mathrm{Ker}(i\pm I)$. Then $V_{\pm}$ are maximal subspaces in V which are isotropic (Lagrange) with respect to the form $\{\cdot\,,\cdot\}$, and where $V=V_+\oplus V_-$. In addition, the operation of taking conjugates is defined on V in the same way as on the complexification of $V_{\mathbf{R}}$. Obviously $V_+=\overline{V}_-$. Finally we note that the concept of the transpose of an operator $A^t v=\overline{A^*\overline{v}}$ is defined on V. If $v\in V$, we define v^t so that $\langle v\,,w\rangle=\Lambda(v^t\,,w)$ for any $w\in V$.

Let W be a Hilbert space canonically identified with V. We introduce on $V\oplus W$ the symplectic form

$$\{(v_1\,,w_1)\,,(v_2\,,w_2)\}'=\{v_1\,,v_2\}-\{w_1\,,w_2\}. \tag{2.1}$$

We shall say that a Lagrange subspace P in $V\oplus W$ is *correct* if it is the graph of an operator $\Omega_P: V_+\oplus V_-\to V_-\oplus W_+$ where the matrix

$$\Omega_P=\begin{pmatrix} K & L\\ L^t & M\end{pmatrix}$$

has the following properties:

1°. $K=K^t$ and $M=M^t$.

2°. $\|\Omega_P\|\le 1$.

3°. $\|K\|<1$ and $\|M\|<1$.

4°. K and M are Hilbert-Schmidt operators.

We shall say that the matrix Ω_P is associated with the relation P.

REMARK 1. Condition 1° is equivalent to the fact that the subspace P is Lagrange. Condition 2° means that the form

$$\Lambda'((v_1\,,w_1)\,,(v_2\,,w_2))=\Lambda(v_1\,,v_2)-\Lambda(w_1\,,w_2),$$

is defined on the space $V\oplus W$ and is nonnegative on P. Finally, condition 3° means that the form Λ is positive definite on $P\cap V$ and $P\cap W$.

REMARK 2. If the matrix L is invertible, then P is the graph of a symplectic operator $R_P: V\to W$, where $\Lambda(R_Pv\,,R_Pw)\le\Lambda(v\,,w)$. Let $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)$ be the matrix of the operator $R_P: V_+\oplus V_-\to W_+\oplus W_-$. Then

$$\Omega_P=\begin{pmatrix} CA^{-1} & A^{t-1}\\ A^{-1} & -A^{-1}B\end{pmatrix}.$$

This is a fairly standard way to writing such an operator, and it is sometimes called the Potapov-Ginsburg transformation (see [1]).

We now note that subspaces in $V\oplus V$ may be multiplied as linear relations. It turns out that the set $\Gamma\mathrm{Sp}$ of all correct linear relations forms a semigroup. On the level of the associated matrix the product is given by

$$\begin{pmatrix} K & L\\ L^t & M\end{pmatrix}*\begin{pmatrix} P & Q\\ Q^t & R\end{pmatrix}=\begin{pmatrix} K+LP(1-MP)^{-1}L^t & L(1-PM)^{-1}Q\\ Q^t(1-MP)^{-1}L^t & R+Q^t(1-MP)^{-1}MQ\end{pmatrix}. \tag{2.2}$$

2.3. The groups Sp **and** SpH. We define a group Sp as a group of real-linear operators in $V_{\mathbf{R}}$ that can be represented as $U(1+T)$, where U is a unitary complex-linear operator and T is a Hilbert-Schmidt operator. The group SpH consists of the affine transformations of $V_{\mathbf{R}}$ with linear part in Sp.

The following conditions on an element $P \in \Gamma\mathrm{Sp}$ are equivalent:

1. P is the graph of an operator in Sp.
2. The subspace P is isotropic with respect to the form $\Lambda(\cdot\,,\,\cdot)$.
3. The matrix Ω_P is unitary.

Thus the group Sp forms something like the "skeleton" or the "Shilov boundary" of the domain ΓSp.

2.4. The semigroup ΓSpH. We let ΓSpH denote the set of all linear manifolds in $V \oplus W$ represented in the form $Q = h + P$, where $h \in V \oplus W$ and $P \in \Gamma\mathrm{Sp}$. Clearly one may suppose $h \in V_- \oplus W_+$. Then we shall say that the relation Q is associated with the matrix $\Lambda_Q = [\Omega_P | h^t]$. A product in ΓSpH is defined as the composition of relations, and on the level of the associated matrices it is given by

$$\begin{aligned}&\left[\begin{array}{cc|c} K & L & \lambda^t \\ L^t & M & \mu^t \end{array}\right] \circ \left[\begin{array}{cc|c} P & Q & \pi^t \\ Q^t & R & \kappa^t \end{array}\right] \\ &\qquad = \left[\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} * \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix} \begin{array}{c} \lambda + L(1-PM)^{-1}(\pi + P\mu) \\ \kappa + Q^t(1-MP)^{-1}(M\pi + \mu) \end{array}\right]. \end{aligned} \tag{2.3}$$

The group SpH is obvious embedded in ΓSpH.

2.5. The Segal-Bargmann-Berezin model of the Fock boson space. Let V be the same as in §2.5. Let F' be the space of polynomials on V_+ (that is, polynomials in the linear forms on V_+) with the inner product

$$\langle f(z)\,,\, g(z)\rangle_F = \iint f(z)\overline{g(z)}\exp(-\langle z\,,\, z\rangle)\, dz\, d\overline{z}.$$

The Fock boson space $F(V_+)$ is the completion of F' with respect to the norm defined by this inner product. It consists of holomorphic functions on V_+. The function $f(z) = 1$ is called the *vacuum vector* (for details of the definition see [2]).

Let e_j be an orthonormal basis in V_+, and let

$$v = \sum v_j^+ e_j + \sum v_j^- \overline{e}_j \in V = V_+ \oplus V_-.$$

Then the creation-annihilation operator $\widehat{A}(v)$ is given in $F(V_+)$ by

$$\widehat{A}(v) f(z) = \left(\sum v_j^+ z_j + \sum v_j^- \frac{\partial}{\partial z_j}\right) f(z).$$

We note that the definition of $\widehat{A}(v)$ does not depend on the choice of basis in V_+.

2.6. The vectors $b[M|l^t]$. Let M be a symmetric ($M = M^t$) Hilbert-Schmidt operator from V_+ into V_-. Let V_+ be realized as the space l_2 whose elements are considered to be row-matrices of dimension $\infty \times 1$. We define the vector

$$b[M|l^t] = \exp\{\tfrac{1}{2} zMz^t + zl^t\} \in F(V_+).$$

REMARK. If it is not convenient for us to suppose that $V_+ = l_2$, then the notation zMz^t should be taken to mean $\{z\,,\, Mz\}$, where $z \in V_+$.

The set of all vectors of the form $b[M|l^t]$ is denoted by FZ. We let F_0 denote the (finite) linear space spanned by all products of the type qb, where $q \in F'$ and $b \in FZ$.

2.7. The operators $B[\Omega|m^t]$. Again let $V_+ = l_2$. Let $T \in \Gamma\mathrm{SpH}$, and let Λ_T be the associated matrix. The operators $B(T) = B[\Lambda_T]$ are defined by

$$B\left[\begin{matrix} K & L \\ L^t & M \end{matrix}\middle|\begin{matrix} \lambda^t \\ \mu^t \end{matrix}\right] f(z) = \iint \exp\{\tfrac{1}{2}(zKz^t + 2zL\overline{u}^t + \overline{u}M\overline{u}^t) + z\lambda^t + \overline{u}\mu^t\} \times \exp(-\langle u, u\rangle) f(u)\, du\, d\overline{u}.$$

It can be shown that for any $T \in \Gamma\mathrm{SpH}$ the operator $B(T)$ takes the subspace $F_0 \subset F(H)$ into itself and extends to a closed, generally speaking unbounded, operator in $F(V_+)$.

2.8. The Weyl representation of the semigroup $\Gamma\mathrm{SpH}$.

THEOREM 1. a) *The map $T \to B(T)$ is a projective representation of the semigroup* $\Gamma\mathrm{SpH}$.

b) *Let $\Lambda_T = (\Omega_T|l^t)$. If $\|\Omega_T\| < 1$, then $B(T)$ is bounded.*

c) *If T is the graph of a transformation in* SpH, *then $B(T)$ is unitary (up to multiplication by a constant).*

d) *The operators $B(T)$ translate the cone $\mathbf{C}\cdot FZ$ into themselves.*

We need some explicit formulas:

$$B\left[\begin{matrix} K & L \\ L^t & M \end{matrix}\middle|\begin{matrix} \lambda^t \\ \mu^t \end{matrix}\right] b[P|\pi^t] = c(M, P, \mu, \pi) b[K + LP(1-MP)^{-1}L^t | \lambda^t + L(1-PM)^{-1}(\pi^t + P\mu^t)],$$

$$B\left[\begin{matrix} K & L \\ L^t & M \end{matrix}\middle|\begin{matrix} \lambda^t \\ \mu^t \end{matrix}\right] B\left[\begin{matrix} P & Q \\ Q^t & R \end{matrix}\middle|\begin{matrix} \pi^t \\ \kappa^t \end{matrix}\right] = c(M, P, \mu, \pi) B\left[\left(\begin{matrix} K & L \\ L^t & M \end{matrix}\middle|\begin{matrix} \lambda^t \\ \mu^t \end{matrix}\right) \circ \left(\begin{matrix} P & Q \\ Q^t & R \end{matrix}\middle|\begin{matrix} \pi^t \\ \kappa^t \end{matrix}\right)\right], \tag{2.4}$$

where in both cases the constant $c(M, P, \mu, \pi)$ is equal to

$$c(M, P, \mu, \pi) = \det[(1 - MP)^{-1/2}] \exp\left\{\frac{1}{2}(\pi\mu)\begin{pmatrix} -P & 1 \\ 1 & -M \end{pmatrix}^{-1}\begin{pmatrix} \pi^t \\ \mu^t \end{pmatrix}\right\}.$$

REMARK. Theorem 1 is a generalization of the classical Friedrichs-Segal-Shail-A. Weil-Berezin theorem about automorphisms of canonical commutation relations (see [2], §4). In fact, let $v_1, v_2 \in V$ be connected by the linear relation $T \in \Gamma\mathrm{Sp}$. It is not difficult to check that

$$\widehat{A}(v_1)B(T) = B(T)\widehat{A}(v_2)$$

(in the case $T \in \Gamma\mathrm{SpH}$ a similar equation holds, only a term of the form $cB(T)$ is added to the right-hand side).

2.9. The bounded symmetric domain $\mathscr{Z}$ consists of all the symmetric Hilbert-Schmidt operators $V_+ \to V_-$ with norm less than 1. The map $M \to b[M|0]$ gives an embedding of $\mathscr{Z}$ in $F(V_+)$. The action of the semigroup $\Gamma\mathrm{Sp}$ on $\mathbf{C}\cdot FZ$ (see Theorem 1d) and (2.3)) thus extends the action of $\Gamma\mathrm{Sp}$ to $\mathscr{Z}$:

$$\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} : Z \to K + LZ(1 - MZ)^{-1}L^t,$$

where $z \in \mathscr{Z}$.

Now let $f \in F(V_+)$. We make it correspond to a holomorphic function $\mathscr{L}(f)$ on $\mathscr{Z}$ constructed according to the formula

$$\mathscr{L}(f) = \langle f, b[\overline{M}|0]\rangle_F .$$

Then, obviously, the operator $\mathscr{L}(f)$ links the Weil representation of the semigroup $\Gamma\mathrm{Sp}$ with the representation $T \to R(T)$ in the space of holomorphic functions on $\mathscr{Z}$, where $R(T)$ is given by

$$R\begin{bmatrix} K & L \\ L^t & M \end{bmatrix} g(Z) = g(M + L^t Z(1-KZ)^{-1}L)\det[(1-KZ)^{-1/2}].$$

We note that the function $\mathscr{L}(f)$ is not arbitrary but must satisfy

$$(\partial_{ij}\partial_{kl} - \partial_{ik}\partial_{jl})\mathscr{L}(f) = 0,$$

where $\partial_{ii} = \frac{1}{2}\partial/\partial z_{ii}$ and $\partial_{ij} = \partial/\partial z_{ij}$ when $i \neq j$ (the z_{ij} denote elements of the matrix of Z).

2.10. The domain $\mathscr{Z}H$ consists of the pairs (M, l^t), where $M \in \mathscr{Z}$ and $l^t \in V_+$. The map $(M, l) \to b[M|l^t]$ gives an embedding of $\mathscr{Z}H$ into the cone $\mathbf{C}\cdot FZ$. The construction, repeated word for word from 2.9, shows that the Weil representation of $\Gamma\mathrm{SpH}$ may be realized as holomorphic functions on $\mathscr{Z}H$. It is easy to write explicit formulas with the help of (2.3) and (2.4).

§3. The Virasoro algebra

This section contains a summary of the results about representations of the Virasoro algebra that are indispensable below. For more details see, for example, [14] and [20].

3.1. The Virasoro algebra. The Virasoro algebra $\mathscr{L}$ is a Lie algebra with basis η and L_n, $n \in Z$, and with the commutation relations

$$[L_n, L_m] = (m-n)L_{m+n} + \tfrac{1}{12}(n^3 - n)\delta_{n-m}\eta, \qquad [L_n, \eta] = 0,$$

where $\delta_{k,l} = 0$ when $k \neq l$, and $\delta_{k,k} = 1$. Obviously the map

$$\alpha\eta + \sum p_k L_k \to \left(\sum p_k e^{ik\varphi}\right)\partial/i\partial\varphi$$

gives a homomorphism from $\mathscr{L}$ into $\mathrm{Vect}_{\mathbf{C}}$, and its kernel is $\mathbf{C}\eta$. It is convenient to say, allowing a little imprecision, that $\mathscr{L}$ is a central extension of $\mathrm{Vect}_{\mathbf{C}}$.

3.2. The modules $M(h, c)$ **and** $L(h, c)$. A vector v in the $\mathscr{L}$-module V is called a *singular vector of weight* (h, c) if $L_n v = 0$ when $n < 0$ and $L_0 v = hv$, $\eta v = cv$.

Let $\mathscr{O}_{h,c}$ be the set of all $\mathscr{L}$-modules having a nonzero cyclic singular vector of weight (h, c). It is easy to see that there exists a universal (repelling) Verma module $M(h, c)$ such that any module $V \in \mathscr{O}_{h,c}$ is a quotient module of $M(h, c)$. There also exists a universal attracting module $L(h, c)$ which is a quotient module of any module $V \in \mathscr{O}_{h,c}$. It is easy to see that $L(h, c)$ is the quotient module of $M(h, c)$ modulo the unique maximal submodule. If (h, c) is a generic point, then $M(h, c) = L(h, c)$ (we shall call such modules nondegenerate). For details of the definition and the Kats-Feĭgin-Fuks reducibility condition for $M(h, c)$ see, for example, [14].

We note that, for any $n \geq 0$, the dimension of the weighted L_0-space with eigenvalue $n + h$ is equal to $p(n)$, where $p(n)$ is the number of partitionings of n into a sum of positive integers. Below we shall meet two other families of $\mathscr{L}$-modules, $N(\alpha, \beta)$ and $K(h, c)$ having this property.

3.3. Condition for the modules $L(h, c)$ **to be unitarizable.** We say that the module V over $\mathscr{L}$ is *unitarizable* if there exists a positive definite Hermitian form $\langle \cdot, \cdot \rangle$ on V such that $\langle L_n v, w\rangle = \langle v, L_{-n} w\rangle$ for any $v, w \in V$.

The module $L(h, c)$ is unitarizable if and only if one of the following conditions is satisfied:

1. $h \geq 0$ and $c \geq 1$.
2. $c = 1 - 6/p(p+1)$ and $h = [(\alpha p - \beta(p+1))^2 - 1]/4p(p+1)$, where $p, \alpha, \beta \in \mathbf{Z}$, $p \geq 2$, $1 \leq \alpha \leq p$, and $1 \leq \beta \leq p-1$.

Necessity was proved independently in [11] and [18], while sufficiency was proved in [17] (see also [14], §§5 and 7).

Any representation of $L(h, c)$ can be integrated to give a projective representation of the group Diff [12], and any unitarizable representation $L(h, c)$ integrates to give a unitary projective representation of Diff [19] (a weaker result was obtained in [10]; see also [14], §§5 and 9).

3.4. The embedding of Diff **in** Sp. Let H be a space of smooth real functions on the circle $S^1 : z = e^{i\varphi}$, with nonnegative quadratic form

$$(f, g) = \frac{1}{\pi^2} \int_0^{2\pi} \text{p.v.} \int_0^{2\pi} \cot\left(\frac{\varphi - \psi}{2}\right) f(\varphi) g(\psi') \, d\varphi \, d\psi.$$

We factor out H by the subspace of constants and we take the completion with respect to this inner product. We introduce a complex structure in the space $V_{\mathbf{R}}$ thus obtained with the help of the operator I; this is the Hilbert transform

$$If(\varphi) = \frac{1}{\pi} \int_0^{2\pi} \cot\left(\frac{\varphi - \psi}{2}\right) f(\psi) \, d\psi.$$

It is easy to see that $I^2 = -1$. The inner product in $V_{\mathbf{R}}$ is defined by

$$\langle f, g\rangle = (f, g) + i(f, Ig).$$

Further, we apply to $V_{\mathbf{R}}$ the procedure described in §2.2. Then V consists of functions on S^1 determined up to the addition of a constant. Furthermore, the orthogonal subspaces V_+ and V_- consist of functions which extend holomorphically into the interior and the exterior of the disc $D_+ : |z| \leq 1$ respectively. The inner product in $V_\pm$ is defined by:

$$\langle f(\varphi), g(\varphi)\rangle = \mp \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) \overline{g(\psi)'} \, d\varphi \, d\psi,$$

where $f, g \in V_\pm$. The symplectic form in V is equal to

$$\{f(z), g(z)\} = \frac{1}{2\pi i} \int_{|z|=1} f(z) \, dg(z).$$

Let $q \in \text{Diff}$. Then the operator $T(q) : f(\varphi) \to f(q(\varphi))$ belongs to the group Sp (see also [21], [9], and [14]).

3.5. The representations $N(\alpha, \beta)$. Let $\alpha, \beta \in \mathbf{R}$. We consider the affine transformation $T_{\alpha,\beta}(q)$ in V:

$$T_{\alpha,\beta}(q) f(\varphi) = f(q(\varphi)) + \alpha(q(\varphi) - \varphi) + \beta \ln q'(\varphi), \tag{3.1}$$

where $q \in \mathrm{Diff}$. It is not hard to verify that

$$T_{\alpha,\beta}(q \circ p) = T_{\alpha,\beta}(q) T_{\alpha,\beta}(p).$$

Thus we have realized Diff as a subgroup in SpH. If we restrict the Weyl representation of SpH to Diff we get a two-parameter family of unitary projective representations $N(\alpha, \beta)$ of the group Diff.

Let us describe the representations thus constructed at the level of the Lie algebra $\mathscr{L}$. We introduce into the space of Fock functions in the variables $z_1, z_2, \ldots$ the operators

$$a_k f = \sqrt{k} z_k f, \qquad a_{-k} f = \sqrt{k} \frac{\partial}{\partial z_k} f,$$

where $k \geq 0$. Then the generators of $\mathscr{L}$ act according to the formulas

$$L_n = \frac{1}{2} : \sum_j a_{n+j} a_{-j} : + (\alpha + i\beta n) a_n, \qquad n \neq 0,$$

$$L_0 = \sum_{j>0} a_j a_{-j} + \frac{1}{2}(\alpha^2 + \beta^2), \qquad \eta = 1 + 12\beta^2 \tag{3.2}$$

(the colon denotes the "normal order"; that is, the annihilation operators a_{-k} are satisfied before the creation operators a_k).

The vacuum vector, consequently, will be a singular vector of weight

$$h = \tfrac{1}{2}(\alpha^2 + \beta^2), \qquad c = 1 + 12\beta^2. \tag{3.3}$$

For real values of (α, β) the parameters (h, c) run over the domain $\{c \geq 1, h < 1 + 24c\}$. If $c = 1$ and $h = k^2/4$, these modules are isomorphic to $\bigoplus_{j \geq 0} L((k+2j)^2/4, 1)$. Otherwise they are irreducible and coincide with $L(h, c) = M(h, c)$.

Of course the formulas (3.2) make sense also for complex (α, β). As before, (3.1) gives an embedding of Diff into the group of invertible elements of the semigroup ΓSpH, and, also as before, we get a projective representation of Diff in $F(V_+)$. However, the operators giving the representation will here be unbounded. At generic points the representations $N(\alpha, \beta)$ are isomorphic to the corresponding $M(h, c)$ (see (3.3)). If the module $M(h, c)$ is reducible, then $M(h, c)$ and $N(\alpha, \beta)$ have similar Jordan-Hölder series.

§4. Holomorphic representations of the semigroup Γ

4.1. THEOREM 2. *Any unitary representation of the group* Diff *with highest weight extends to a representation of* Γ.

We shall prove this theorem in the following form.

THEOREM $2'$. *Let* ρ *be a unitary projective representation of* Diff *which is the sum of representations with highest weight. Let* $\gamma = p \cdot A(t) \cdot q$ *(see* §1.4*). Then the following formula gives a projective representation of the semigroup* Γ:

$$\hat{\rho}(p \cdot A(t) \cdot q) = \rho(p) \exp(-tL_0) \rho(q), \tag{4.1}$$

where the generator of the Virasoro algebra L_0 *is defined in* §3.1.

4.2. THEOREM $2''$. *Theorem* $2'$ *is true for* $\rho = N(0, 0)$.

We shall prove Theorem $2''$ in §§4.3 and 4.4 by giving the explicit construction of the operators of $\hat{\rho}$. Meanwhile we suppose that it is true and we shall prove Theorem $2'$ with the help of a slightly modified Goodman-Wallach triple [19].

PROOF OF THEOREM $2'$. The following lemmas are obvious:

LEMMA 4.1. *If Theorem $2'$ is true for the unitary representations ρ_1 and ρ_2, then it is also true for $\rho_1 \otimes \rho_2$.*

LEMMA 4.2. *Let ρ_1 and ρ_2 be unitary representations of* Diff *with highest weight. If Theorem $2'$ is true for $\rho_1 \otimes \rho_2$, then it is also true for ρ_1 and ρ_2.*

We now note that $N(0, 0)$ contains the subrepresentation $L(1, 1)$, and $L(1, 1)\otimes L(1, 1)$ contains the Verma module $L(2, 2)$ as a submodule. We note further that if $M(h_1, c_1)$ and $M(h_2, c_2)$ are unitarizable Verma modules, then

$$M(h_1, c_1) \otimes M(h_2, c_2) = \bigoplus_{n\geq 0} p(n) M(h_1 + h_2 + n, c_1 + c_2). \tag{4.2}$$

This immediately implies that Theorem $2'$ is true for the modules

$$A_n = \bigoplus_{j\geq 0} L(4n + j, 4n).$$

All that is left is to note that for any unitarizable $L(h_1, c_1)$ and $L(h_2, c_2)$ we have

$$L(h_1, c_1) \otimes L(h_2, c_2) = \sum_{k\geq 0} \lambda_k L(h_1 + h_2 + k, c_1 + c_2)$$

and to apply Lemma 4.2 and the conditions in §3.3 for a module to be unitarizable.

4.3. The embedding of $\overline{\Gamma}$ in $\Gamma\mathrm{Sp}$. Let $\gamma = (S_\gamma, p, q) \in \overline{\Gamma}$ (see §1.3). We say that the two functions $f, g \in V$ are connected by the relation $L_0(\gamma)$ if there exists a function $F(z)$ which is holomorphic up to the boundary on the annulus $S_\gamma \setminus (p(D_-^0) \cup q(D_+^0))$ and such that

$$F(p(e^{i\varphi})) = f(e^{i\varphi}), \qquad F(q(e^{i\varphi})) = g(e^{i\varphi}).$$

Let $L(\gamma)$ be the closure of $L_0(\gamma)$ in the space $V \oplus V$.

THEOREM 3. *The map $\gamma \to L(\gamma)$ is a homomorphism from $\overline{\Gamma}$ into $\Gamma\mathrm{Sp}$.*

PROOF. a) We prove that $L(\gamma) \in \Gamma\mathrm{Sp}$. Let $\gamma = \varphi \cdot A(t) \cdot \psi$ (see §1.4). Then, by §3.4, the linear relations $L(\varphi)$ and $L(\psi)$ lie in $\mathrm{Sp} \subset \Gamma\mathrm{Sp}$. The assertion $L(A(t)) \in \Gamma\mathrm{Sp}$ is verified by an easy computation.

b) We prove that $L(\gamma_1\gamma_2) = L(\gamma_1)L(\gamma_2)$. We have

$$L(\gamma_1)L(\gamma_2) \supset L(\gamma_1\gamma_2) \supset L_0(\gamma_1\gamma_2) = L_0(\gamma_1)L_0(\gamma_2).$$

We now note that the subspace $L(\gamma_1)L(\gamma_2)$ lies in $\Gamma\mathrm{Sp}$ and consequently is a closed Lagrange subspace. But the closure of $L_0(\gamma_1\gamma_2)$ that it contains is the Lagrange subspace $L(\gamma_1\gamma_2)$. Assertion b) is proved.

4.4. The semigroup extension of the representation $N(0, 0)$. If we embed $\overline{\Gamma}$ in $\Gamma\mathrm{Sp}$ (see §4.3) and take the restriction of the Weyl representation to $\overline{\Gamma}$, we get a projective representation $\gamma \to B(L(\gamma))$ of the semigroup $\overline{\Gamma}$ in the Fock space $F(V_+)$ which is unitary, up to multiplication by a constant, on $\mathrm{Diff} \subset \overline{\Gamma}$.

THEOREM 4. *The operators* $B(L(\gamma))$ *are bounded if* $\gamma \in \overline{\Gamma}$. *If* $\gamma \in \Gamma$, *these operators are trace class.*

PROOF. Let $\gamma = \varphi \cdot A(t) \cdot \psi$ (see §1.4). If we take into account the fact that the operators $B(L(\varphi))$ and $B(L(\psi))$ are unitary, up to multiplication by a constant, it is sufficient to verify our assertion for $B(L(A(t))) = \exp(-tL_0)$ (see 3.2). But L_0 is a selfadjoint operator with discrete spectrum $0, 1, 2, \dots$, where the eigenvalue n occurs with multiplicity $p(n)$.

4.5. Logarithmic forms. We shall say that a logarithmic form F of type λ is given on the Riemann surface R if, for any chart $D \subset R$, an expression $f(z) + \lambda \ln dz$ is given, where f is a function from D into the group $\mathbf{C}/2\pi i\lambda\mathbf{Z}$ (where $\mathbf{C}$ is the additive group of complex numbers) such that if we make the substitution $z = p(u)$ the expression $f(z) + \lambda \ln dz$ becomes $f(p(u)) + \lambda \ln p'(u) + \lambda \ln du$. Clearly for any nonzero 1-form G the logarithmic form $\ln G$ can be defined; namely, if G is $g(z)dz$ in the chart D, then $\ln G$ has the form $\ln g(z) + \ln dz$. The definition of the preimage of a logarithmic form under a map is defined in an obvious way.

4.6. The embedding of $\overline{\Gamma}$ in $\Gamma\mathrm{SpH}$ and the semigroup extensions of the representations $N(\alpha, \beta)$. Let $\alpha, \beta \in \mathbf{C}$ and let $\gamma = (S_\gamma, p, q) \in \Gamma$ (see §1.3). We shall say that the functions $f, g \in V$ are connected by a linear relation $L^0_{\alpha,\beta}(\gamma)$ if there exists a logarithmic form $\ln H$ of type β on $S_\gamma \setminus (p(D^0_-) \cup q(D^0_+))$ which is holomorphic up to the boundary, but may be multi-valued, such that

1) the preimage of $\ln H$ under the map $e^{i\varphi} \to p(e^{i\varphi})$ is $f(e^{i\varphi}) + \alpha\varphi + \beta \ln d\varphi$, and

2) the preimage of $\ln H$ under the map $e^{i\psi} \to q(e^{i\psi})$ is $g(e^{i\psi}) + \alpha\psi + \beta \ln d\psi$.

We let $L_{\alpha,\beta}(\gamma)$ denote the closure of $L^0_{\alpha,\beta}(\gamma)$ in $V \oplus V$.

THEOREM 3′. *The map* $\gamma \to L_{\alpha,\beta}(\gamma)$ *is a homomorphism from* $\overline{\Gamma}$ *into* $\Gamma\mathrm{SpH}$.

THEOREM 4′. a) *The map* $\gamma \to B(L_{\alpha,\beta}(\gamma))$ *gives a projective representation of* $\overline{\Gamma}$ *in* $F(V)$.

b) *If* $\gamma \in \mathrm{Diff}$, *then the operator* $B(L_{\alpha,\beta}(\gamma))$ *is trace class.*

c) *Suppose* $\gamma \in \mathrm{Diff} = \overline{\Gamma} \setminus \Gamma$. *If* $(\alpha, \beta) \in \mathbf{R}^2$, *then the operator* $B(L_{\alpha,\beta}(\gamma))$ *is unitary, to within multiplication by a constant.*

The proofs of these assertions are repeats of the proofs of Theorems 3 and 4, so we omit them.

REMARK. If $\gamma \in \mathrm{Diff}$ and $(\alpha, \beta) \notin \mathbf{R}^2$, then the operators $B(L_{\alpha,\beta}(\gamma))$ are, as a rule, not bounded.

4.7. Explicit formulas for the representations $N(\alpha, \beta)$. In order to obtain explicit formulas for the operators $B(L_{\alpha,\beta}(\gamma))$ it is sufficient to write $L_{\alpha,\beta}(\gamma)$ in the form of §2.4.

So let $\gamma = (\mathbf{C}, p, q) \in \overline{\Gamma}$, where $\overline{\mathbf{C}}$ is the Riemann sphere and $p : D_- \to \overline{\mathbf{C}}$ and $q : D_+ \to \overline{\mathbf{C}}$ are single-valued maps. Without loss of generality we may suppose that $p(\infty) = \infty$ and $q(0) = 0$. Then

$$B(L_{\alpha,\beta}(\gamma)) = B\left[\begin{array}{cc|c} K(p) & L(p, q) & \alpha l_1^t(p) + \beta m_1^t(p) \\ L^t(p, q) & M(q) & \alpha l_2^t(q) + \beta m_2^t(q) \end{array}\right], \tag{4.3}$$

where the matrix functions K, L, and M and the vector-valued functions l_1^t, l_2^t, m_1^t, and m_2^t are defined below.

Let $f \in V_+$. Then the function $f \circ p^{-1}$, defined on the contour $p(e^{i\varphi})$, can be represented in the form $(f \circ p^{-1})_+ + (f \circ p^{-1})_-$, where the first term is holomorphic in the domain $p(D_-)$ and the second in $\overline{\mathbf{C}} \setminus p(D_-^0)$. Then

$$Kf = (f \circ p^{-1})_+ \circ p, \qquad Lf = (f \circ p^{-1})_- \circ q; \tag{4.4}$$

where the first function is holomorphic in D_- and the second in D_+.

Let $g \in V_-$. Then the function $g \circ q^{-1}$, defined on the contour $q(e^{i\varphi})$, can be represented in the form $(g \circ q^{-1})_+ + (g \circ q^{-1})_-$, where the first term is holomorphic in $q(D_+)$ and the second in $\overline{\mathbf{C}} \setminus p(D_+^0)$. Then

$$Mg = (f \circ q^{-1})_+ \circ q, \qquad L^t g = (g \cdot q^{-1})_- \circ p; \tag{4.5}$$

the first function is holomorphic in D_+ and the second in D_-. Finally

$$\begin{aligned} l_1^t(p) &= \ln(p(z)/z), \qquad & l_2^t(q) &= \ln(q(z)/z), \\ m_1^t(p) &= \ln(p'(z)), \qquad & m_2^t(q) &= \ln(q'(z)). \end{aligned} \tag{4.6}$$

REMARK 1. The right-hand sides of (4.4)–(4.6) are only defined up to the addition of a constant. However, in view of the definition of the space V (see §3.4) this is not crucial for us.

REMARK 2. Let f be an analytic function on a closed contour K in $\mathbf{C}$. Let $f = f_+ + f_-$, where f_+ is holomorphic inside K and f_- outside K. Then, as is well known,

$$f_\pm(v) = \pm \int_K \frac{f(u)\,du}{u - v}.$$

4.8. The Grunsky matrix. As Yur′ev showed, the operator $K(p)$ (§4.7)

$$K(p)f(z) = \int_{|z|=1} \ln \frac{p(z) - p(u)}{z - u} f(u)\,du$$

is none other than the Grunsky matrix—a classical object of the geometric theory of functions (see, for example, [16]). This matrix is determined for any function $p(z)$ of the form

$$p(z) = z + \sum_{k>0} c_k z^{-k}$$

that is single-valued in the domain $|z| > 1$.

If $p(z)$ is single-valued up to the boundary, then by condition 3° in §2.2 we have $\|K(p)\| < 1$. For arbitrary $p(z)$, by arguments about weak convergence, we get that $\|K(p)\| \leq 1$. This assertion is called the Grunsky inequality (see [16] or [3], Chapter IV, §2). It is also known that $\|K(p)\| = 1$ if and only if the area of $\overline{\mathbf{C}} \setminus q(D_-^0)$ is 0, and in this case the matrix $K(p)$ is unitary (see [16]).

4.9. The canonical cocycles. Let $T(\gamma)$ be a projective representation of $\overline{\Gamma}$ with highest weight (h, c), and let v be the highest weight vector. It is obvious from (4.3) that $\langle T(\gamma)v, v\rangle$ is nowhere 0 on $\overline{\Gamma}$. Since the operators $T(\gamma)$ are defined only up to multiplication by a constant, we can choose this constant so that

$$\langle T(\gamma), v, v\rangle = 1. \tag{4.7}$$

The operators $T(\gamma)$ are defined uniquely by this equation. A canonical cocycle is a function $\kappa_{h,c}(\gamma_1, \gamma_2) : \overline{\Gamma} \times \overline{\Gamma} \to \mathbf{C}^*$ given by

$$T(\gamma_1)T(\gamma_2) = \kappa_{h,c}(\gamma_1, \gamma_2)T(\gamma_1\gamma_2).$$

Obviously,

$$\kappa_{h_1+h_2, c_1+c_2}(\gamma_1, \gamma_2) = \kappa_{h_1, c_1}(\gamma_1, \gamma_2)\kappa_{h_2, c_2}(\gamma_1, \gamma_2). \tag{4.8}$$

Since $\kappa_{h,c}$ is holomorphic in h and c, the function $\kappa_{h,c}$ has the form

$$\kappa_{h,c}(\gamma_1, \gamma_2) = \exp(h\lambda(\gamma_1, \gamma_2) + c\mu(\gamma_1, \gamma_2)). \tag{4.9}$$

4.10. Computing the canonical cocycles. Let $\gamma_1 = (\overline{\mathbf{C}}, p_1, q_1)$ and $\gamma_2 = (\mathbf{C}, p_2, q_2)$. We note that the operators $B(L_{\alpha,\beta}(\gamma))$ have the property (4.7) if the vacuum vector is taken as v. Therefore in view of (2.4) and (2.5) it is sufficient for us to find the constant

$$c_{\alpha,\beta}(q_1, q_2) = \det[(1 - MK)^{-1/2}] \times \exp\left\{\frac{1}{2}(\alpha l_1 + \beta m_1, \alpha l_2 + \beta m_2)\begin{pmatrix} -K & 1 \\ 1 & -M \end{pmatrix}^{-1} \begin{pmatrix} \alpha l_1^t + \beta m_1^t \\ \alpha l_2^t + \beta m_2^t \end{pmatrix}\right\}, \tag{4.10}$$

where, in the notation of §4.7, $M = M(q_1)$, $K = K(p_2)$, $l_1 = l_1(p_2)$, $l_2 = l_2(q_1)$, $m_1 = m_1(p_2)$, and $m_2 = m_2(q_1)$. If we take (4.9), (4.10), and (3.3) into account, we get two corollaries:

1. The coefficient of $\alpha\beta$ in the curly brackets in (4.10) is 0.
2. When $\alpha^2 = 1/12$ and $\beta^2 = -1/12$ (that is, $h = c = 0$; see (3.3)) the expression (4.10) is equal to 1. This enables us to express the Fredholm determinant $\det[(1 - MK)^{-1/2}]$ in terms of the second factor.

The computation for the coefficients of α^2 and β^2 in (4.10) is carried out similarly. We only give the arguments for α^2.

First we note that

$$\begin{pmatrix} -K & 1 \\ 1 & -M \end{pmatrix}^{-1} = \begin{pmatrix} M(1-KM)^{-1} & (1-MK)^{-1} \\ (1-KM)^{-1} & K(1-MK)^{-1} \end{pmatrix},$$

so we have to compute the expression

$$l_1[M(1-KM)^{-1}l_1^t + (1-MK)^{-1}l_2^t] + l_2\{(1-KM)^{-1}l_1^t + K(1-MK)^{-1}l_2^t\}. \tag{4.11}$$

We now use the fact that the function $\gamma_1 \to (l_1(p), l_2(q))$ is not arbitrary but is in its own way a "cocycle" on the semigroup $\overline{\Gamma}$. None of our expressions depend on p_1 and q_2; therefore we can choose p_1 and q_2 so that $\gamma_1, \gamma_2 \in \mathrm{Diff}$. Let $\tau = \gamma_1\gamma_2 = (\overline{\mathbf{C}}, p_3, q_3)$. Then it is obvious from (2.2) that the expression in curly brackets in (4.11) is none other than

$$L^{-1}(p_1, q_1)(l_1(p_3) - l_1(p_1)),$$

and the expression in the square brackets is

$$L^{t-1}(p_2, q_2)(l_2(q_3) - l_2(q_2)).$$

Furthermore, it is obvious from the formula for the Potapov-Ginzburg transformation (see §2.2) that $L^{-1}f = P_+(f \circ \gamma)$, where P_+ is the projection of the space V

onto V_+. Finally, if we take into account the fact that for a function g that is holomorphic outside the disc

$$\int_{|z|=1} g\,d(P_+f) = \int_{|z|=1} g\,df,$$

we get an expression for the coefficient of α^2 in (4.10):

$$\nu(q_1, p_2) = \frac{1}{2\pi i}\int_{|z|=1} \ln\frac{p_2(z)}{z}\,d\ln\frac{p_3(\gamma_1(z))}{p_1(\gamma_1(z))} + \frac{1}{2\pi i}\int_{|z|=1} \ln\frac{q_2(\gamma_2(z))}{\gamma_2(z)}\,d\ln\frac{q_3(z)}{q_1(z)}$$

and an expression for the coefficient of β^2:

$$\eta(q_1, p_2) = \frac{1}{2\pi i}\int_{|z|=1} \ln p_2'(z)\,d\ln\frac{p_3'(\gamma_1(z))}{p_1'(\gamma_1(z))} + \frac{1}{2\pi i}\int_{|z|=1} \ln q_2'(\gamma_2(z))\,d\ln\frac{q_3'(z)}{q_1'(z)}.$$

Finally in the formula (4.9) for the canonical cocycle we have

$$\lambda(\gamma_1, \gamma_2) = \lambda(q_1, p_2) = 2\nu(q_1, p_2), \tag{4.12}$$
$$\mu(\gamma_1, \gamma_2) = \mu(q_1, p_2) = \tfrac{1}{12}(\eta(q_1, p_2) - \nu(q_1, p_2)).$$

It is worth noting that our computations imply the following beautiful identity for Grunsky matrices:

$$\det[(1 - M(q_1)K(p_2))^{-1/2}] = \exp\{\mu(q_1, p_2)\}.$$

4.11. The equivariant embeddings $\Omega_{0,0}$ in $\mathcal{Z}$ and K in $\mathcal{Z}H$. We consider the representation $N(0, 0)$. Let v be the vacuum vector. Then it is easy to see that the following sets in $F(V_+)$ coincide:

1. the set of vectors of the form $B(L(\gamma))v$, where $\gamma \in \Gamma$;
2. the set of vectors of the form $B(L(\gamma))v$, where $\gamma \in \mathrm{Diff}$; and
3. the set of vectors of the form $b[K(p)|0]$, where $p(z)$ is single-valued up to the boundary.

Hence it is not hard to see that the map $p \to K(p)$ is an equivariant embedding of $\Omega_{0,0}$ in $\mathcal{Z}$ (the explicit formula for this embedding was first discovered by Yur′ev).

Similarly, it can be shown that the map $p \to (K(p)|\alpha l_1^t(p) + \beta m_1^t(p))$ is an equivariant embedding of K (see §1.7) into $\mathcal{Z}H$ (see §3.10).

If we apply the construction of §§2.9 and 2.10, we quickly get that any irreducible representation $M(h, c)$ of the semigroup $\overline{\Gamma}$ may be realized in the space of holomorphic functions on K by operators such as shift operators and multiplication by a function. In addition, any irreducible representation of the form $M(0, c)/M(1, c)$ may be realized in the space of holomorphic functions on $\Omega_{0,0}$.

The construction itself is completely obvious. It is remarkable, however, that it may be reduced to explicit formulas. All the computations necessary for this have already been done in §4.10, and in the following subsection we give the conclusive answer.

4.12. The representation $K(h, c)$. Let $x \in (\overline{\mathbf{C}}, \infty, p) \in K$, $\gamma = (\overline{\mathbf{C}}, r, q) \in \overline{\Gamma}$. Then the representations $K(h, c)$ of the semigroup $\overline{\Gamma}$ in the space of holomorphic functions on K are given by

$$T_{h,c}(\gamma)f(x) = f(\gamma x)\exp(h\lambda(p, r) + c\mu(p, r)),$$

where the functions λ and μ are determined by (4.12).

Now let the points of the space K be given as functions of the form $f(z) = z + c_1 z^2 + \cdots$ which are single-valued in D_+. Then the action of the vector fields $B_n = e^{in\varphi}\partial/i\partial\varphi$ on K, according to [5], is determined by $(p > 0)$

$$B_{-p} = \frac{\partial}{\partial c_p} + \sum_{k>0}(k+1)c_k\frac{\partial}{\partial c_{k+p}}, \qquad B_0 = \sum_{k\geq 1} kc_k\frac{\partial}{\partial c_k},$$

$$B_p = \sum_{k\geq 1}(k+p+1)c_{k+p} - \frac{1}{(2\pi i)^2}\int_{|z|=1-\varepsilon}\int_{|h|=\delta<1/4}\frac{z^{-k-1}f^2(z)f'(h)^2\,dh\,dz}{h^{p-1}(f(h)-f(z))f^2(h)}.$$

Then the action of the generators of the Virasoro algebra on the module $K(h, c)$ is given by $L_{-p} = B_{-p}$ and $L_n + B_n + Q_n$, where $p > 0$, $n \geq 0$, and

$$\sum_{j=0}^{\infty} Q_j z^j = z^2\left[h\frac{f'(z)^2}{f(z)^2} + \frac{c}{24}\left(\frac{2f'''(z)}{f'(z)} - \frac{3f''(z)^2}{f'(z)^2}\right)\right].$$

4.13. Characters. By Theorem 4$'$ the concept of the character makes sense for the representations $N(\alpha, \beta)$:

$$\chi_{\alpha,\beta}(\gamma) = \operatorname{tr} B(L_{\alpha,\beta}(\gamma)).$$

We also let the function $\chi_{\alpha,\beta}$ be denoted by $\chi^{h,c}$ if

$$h = \tfrac{1}{2}(\alpha^2 + \beta^2), \qquad c = 1 + 12\beta^2.$$

The trace of the operator $B(T)$ may be computed by the usual procedure of integrating the kernel over the diagonal (see [2], §2.8). For the operator (4.3) we get

$$\operatorname{tr} B(L_{\alpha,\beta}(\gamma)) = \det(iR)\exp\{\tfrac{1}{2}(\alpha l + \beta m)R(\alpha l^t + \beta m^t)\}, \tag{4.13}$$

where $l = (l_1, l_2)$, $m = (m_1, m_2)$, and

$$R = \begin{pmatrix} -K & 1-L \\ 1-L^t & -M \end{pmatrix}^{-1}.$$

By (4.2) we have

$$\chi^{h_1,c_1}(\gamma)\chi^{h_2,c_2}(\gamma) = \sum_{n\geq 0} p(n)\chi^{h_1+h_2+n,\,c_1+c_2}(\gamma).$$

Comparing this with (4.13), we get, first, $lRm^t = 0$, and, second, an identity for the Fredholm determinant which appears interesting in itself:

$$\det(iR) = \exp\{\tfrac{1}{24}[lRl^t - mRm^t]\}P(\exp\{lRl^t\}),$$

where

$$P(t) = \sum_{n\geq 0} p(n)t^n = \prod_{n>0}(1-t^n)^{-1}.$$

Finally, we get

$$\chi_{\alpha,\beta}(\gamma) = \exp\{(\alpha^2 + \tfrac{1}{24})lRl^t + (\beta^2 - \tfrac{1}{24})mRm^t\}\cdot P(\exp\{lRl^t\}).$$

4.14. The area theorem. Let $\mathfrak{p} = (P, p_i^+, \pi)$ be a morphism from k into 0 in the category $\mathrm{Shtan}^\sim$ (see §1.15). The space V introduced in §3.4 is, by §2.2, furnished with, in addition to the structures listed there, an indefinite Hermitian form:

$$\Lambda(f, g) = \int_{|z|=1} f\, d\overline{g}.$$

We let $p^*\mu$ denote the preimage of the 1-form μ under the map p.

AREA THEOREM. *Let μ be a holomorphic 1-form on $P \setminus \bigcup p_i^+(D_+^0)$, where the integrals of μ over all the cycles in the lattice π are 0. Then*

$$\sum_i \Lambda((p_i^+)^*\mu, (p_i^+)^*\mu) > 0.$$

This theorem was obtained by Milin and Lebedev (1951) in the case when P is a Riemann sphere and $k = 1$, and was generalized by Lebedev to the case of arbitrary k. We omit the proof, since it is almost the same as the classical proof (see [24], I.1 and III.1).

4.15. The embedding of the category $\mathrm{Shtan}^\sim$ **in the symplectic category.** In the construction in §2.2 we supposed that the spaces V and W are the same. In fact nothing stops us from supposing them to be different. For each correct linear relation T we can construct in exactly the same way a linear operator $B(T): F(V_+) \to F(W_+)$, and, in exactly the same way, the product of the operators $B(T)B(S) = c(T, S)B(TS)$, $c \in \mathbf{C}$, will correspond to the product of the relations T and S.

Let $\mathfrak{p} = (P, p_i^+, p_j^-, \pi) : m \to n$ be a morphism in $\mathrm{Shtan}^\sim$. Let V be the same as in §3.4. We construct with respect to $\mathfrak{p}$ the linear relation $T(\mathfrak{p})$ in $V^m + V^n$. Namely, $(f_1^+, \dots, f_m^+, f_1^-, \dots, f_n^-) \in T(\mathfrak{p})$ if there exists a holomorphic 1-form F on $P \setminus \bigcup p_\alpha^\pm(D_\pm)$ such that $(p_\alpha^\pm)^* F = df_\alpha^\pm$. Then the area theorem provides precisely the condition that the Hermitian form $\Lambda'(\cdot, \cdot)$ (see §2.2) be nonnegative; that is, condition 2° for "correctness" of the linear relation. The remaining conditions are obvious, and we get that the relation $T(\mathfrak{p})$ is correct.

We now make the Fock space $F((V_+)^m)$ correspond to the object m of the category $\mathrm{Shtan}^\sim$ and make the operator $A(\mathfrak{p}) = B(T(\mathfrak{p})) : F((V_+)^m) \to F((V_+)^n)$ correspond to the morphism $\mathfrak{p} : m \to n$. Then

$$A(\mathfrak{p})A(\mathfrak{q}) = c(\mathfrak{p}, \mathfrak{q})A(\mathfrak{p}\mathfrak{q}),$$

where $c \in \mathbf{C}$, and thus we have obtained a representation of the category $\mathrm{Shtan}^\sim$.

4.16. The formula for the representation of $\mathrm{Shtan}^\sim$. In order to write down the explicit formula for the representation of the category $\mathrm{Shtan}^\sim$ that we have just constructed (it is written in terms of Cauchy projections) it is sufficient to write down the matrix Ω associated with $T(\mathfrak{p})$:

$$\Omega : (V_-)^m \oplus (V_+)^n \to (V_+)^m \oplus (V_-)^n.$$

We describe what the block Ω_{ij} of the matrix Ω looks like:

a) Let $i \le m$, let $f \in V_-$, and let $q = ((p_i^+)^{-1})^* df$. We represent the 1-form q, defined on the contour $p_i^+(e^{i\varphi})$, as a sum $q = q_+ + q_-$, where q_+ is holomorphic in $p_i(D_+^0)$ and q_- is holomorphic in $P \setminus p_i(D_+^0)$, and where the integrals of q_- over all cycles in π are 0. Then when $j \ne i$, $j \le m$, and $k > m$ we have

$$\Omega_{ii} f = (p_i^+)^* q_+ \Omega_{ij} f = (p_j^+)^* q_- \Omega_{ik} = (p_k^-)^* q_-.$$

b) Let $i > m$. The construction is analogous; it is only necessary to interchange the indices $+$ and $-$ and to suppose that $k < m$, $j > m$, and $j \neq i$.

The area theorem implies that the decomposition $q = q_+ + q_-$ is unique, and also implies the following assertion:

THEOREM ("BASIC AREA THEOREM"). $\|\Omega\| < 1$.

In the case when P is a sphere and $k = m + n = 1$, this is just the Grunsky inequality (1939); for arbitrary k it is the Lebedev theorem (1961) (see [24], III.1). Our generalization of the old theorem has perhaps some interest of its own. It is remarkable, however, that the Grunsky-Lebedev area theorem which is used for a solution of extremal problems finds an unexpected application in the theory of representations (and, in addition, itself becomes a theorem in the theory of representations). It is also interesting that the matrix-valued function Ω on the spaces $\mathrm{Mor}(m, n)$ has nontrivial algebraic properties (namely, the $*$-multiplication of (2.2) is defined on the matrices Ω). Finally it is worth noting that in the case when $k = 1$ and the surface P is arbitrary, the matrix A has appeared in the theory of integrable systems (see [25]).

Moscow Institute of Electrical Machine Building
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