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Conformal geometry of symmetric spaces and generalized linear-fractional maps of Kreĭn–Shmul'yan

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Abstract. The matrix balls $\mathrm{B}_{p,q}$ consisting of $(p \times q)$-matrices of norm < 1 over $\mathbb{C}$ are considered. These balls are one possible realization of the symmetric spaces $\mathrm{B}_{p,q} = \mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$. Generalized linear-fractional maps are maps $\mathrm{B}_{p,q} \to \mathrm{B}_{r,s}$ of the form $Z \mapsto K + LZ(1-NZ)^{-1}$ (they are in general neither injective nor surjective). Characterizations of generalized linear-fractional maps in the spirit of the 'fundamental theorem of projective geometry' are obtained: for a certain family of submanifolds of $\mathrm{B}_{p,q}$ ('quasilines')it is shown that maps taking quasilines to quasilines are generalized linear-fractional. In addition, for the standard field of cones on $\mathrm{B}_{p,q}$ (described by the inequality $\operatorname{rk} dZ \leqslant 1$) it is shown that maps taking cones to cones are generalized linear-fractional.

Bibliography: 21 titles.

Introduction

Consider any one of the 10 series of classical Riemannian non-compact symmetric spaces

$$\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n), \quad \mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n), \quad \mathrm{GL}(n,\mathbb{H})/\mathrm{Sp}(n), \quad \mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q),$$
$$\mathrm{O}(p,q)/\mathrm{O}(p)\times\mathrm{O}(q), \qquad \mathrm{Sp}(p,q)/\mathrm{Sp}(p)\times\mathrm{Sp}(q),$$
$$\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n), \quad \mathrm{Sp}(2n,\mathbb{C})/\mathrm{Sp}(n), \quad \mathrm{SO}^*(2n)/\mathrm{U}(n), \quad \mathrm{O}(n,\mathbb{C})/\mathrm{O}(n,\mathbb{R}).$$

It turns out that there exist natural maps between distinct spaces in one series, the so-called generalized linear-fractional maps of Kreĭn–Shmul'yan. For the series $\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q)$ such maps have been known in operator theory since the 1950s at least (they have appeared in papers by Potapov, M. Kreĭn, Yu. Ginzburg, and Shmul'yan; see, for instance, [1]–[4]). For the series $\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$ they have been introduced in [2] and for other series in [5]. In the 1980s these maps found their way into representation theory (see [5]–[7]). The aim of the present paper is to demonstrate that generalized linear-fractional maps are very natural from the geometric standpoint.

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We now explain just what we mean. For example. consider the series of spaces

$$\mathrm{U}(p,q)/\,\mathrm{U}(p)\times\mathrm{U}(q).$$

They have a convenient realization as *Cartan domains* or *matrix balls*. Namely, let $\mathrm{B}_{(p,q)}$ be the set of complex $p\times q$-matrices with norm <1 (by the *norm of a matrix* we mean the norm of the corresponding linear operator from the Euclidean space $\mathbb{C}^p$ into the Euclidean space $\mathbb{C}^q$). We consider the pseudo-unitary group $\mathrm{U}(p,q)$, that is, the group of block matrices $\begin{pmatrix}A&B\\C&D\end{pmatrix}$ of size $(p+q)\times(p+q)$ preserving the Hermitian form with matrix $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. The group $U(p,q)$ acts in $\mathrm{B}_{p,q}$ by linear-fractional maps of the form

$$Z\mapsto(A+ZC)^{-1}(B+ZD), \tag{0.1}$$

where $\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\mathrm{U}(p,q)$. The stabilizer of the point $Z=0$ consists of the matrices

$$\begin{pmatrix}A&0\\0&D\end{pmatrix},$$

where $A\in\mathrm{U}(p)$, $D\in\mathrm{U}(q)$. Thus, $\mathrm{B}_{p,q}=\mathrm{U}(p,q)/\,\mathrm{U}(p)\times\mathrm{U}(q)$.

Let $Z_1,Z_2\in\mathrm{B}_{p,q}$. We consider now the matrix

$$\Lambda(Z_1,Z_2)=(1-Z_1^*Z_1)^{-1/2}(1-Z_1^*Z_2)(1-Z_2^*Z_2)^{-1/2}.$$

Let $\lambda_1\geqslant\lambda_2\geqslant\cdots$ be the singular values of this matrix (recall that *singular values of a matrix* Λ are eigenvalues of the matrix $|\Lambda|=\sqrt{\Lambda^*\Lambda}$). It turns out that this collection of numbers is a complete set of invariants of the pair $Z_1,Z_2\in\mathrm{B}_{p,q}$ under the action of $\mathrm{U}(p,q)$. The quantities

$$\varphi_1:=\cosh^{-1}(\lambda_1),\quad \varphi_2=\cosh^{-1}(\lambda_2),\quad \dots$$

are called the *complex* (*compound*) *distance* or the *stationary angles* (see [8], [9] for fuller detail). The standard distance in the symmetric space $\mathrm{U}(p,q)/\,\mathrm{U}(p)\times\mathrm{U}(q)$ induced by the invariant Riemannian metric can be expressed through the complex distance by the formula

$$\rho(Z_1,Z_2)=\Bigl(\sum\varphi_j^2\Bigr)^{1/2}.$$

We consider now the block $\bigl((r+q)\times(s+p)\bigr)$-matrix

$$S=\begin{pmatrix}K&L\\M&N\end{pmatrix}$$

(note the size) satisfying the conditions

(a) $\|S\|\leqslant 1$,
(b) $\|K\|<1$.

It can be shown that the formula

$$\tau_S(Z) = K + LZ(1 - NZ)^{-1}M \tag{0.2}$$

defines in this case a map

$$\tau(S)\colon \mathrm{B}_{p,q} \to \mathrm{B}_{r,s}.$$

Note that if $(p,q) = (r,s)$, then the matrices S corresponding to motions (0.1) of the symmetric space $\mathrm{B}_{p,q}$ are unitary, that is, even in this case there exist more maps (0.2) than there are motions (0.1). Maps of the form (0.2) are called *generalized linear-fractional maps of Kreĭn–Shmul'yan.*

The following result can be found in [8], [9] (in the form of a lemma in the proof of boundedness for a class of integral operators).

Theorem 0.1. *Let $\tau_S\colon \mathrm{B}_{p,q} \to \mathrm{B}_{r,s}$ be a map of the form* (0.2). *Let $\varphi_1 \geqslant \varphi_2 \geqslant \cdots$ be the compound distance between points $Z_1, Z_2 \in \mathrm{B}_{p,q}$ and let $\psi_1 \geqslant \psi_2 \geqslant \cdots$ be the compound distance between their images. Then*

$$\varphi_1 \geqslant \psi_1, \quad \varphi_2 \geqslant \psi_2, \quad \varphi_3 \geqslant \psi_3, \quad \dots \tag{0.3}$$

(see the first inequality in [10]).

Of course, if follows from (0.3) that generalized linear-fractional maps decrease the standard Riemann distance in $\mathrm{B}_{p,q}$. However, Theorem 0.1 states in fact that generalized linear-fractional maps are contractions in an incomparably stronger — and fairly unusual — sense.

The aim of the present paper is to characterize generalized linear-fractional maps (in the spaces $\mathrm{U}(p,q)/\,\mathrm{U}(p)\times\mathrm{U}(q)$) in terms of the geometry of symmetric spaces (Theorems 2.2–2.4). First, we give their characterizations in the spirit of the 'fundamental theorem of projective geometry': an (in general, not injective) map taking some family of submanifolds of one symmetric space to a similar family of submanifolds of another is (under certain additional assumptions) generalized linear-fractional.

We also give a characterization of generalized linear-fractional maps in the spirit of so-called 'conformal geometry'. Namely, we consider a standard field of cones in the tangent spaces to a symmetric space (see [11]–[14]). We show that (under certain additional provisos) maps taking cones into cones (not necessarily in a one-to-one way) are generalized linear-fractional maps. As a consequence we derive Theorem 2.6, a result converse to Theorem 0.1: a map $\mathrm{B}_{p,q} \to \mathrm{B}_{r,s}$ satisfying (0.3) is generalized linear-fractional.

All these results are stated in § 2 and proved in § 3. In § 1 we present preliminary information. In § 4 we discuss other series of symmetric spaces.

I am indebted to G. I. Ol'shanskiĭ, who asked me about possible geometric characterizations of generalized linear-fractional maps.

§ 1. Results similar to the 'fundamental theorem of projective geometry'

1.1. Grassmannian. Let $\mathrm{Gr}_{p,q}$ be the set of all p-dimensional subspaces of $\mathbb{C}^{p+q} = \mathbb{C}^p \oplus \mathbb{C}^q$. By $\mathrm{Mat}_{p,q}$ we shall mean the space of $(p\times q)$-matrices or, equivalently, the space of linear operators $\mathbb{C}^p \to \mathbb{C}^q$.

Let $A \in \mathrm{Mat}_{p,q}$. Then $\mathrm{graph}(A)$, the graph of the operator A, is a p-dimensional subspace of $\mathbb{C}^p \oplus \mathbb{C}^q$, that is, an element of $\mathrm{Gr}_{p,q}$. Thus, the map $A \mapsto \mathrm{graph}(A)$ is an embedding

$$\mathrm{Mat}_{p,q} \to \mathrm{Gr}_{p,q}\,.$$

It is easy to see that the image of $\mathrm{Mat}_{p,q}$ is an open dense subset of $\mathrm{Gr}_{p,q}$.

The group $\mathrm{GL}(p+q,\mathbb{C})$ acts in $\mathbb{C}^{p+q}$, therefore it acts also in $\mathrm{Gr}_{p,q}$. Using the variable $Z \in \mathrm{Mat}_{p,q}$ we can express the action of $\mathrm{GL}(p+q,\mathbb{C})$ as follows:

$$Z \mapsto (A+ZC)^{-1}(B+ZD),$$

where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is an invertible $((p+q)\times(p+q))$-matrix.

1.2. Line structures. Let T and S be subspaces of $\mathbb{C}^{p+q}$ such that

$$\dim T = p+1, \qquad \dim S = p-1, \quad \text{and} \quad T \supset S.$$

We consider the set $\ell_{T,S}$ of $H \in \mathrm{Gr}_{p,q}$ such that

$$S \subset H \subset T.$$

Clearly, $\ell_{T,S}$ is a one-dimensional complex manifold holomorphically equivalent to the projective line $\mathbb{CP}^1$. We shall call the submanifolds $\ell_{T,S}$ *quasilines*.

In the chart $\mathrm{Mat}_{p,q} \subset \mathrm{Gr}_{p,q}$ quasilines are described by the formulae

$$A + tB,$$

where $A, B \in \mathrm{Mat}_{p,q}$, $\mathrm{rk}\, B = 1$, and t ranges in $\mathbb{C}$.

We now consider a subspace S of $\mathbb{C}^{p+q}$ of dimension $p-1$. Let V_S be the set of all $H \in \mathrm{Gr}_{p,q}$ such that $H \supset S$. Clearly, V_S is a complex manifold biholomorphically equivalent to the projective space $\mathbb{CP}^q$. We shall call the submanifolds $V_S \subset \mathrm{Gr}_{p,q}$ *quasiplanes of the first kind*.

Let $T \subset \mathbb{C}^{p+q}$ be a subspace of dimension $p+1$ and let W_T be the set of $H \in \mathrm{Gr}_{p,q}$ such that $H \subset T$. Clearly, W_T is a complex manifold that is holomorphically equivalent to the projective space $\mathbb{CP}^p$. We shall call the submanifolds $W_T \subset \mathrm{Gr}_{p,q}$ *quasiplanes of the second kind*.

In the variables $Z \in \mathrm{Mat}_{p,q}$ a quasiplane of the first kind can be expressed as the set of matrices of the form

$$Z(t_1,\dots,t_q) = A + \begin{pmatrix} t_1\alpha_1 & t_2\alpha_1 & \dots & t_q\alpha_1 \\ \multicolumn{4}{c}{\dotfill} \\ t_1\alpha_p & t_2\alpha_p & \dots & t_q\alpha_p \end{pmatrix},$$

where $t_1,\dots,t_q$ range over $\mathbb{C}$, while the matrix A and the numbers $\alpha_1,\dots,\alpha_p$ are fixed. Accordingly, quasiplanes of the second kind can be parametrized in accordance with the formula

$$Z(s_1,\dots,s_p) = A + \begin{pmatrix} s_1\beta_1 & s_1\beta_2 & \dots & s_1\beta_q \\ \multicolumn{4}{c}{\dotfill} \\ s_p\beta_1 & s_p\beta_2 & \dots & s_p\beta_q \end{pmatrix}.$$

Lemma 1.1. (a) *For $S \neq S'$ the quasiplanes V_S and $V_{S'}$ intersect in one point at most. The same holds for the quasiplanes W_T and $W_{T'}$.*

(b) *For $T \not\supset S$ the quasiplanes W_T and V_S are disjoint, and for $T \supset S$ they intersect in the quasiline $\ell_{T,S}$.*

(c) *For each quasiline ℓ there exist a unique quasiplane of the first kind and a unique quasiplane of the second kind containing ℓ.*

Lemma 1.2. (a) *Two arbitrary points in a quasiplane can be joined by a quasiline.*

(b) *Let M be a subset of $\mathrm{Gr}_{p,q}$ such that for two arbitrary points in M there exists a quasiline through them. Then M lies in some quasiplane.*

All these assertions are obvious.

Remark. The Plücker embedding

$$\mathrm{Gr}_{p,q} \to \mathbb{CP}^{C^p_{p+q}-1}$$

takes quasilines into straight lines (see [15]).

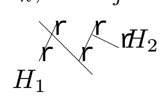
1.3. Determinant submanifolds. We can define the so-called *integral distance* $n(H_1, H_2)$ in $\mathrm{Gr}_{p,q}$ by the formula

$$n(H_1, H_2) = \text{codimension of } H_1 \cap H_2 \text{ in } H_1.$$

It is easy to see that it satisfies in fact all the axioms of a metric. In terms of matrices this distance can be expressed as follows:

$$n(Z_1, Z_2) = \mathrm{rk}(Z_1 - Z_2).$$

Remark. The quantity $n(H_1, H_2)$ is the minimum length of a chain $\ell_1, \dots, \ell_n$ of quasilines such that $H_1 \in \ell_1, H_2 \in \ell_n$, and ℓ_j intersects ℓ_{j+1}.



Let $\alpha = 0, 1, \dots, p$. We define the *determinant submanifold* $\mathrm{D}_\alpha(H)$ of $\mathrm{Gr}_{p,q}$ as the set of $R \in \mathrm{Gr}_{p,q}$ such that $n(H, R) \leqslant \alpha$. In terms of matrices $\mathrm{D}_\alpha(A)$ is described by the condition

$$\mathrm{rk}(Z - A) \leqslant \alpha.$$

1.4. Projective characterization of the group GL($p+q, \mathbb{C}$). We introduce in $\mathbb{C}^{p+q}$ the operation of component-wise complex conjugation

$$(x_1, \dots, x_{p+q}) \mapsto (\overline{x_1}, \dots, \overline{x_{p+q}}).$$

For arbitrary $H \in \mathrm{Gr}_{p,q}$ let $\overline{H}$ be the image of H under this conjugation. Further, we consider in $\mathbb{C}^{p+q}$ the non-degenerate symmetric bilinear form

$$(x, y) = \sum_{j \leqslant p} x_j y_j - \sum_{j > p} x_j y_j.$$

Let $H^\perp$ be the orthogonal complement of $H \in \mathrm{Gr}_{p,q}$ with respect to this form. In the variables $Z \in \mathrm{Mat}_{p,q}$ this operation can be expressed by the formula $Z \mapsto Z^t$, where $Z^t \in \mathrm{Mat}_{q,p}$ is the transpose of Z.

Theorem 1.3 (Hua Loo Keng [16]). *Let* $\lambda\colon \mathrm{Gr}_{p,q} \to \mathrm{Gr}_{p,q}$, *where* $\min\{p;q\} > 1$, *be a continuous bijection*[1] *preserving the integral distance. If* $p \neq q$, *then the map* λ *has one of the following forms*:

$$\lambda(H) = gH \quad \text{or} \quad \lambda(H) = g\overline{H}, \quad \text{where} \quad g \in \mathrm{GL}(p+q, \mathbb{C}), \quad H \in \mathrm{Gr}_{p,q}\,. \tag{1.1}$$

If $p = q$, *then* λ *can also have one of the following forms*:

$$\lambda(H) = gH^{\perp} \quad \text{or} \quad \lambda(H) = g\overline{H}^{\perp}, \quad \text{where} \quad g \in \mathrm{GL}(p+q, \mathbb{C}), \quad H \in \mathrm{Gr}_{p,q}\,. \tag{1.2}$$

An equivalent result was obtained by Chow in [15] (1949).

Theorem 1.4. *Let* $\lambda\colon \mathrm{Gr}_{p,q} \to \mathrm{Gr}_{p,q}$, *where* $\max\{p;q\} > 1$, *be a continuous bijection taking quasilines to quasilines. Then for* $p \neq q$ *the map* λ *has the form* (1.1) *and for* $p = q$ *it has the form* (1.1) *or* (1.2).

The proof of these results and their various versions can be found in Dieudonné's book [17]; see also [18] and the papers in the 'Geometry of matrices' section in Hua Loo Keng's collection of papers [19]

1.5. Conformal structure. Let $H \in \mathrm{Gr}_{p,q}$ and let T_H be the tangent space to $\mathrm{Gr}_{p,q}$ at H. We denote by Cone_H the cone in T_H formed by the vectors tangent to the determinant submanifold $\mathrm{D}_1(H)$. We obtain in this way a field of cones on $\mathrm{Gr}_{p,q}$.

Operating in the space $\mathrm{Mat}_{p,q}$ we shall identify the tangent space at a point $A \in \mathrm{Mat}_{p,q}$ and $\mathrm{Mat}_{p,q}$ itself. A vector dZ lies in Cone_A if

$$\operatorname{rk} dZ \leqslant 1.$$

Theorem 1.5 (Goncharov [11], [12]). *Assume that* $\min\{p;q\} > 1$ *and let* λ *be a holomorphic diffeomorphism of an open subset* $\mathcal{O}_1$ *of* $\mathrm{Gr}_{p,q}$ *onto another open subset* $\mathcal{O}_2$ *of* $\mathrm{Gr}_{p,q}$ *taking the field of cones* Cone_H *into itself. Then* λ *can be extended to a map* $H \mapsto gH$, *where* $g \in \mathrm{GL}(p+q,\mathbb{C})$ (*if* $p = q$, *then maps* $H \mapsto gH^{\perp}$ *are also possible*).

Remark. A similar result is proved in [11] and [12] for all Hermitian symmetric spaces. It has been extended to other Riemannian symmetric spaces in [14].

1.6. Comments. (a) Of course, results similar to Theorems 1.3–1.5 are mostly of aesthetic value (which has merits on its own). Nevertheless, results similar to Theorems 1.3–1.4 find fairly sensible applications in the structure theory of classical groups over fields distinct from $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (see [17], Ch. 4).

(b) For $p = 1$ both integral distance and conformal structure are meaningless (the integral distance is always equal to 1 and the cone Cone_H coincides with the entire space). Hence Theorems 1.3 and 1.5 do not hold. Theorem 1.4 turns into the so-called '*fundamental theorem of projective geometry*' (or *von Staudt's theorem*).

[1] In fact, measurability is sufficient. We could also do without measurability, but in this case the complex conjugation in the statement of the theorem must be replaced by some automorphism of the field $\mathbb{C}$ (see [17]).

Theorem. *Assume that $q > 1$. Then a continuous one-to-one map of the q-dimensional projective space $\mathbb{CP}^q$ into itself that takes lines into lines is induced by a linear or an antilinear map of $\mathbb{CP}^{q+1}$.*

This result, in its turn, is void for $q = 1$.

(c) Theorem 1.5 is a consequence of Cartan's theorem on primitive pseudogroups of holomorphic diffeomorphisms (see [20]). Indeed, the local group $\mathfrak{G}$ of diffeomorphisms preserving the field of cones contains the local group $\mathrm{GL}(p{+}q, \mathbb{C})$. Hence it cannot preserve any partitioning into subsets. On the other hand it is clearly distinct from all primitive infinite-dimensional diffeomorphism pseudogroups (the full diffeomorphism pseudogroup, the symplectic, the contact, the pseudogroup preserving the volume up to a constant factor, and the pseudogroup preserving a symplectic form up to a function coefficient). Hence the local group $\mathfrak{G}$ is finite-dimensional by Cartan's theorem.

Next, we consider the stabilizer $\mathfrak{G}_x$ of a point x in $\mathfrak{G}$. It contains a parabolic subgroup $(\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C})) \ltimes N \subset \mathrm{GL}(p+q, \mathbb{C})$, where N is an Abelian group isomorphic to the additive group $\mathrm{Mat}_{p,q}$. The natural filtration in the group of jets induces a filtration in $\mathfrak{G}_x$. Its lowest level is an Abelian group commuting with N. However, N acts transitively in $\mathrm{Gr}_{p,q} \setminus \{x\}$, therefore each local diffeomorphism commuting with N belongs to N.

In the case of real diffeomorphisms the status of Cartan's theorem is apparently not completely clear, therefore it can hardly be regarded as a universal key to results similar to Theorem 1.5.

§ 2. Statements of results

2.1. Generalized linear-fractional maps. A *generalized linear-fractional map*

$$\mathrm{Mat}_{p,q} \to \mathrm{Mat}_{r,s}$$

is a map of the following form:

$$\Psi(Z) = K + LZ(1 - NZ)^{-1}M, \tag{2.1}$$

where $\begin{pmatrix} K & L \\ M & N \end{pmatrix}$ is a block matrix of size $(r{+}q)\times(s{+}p)$. Of course, the map (2.1) is not defined on the whole of $\mathrm{Mat}_{p,q}$, but rather on the dense open subset described by the condition $\det(1 - NZ) \neq 0$. This subset clearly contains the point 0.

Remark. It is useful to bear in mind the formula

$$Z(1 - NZ)^{-1} = (1 - ZN)^{-1}Z.$$

Remark. Consider the map

$$Z \mapsto (A + ZC)^{-1}(B + ZD) \tag{2.2}$$

from $\mathrm{Mat}_{p,q}$ into $\mathrm{Mat}_{p,s}$. Assume that it is well defined in a neighbourhood of 0 (which is equivalent to the invertibility of A). Then (2.2) can be written also in the form (2.1); namely, it coincides with the map

$$Z \mapsto A^{-1}B + A^{-1}(E + ZCA^{-1})^{-1}Z(D - CA^{-1}B).$$

Conversely, if the matrix L in (2.1) is invertible (which means, by the way, that it is a square matrix, that is, $r = p$), then the transformation (2.1) can also be written in the form (2.2).

Remark. Consider now the map

$$Z \mapsto (PZ + Q)(RZ + T)^{-1} \tag{2.3}$$

from $\mathrm{Mat}_{p,q}$ into $\mathrm{Mat}_{r,q}$. Assume that it is well defined in a neighbourhood of 0 (which is equivalent to the invertibility of T). Then (2.3) can be written also in the form (2.1); namely, as the map

$$Z \mapsto QT^{-1} + (P - QT^{-1}R)Z(E + T^{-1}RZ)^{-1}T^{-1}.$$

Conversely, if the matrix M in (2.1) is invertible (in particular, $s = q$), then (2.1) can also be written in the form (2.3).

2.2. Linear relations. We now explain the meaning of generalized linear-fractional maps in the language of Grassmannians (see [9] for further detail). We consider an $(r + q)$-dimensional subspace S (a '*linear relation*') of the space

$$\mathbb{C}^{p+q} \oplus \mathbb{C}^{r+s}.$$

For each $H \in \mathrm{Gr}_{p,q}$ we can define the subspace $SH \subset \mathbb{C}^{r+s}$ of all vectors $w \in \mathbb{C}^{r+s}$ for which there exists $v \in H$ such that $(v, w) \in S$. A calculation of dimensions demonstrates that if $H \in \mathrm{Gr}_{p,q}$ is in general position, then SH belongs to $\mathrm{Gr}_{r,s}$.

Proposition 2.1. *The map* (2.1) *is defined by some* $(r + q)$*-dimensional linear relation* $S \subset \mathbb{C}^{p+q} \oplus \mathbb{C}^{r+s}$.

Proof. The linear relation S is the graph of the operator

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} : \mathbb{C}^r \oplus \mathbb{C}^q \to \mathbb{C}^s \oplus \mathbb{C}^p$$

(note the order of the variables!); see [9] for greater detail.

2.3. Statements of local theorems. We say that a holomorphic map Ψ of a connected open subset $\mathcal{O}$ of $\mathrm{Mat}_{p,q}$ into $\mathrm{Mat}_{r,s}$ is *non-degenerate* if

(1) the dimension of the range of Ψ is larger than 1;
(2) the range of Ψ does not entirely lie in a quasiplane.

We say that a holomorphic map of $\mathcal{O} \subset \mathrm{Mat}_{p,q}$ into $\mathrm{Mat}_{r,s}$ is *conformal* if for each point $A \in \mathcal{O}$ the image of the cone Cone_A under the action of the differential of the map Ψ lies in the cone $\mathrm{Cone}_{\Psi(A)}$.

Theorem 2.2. *Let* $\mathcal{O}$ *be a connected open subset of* $\mathrm{Mat}_{p,q}$ *containing* 0. *Let* $\Psi\colon \mathcal{O} \to \mathrm{Mat}_{r,s}$ *be a non-degenerate holomorphic map satisfying at least one of the following conditions*:

(A) Ψ *is conformal*;
(B) *for each quasiline* $\ell \subset \mathrm{Mat}_{p,q}$ *the set* $\Psi(\ell \cap \mathcal{O})$ *lies entirely in a quasiline*;
(C) *for each quasiplane* $Y \subset \mathrm{Mat}_{p,q}$ *the set* $\Psi(Y \cap \mathcal{O})$ *lies entirely in a quasiplane*;
(D) *for arbitrary* $Z_1, Z_2 \in \mathcal{O}$,

$$n(\Psi(Z_1), \Psi(Z_2)) \leqslant n(Z_1, Z_2). \tag{2.4}$$

Then Ψ *has the following form*:

$$\Psi(Z) = K + LZ(1 - NZ)^{-1}M \tag{2.5}$$

or

$$\Psi(Z) = P + QZ^t(1 - RZ^t)^{-1}T. \tag{2.6}$$

Remark. Clearly, each of conditions (B)–(D) yields (A). Hence we shall deduce the conclusion of the theorem from (A).

Remark. The condition that the holomorphic map is non-degenerate has roughly the same significance as the condition $q > 1$ in von Staudt's theorem or the conditions $\min\{p; q\} > 1$, $\max\{p; q\} > 1$ in Hua Loo Keng's, Chow's, and Goncharov's results.

(1) If the range of a map lies entirely in some quasiplane, then conditions (A), (C), and (D) hold automatically.

(2) Consider an affine (that is, linear inhomogeneous) map from $\mathrm{Mat}_{p,q}$ into a quasiplane in $\mathrm{Mat}_{r,s}$. Then all conditions (A)–(D) must hold.

(3) Conditions (A)–(D) hold for each map of $\mathrm{Mat}_{p,q}$ into a quasiline.

(4) Let L be a holomorphic curve tangent at each point Q to the cone Cone_Q. Let $\Psi(\mathcal{O}) \subset L$. Then condition (A) is clearly satisfied.

The elimination of all these 'stupid' cases is an essential (and unpleasant) part of the proof of the theorem.

We now present another formulation of the same result.

Theorem 2.3. *Let* $\mathcal{O} \subset \mathrm{Mat}_{p,q}$ *be an open connected subset containing* 0. *Let* Ψ *be a non-degenerate holomorphic map* $\mathcal{O} \to \mathrm{Mat}_{r,s}$ *such that its differential* $d\Psi$ *has the following form at each point* $Z \in \mathcal{O}$:

$$d\Psi = A(Z) \cdot dZ \cdot B(Z).$$

Then Ψ *is a generalized linear-fractional map.*

Remark. In this connection one could put a question the reasonableness of which is not entirely clear to this author. Let $\Psi\colon \mathrm{Mat}_{p,q} \to \mathrm{Mat}_{r,s}$ be a holomorphic map. Then its differential can be written as follows:

$$d\Psi = \sum_{j=1}^{\alpha} A_j(Z) \cdot dZ \cdot B_j(Z),$$

where $A_j(Z), B_j(Z)$ are some matrices. For maps in general position the least number of terms α is equal to $\min\{p; q; r; s\}$. The question is as follows: is there any sensible way to classify maps with fixed small α ($\alpha < \min\{p; q; r; s\}$)?

2.4. Coordinate-free version of the theorem.

Theorem 2.4. *Let* $\mathcal{O}$ *be a connected open subset of* $\mathrm{Gr}_{p,q}$ *and let* $\Psi\colon \mathcal{O} \to \mathrm{Gr}_{r,s}$ *be a non-degenerate holomorphic map satisfying one of conditions* (A)–(D) *in Theorem* 2.2. *Then either the map* $H \to \Psi(H)$ *is induced by some* $(r+q)$*-dimensional linear relation in* $\mathbb{C}^{p+q} \oplus \mathbb{C}^{r+s}$ *or the map* $H \to \Psi(H^\perp)$ *is induced by some* $(r+p)$*-dimensional linear relation in* $\mathbb{C}^{p+q} \oplus \mathbb{C}^{r+s}$.

2.5. Metric version of the theorem. We return to the notation used in the introduction.

Theorem 2.5 (Shmul'yan [4]). *Let*

$$\Psi(Z) = K + LZ(1-NZ)^{-1}M$$

be a generalized linear-fractional map from the matrix ball $\mathrm{B}_{p,q}$ *into* $\mathrm{B}_{r,s}$. *Then for some* $\lambda \in \mathbb{R}$ *the matrix*

$$S_\lambda = \begin{pmatrix} K & \lambda L \\ \lambda^{-1} M & N \end{pmatrix} \tag{2.7}$$

satisfies the condition $\|S_\lambda\| \leqslant 1$, $K < 1$.

Note that generalized linear-fractional maps τ_{S_λ} corresponding to distinct λ are the same.

We point out also that the composite of two generalized linear-fractional maps, $\Phi\colon \mathrm{B}_{p,q} \to \mathrm{B}_{r,s}$ and $\Psi\colon \mathrm{B}_{r,s} \to \mathrm{B}_{u,v}$, is also a generalized linear-fractional map (this is not that obvious).

As in the introduction, let

$$\varphi_1(Z_1, Z_2) \geqslant \varphi_2(Z_1, Z_2) \geqslant \cdots$$

be the compound distance between points Z_1 and Z_2.

Theorem 2.6. *Let* R *be a non-degenerate holomorphic map* $\mathrm{B}_{p,q} \to \mathrm{B}_{r,s}$. *Assume that*

$$\varphi_j(R(Z_1), R(Z_2)) \leqslant \varphi_j(Z_1, Z_2) \tag{2.8}$$

for all $Z_1, Z_2 \in \mathrm{B}_{p,q}$ *and each* j. *Then* R *is generalized linear-fractional.*

Proof. The following result is easy to verify.

> *The integral distance* $n(Z_1, Z_2)$ *is equal to the number of elements in the collection* $\varphi_1(Z_1, Z_1), \varphi_2(Z_1, Z_2), \dots$ *that are distinct from zero.*

Hence (2.8) yields (2.4).

§ 3. Proof of Theorem 2.2

The main ingredients in the proof of the theorem are Lemma 3.3 (this is a well-known result (see [11], [12]) and we present its proof merely for completeness) and the lemmas in subsections 3.5–3.6 on the geometry of the sphere in the integral metric. The rest is very simple, but the author feels obliged to give the details of the proof because of its many slippery places.

Throughout this section $\mathcal{O}$ is a connected open subset of $\mathrm{Mat}_{p,q}$ containing 0. We shall assume that the intersection of $\mathcal{O}$ with each quasiline is connected. This can be obtained, for instance, by shrinking this set and taking for $\mathcal{O}$ a sufficiently small ε-neighbourhood of zero (with respect to an arbitrary norm).

The phrase '*in general position*' means '*in some open* (in the sense of the standard topology) *dense set*' (which will be fixed throughout).

3.1. Locally projective maps. Let $\mathcal{U} \subset \mathbb{CP}^n$ be a connected open subset and let $\psi : \mathcal{U} \to \mathbb{CP}^k$ be a holomorphic map. We say that ψ is *locally projective* if

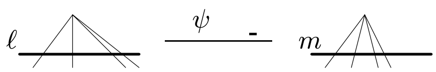
(a) for each line $\ell \subset \mathbb{CP}^n$ the ψ-image of the set $\ell \cap \mathcal{U}$ lies in some line $m \subset \mathbb{CP}^k$;
(b) the set $\psi(\mathcal{U})$ does not entirely lie in a single line.

Let $A\colon \mathbb{C}^{n+1} \to \mathbb{C}^{k+1}$ be a linear operator and let $\mathbb{P}\ker A \subset \mathbb{CP}^k$ be the projectivization of its kernel. Then A induces in a natural way a locally projective map $\mathbb{CP}^n \setminus \mathbb{P}\ker A \to \mathbb{CP}^k$.

Lemma 3.1. *Each locally projective map is induced by a linear operator.*

Proof. (1) We can assume without loss of generality that the range of ψ does not lie in a proper subspace of $\mathbb{CP}^k$.

(2) Clearly, the image of each s-dimensional subspace of $\mathbb{CP}^n$ lies in some s-dimensional subspace (see the figure: the structure on the left-hand side is transformed into the structure on the right-hand side).



In particular, $k \leqslant n$.

(3) We now consider $(k+1)$ points $P_0, P_1, \ldots, P_k$ in the range of ψ not lying in the same $(k-1)$-dimensional subspace of $\mathbb{CP}^k$. Choosing appropriate coordinates we can assume without loss of generality that

$$P_0 = (1:0:0:\ldots),\ \ P_1 = (0:1:0:\cdots:0),\ \ \ldots,\ \ P_k = (0:\cdots:0:1).$$

(4) We choose points $Q_0, Q_1, \ldots, Q_k$ such that $\psi(Q_j) = P_j$. By (2) they do not lie in one $(k-1)$-dimensional subspace. Choosing suitable coordinates we can assume that

$$Q_0 = (1:0:\cdots:0),\ Q_1 = (0:1:0:\cdots:0),\ \ldots,\ \ Q_k = (\underbrace{0:\cdots:0}_{k}:1:\underbrace{0:\cdots:0}_{n-k}).$$

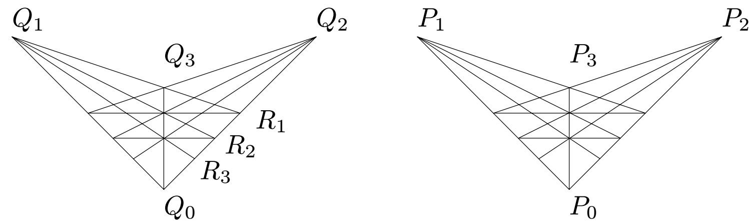
(5) For each element

$$(x_0 : x_1 : \cdots : x_k : 0 : \cdots : 0) \in \mathcal{U}$$

we have

$$\psi(x_0 : x_1 : \cdots : x_k : 0 : \cdots : 0) = (x_0 : x_1 : \cdots : x_k).$$

See the figure on the next page: the structure on the left-hand side is transformed into the one on the right-hand side. Once we know the images of Q_0, Q_1, Q_2, Q_3, we know also the images of the points $R_1, R_2, R_3, \ldots$. However, the sequence $R_1, R_2, \ldots$ converges to Q_0, and we can use uniqueness results for holomorphic functions.



(6) Written in coordinate-free form, (5) means that the map ψ is projective in the usual sense of this term on k-dimensional subspaces of $\mathbb{CP}^n$ in general position.

(7) Let m be a line in $\mathbb{CP}^k$ intersecting the range of ψ. Then its inverse image $\psi^{-1}(m) \subset \mathbb{CP}^n$ is an open subset of some subspace of $\mathbb{CP}^n$. For let $a, b \in \psi^{-1}(m)$ be points in general position and let ℓ be the line through these points. Then $\psi(a) \neq \psi(b)$. Hence $\psi(\ell) \subset m$ and therefore $\ell \cap \mathcal{U} \subset \psi^{-1}(m)$. We leave out the final trivial steps.

(8) Consider a point $x \in \psi(\mathcal{U})$. We claim that its inverse image $\psi^{-1}(x)$ is an open subset of some subspace of $\mathbb{CP}^n$. We consider two straight lines m_1 and m_2 intersecting at x. Then

$$\psi^{-1}(x) = \psi^{-1}(m_1) \cap \psi^{-1}(m_2),$$

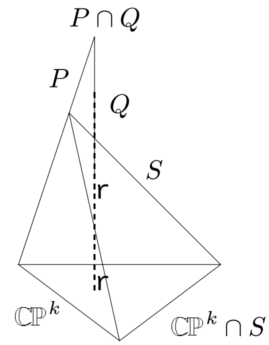
and the result is now obvious.

(9) We shall now treat $\mathbb{CP}^k$ as the subspace of $\mathbb{CP}^n$ consisting of the points with coordinates

$$(x_0 : x_1 : \cdots : x_k : 0 : \cdots : 0).$$

(10) We consider now a line m in $\mathbb{CP}^k$ in general position. Its inverse image $\psi^{-1}(m)$ is an open subset of some $(n-k+1)$-dimensional subspace $M \subset \mathbb{CP}^n$. We consider two points on m, p and q in general position. Their inverse images are open subsets of some $(n-k)$-dimensional subspaces $P, Q \subset M \subset \mathbb{CP}^n$. Hence $P \cap Q$ is an $(n-k-1)$-dimensional subspace of M.

Let φ be the operation of projection of $\mathbb{CP}^n$ onto $\mathbb{CP}^k$ from $P \cap Q$. Clearly, φ coincides with ψ on the sets $\mathbb{CP}^k \cap \mathcal{U}$, $P \cap \mathcal{U}$, and $Q \cap \mathcal{U}$. We claim that $\varphi = \psi$ everywhere in $\mathcal{U}$. We consider a k-dimensional subspace of S in general position that has a $(k-1)$-dimensional intersection with $\mathbb{CP}^k \cap \mathcal{U}$ and also intersects $P \cap \mathcal{U}$ and $Q \cap \mathcal{U}$. As we saw in (6), ψ is a projective map in S. On the other hand it coincides with φ on $P \cap S$, $Q \cap S$, and on the $(k-1)$-dimensional subspace $S \cap \mathbb{CP}^k$. Hence $\varphi = \psi$ everywhere in S.



The required result is now obvious.

We shall use also the following technical lemma.

Lemma 3.2. *Let $\ell_1, \dots, \ell_n \subset \mathbb{CP}^n$ be a collection of straight lines that intersect at some point u and do not all lie in the same $(n-1)$-dimensional subspace and let $m_1, \dots, m_n \subset \mathbb{CP}^k$ be an arbitrary collection of straight lines intersecting at a point v. Let $\varphi_j \colon \ell_j \to m_j$ be a collection of projective maps such that $\psi_j(u) = v$ for each j. Then there exists a unique locally projective map $\Psi \colon \mathbb{CP}^n \to \mathbb{CP}^k$ equal to ψ_j on the line ℓ_j for each j.*

Proof. The point u corresponds to some line $\widetilde{u} \subset \mathbb{C}^{n+1}$, and the lines $\ell_1, \dots, \ell_n$ correspond to planes $\widetilde{\ell}_1, \dots, \widetilde{\ell}_n \subset \mathbb{C}^{n+1}$ passing through $\widetilde{u}$. Further, we have linear maps $\widetilde{\ell}_j \to \mathbb{CP}^k$ coinciding on $\widetilde{u}$. The required result is now trivial.

3.2. Integral manifolds of the field Cone. Let R be a complex submanifold of $\mathrm{Gr}_{p,q}$. We call R an *integral manifold of the field of cones* Cone if for each $Z \in R$ the tangent space $T_Z(R)$ to R at Z lies in Cone_Z.

Lemma 3.3. *Each manifold R of dimension $\geqslant 2$ that is an integral manifold of the field* Cone *lies in some quasiplane.*

Proof. The cone Cone_Z contains two families of linear subspaces, $\mathcal{V}_a$ and $\mathcal{W}_b$, defined as follows. The subspaces $\mathcal{V}_a$ consist of the matrices of the form

$$dZ = a \cdot x,$$

where a is a fixed matrix column and x ranges over the set of matrix rows. The subspaces $\mathcal{W}_b$ consist of the matrices of the form

$$dZ = y \cdot b,$$

where b is a fixed matrix row and y ranges over the set of matrix columns. Of course, these spaces are the tangent spaces of certain quasiplanes through Z.

Each linear subspace of Cone_Z lies in some space $\mathcal{V}_a$ or $\mathcal{W}_b$. Clearly, the tangent space $T_Z(R)$ of the integral manifold R at an arbitrary point Z lies in one of the maximal linear subspaces $\mathcal{V}_a$ or $\mathcal{W}_b$. Arguments based on holomorphy show that either for all $Z\in R$ the space $T_Z(R)$ lies in the subspaces $\mathcal{V}_a$ or for all $Z\in R$ it lies in the subspaces $\mathcal{W}_b$.

For definiteness, let $T_Z(R)\subset\mathcal{W}_b$. It suffices to consider the case $\dim R=2$. Let $R=R(t,s)$ be a parametrization of R in a neighbourhood of a non-singular point. Then the matrices $\dfrac{\partial R}{\partial t}$ and $\dfrac{\partial R}{\partial s}$ lie in a common subspace $\mathcal{W}_b$ (with b dependent on t and s). Hence the entries r_{ij} of the matrix R satisfy equations of the following form:

$$\begin{cases} \dfrac{\partial}{\partial t} r_{ij} = a_i(t,s) b_j(t,s), \\ \dfrac{\partial}{\partial s} r_{ij} = c_i(t,s) b_j(t,s). \end{cases}$$

We emphasize that the matrices $\dfrac{\partial R}{\partial t}$ and $\dfrac{\partial R}{\partial s}$ must be linearly independent, that is, the vectors $(a_1,\dots,a_p)$ and $(c_1,\dots,c_p)$ cannot be collinear.

We claim that the quantity

$$\frac{b_1(t,s)}{b_2(t,s)}$$

is independent of t and s. To this end we write down the relation $\dfrac{\partial^2}{\partial t\partial s}=\dfrac{\partial^2}{\partial s\partial t}$ in our case:

$$\frac{\partial}{\partial s}\bigl(a_i(t,s)b_j(t,s)\bigr)=\frac{\partial}{\partial t}\bigl(c_i(t,s)b_j(t,s)\bigr),$$

or equivalently,

$$\left(\frac{\partial a_i}{\partial s}-\frac{\partial c_i}{\partial t}\right)b_j = c_i\frac{\partial b_j}{\partial t}-a_i\frac{\partial b_j}{\partial s}\,.$$

Dividing both sides by b_j we obtain

$$\frac{\partial a_i}{\partial s}-\frac{\partial c_i}{\partial t}=c_i\frac{\partial}{\partial t}(\ln b_j)-a_i\frac{\partial}{\partial s}(\ln b_j). \tag{3.1}$$

Subtracting (3.1) with $j=2$ from the same equation with $j=1$ we obtain

$$c_i\frac{\partial}{\partial t}\left(\ln\frac{b_2}{b_1}\right)-a_i\frac{\partial}{\partial s}\left(\ln\frac{b_2}{b_1}\right)=0$$

for all i. However, the vectors $(a_1,a_2,\dots)$ and $(c_1,c_2,\dots)$ are not collinear, so that

$$\frac{\partial}{\partial t}\left(\ln\frac{b_2}{b_1}\right)=\frac{\partial}{\partial s}\left(\ln\frac{b_2}{b_1}\right)=0,$$

that is, $b_2/b_1=\mathrm{const}$. In a similar way, for all k and l we have $b_k/b_l=\mathrm{const}$. Hence the matrices $\dfrac{\partial R}{\partial t}$ and $\dfrac{\partial R}{\partial s}$ lie in some subspace $\mathcal{W}_b$ with b independent of t and s. The required result is now obvious.

3.3. Elimination of degenerate cases. Let $\mathcal{O}$ be an open connected subset of $\mathrm{Mat}_{p,q}$.

Lemma 3.4. *Let Ψ be a non-degenerate conformal map from $\mathcal{O}$ into* $\mathrm{Mat}_{r,s}$. *Then*

(a) *the image of each quasiplane lies in some quasiplane*;
(b) *the image of each quasiline lies in some quasiline*;
(c) *either the image of each quasiplane of the* 1*st kind lies in a quasiplane of the* 1*st kind and the image of each quasiplane of the* 2*nd kind lies in a quasiplane of the* 2*nd kind or, on the contrary, the image of each quasiplane of the* 1*st kind lies in a quasiplane of the* 2*nd kind and the image of each quasiplane of the* 2*nd kind lies in a quasiplane of the* 1*st kind*;
(d) *the image of a quasiplane in general position has dimension* $\geqslant 2$.

Proof. (1) Quasiplanes are integral manifolds of the field of cones. Hence the image of a quasiplane is also an integral manifold of the field of cones. That is, the image of a quasiplane lies either in a quasiplane or in a (complex) curve.

The rest of the proof is obvious but becomes quite lengthy when written down.

(2) We consider a point $X \in \mathcal{O}$ such that the differential d_X of Ψ does not identically vanish. The cone Cone_X spans the entire tangent space T_X. Hence the value of the linear operator d_X on the general vector in Cone_X is not zero and therefore the map Ψ takes a quasiline in general position through X into a curve (rather than a point).

(3) Assume that all quasiplanes of the 1st and the 2nd kind are mapped into curves and consider a quasiline ℓ such that its Ψ-image is not a point. Then close quasilines are not taken into points either. We consider now a quasiplane of the 1st kind V and a quasiplane of the 2nd kind W such that $V \cap W = \ell$. Then the images of V, W, and ℓ are curves, and therefore the sets $\Psi(V)$, $\Psi(W)$, and $\Psi(\ell)$ lie on the same curve.

Next, we consider a quasiplane V' of the 1st kind intersecting W in a quasiline ℓ' close to ℓ. Then $\Psi(\ell')$ is a curve that, on the one hand, coincides (locally) with $\Psi(\ell) \simeq \Psi(W)$ and, on the other, must coincide with $\Psi(V')$. Hence $\Psi(V')$ coincides with $\Psi(\ell)$.

Repeating these arguments we see that all quasilines close to ℓ are mapped into $\Psi(\ell)$. Hence the rank of the differential at each point in ℓ or in a quasiline close to ℓ (that is, at all points in a neighbourhood of ℓ) is equal to 1. Thus, Ψ maps the entire set $\mathcal{O}$ into a single curve, which contradicts the non-degeneracy of Ψ. This proves (d).

(4) Assume that the map Ψ takes some quasiplane of the 1st kind V into a manifold of dimension $\geqslant 2$, while the image of each quasiplane of the 2nd kind lies on a curve. We claim that the entire range of Ψ lies in one quasiplane in this case.

Let S be a quasiplane containing $\Psi(V)$. We consider a point $A \in V$ such that the restriction of the differential d_A to V has rank $\geqslant 2$. Let $\ell \subset V$ be a quasiline through A such that d_A does not annihilate the direction vector of the quasiline ℓ. Let W be a quasiplane of the 2nd kind such that $\ell = V \cap W$. Then the curve $\Psi(W)$ coincides with $\Psi(\ell)$, so that $\Psi(W)$ lies in the quasiplane S.

Further, we consider in W a quasiplane ℓ' close to ℓ. We consider a quasiplane of the 1st kind V' such that $\ell' = V' \cap W$. Then ℓ' is mapped into a curve, which must be $\Psi(\ell)$. Thus, the quasiplanes containing $\Psi(V)$ and $\Psi(V')$ must contain a common curve (namely, $\Psi(\ell)$), therefore they must be the same (see Lemma 1.1).

Thus, for each quasiplane V' of the 1st kind and in general position that passes through A we have the inclusion $\Psi(V') \subset S$. By continuity the same must hold for each quasiplane through A. Hence the entire determinant submanifold $\mathrm{D}_1(A)$ is mapped into the quasiplane S.

Next, we consider a point $A' \in \mathrm{D}_1(A)$ close to A. The same arguments show that $\Psi(\mathrm{D}_1(A')) \subset S$. Hence we immediately obtain that $\Psi(\mathrm{D}_2(A)) \subset S$, and so forth.

(5) We have thus proved (a).

(6) Assume that all quasiplanes (of the 1st and 2nd kinds) are mapped, for instance, into quasiplanes of the 1st kind. We consider now a quasiline ℓ in general position; let ℓ be the intersection of quasiplanes V and W. Then the sets $\Psi(V)$ and $\Psi(W)$ intersect in the curve $\Psi(\ell)$, therefore (see Lemma 1.1) the quasiplanes of the 1st kind containing $\Psi(V)$ and $\Psi(W)$ are the same. It is now easy to deduce that the entire range of Ψ lies in one hyperplane.

(7) Thus, we have established (c). This immediately gives us assertion (b). For let ℓ be a quasiline in general position and let V and W be quasiplanes containing it (that is, $\ell = V \cap W$). Let S and Q be quasiplanes containing $\Psi(V)$ and $\Psi(W)$. Then the set $\Psi(\ell)$ lies in $\Psi(V) \cap \Psi(W) \subset S \cap Q$. Hence the intersection $S \cap Q$ is non-empty and therefore it is a quasiline (see Lemma 1.1). This proves (b) for quasilines in general position, and by continuity (b) holds for all quasilines.

Corollary 3.5. *Each non-degenerate conformal map is locally projective on quasiplanes in general position.*

Proof. In fact, such a map takes quasilines to quasilines.

Corollary 3.6. *Let $\ell \subset \mathrm{Mat}_{p,q}$ be a quasiline such that the set $\Psi(\ell)$ contains more than one point. Then, as a map between the projective lines ℓ and $\Psi(\ell)$, Ψ is projective.*

Proof. In fact, the maps of quasiplanes in general position are locally projective, therefore they are projective on quasilines.

Corollary 3.7. *For each point A the image of the determinant submanifolds $\mathrm{D}_1(A)$ lies in the determinant manifold $\mathrm{D}_1(\Psi(A))$.*

We say that a non-degenerate conformal map '*respects*' *the kind of hyperplanes* if the image of each quasiplane of the 1st kind lies in a quasiplane of the 1st kind and the image of each quasiplane of the 2nd kind lies in a quasiplane of the 2nd kind.

Assume that a map $\Psi\colon \mathcal{O} \to \mathrm{Mat}_{r,s}$ does not respect the kind of hyperplanes. Then the map

$$Z \mapsto \Psi(Z)^t$$

from $\mathcal{O}$ into $\mathrm{Mat}_{s,r}$ respects the kind of hyperplanes. For that reason we can content ourselves with maps respecting the kind of hyperplanes.

3.4. Maps of determinant submanifolds $D_1(A)$. Let Ψ be a non-degenerate conformal map respecting the kind of hyperplanes. We say that the differential d_A of Ψ is *non-degenerate at a point A* if the operator d_A has rank > 1 on some quasiplane of the 1st kind and on some quasiplane of the 2nd kind passing through A.

Lemma 3.8. *Let Ψ be a non-degenerate conformal map respecting the kind of quasiplanes. Assume that the differential of Ψ is non-degenerate at* 0*. Then Ψ coincides on* $\mathrm{D}_1(0)$ *with some generalized linear-fractional map.*

Proof. (1) Without loss of generality we can set $\Psi(0) = 0$. We denote the cone $\mathrm{D}_1(0) \subset \mathrm{Mat}_{p,q}$ by D_1 and the cone $\mathrm{D}_1(0) \subset \mathrm{Mat}_{r,s}$ by $\widetilde{\mathrm{D}}_1$.

(2) We fix a quasiplane of the 1st kind V° in $\mathrm{Mat}_{p,q}$ passing through 0 and a quasiplane of the 2nd kind W° passing through 0. Let $\mathbb{P}V^\circ$ be the set of quasilines passing through 0 and lying in V°. We define in a similar way $\mathbb{P}W^\circ$. Clearly, $\mathbb{P}V^\circ$ as a complex manifold is isomorphic to $\mathbb{CP}^{q-1}$ and $\mathbb{P}W^\circ$ is isomorphic to $\mathbb{CP}^{p-1}$.

We observe now that quasiplanes of the 2nd kind passing through 0 are in one-to-one correspondence with points in $\mathbb{P}V^\circ \simeq \mathbb{CP}^{q-1}$. Namely, with each quasiplane of the 2nd kind W we associate the intersection $W \cap V^\circ$. It is a quasiline, that is, a point in $\mathbb{P}V^\circ \simeq \mathbb{CP}^{q-1}$. In a similar way, quasiplanes of the 1st kind passing through 0 are in one-to-one correspondence with points in $\mathbb{P}W^\circ \simeq \mathbb{CP}^{p-1}$.

(3) The manifold D_1 is a cone with base

$$\mathbb{CP}^{p-1} \times \mathbb{CP}^{q-1}.$$

For consider a generator of this cone, that is, a quasiline through 0. It can be represented in a unique way as the intersection of a quasiplane of the 1st kind and a quasiplane of the 2nd kind (see Lemma 1.1). As we saw in (2), quasiplanes of the 1st kind are indexed by points in $\mathbb{CP}^{p-1}$ and quasiplanes of the 2nd kind are indexed by planes in $\mathbb{CP}^{q-1}$.

(4) A map between cones $\Psi\colon \mathrm{D}_1 \to \widetilde{\mathrm{D}}_1$ induces a map between their bases:

$$\Lambda\colon \mathbb{CP}^{p-1} \times \mathbb{CP}^{q-1} \to \mathbb{CP}^{r-1} \times \mathbb{CP}^{s-1},$$

which is defined in an open dense subset of $\mathbb{CP}^{p-1} \times \mathbb{CP}^{q-1}$. Moreover, it follows from (3) that Λ is a product of maps

$$\lambda_1\colon \mathbb{CP}^{p-1} \to \mathbb{CP}^{r-1} \quad \text{and} \quad \lambda_2\colon \mathbb{CP}^{q-1} \to \mathbb{CP}^{s-1}.$$

However, Ψ is locally projective on the quasiplanes V° and W°. Hence the maps λ_1 and λ_2 are induced by certain linear maps

$$A\colon \mathbb{C}^p \to \mathbb{C}^r \quad \text{and} \quad B\colon \mathbb{C}^q \to \mathbb{C}^s.$$

(5) The next question is as follows: to what extent is a map between cones $\mathrm{D}_1 \to \widetilde{\mathrm{D}}_1$ defined by the map between their bases?

(6) We claim that the differential of Ψ at zero is defined by the maps λ_1 and λ_2 to within a coefficient.

We consider a quasiplane Y in general position passing through 0. Let y be an arbitrary tangent vector to Y at 0. We know the value of the differential on y to within a coefficient (for we know the image of the corresponding quasiline). Since the rank of the differential at 0 is larger than 1, the operator d is defined on Y up to multiplication by a constant.

We now fix some possible value of the differential on the quasiplane Y. Let X be a quasiplane of another kind through 0. The differential on it is also defined up to multiplication by a constant. However, we have already fixed the value of the differential on the direction vector of $\ell = X \cap Y$ (for we have fixed it on Y). Thus, the operator d is uniquely defined on X, and such quasiplanes X sweep out the entire cone Cone_0. Hence the differential d is fixed on the cone Cone_0 to within a coefficient. Since this cone spans the entire tangent space, this gives us the required result.

(7) Thus, the differential of Ψ has the following form at the origin:

$$dZ \mapsto A \cdot dZ \cdot B, \tag{3.2}$$

where A and B are $(r \times p)$- and $(q \times s)$-matrices, respectively.

Let E_{ij} be the matrix having 1 at the intersection of the ith row and the jth column and having zeros at other places. We can assume that the differential does not annihilate any of the matrix units E_{ij}. For otherwise we could make a change of variables of the form $Z \mapsto SZT$, where S and T are non-singular square matrices.

(8) We claim that the map $\Psi\colon \mathrm{D}_1 \to \widetilde{\mathrm{D}}_1$ is uniquely defined by its restrictions to the quasilines $t \cdot E_{ij}$, where $t \in \mathbb{C}$.

We consider a quasiplane of the 2nd kind V_α consisting of all matrices of the form $\sum_i t_{i\alpha} E_{i\alpha}$. By Lemma 3.2 the restriction of Ψ to the quasiplane V_α is well defined.

We consider now a quasiplane W of the 1st kind; let $\ell_\alpha = W \cap V_\alpha$. Clearly, for quasiplanes W in general position the value of d on the direction vector of ℓ_α does not vanish, therefore the map Ψ is well defined on each quasiplane W of the 1st kind by Lemma 3.2. However, such quasiplanes sweep out $\mathrm{D}_1(0)$.

(9) Assume now that the differential (3.2) is fixed. Let ℓ_{ij} be the quasiline consisting of the points $t \cdot E_{ij}$. Let m_{ij} be the image of this quasiline under the map Ψ, and let ψ_{ij} be the restriction of Ψ to ℓ_{ij}. Note that, given the differential, we also know the quasilines m_{ij} and the first derivatives

$$\psi'_{ij}(0) := \frac{d}{dt}\psi_{ij}(t)\Big|_{t=0}.$$

On the other hand, a projective map $\psi_{ij}\colon \ell_{ij} \to m_{ij}$ is uniquely determined by the quantities

$$\psi_{ij}(0) = 0, \qquad \psi'_{ij}(0), \qquad \psi''_{ij}(0).$$

It remains to show that each collection of second derivatives can in fact be obtained from some generalized linear-fractional map

$$\Phi(Z) = AZ(1 - NZ)^{-1}B.$$

Setting $Z = t \cdot E_{ij}$ we obtain

$$\Phi(t \cdot E_{ij}) = (t - n_{ji}t^2)AE_{ij}B,$$

where the $n_{\alpha\beta}$ are the entries of the matrix N. We have $AE_{ij}B \neq 0$ (this means that the differential does not vanish at E_{ij}), therefore choosing a suitable matrix N we can obtain an arbitrary collection of second derivatives.

3.5. Intersection of determinant manifolds. Let $A \in \mathrm{Mat}_{p,q}$ be a matrix of rank $k > 1$. We are interested in the intersection of the determinant manifolds

$$\mathrm{D}_{k-1}(0) = \{Z : \operatorname{rk} Z = k-1\}$$

and

$$\mathrm{D}_1(A) = \{Z : \operatorname{rk}(Z - A) = 1\}.$$

We now explain how one can describe all points in their intersection.

Thus, let A be an operator $\mathbb{C}^p \to \mathbb{C}^q$ of rank k. Let f be a non-trivial linear functional vanishing at the kernel $\ker A$ of A. We choose $v \in \mathbb{C}^p$ such that

$$f(v) = 1.$$

Further, we define an operator Y of rank 1 from $\mathbb{C}^p$ into $\mathbb{C}^q$ by the formula

$$Yx = f(x)\, Av.$$

Then the operator

$$B = A - Y$$

lies in the required intersection, and one can obtain in this way all points in this intersection. It is easy to see that

$$\dim\bigl(\mathrm{D}_{k-1}(0) \cap \mathrm{D}_1(0)\bigr) = 2k - 1.$$

Lemma 3.9. *Let* $\operatorname{rk} A = \operatorname{rk} A' = k$ *and assume that* $A \neq A'$. *Then the sets*

$$\mathrm{D}_{k-1}(0) \cap D_1(A) \quad \textit{and} \quad \mathrm{D}_{k-1}(0) \cap D_1(A')$$

are distinct.

Proof. We must describe the recovery of the operator A from $\Xi = \mathrm{D}_{k-1}(0) \cap D_1(A)$.

(1) It is easy to see that

$$\ker A = \bigcap_{B \in \Xi} \ker B.$$

Hence we can assume without loss of generality that $\ker A = 0$.

(2) We fix a subspace H of $\mathbb{C}^p$ of codimension 1. For each operator $B \in \Xi$ we consider its restriction $\mathrm{Res}_H(B)$ to H. We are interested in the level sets (that is, inverse images of points) of the map

$$B \mapsto \mathrm{Res}_H(B)$$

from Ξ into the space of operators $H \to \mathbb{C}^q$.

(3) Let φ be a linear functional in $\mathbb{C}^q$ annihilating H. Then for all vectors v such that $\varphi(v) = 1$ the operators of the form

$$Bx = Ax - \varphi(x)\, Av \tag{3.3}$$

coincide on H. These operators are indexed by a linear functional φ (defined to within a coefficient) and a vector v. Thus, the operators of the form (3.3) make up a k-dimensional family. For all operators B obtained in this way the corresponding operators $\mathrm{Res}_H(B)$ are the same and are equal to $\mathrm{Res}_H(A)$.

(4) We consider now the two operators,

$$Bx = Ax - f(x)\,Av$$

and

$$Cx = Ax - g(x)\,Aw,$$

that are not of the form described in (3) (that is, neither f nor g annihilates H). The equality $\mathrm{Res}_H(B) = \mathrm{Res}_H(C)$ is possible only in the case when

$$f(x)\,Av = g(x)\,Aw \quad \text{for all } x \in H.$$

Hence the vectors Av and Aw are proportional. The matrix A is non-degenerate, therefore v and w are proportional. Multiplying g by a coefficient we can assume that $v = w$.

Now, the functionals f and g satisfy the conditions

$$f(x) = g(x) \quad \text{for all } x \in H,$$
$$f(v) = g(v) = 1.$$

If $v \notin H$, then $f = g$ on the whole of $\mathbb{C}^p$, that is, $B = C$, and the level set is a singleton.

Let $v \in H$ and let φ be a linear functional annihilating H. Then there exists $\lambda \in \mathbb{C}$ such that

$$f(x) = g(x) + \lambda\varphi(x),$$

that is, the level set is one-dimensional.

(5) Thus, the map $\mathrm{Res}_H(\,\cdot\,)$ has a unique level set of dimension k (recall that $k \geqslant 2$). Hence we know (see (3)) the restriction of A to an (arbitrary) subspace $H \subset \mathbb{C}^p$ of codimension 1. The proof is complete.

3.6. Induction on the index of a determinant submanifold. Let Ψ be a non-degenerate conformal map respecting the kind of quasiplanes. Assume that the differential of Ψ is non-degenerate at zero (see § 3.4), and let $\Psi(0) = 0$. We have already seen that Ψ coincides on the determinant submanifold $\mathrm{D}_1(0)$ with some generalized linear-fractional map Φ. Now, using Lemmas 3.10 and 3.11 we shall successively show that $\Phi = \Psi$ on $\mathrm{D}_2(0), \mathrm{D}_3(0), \dots$.

Let us first of all introduce possibly simpler coordinates. Consider a generalized linear-fractional map

$$Y = \Phi(Z) = AZ(1 - NZ)^{-1}B.$$

We introduce the variable

$$\widetilde{Z} := Z(1 - NZ)^{-1}, \tag{3.4}$$

and reduce our map to the form

$$Y = \Phi(\widetilde{Z}) = A\widetilde{Z}B.$$

Further, we can change coordinates in $\mathrm{Mat}_{p,q}$ and $\mathrm{Mat}_{r,s}$ by the formulae

$$Z \mapsto S_1 Z S_2, \qquad Y \mapsto T_1 Y T_2, \tag{3.5}$$

where S_1, S_2, T_1, and T_2 are invertible square matrices.

Thus, we can assume without loss of generality that Φ has the following form:

$$\Phi(Z) = PZQ, \tag{3.6}$$

where

$$P = \begin{pmatrix} E_\alpha & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} E_\beta & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.7}$$

and E_α, E_β are the identity matrices of sizes $\alpha \times \alpha$ and $\beta \times \beta$, respectively.

Lemma 3.10. *Let Ψ be a non-degenerate conformal map with non-degenerate differential at zero. Assume that Ψ coincides with a generalized linear-fractional map Φ on $\mathrm{D}_{k-1}(0)$. Let α and β be as above and assume that $2 \leqslant k \leqslant \min\{\alpha;\beta\}$. Then $\Phi = \Psi$ on $\mathrm{D}_k(0)$.*

Proof. (1) Let $A \in \mathrm{D}_k(0)$ be a matrix in general position. Then $\operatorname{rk}\Psi(A) = k$ and $\operatorname{rk}\Phi(A) = k$ (the second relation is obvious, and the first follows from the explicit formula for the differential).

(2) Let us introduce our notation:

$$\begin{aligned} \Xi &= \mathrm{D}_{k-1}(0) \cap \mathrm{D}_1(A) \subset \mathrm{Mat}_{p,q}, \\ \Pi_\Psi &= \mathrm{D}_{k-1}(0) \cap \mathrm{D}_1(\Psi(A)) \subset \mathrm{Mat}_{r,s}, \\ \Pi_\Phi &= \mathrm{D}_{k-1}(0) \cap \mathrm{D}_1(\Phi(A)) \subset \mathrm{Mat}_{r,s}\,. \end{aligned}$$

The dimension of all these sets is $2k-1$. Since both Ψ and Φ decrease the integer distance, the set

$$\Phi(\Xi) = \Psi(\Xi)$$

lies in both Π_Ψ and Π_Φ.

(3) We claim that if A is a matrix in general position, then the dimension of the set $\Phi(\Xi) = \Psi(\Xi)$ is $2k-1$. It suffices to show that this holds for one matrix A. However, each matrix of the form

$$A = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix},$$

where the upper left corner is of size $\alpha \times \beta$, has these properties.

(4) Thus, the $(2k-1)$-dimensional sets Π_Ψ and Π_Φ contain a $(2k-1)$-dimensional subset $\Psi(\Xi) = \Phi(\Xi)$. In view of the analyticity, the sets Π_Ψ and Π_Φ are the same. By Lemma 3.9 we obtain the equality $\Psi(A) = \Phi(A)$.

Lemma 3.11. *Let α and β be as above and assume that $k > \min\{\alpha;\beta\} \geqslant 2$. Let Ψ be a non-degenerate conformal map with non-degenerate differential at zero. Assume that Ψ coincides with a generalized linear-fractional map Φ on $\mathrm{D}_{k-1}(0)$. Then $\Phi = \Psi$ non $\mathrm{D}_k(0)$.*

Proof. For definiteness, assume that $\alpha \leqslant \beta$.

(1) We express an element $Z \in \mathrm{Mat}_{p,q}$ as a block matrix

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}$$

of size $(\alpha + (p-\alpha)) \times (\beta + (q-\beta))$. Then

$$\Phi(Z) = \begin{pmatrix} Z_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Moreover, if $\operatorname{rk} Z < k$, then $\Psi(Z) = \Phi(Z)$.

(2) Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathrm{Mat}_{p,q}.$$

Let $\operatorname{rk} A = k$ and assume that the first k rows in the matrix A are linearly independent (which is true for matrices $A \in \mathrm{D}_k(0)$ in general position). We claim that $\Psi(A) = \Phi(A)$. Once this is established, continuity arguments show that $\Psi(A) = \Phi(A)$ for all $A \in \mathrm{D}_k(0)$.

(3) Let Θ be the subset

$$\Theta = \mathrm{D}_{\alpha-1}(0) \cap \mathrm{D}_1(E_\alpha)$$

of the space of $(\alpha \times \alpha)$-matrices.

We fix $S \in \Theta$ and consider the matrix

$$U = \begin{pmatrix} S & 0 \\ T & E_{p-\alpha} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where the first factor is a block $((\alpha + (p-\alpha)) \times (\alpha + (p-\alpha)))$-matrix. Clearly, $\operatorname{rk} U \leqslant k$. We claim that we can choose the matrix T such that $\operatorname{rk} U = k-1$ and $\operatorname{rk}\left(\begin{pmatrix} S & 0 \\ T & E_{p-\alpha} \end{pmatrix} - E_p\right) = 1$.

(4) We denote the rows of A by $a_1, \dots, a_p$. We can choose a basis in $\mathbb{C}^\alpha$ such that S takes the following form:

$$\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}.$$

Assume that $h > k$. We expand the row a_h in terms of $a_1, \dots, a_k$:

$$a_h = \sum_{j=1}^{k} \nu_{h_j} a_j.$$

Next, we set the entries $t_{(h-\alpha)1}$ of T equal to $-\nu_{h1}$ and we set all other entries equal to 0. The construction of T is complete.

(5) Thus, $\operatorname{rk} U = k-1$, therefore $\Psi(U) = \Phi(U)$. Hence

$$\Psi(U) = \begin{pmatrix} SA_{11} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.8}$$

As S ranges over Θ the set of matrices of the form (3.8) ranges over

$$\mathrm{D}_{k-1}(0) \ \cap \ \mathrm{D}_1 \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.9}$$

On the other hand $\operatorname{rk}(A-U) = 1$, therefore $\operatorname{rk}(\Psi(A) - \Psi(U)) \leqslant 1$, that is, $\Psi(U)$ belongs to $\mathrm{D}_{k-1}(0) \cap \mathrm{D}_1(\Psi(A))$. Hence $\mathrm{D}_{k-1}(0) \cap \mathrm{D}_1(\Psi(A))$ coincides with (3.9). By Lemma 3.9,

$$\Psi(A) = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

as required.

3.7. Non-degeneracy of the differential. We have shown that, by means of linear-fractional changes of variables (3.4), (3.5), a conformal map can be brought into the form (3.6), (3.7). However, the map (3.6), (3.7) is generalized linear-fractional, therefore, by Proposition 2.1, it is defined by an $(r+q)$-dimensional linear relation in $\mathbb{C}^{p+q} \oplus \mathbb{C}^{r+s}$. Linear-fractional changes of variables in $\mathrm{Mat}_{p,q}$ and $\mathrm{Mat}_{r,s}$ correspond to the transformations of the Grassmannians $\mathrm{Gr}_{p,q}$ and $\mathrm{Gr}_{r,s}$ induced by linear transformations of $\mathbb{C}^{p+q}$ and $\mathbb{C}^{r+s}$.

Thus, each conformal map is induced by a linear relation, that is, we have established Theorem 2.4. As regards Theorem 2.2, we have proved it only under the assumption that the differential at zero of the map Ψ is non-degenerate. We now show that this restriction is not important.

Lemma 3.12. *Let P be a linear relation in $\mathbb{C}^{p+q} \oplus \mathbb{C}^{r+s}$ of dimension $r+q$, let $\ker P$ be the subspace $P \cap \mathbb{C}^{p+q}$ and let $\operatorname{dom} P$ be the projection of P onto $\mathbb{C}^{p+q}$. Further, let $\mathcal{M}$ be the set of all $H \in \mathrm{Gr}_{p,q}$ such that*

$$\ker P \cap H = 0 \quad \textit{and} \quad \operatorname{dom} P + H = \mathbb{C}^{p+q}. \tag{3.10}$$

Then the map $H \mapsto PH$ is holomorphic in $\mathcal{M}$ and has non-removable singularities at all points in $\mathrm{Gr}_{p,q} \setminus \mathcal{M}$.

The proof is trivial, and we leave it out.

It is easy to verify that the differential of the map $H \mapsto PH$ is either degenerate at all points in $\mathcal{M}$ or is non-degenerate at all points in this set.

This observation completes the proof of Theorem 2.2.

§ 4. Other symmetric spaces

4.1. Matrix spaces. By a *matrix space* we mean a space in one of the following 10 series:

- the spaces of $(p \times q)$-matrices over a division ring $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$;
- the spaces of real symmetric (skew-symmetric) matrices of size $n \times n$;
- the spaces of complex symmetric (skew-symmetric, Hermitian) matrices of size $n \times n$;
- the spaces of quaternion Hermitian (anti-Hermitian) $(n \times n)$-matrices.

The list of all these spaces is presented in the table opposite. We indicate the division ring $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ in the second column and, in the fourth, the additional condition imposed on the matrices.

4.2. Grassmannians. Consider one of the above-listed matrix spaces Mat°. We shall regard elements $Z \in \mathrm{Mat}^\circ$ as the matrices of the linear operators $v \mapsto v\,Z$ from the linear space V^+ into the linear space V^- (using the notation of the previous subsection, $V^+ = \mathbb{K}^p$ and $V_- = \mathbb{K}^q$, or $V^+ = \mathbb{K}^n$ and $V_- = \mathbb{K}^n$).

We consider $\mathrm{graph}(Z)$, the graph of the operator Z. In cases 1, 5, and 8 (see the first column of the table) this can be an arbitrary p-dimensional linear subspace of $V^+ \oplus V^-$ disjoint from V^-. In cases 2–4, 6–7, 9–10 $\mathrm{graph}(Z)$ is a maximal isotropic subspace with respect to the form $\Lambda = \Lambda((v^+, v^-); (w^+, w_-))$ (which can be symmetric, skew-symmetric, Hermitian, or anti-Hermitian) indicated in column 5.

Let Gr° be the Grassmannian of all subspaces that are maximal isotropic with respect to the form Λ indicated in column 5 (in cases 1, 5, and 8, when there is no form indicated, we merely take the Grassmannian of all p-dimensional subspaces). The range of the embedding graph: $\mathrm{Mat}^\circ \to \mathrm{Gr}^\circ$ is dense in Gr° except for cases 3 and 7. In these cases the Grassmannian has two connected components and the range of this embedding is dense in one of them.

We consider now the group G^* indicated in column 6. It consists of all linear operators in $V^+ \oplus V^-$ preserving Λ. This group acts in Gr° in an obvious way. We shall write elements $g \in G^*$ as block matrices

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : V^+ \oplus V^- \to V^+ \oplus V^-.$$

In cases 2–4, 6–7, 9–10 the matrix g preserves the form Λ, which imposes on it a condition of the following form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & E \\ \pm E & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^\sigma = \begin{pmatrix} 0 & E \\ \pm E & 0 \end{pmatrix},$$

where σ signifies transposition or conjugation.

Using the variable $Z \in \mathrm{Mat}^\circ$ in Gr° we can express the action of Gr° on the Grassmannian as follows:

$$Z \mapsto (A + ZC)^{-1}(B + ZD).$$

1	2	3	4	5	6	7	8	9	10
	$\mathbb{K}$	Size	Condition on matrix	form Λ	G^*	G_{nc}	G_c	K	
1	$\mathbb{C}$	$p \times q$	—	—	$\mathrm{GL}(p+q, \mathbb{C})$	$\mathrm{U}(p,q)$	$\mathrm{U}(p+q)$	$\mathrm{U}(p) \times \mathrm{U}(q)$	—
2	$\mathbb{C}$	$n \times n$	$Z = Z^t$	$\sum_i (v_i^+ w_i^- - v_i^- w_i^+)$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{Sp}(2n, \mathbb{R})$	$\mathrm{Sp}(n)$	$\mathrm{U}(n)$	$S = S^t$
3	$\mathbb{C}$	$n \times n$	$Z = -Z^t$	$\sum_i (v_i^+ w_i^- + v_i^- w_i^+)$	$\mathrm{O}(2n, \mathbb{C})$	$\mathrm{SO}^*(2n, \mathbb{R})$	$\mathrm{O}(2n)$	$\mathrm{U}(n)$	$K = -K^t$, $N = -N^t$, $M = L^t$
4	$\mathbb{C}$	$n \times n$	$Z = Z^*$	$\sum_i (v_i^+ \overline{w}_i^- - v_i^- \overline{w}_i^+)$	$\mathrm{U}(n,n)$	$\mathrm{GL}(n, \mathbb{C})$	$\mathrm{U}(n) \times \mathrm{U}(n)$	$\mathrm{U}(n)$	$S = S^*$
5	$\mathbb{R}$	$p \times q$	—	—	$\mathrm{GL}(p+q, \mathbb{R})$	$\mathrm{O}(p,q)$	$\mathrm{O}(p+q)$	$\mathrm{O}(p) \times \mathrm{O}(q)$	—
6	$\mathbb{R}$	$n \times n$	$Z = Z^t$	$\sum_i (v_i^+ w_i^- - v_i^- w_i^+)$	$\mathrm{Sp}(2n, \mathbb{R})$	$\mathrm{GL}(n, \mathbb{R})$	$\mathrm{U}(n)$	$\mathrm{O}(n)$	$S = S^t$
7	$\mathbb{R}$	$n \times n$	$Z = -Z^t$	$\sum_i (v_i^+ w_i^- + v_i^- w_i^+)$	$\mathrm{O}(n,n)$	$\mathrm{O}(n, \mathbb{C})$	$\mathrm{O}(n) \times \mathrm{O}(n)$	$\mathrm{O}(n)$	$K = -K^t$, $N = -N^t$, $M = L^t$
8	$\mathbb{H}$	$p \times q$	—	—	$\mathrm{GL}(p+q, \mathbb{H})$	$\mathrm{Sp}(p,q)$	$\mathrm{Sp}(p+q)$	$\mathrm{Sp}(p) \times \mathrm{Sp}(q)$	—
9	$\mathbb{H}$	$n \times n$	$Z = Z^*$	$\sum_i (v_i^+ \overline{w}_i^- - v_i^- \overline{w}_i^+)$	$\mathrm{SO}^*(4n)$	$\mathrm{GL}(n, \mathbb{H})$	$\mathrm{U}(2n)$	$\mathrm{Sp}(n)$	$S = S^*$
10	$\mathbb{H}$	$n \times n$	$Z = -Z^*$	$\sum_i (v_i^+ \overline{w}_i^- + v_i^- \overline{w}_i^+)$	$\mathrm{Sp}(n,n)$	$\mathrm{Sp}(2n, \mathbb{C})$	$\mathrm{Sp}(n) \times \mathrm{Sp}(n)$	$\mathrm{Sp}(n)$	$K = -K^*$, $N = -N^*$, $M = L^*$

4.3. Symmetric spaces. We consider the following three subgroups of G^*.

(a) The non-compact subgroup G_{nc} of operators in $V^+ \oplus V^-$ preserving the indefinite Hermitian form

$$M((v^+, v^-); (w^+, w^-)) := \sum v_i^+ \overline{w_i^+} - \sum v_j^- \overline{w_j^-}$$

(see column 7). In other words, G_{nc} is a subgroup of G^* described by the additional condition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}.$$

(b) The compact subgroup G_{c} of operators in $V^+ \oplus V^-$ preserving the positive-definite Hermitian form

$$E\big((v^+, v^-); (w^+, w^-)\big) := \sum v_i^+ \overline{w_i^+} + \sum v_j^- \overline{w_j^-}$$

(see column 8). In other words, G_{nc} is the subgroup of G^* described by the additional condition

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^* = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}.$$

(c) The compact subgroup K of matrices of the following form:

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

where A is an unitary operator in V^+ (that is, $AA^* = E$), and D is unitary in V^-.

It is easy to see that

$$K = G_{\mathrm{nc}} \cap G_{\mathrm{c}}.$$

In all 10 cases the Grassmannian G° is a G_{c}-homogeneous space, and the group K is always the stabilizer of the subspace $V^+ \in \mathrm{Gr}^\circ$. Thus,

$$\mathrm{Gr}^\circ = G_c/K.$$

It is easy to see that the 10 series of the spaces $G_{\mathrm{c}/K}$ cover all 10 series of *compact* symmetric spaces (up to centres and coverings; for instance, the space $\mathrm{U}(n)/\,\mathrm{O}(n)$ gets into our table, but $\mathrm{SU}(n)/\,\mathrm{SO}(n)$ does not).

Further, we consider the subset B° of Gr° consisting of the subspaces such that the restriction to them of the Hermitian form M is positive-definite. Then (see [9]) the set B° is a G_{nc}-homogeneous space and the stabilizer of the point $Z = 0$ is equal to K. Thus,

$$B^\circ = G_{\mathrm{nc}}/K.$$

It is easy to see that, among the 10 series of the spaces G_{nc}/K we can find the entire 10 series of *non-compact Riemannian* symmetric spaces (up to centres: for instance, the space $\mathrm{GL}(n, \mathbb{R})/\,\mathrm{O}(n)$ gets into our table, whereas $\mathrm{SL}(n, \mathbb{R})/\,\mathrm{SO}(n)$ is not there).

Further, it is easy to verify (see [9]) that each $P \in \mathrm{B}^\circ$ is the graph of an operator $Z\colon V^+ \to V^-$, where $\|Z\| < 1$. Conversely, the condition $\|Z\| < 1$ yields the inclusion $\mathrm{graph}(Z) \in \mathrm{B}^\circ$. For that reason we shall make no difference between the domain $B^\circ \in \mathrm{Gr}^\circ$ and the matrix ball $\|Z\| < 1$ in Mat°.

4.4. Generalized linear-fractional maps. These are defined for two Grassmannians from the same series and (in the coordinate form) are described by the formula

$$Z \mapsto K + LZ(1 - NZ)^{-1}M, \tag{4.1}$$

where the matrix $S = \begin{pmatrix} K & L \\ M & N \end{pmatrix}$ satisfies the condition indicated in the last column of our table.

If $\|S\| \leqslant 1$ and $\|K\| < 1$, then the map (4.1) takes the matrix ball into the matrix ball.

Remark. For the Hermitian symmetric spaces

$$\mathrm{B}^\circ = \mathrm{U}(p,q)/\,\mathrm{U}(p) \times \mathrm{U}(q), \qquad \mathrm{Sp}(2n,\mathbb{R})/\,\mathrm{U}(p), \qquad \mathrm{SO}^*(2n)/\,\mathrm{U}(n)$$

the semigroup of generalized linear-fractional maps of the unit ball into itself contains the Ol'shanskiĭ semigroup (see [21]).

Theorem 4.1. *Theorem* 0.1 *on the decrease of the compound distance holds for all* 10 *series of matrix balls* (= *Riemannian non-compact symmetric spaces*).

Proof. The proof in [9] is suitable for all 10 series of symmetric spaces.

4.5. Linear structures. (a) One can define *quasilines* for all 10 types of Grassmannian Gr°.

For the full Grassmannians over $\mathbb{R}$ or $\mathbb{H}$ (cases 5 and 8) they are defined in precisely the same way as for the full Grassmannian over $\mathbb{C}$ (case 1).

In cases 2, 4, 6, 9, and 10 we must consider an isotropic subspace S of dimension $n-1$ and its orthogonal complement $S^\perp$. We define the quasiline ℓ_S as the set of all subspaces $H \in \mathrm{Gr}^\circ$ such that $S \subset H \subset S^\perp$.

Finally, the definition in cases 3 and 7 (orthogonal Grassmannians) is similar, but one must consider an isotropic subspace S of dimension $n-2$.

In all cases quasilines are submanifolds homeomorphic to the projective line $\mathbb{PK}^1$.

(b) The *integral distance* $n(H_1, H_2)$ is defined as the codimension of $H_1 \cap H_2$ in H_1 and H_2.

(c) The *determinant submanifold* $\mathrm{D}_k(A)$ consists of the points in H such that $n(H, A) \leqslant k$.

4.6. Conformal structures. We set α to be equal to 2 for the orthogonal Grassmannians in cases 3 and 7 and to 1 in the other cases. In the tangent space at an arbitrary point $H \in \mathrm{Gr}^\circ$ we consider the cone Cone_H consisting of the vectors tangent to the determinant submanifold $\mathrm{D}_\alpha(H)$.

In the coordinate notation the cone Cone_Z in the tangent space at $Z \in \mathrm{Mat}^\circ$ is described by the condition

$$\mathrm{rk}\, dZ \leqslant \alpha.$$

4.7. Conformal maps. Analogues of Theorems 1.3–1.5 hold for all 10 series of Grassmannians (and are already known; see [11], [12], [14], [18], [19]).

It seems plausible that analogues of Theorems 2.3–2.6 also hold for all series. However, this author does not know of any proof of this assertion.

Our proof in § 3 runs into small or big trouble on any attempt to use it for other series. A smooth version of Theorem 2.2 for the full Grassmannian $\mathrm{Gr}_{p,q}$ over $\mathbb{C}$ (recall that we considered holomorphic maps in the original version of that theorem) requires only slightly more complex arguments in the case of $\mathbb{R}$ or $\mathbb{H}$. The Lagrangian Grassmannian over $\mathbb{C}$ calls for additional ideas (which makes the already lengthy proof longer still). The author knows no proofs for other series.

4.8. Exceptional groups. The question of possible analogues for exceptional groups of the phenomena discussed in this paper is not an unreasonable one. Still, it must be pointed out that we discussed above only the *classical groups* $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{GL}(n,\mathbb{H})$, $\mathrm{GL}(n,\mathbb{R})$, and $\mathrm{U}(p,q)$, not the *semisimple classical* groups $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{H})$, $\mathrm{SL}(n,\mathbb{R})$, and $\mathrm{SU}(p,q)$, and this faint distinction was significant for us. It is not even clear how one could pose the problem in the case of *semisimple* exceptional groups, and this author does not know the answer.

However, for the Hermitian symmetric spaces

$$G/K = \mathrm{EIII}\,/\,\mathrm{SO}(10)\times\mathrm{U}(1) \quad\text{and}\quad \mathrm{EVII}\,/E_6\times\mathrm{U}(1)$$

there are the Ol'shanskiĭ semigroups [21] acting by injective maps $G/K \to G/K$. Further, one can consider the closure of an Ol'shanskiĭ semigroup in the set of all holomorphic maps from G/K into itself. Elements of this closure are natural candidates for analogues of generalized linear-fractional maps.

The situation with symmetric spaces of the type

$$G/K = \mathrm{O}(n,2)/\,\mathrm{O}(n)\times\mathrm{O}(2)$$

(*Cartan domains of type* IV or *future tubes*) is not very transparent. In this case there are two distinct semigroups acting in G/K: one is the same as for all spaces $\mathrm{O}(p,q)/\,\mathrm{O}(p)\times\mathrm{O}(q)$, and the other is the Ol'shanskiĭ semigroup. The question of the closure of the latter in the space of holomorphic maps is unclear.

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