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ON COMBINATORIAL ANALOGS OF THE GROUP OF DIFFEOMORPHISMS OF THE CIRCLE

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ABSTRACT. The goal of this article is to construct and study groups which, from the point of view of the theory of representations, should resemble the group of diffeomorphisms of the circle. The first type of such groups are the diffeomorphism groups of p-adic projective lines. The second type are groups consisting of diffeomorphisms (satisfying certain conditions) of the absolutes of Bruhat-Tits trees; they can be regarded as precisely the diffeomorphism groups of Cantor perfect sets. Several series of unitary representations of these groups are constructed, including the analogs of highest-weight representations.

From the point of view of the theory of representations, the group Diff of diffeomorphisms of the circle is an object that is very important and very unusual. Moreover, Diff is an object that is highly complex. (For example, at present it remains practically the only large (=infinite-dimensional) group for which mantles and trains [18] still have not been constructed.) The desire to generalize it is completely natural (if only to obtain an additional way of looking at the group itself), and this desire is evidently shared by the majority of people who have dealt with large groups. However, although the group itself (or its Lie algebra) is included in various series, the theory of representations of Diff turns out to be unique in its own way. This statement is not exactly precise: there are several series of groups with a similar theory of representations, but these groups are more likely different manifestations of Diff than different essences. This was first studied a lot (see [17], [21], and [13]) in semidirect products of Diff and loop groups, as well as the combinatorial analog of Diff discussed here and the group of almost periodic diffeomorphisms of the line recently investigated by Ismagilov [5].

The combinatorial analogs $\mathrm{Diff}(A_p)$ of the group of diffeomorphisms of the circle were constructed by the author in 1983 (see [10]). In the same place it was shown that the constructions of the representations of Diff connected with almost invariant structures (see [8], [9], [12], [13], and [19]) can be partially carried over to $\mathrm{Diff}(A_p)$.

Evidently, our groups are somehow connected with "non-Archimedean field theory" (references can be found in [24]).

I thank G. I. Ol'shanskiĭ for discussing this subject.

§1. CLASSICAL GROUPS

This section contains a summary of the necessary results on infinite-dimensional classical groups. For more details on representations of (G, K)-pairs see [15] and [20], and on the spinor representation of $(O(2\infty, \mathbf{C}), \mathrm{GL}(\infty, \mathbf{C}))$ see [12].

1.1. (G, K)**-pairs.** We denote by $U(\infty)$ the full unitary group of Hilbert space, by

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$O(\infty)$ the full orthogonal group, by $\mathrm{GL}(\infty)$ the full linear group, etc. Let $G \supset K$ be two groups of similar type. We denote by (G, K) the group of all operators $A \in G$ representable in the form $A = (1+S)B$, where $B \in K$ and S is a Hilbert-Schmidt operator.

1.2. **Universal groups.** A. The *symplectic group* (automorphism group of the canonical commutational relations) $(\mathrm{Sp}(2\infty, \mathbf{R}), U(\infty))$ consists of invertible matrices of the form $\left(\begin{smallmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{smallmatrix}\right)$ that preserve the skew-symmetric bilinear form with matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and such that Ψ is a Hilbert-Schmidt operator (the subgroup $U(\infty)$ in our case consists of matrices of the form $\left(\begin{smallmatrix} \Phi & \\ & \overline{\Phi} \end{smallmatrix}\right)$).

B. The *affine symplectic group* consists of affine transformations of the form $v \mapsto Av + b$, where $A \in (\mathrm{Sp}(2\infty, \mathbf{R}), U(\infty))$ and $v = \left(\begin{smallmatrix} p \\ \overline{p} \end{smallmatrix}\right)$.

C. The *complex orthogonal group* $(O(2\infty, \mathbf{C}), \mathrm{GL}(\infty, \mathbf{C}))$ consists of invertible matrices of the form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ that preserve the symmetric bilinear form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and are such that B and C are Hilbert-Schmidt operators (the subgroup $\mathrm{GL}(\infty, \mathbf{C})$ consists of matrices of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & A^{-1} \end{smallmatrix}\right)$).

D. The *real orthogonal group* $(O(2\infty), U(\infty))$ (automorphism group of the canonical commutational relations) is the subgroup in $(O(2\infty, \mathbf{C}), \mathrm{GL}(\infty, \mathbf{C}))$ distinguished by the condition $D = \overline{A}$, $C = \overline{B}$.

The "Weyl representation" of the group $(\mathrm{Sp}(2\infty, \mathbf{R}), U(\infty))$ and the spinor representation of $(O(2\infty, U(\infty))$ are old mathematical objects (see [1], [13], [22], and [23]). It is also well known that the "Weyl representation" of the group $(\mathrm{Sp}(2\infty, \mathbf{R}), U(\infty))$ can be extended to a representation of the affine symplectic group. All these representations are unitary. The spinor representation of $(O(2\infty, \mathbf{R}), U(\infty))$ was constructed in [11].

We assume that all these constructions are known. (Actually, to understand this article it is enough to believe that such constructions exist.)

1.3. **The group** $(\mathrm{GL}(\infty, \mathbf{R}), O(\infty))$. This group consists of the bounded operators in a real Hilbert space representable in the form $A(1+T)$, where $A \in O(\infty)$ and T is a Hilbert-Schmidt operator.

We construct a series of imbeddings $\tau_S\colon (\mathrm{GL}(\infty, \mathbf{R}), O(\infty)) \to (\mathrm{Sp}(2\infty, \mathbf{R}), U(\infty))$ by the formula

$$\tau_s(g) = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \frac{1}{2} \begin{pmatrix} g + g^{t-1} & i(g - g^{t-1}) \\ -i(g - g^{t-1}) & g + g^{t-1} \end{pmatrix} \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}^{-1},$$

where $s \in \mathbf{R}$ and g^t denotes the transpose matrix. Restricting the Weyl representation of $(\mathrm{Sp}(2\infty, \mathbf{R}), U(\infty))$ to $(\mathrm{GL}(\infty, \mathbf{R}), O(\infty))$, we obtain a series of unitary representations of $(\mathrm{GL}(\infty, \mathbf{R}), O(\infty))$ that depends on the parameter s.

1.4. **The group** $(U(\infty), O(\infty))$. Let H be a real Hilbert space and $H_{\mathbf{C}}$ its complexification. The group $(U(\infty), O(\infty))$ consists of the unitary operators in $H_{\mathbf{C}}$ representable in the form $A(1+T)$, where A is an orthogonal operator (i.e., a unitary operator that leaves the real subspace $H \subset H_{\mathbf{C}}$ invariant) and T is a Hilbert-Schmidt operator.

We construct a series of imbeddings $\alpha_S\colon (U(\infty), O(\infty)) \to (\mathrm{Sp}(2\infty, \mathbf{R}), U(\infty))$ by the formula

$$\alpha_s(g) = \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & \overline{g} \end{pmatrix} \begin{pmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{pmatrix}^{-1},$$

where $s \in \mathbf{R}$. Restricting the Weyl representation of $(\mathrm{Sp}(2\infty, \mathbf{R}), U(\infty))$ to $(U(\infty),$

$O(\infty))$, we obtain a series of unitary representations of $(U(\infty), O(\infty))$ that depends on s.

There also exists a series of imbeddings of $(U(\infty), O(\infty))$ into $(O(4\infty), U(2\infty))$ (see [20]), but its construction is somewhat more complicated.

1.5 **The groups** $(U(2\infty), U(\infty)\times U(\infty))$ **and** $(\mathrm{GL}(2\infty, \mathbf{C}), \mathrm{GL}(\infty, \mathbf{C})\times\mathrm{GL}(\infty, \mathbf{C}))$. Let H be a Hilbert space. The group

$$\mathrm{GL}_\infty = (\mathrm{GL}(2\infty, \mathbf{C}),\ \mathrm{GL}(\infty, \mathbf{C}) \times \mathrm{GL}(\infty, \mathbf{C}))$$

consists of bounded invertible operators in $H \oplus H$ representable in the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + T\right), \quad \text{where } A, B \in \mathrm{GL}(\infty, \mathbf{C})$$

(i.e., A and B are bounded operators in H), and T is a Hilbert-Schmidt operator. Its subgroup $(U(2\infty), U(\infty) \times U(\infty))$ consists of unitary operators that belong to GL_∞.

We construct an imbedding of GL_∞ into $(O(4\infty, \mathbf{C}), \mathrm{GL}(2\infty, \mathbf{C}))$ by the formula

$$\nu\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \\ & \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}.$$

Restricting the spinor representation of $(O(4\infty, \mathbf{C}), \mathrm{GL}(2\infty, \mathbf{C}))$ to GL_∞, we obtain a holomorphic representation of GL_∞ (it splits into a countable sum of irreducible ones).

The same formula (1.1) defines an imbedding of $(U(2\infty), U(\infty) \times U(\infty)$ into $(O(4\infty), U(2\infty))$. Restricting the spinor representation of $(O(4\infty), U(2\infty))$ to $(U(2\infty), U(\infty) \times U(\infty))$, we obtain a unitary representation of $(U(2\infty, U(\infty) \times U(\infty))$.

§2. The p-adic analog of the group of diffeomorphisms of the circle

Let $\mathbf{Q}_p$ be the p-adic number field, $\mathbf{Q}_p^*$ its multiplicative group, $\mathbf{Z}_p$ the ring of p-adic integers, and $\mathbf{F}_p$ the field of p elements. We endow $\mathbf{Q}_p$ with the canonical Haar metric $d\mu(z)$ so that the measure of $\mathbf{Z}_p$ is equal to 1. We denote by $\mathbf{Q}_pP^1$ the p-adic projective line and by An_p the group of analytic diffeomorphisms of $\mathbf{Q}_p$.

2.1. **Complementary series of unitary representations of** $\mathrm{SL}_2(\mathbf{Q}_p)$. Let $0 < s < 1$. Let H_s be the space of real functions on $\mathbf{Q}_p$ with scalar product

$$\langle f, g\rangle = \int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p} |z_1 - z_2|^{s-1} f(z_1) g(z_2)\, dz_1\, dz_2.$$

The unitary representations T_s of the group $\mathrm{SL}_2(\mathbf{Q}_p)$ of the complementary series are realized in the space H_s by the formula (see [4])

$$T_s\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) f(z) = f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) |\gamma z + \delta|^{-s-1}. \tag{2.1}$$

2.2. **Imbeddings of** An_p **in** $(\mathrm{GL}(\infty, \mathbf{R}), O(\infty))$. We extend the representation (2.1) of the group $\mathrm{SL}_2(\mathbf{Q}_p)$ to the group An_p. Let $q \in \mathrm{An}_p$. Then

$$T_s(q) f(z) = f(q(z))|q'(z)|^{(1+s)/2}.$$

The operators $T_s(q)$ no longer need to be orthogonal. However, the following theorem is valid:

Theorem 2.1. $T_s(q) \in (\mathrm{GL}(\infty, \mathbf{R}), O(\infty))$.

This theorem is a consequence of the following lemma.

Lemma 2.1. *The operators* $T_s^*(q)T_s(q) - E$ *have finite rank.*

Proof. Let us show that the bilinear form

$$\langle (T_s(q)^* T_s(q) - E)f_1, f_2\rangle = \langle T_s(q)f_1, T_s(q)f_2\rangle - \langle f_1, f_2\rangle \tag{2.2}$$

has finite rank. The expression $\langle T_s(q)f_1, T_s(q)f_2\rangle$ is equal to

$$\int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p} \frac{f_1(q(z_1))f_2(q(z_2))|q'(z_1)|^{(1+s)/2}|q'(z_2)|^{(1+s)/2}\,dz_1\,dz_2}{|z_1 - z_2|^{1-s}}.$$

Let r be the diffeomorphism inverse to q. Making the change of variables $u_1 = q(z_1)$, $u_2 = q(z_2)$, we obtain

$$\int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p} \frac{|r'(u_1))|^{(1-s)/2}|r'(u_2)|^{(1-s)/2}}{|r(u_1) - r(u_2)|^{1-s}} f_1(u_1)f(u_2)\,du_1\,du_2.$$

Thus, (2.2) is equal to

$$\int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p} \left[\frac{|r'(u_1)|^{(1-s)/2}|r'(u_2)|^{(1-s)/2}}{|r(u_1) - r(u_2)|^{1-s}} - \frac{1}{|u_1 - u_2|^{1-s}}\right] f_1(u_1)f_2(u_2)\,du_1\,du_2.$$

As is easy to see, the expression in square brackets is locally constant and equal to 0 in a neighborhood of the line $u_1 = u_2$. This proves the lemma.

Restricting the unitary representations (see §1.3) of $(\mathrm{GL}(\infty, \mathbf{R}), O(\infty))$ to An_p, we obtain a series of unitary representations of An_p.

2.3. **The singular representation of** $\mathrm{SL}_2(\mathbf{Q}_p)$. Let H_1 be the space of real functions on $\mathbf{Q}_p$ such that

$$\int_{\mathbf{Q}_p} \varphi(z)\,dz = 0$$

with scalar product

$$\langle f_1, f_2\rangle = \int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p} \ln|z_1 - z_2| f(z_1)f(z_2)\,dz_1\,dz_2.$$

The group $\mathrm{SL}_2(\mathbf{Q}_p)$ acts in H by the formula

$$T_1\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} f(z) = f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right)|\gamma z + \delta|^{-2}.$$

This (unitary) representation of $\mathrm{SL}_2(\mathbf{Q}_p)$ is commonly called *singular* [4]. The representation T_1 is properly understood as the limit of the representations T_s as $s \to 1$.

The action of T_1 can be extended to an action of An_p in H_1 by the formula

$$T_1(q)f(z) = f(q(z))|q'(z)|.$$

Theorem 2.1′. $T_1(q) \in (\mathrm{GL}(\infty, \mathbf{R}), O(\infty))$.

The proof is similar to that of Theorem 2.1.

Here, however, one can obtain somewhat more. To wit, we now construct a series of imbeddings of An_p into the group of affine transformations of the form $f \mapsto Af + b$, where $A \in (\mathrm{GL}(\infty, \mathbf{R}), O(\infty))$ and $b \in H_1$. This group, in turn, can be imbedded into the affine symplectic group, and we obtain the possibility of restricting the Weyl representation to An_p. This affine action is defined by the formula

$$f(z) \mapsto f(q(z))|q'(z)| + \lambda(|q'(z)| - 1).$$

2.4. **The even fundamental series of representations of $\mathrm{SL}_2(\mathbf{Q}_p)$.** Let χ be a unitary character of the group $\mathrm{SL}_2(\mathbf{Q}_p)$ (i.e., a homomorphism of $\mathbf{Q}_p^*$ into the group of complex numbers equal to 1 in absolute value). The representations T_χ of the even fundamental series are realized in the space $L_2(\mathbf{Q}_p)$ by the formula

$$T_\chi \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} f(z) = f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) \chi((\gamma z + \delta)^2)|\gamma z + \delta|^{-1}.$$

The representation T_χ is equivalent to $T_{\chi^{-1}}$. The operator that intertwines T_χ and $T_{\chi^{-1}}$ is defined by

$$A_\chi f(z) = \int_{\mathbf{Q}_p} \frac{f(z)\,dz}{|z-u|\chi^2(z-u)} \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \int_{\mathbf{Q}_p} \frac{f(z)\,dz}{|z-u|^{1-\varepsilon}\chi^2(z-u)}.$$

But the representation $T_{\chi^{-1}}$ is complex-conjugate to T_χ; that is, T_χ is equivalent to its conjugate. Hence, T_χ has either real or quaternionic type ([6], §7). Consider the real-linear operator I_χ that intertwines T_χ with itself:

$$I_\chi f(z) = A_\chi \overline{f(z)}.$$

A direct calculation shows that $I_\chi^2 = \lambda E$, where $\lambda > 0$. (For the calculation it is useful to carry out a Fourier transform; all the necessary calculations are contained in [4], II.3.3.) It follows that T_χ has real type (if $\lambda < 0$, then we would have quaternionic type). Thus, $L^2(\mathbf{Q}_p, \mathbf{C})$ contains two real $\mathrm{SL}_2(\mathbf{Q}_p)$-invariant spaces V_+ and V_-:

$$V_\pm = \{v \in L^2 \colon I_\chi v = \pm\sqrt{\lambda} v\}.$$

Multiplication by i interchanges these subspaces.

In particular, L^2 is the complexification of V_+, and so we can define the subgroup $(U(\infty), O(\infty))$ in $U(\infty)$ (see §1.4).

Suppose that the group An_p acts in $L^2(\mathbf{Q}_p)$ by unitary operators according to the formula

$$T_\chi(q)f(z) = f(q(z))\chi(q'(z))|q'(z)|^{1/2}.$$

Theorem 2.2. $T_\chi(q) \in (U(\infty), O(\infty))$.

The theorem is a consequence of the following lemma.

Lemma 2.2. *The operator $A_\chi(q) = I_\chi T_\chi(q) - T_{\chi^{-1}}(q) I_\chi$ has finite rank.*

Proof. We have

$$A_\chi(q)f(u) = \int \frac{\overline{f(q(z))}|q'(z)|^{1/2}\chi(p'(z))\,dz}{|z-u|\chi^2(z-u)} - \int \frac{\overline{f(z)}\,dz \cdot \chi^{-1}(p'(u))|p'(u)|^{1/2}}{|z-p(u)|\chi^2(z-p(u))}.$$

Making the change of variable $z = p(w)$ in the second integral, we obtain

$$A_\chi(q)f(u) = \int \overline{f(p(u))}|p'(w)|\chi(p'(w)) \times \left[\frac{1}{|w-u|\chi^{-2}(w-u)} - \frac{|p'(w)|^{1/2}|p'(u)|^{1/2}\chi^{-1}(p'(w))\chi^{-1}(p'(u))}{|p(w)-p(u)|\chi^{-2}(p(w)-p(u))}\right] du.$$

The expression is square brackets is locally constant and equal to 0 in a neighborhood of the diagonal. This proves the lemma.

We now obtain the possibility of restricting the unitary representations of the group $(U(\infty), O(\infty))$ to An_p.

§3. Combinatorial structures

3.1. **Bruhat-Tits trees.** A *Bruhat-Tits tree* J_n is a tree (=cycle-free connected graph) at each vertex of which $n+1$ edges converge. By a *path on a tree* we mean a sequence $a_1, a_2, \dots$ such that a_j and a_{j+1} have a common vertex and are distinct. We call two paths $a_1, a_2, \dots$ and $b_1, b_2, \dots$ *equivalent* if there exists a k such that $a_j = b_{j+k}$ for sufficiently large j. The set of equivalence classes of paths is called the *absolute* A_p (this is the formal definition, but it is necessary to understand the absolute as the set of a tree's points at infinity).

Remark. Bruhat-Tits trees are a special case of Bruhat-Tits buildings [2], [7].

3.2. **Bruhat-Tits trees and the projective line.** The Bruhat-Tits tree J_p and $\mathbf{Q}_pP^1$ are related to each other in roughly the same way as the Lobachevsky plane and the circle—its absolute.

We call sets of the form

$$B(a, p^k) = \{z \in \mathbf{Q}_p : |z-a| \le p^k\},$$

where $k \in \mathbf{Z}$, *spheres* in $\mathbf{Q}_p$. Let $B(a, p^k)$ be a sphere. We call the sphere $B(a, p^{k+1})$ an *upper neighbor* of it. It is clear that each sphere has exactly one upper neighbor and exactly p lower neighbors (a sphere B_1 is a lower neighbor of a sphere B_2 if B_2 is an upper neighbor of B_1).

Let us construct a graph whose vertices are numbered by spheres. The vertices of B_1 and B_2 are joined by an edge if and only if they are neighbors. It is easy to see that this graph is precisely the Bruhat-Tits tree J_p.

It is natural to identify the projective line with the absolute. Indeed, consider the path $B_1, B_2, \dots$. Let b_1 be the corresponding point of the absolute. If $B_j \supset B_{j+1}$ for sufficiently large j (for $j \ge N$), then it is natural to identify a point $b \in A_p$ with the point $\bigcap_{j=N}^{\infty} B_j$. But if $B_j \subset B_{j+1}$ for sufficiently large j, then it is natural to identify b with ∞.

By a *cell* in $\mathbf{Q}_pP^1$ we mean either a sphere or the complement of a sphere. The set of vertices of a graph J_p is in one-to-one correspondence with the set of partitionings of $\mathbf{Q}_pP^1$ into $p+1$ pairwise disjoint cells. It is easy to check that elements of $\mathrm{SL}_2(\mathbf{Q}_p)$ carry cells into cells. Therefore, $\mathrm{SL}_2(\mathbf{Q}_p)$ acts in a natural way on the set of vertices of the graph J_p. It is easy to check that indeed $\mathrm{SL}_2(\mathbf{Q}_p)$ acts by automorphisms of J_p (in all it is simpler to check this separately for affine transformations of $\mathbf{Q}_p$, and also for the mapping $z \mapsto 1/z$; the group $\mathrm{PSL}_2(\mathbf{Q}_p)$ is generated by such transformations).

3.3. **The automorphism group of the graph** J_n. Let $\mathrm{Aut}\, J_n$ be the automorphism group of the graph J_n. If $n = p$ is a prime, then $\mathrm{Aut}(J_p) \supset \mathrm{PSL}_2(\mathbf{Q}_p)$. As Cartier observed [16], the group $\mathrm{Aut}(J_p)$ has a sensible theory of representations that largely resembles the theory of representations of $\mathrm{PSL}_2(\mathbf{Q}_p)$. The classification of the representations of $\mathrm{Aut}(J_p)$ is obtained in [14].

3.4. **Spheroids.** We fix an integer $n \ge 2$. By a *spheroid* we mean a compact set in which is distinguished a collection of open-closed subsets, which are called *spheres* and satisfy the following conditions:

(a) M is covered by spheres.

(b) If B and C are spheres, then either $B \supset C$ or $C \supset B$ or $B \cap C = \varnothing$.

(c) Each sphere B can be canonically represented as a union of pairwise disjoint spheres $B_1, \dots, B_n$ (we shall say that the B_j are a *canonical partition of* B).

(d) If $B_1 \supset B_2 \supset \cdots$ is a sequence of imbedded spheres $(B_{j+1} \neq B_j)$, then $\bigcap B_j$ consists of exactly one point.

We call a homeomorphism q of a sphere B into a sphere C *proper* if q carries subspheres into subspheres and canonical partitions into canonical partitions.

We call a homeomorphism r of a spheroid M into a spheroid N a *spheromorphism* if there exists a partition of N into subspheres $N = \bigcup R_j$ such that $r(R_j)$ is a sphere for all R_j and r is a proper sphere homeomorphism $R_j \to r(R_j)$.

Remark [3]. Let M be a spheroid and $M = P_1 \cup \cdots \cup P_N$ a covering of M by pairwise disjoint spheres. Let d be the remainder of the division of N by $n-1$. Then d does not depend on the partition and is the (unique) invariant of the spheroid under spheromorphisms.

Example. The Cantor set is endowed with a spheroid structure in the obvious way.

Another example of a spheroid is the absolute A_n of the Bruhat-Tits tree J_n (spheres are what were called cells above). This example is universal; to wit, any spheroid can be spheromorphically imbedded into A_n.

Proposition 3.1. *Any analytic transformation* $q \in \mathrm{An}_p$ *is a spheromorphism* $A_p \simeq \mathbf{Q}_p P^1$.

Proof. The assertion is local and, by virtue of the action of $\mathrm{SL}_2(\mathbf{Q}_p)$, without loss of generality we can restrict ourselves to a mapping of a sphere of the form $|z-a| \leq p^k$ into a sphere of the form $|z-b| \leq p^n$. Thus, suppose that in a neighborhood of the point a the mapping has the form

$$q(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots .$$

We take a neighborhood $B = \{z : |z-a| < 1/p^N\}$ so small that the series converges in it and $|q'(z) - c_1| < c_1$. Then q is a proper homeomorphism of the sphere B onto the sphere $\{z : |z - c_0| < |c_1|/p^N\}$. This proves the assertion.

3.5. **The group** $\mathrm{Diff}(A_n)$. We define the group $\mathrm{Diff}(A_n)$ as the spheromorphism group of the absolute A_n of the tree J_n. Let us define this group without using the word "spheromorphism".

We take some edge of the tree J_n and cut it in the middle. Then the tree splits into two sets, which we shall call *branches*. To each branch L there naturally corresponds a subset A_L of the absolute, namely, those points to which one can go by moving along paths that lie in this branch (more accurately: A_L consists of equivalence classes of the paths that lie in this branch). We call a set of branches $L_1, \dots, L_k$ such that the L_j are pairwise disjoint and the sets A_{L_j} cover the entire absolute a *broom*.

Let $L_1, \dots, L_k$ and $L'_1, \dots, L'_k$ be two brooms in J_p. Let σ be a permutation of the set $\{1, \dots, k\}$. We map each branch L_j isomorphically onto the branch $L'_{\sigma(j)}$. This set of mappings induces a homeomorphism of the absolute. The group $\mathrm{Diff}(A_n)$ consists of all of the homeomorphisms absolute that can be obtained in this way.

3.6. **Canonical measure on the absolute.** We fix some point ∞ of the absolute A_n. In the set of vertices of the tree J_p we introduce a function h with values in $\mathbf{Z}$ that satisfies the following condition: if $a_1, a_2, \dots$ is a path that leads to ∞, then $h(a_{j+1}) = h(a_j) + 1$. Naturally, this function is unique up to the addition of a constant.

Remark. The "level lines" of the function h are commonly called *horocycles.*

We now cut some edge b of the graph J_n and take, of the two resulting branches, the one that does not contain ∞ as a limit point. We denote this branch by $L(b)$. Let b' be the end of the cut edge b that lies in $L(b)$. the canonical measure μ on the absolute is defined from the condition $\mu(A_{L(b)}) = n^{h(b')}$.

It is easy to see that the measure μ in quasi-invariant under the action of $\mathrm{Diff}(A_n)$; the Radon-Nikodým derivative is locally constant and takes values of the form n^α, where $\alpha \in \mathbf{Z}$. We shall denote the Radon-Nikodým derivative of a mapping $q \in \mathrm{Diff}(A_p)$ at the point $z \in A_p$ by $|q'(z)|$.

We define a metric $\rho(z_1, z_2)$ on the absolute. Consider a path $\dots, a_{-1}, a_0, a_1, \dots$ that leads from z_1 to z_2. Let $\kappa(z_1, z_2)$ be the maximum of the function $h(a_j)$. By definition, we set

$$\rho(z_1, z_2) = n^{\kappa(z_1, z_2)}.$$

Remark. Let $n = p$ be prime. Then μ, up to multiplication by a constant, coincides with Haar measure on $\mathbf{Q}_p$. The metric $\rho(z_1, z_2)$ coincides, up to multiplication by a constant, with $|z_1 - z_2|$.

3.7. **Imbeddings of** $\mathrm{Diff}(A_p)$ **in** $(\mathrm{GL}(\infty, \mathbf{R}), O(\infty))$. Let $0 < s < 1$. Consider the space H_s of real functions on the absolute A_n with scalar product

$$\langle f_1, f_2 \rangle = \int_{A_n} \int_{A_n} \rho(z_1, z_2)^{s-1} f_1(z_1) f_2(z_2)\, dz_1\, dz_2.$$

The group $\mathrm{Diff}(A_n)$ acts in H_s by the formula

$$T_s(q)f(z) = f(q(z))q'(z)^{(1+s)/2}.$$

Theorem 3.1. $T_s(q) \in (\mathrm{GL}(\infty, \mathbf{R}), O(\infty))$.

The proof coincides verbatim with the proof of Theorem 2.1.

Remark. The construction in §2.3 also carries over to $\mathrm{Diff}(A_p)$.

3.8. **Imbeddings of** $\mathrm{Diff}(A_p)$ **in** $(U(\infty), O(\infty))$. Suppose that the group $\mathrm{Diff}(A_p)$ acts in the complex L^2 on A_n by the formula

$$T_{is}(q)f(z) = f(q(z))q'(z)^{(1+is)/2}.$$

A real-linear operator I_s in L^2 is defined by

$$I_s f(z) = \int_{A_n} \rho(z, u)^{is-1} \overline{f(u)}\, du.$$

As before, I_s defines a real structure in L^2.

Theorem 3.2. $T_{is}(q) \in (U(\infty), O(\infty))$.

The proof is similar to that of Theorem 2.2.

§4. Analogs of highest-weight representations

In this section $p \geq 2$ is a prime; the field of p elements is denoted by $\mathbf{F}_p$, and the Legendre symbol is denoted by (a/p) ($(a/p) = 1$ if $a \in \mathbf{F}_p^*$ is a square, and $(a/p) = -1$ otherwise).

4.1. **The p-adic Hilbert transform.** Let $z \in \mathbf{Q}_p^*$, $z = a_k p^k + a_{k+1}p^{k+1} + \cdots$, where $a_k \neq 0$. We set

$$\operatorname{sgn}(z) = (a_k/p).$$

We define the Hilbert transform in $L^2(\mathbf{Q}_p)$ by

$$If(z) = \frac{1}{\sqrt{p}} \int_{\mathbf{Q}_p} \frac{\operatorname{sgn}(z-u) f(u)\, du}{|z-u|}.$$

If f is a finite function that takes only a finite number of values, then this integral is well defined in the sense of principal value:

$$\text{p.v.} \int_{\mathbf{Q}_p} q(z)\, dz \overset{\text{def}}{=} \lim_{N\to\infty} \int_{|z| \geq 1/p^N} f(z)\, dz.$$

In addition, a direct calculation shows that $\langle If, Ig\rangle = \langle f, g\rangle$ for any compactly supported functions f and g taking only a finite number of values. Hence, I can be uniquely extended to a unitary operator in $L^2(\mathbf{Q}_p)$.

It is not complicated to check that $I^2 = -1$. This can be checked directly, but it is more elegant to carry out a Fourier transform $\mathscr{F}$ in $L^2(\mathbf{Q}_p)$:

$$(\mathscr{F} I \mathscr{F}^{-1}) f(u) = i \operatorname{sgn}(u) f(u).$$

In particular, we see that the operator I has two proper subspaces V_+ and V_-, where $V_\pm$ consists of functions whose Fourier transform has support in the set

$$\mathbf{Q}_p^\pm = \{z\mathbf{Q}_p^*\colon \operatorname{sgn} z = \pm 1\}.$$

4.2. **The group** An_p^+. This group consists of analytic transformations of $\mathbf{Q}_p P^1$ such that $\operatorname{sgn} q'(x) = 1$ for all x. If desired, we can interpret An_p^+ as the group of orientation-preserving diffeomorphisms.

We note that $\mathrm{PSL}_2(\mathbf{Q}_p) \subset \mathrm{An}_p^+$.

4.3. **Imbeddings of** An_p^+ **in** GL_∞ **and in** $(U(2\infty), U(\infty) \times U(\infty))$. Let χ be a homeomorphism of $\mathbf{Q}_p^*$ into $\mathbf{C}^*$. We define the representation $T_\chi(q)$ of the group An_p^+ in $L^2(\mathbf{Q}_p)$:

$$T_\chi(q) f(x) = f(q(x)) \chi(q'(x)) |q'(x)|.$$

In $L^2(\mathbf{Q}_p)$ we distinguished the two subspaces V_+ and V_-. The group $\mathrm{GL}_\infty = (\mathrm{GL}(2\infty, \mathbf{C}), \mathrm{GL}(\infty, \mathbf{C}) \times \mathrm{GL}(\infty, \mathbf{C}))$ consists of operators that "almost preserve $V_\pm$" (see §1.5).

Theorem 4.1. (a) $T_\chi(q) \in \mathrm{GL}_\infty$.

(b) *If* $|\chi| = 1$, *then* $T_\chi(q) \in (U(2\infty), U(\infty) \times U(\infty))$.

Proof. Assertion (b) follows from (a), and (a) is a consequence of the following lemma.

Lemma 4.1. $[T_\chi(q), I]$ *has finite rank.*

Proof. We have

$$(IT_\chi(q) - T_\chi(q)I) f(u) = \int_{\mathbf{Q}_p} \frac{f(q(z)) |q'(z)|^{1/2} \chi(q'(z))\, dz}{|z-u| \operatorname{sgn}(z-u)} - \int_{\mathbf{Q}_p} \frac{f(z) |q'(u)|^{1/2} \chi(q'(u))\, dz}{|z-q(u)| \operatorname{sgn}(z-q(u))}.$$

We make the change $z = q(w)$ in the second integral, and $z = w$ in the first. We obtain

$$\int f(p'(w))|p'(w)|^{1/2}\chi(p'(w)) \times \left[\frac{1}{|w-u|\operatorname{sgn}(w-u)|} - \frac{|'p(w)|^{1/2}|p'(u)|^{1/2}\chi(p'(u))\chi(p'(w))}{|w-u|\operatorname{sgn}(p(w)-p(u))}\right] du,$$

and the singularity in square brackets disappears the next time. This proves the lemma.

Restricting the representation of GL_∞ (see §1.5) to An_p^+, we obtain a series of representations of An_p^+. The representations are numbered by the characters χ of the group $\mathbf{Q}_p^*$. If χ is a unitary character, $|\chi| = 1$, then the resulting representation is unitary.

These representations are exact duplicates of the highest-weight representations of the group of diffeomorphisms of the circle (see the "two-fermion construction" in [13]). It is still not clear, to be sure, what is meant by the words "highest weight", since there is no Lie algebra for the group An_p.

4.4. **Combinatorial structures.** A Bruhat-Tits tree J_p has still another interpretation. To wit, the vertices of the tree J_p are in one-to-one correspondence with the lattices of volume 1 in $\mathbf{Q}_p^2$ (a lattice in $\mathbf{Q}_p^2$ is a $\mathbf{Z}_p$-submodule of rank 2). Two vertices are joined by an edge if the intersection of the corresponding lattices has volume p^{-1}.

Let Q be a lattice of volume 1 and $R_1, \dots, R_{p+1}$ neighboring lattices. The space Q/pQ can be naturally identified with the vector space $\mathbf{F}_p^2$. To each lattice R_j we can associate the line $(Q \cap R)/pQ$ in $Q/pQ = \mathbf{F}_p^2$. Thus, the set of edges that emanate from Q is endowed with the structure of the projective line $\mathbf{F}_pP^1$. The group $\mathrm{SL}_2(\mathbf{F}_p)$ acts in the natural way on the set $\{R_1, \dots, R_{p+1}\}$.

We denote by $\widetilde{J}_p$ a Bruhat-Tits tree J_p endowed with the following additional structure: for each vertex v there is defined a bijection of $\mathbf{F}_pP^1$ into the set of edges $l_1, \dots, l_{p+1}$ leading to v. We shall call $\widetilde{J}_p$ an *equipped tree.*

Remark. Earlier (§3.3) we interpreted J_p as a set of spheres in $\mathbf{Q}_p^2$. Let B be a sphere, C its upper neighbor, and $D_1, \dots, D_p$ its lower neighbors. The set of spheres D_j has the form

$$D_j = a + jp^n + p^{n+1}\mathbf{Z}_p,$$

where $j = 0, 1, \dots, p-1$. Hence, the spheres D_j are in one-to-one correspondence with the points of the affine line $\mathbf{F}_p^1$ over the field $\mathbf{F}_p$. We associate the sphere C with the point $\infty \in \mathbf{F}_pP^1$.

4.5. **The combinatorial version of the group** An_p^+. Let $\widetilde{J}_p$ be an equipped tree. Let $L_1, \dots, L_n$ and $L_1', \dots, L_n'$ be two brooms (see §3.5) and σ a permutation of the set $1, \dots, n$. Consider the set of mappings $q_j \colon L_j \to L_{\sigma(j)}$ such that q_j is an isomorphism of equipped branches. The set $\{q_j\}$ defines an absolute homeomorphism (see §3.5). We denote the group of all such homeomorphisms by $\mathrm{Diff}^+(\widetilde{J}_p)$.

We state in passing that the above-mentioned isomorphism q of equipped branches L and L' is one of these. In the first place, this is a tree isomorphism. Further, let v be a vertex and $l_1, \dots, l_{p+1}$ edges that go to it. Then the mapping $\kappa_{v'}$ of the set $l_1', \dots, l_{p+1}'$ onto $\mathbf{F}_pP^1$ is fixed. Thus, we have the composition

$$\mathbf{F}_pP^1 \xrightarrow{\sigma_v^{-1}} \{l_1, \dots, l_{p+1}\} \xrightarrow{q} \{(l_1'), \dots, (l_{p+1}')\} \xrightarrow{\sigma_{v'}} \mathbf{F}_pP^1.$$

We require that this mapping lie in the group $\mathrm{PSL}_2(\mathbf{F}_q)$ (to emphasize the point, it must lie in $\mathrm{PSL}_2(\mathbf{F}_q)$, not just in $\mathrm{PGL}_2(\mathbf{F}_q)$(!)).

4.6. **The combinatorial Hilbert transform.** We fix the point ∞ on the absolute A_p of the tree J_p. Let v be a vertex of the tree. Then among the $p+1$ edges that go to v the edge l_∞ is selected, namely, the one that is directed toward the side of the point $\infty \in A_p$. Let $l_0, \dots, l_{p-1}$ be the remaining edges that go to v. The elements of the set $l_0, l_1, \dots, l_{p-1}, l_\infty$ are in bijective correspondence with the points of the projective line $\mathbf{F}_p P^1$. Without loss of generality we can assume that l_j corresponds to a point $j \in \mathbf{F}_p$. Then the remaining edges $l_0, \dots, l_{p-1}$ are in bijective correspondence with the points of the affine projective line $\mathbf{F}_p^1$. Without loss of generality we can assume that l_j corresponds to a point $j \in \mathbf{F}_p$. Let $i \neq j$. We set

$$\operatorname{sgn}(l_i, l_j) = ((i-j)/p).$$

Remark. It is important to emphasize that the right-hand side of the equality is invariant with respect to the subgroup $B \subset \mathrm{SL}_2(\mathbf{F}_q)$—the stabilizer of the point ∞ in $\mathbf{F}_p P^1$. Indeed, the group B consists of transformations of the projective line of the form $j \mapsto \alpha^2 j + c$.

Let $h(v)$, $\eta(z_1, z_2)$, and $\rho(z_1, z_2)$ be the same as in §3.6. For two distinct points z_1 and z_2 $(z_j \neq \infty)$ of the absolute we also define the quantity $\operatorname{sgn}(z_1, z_2) = \pm 1$. To do so, we join z_1 and z_2 by a path $\dots, a_{-1}, a_0, a_1, \dots$ leading from z_1 to z_2. Let a_s be the vertex at which the maximum of the function $h(a_j)$ is attained. Let l_1 be the edge $[a_j, a_{j-1}]$ and l_2 the edge $[a_j, a_{j+1}]$. Then

$$\operatorname{sgn}(z_1, z_2) := \operatorname{sgn}(l_1, l_2).$$

We define the Hilbert transform in $L^2(A_p)$ by the formula

$$If(z) = \lambda \int_{A_p} \frac{\operatorname{sgn}(z, u)}{\rho(z, u)} f(u)\, du,$$

where λ is chosen from the condition $I^2 = -1$.

Remark. Here we need to use all of the words that we used in §4.1. The integral in the sense of principal value is understood as

$$\text{p.v.} \int_{A_p} f(u)\, du = \lim_{k\to\infty} \int_{A_p \setminus B_k} f(u)\, du,$$

where B_k is a sequence of spheres, containing u_0, such that $\bigcap B_k = u_0$.

If we identify A_p with $\mathbf{Q}_p P^1$, then our Hilbert transform coincides with the Hilbert transform in §4.1.

4.7. **Imbeddings of** $\mathrm{Diff}^+(\widetilde{J}_p)$ **in** GL_∞ **and in** $(U(2\infty, U(\infty)\times U(\infty))$. Let $\alpha \in \mathbf{C}$. We define the action of $\mathrm{Diff}^+(J_p)$ in $L^2(A_p)$ by

$$T_\alpha(q) f(z) = f(q(z)) |q'(z)|^{1/2 + i\alpha}.$$

Theorem 4.2. (a) $T_\alpha(q) \in \mathrm{GL}_\infty$.

(b) If $\alpha \in \mathbf{R}$, then $T_\alpha(q) \in (U(2\infty), U(\infty) \times U(\infty))$.

The proof coincides with that of Theorem 4.1.

Naturally, having such imbeddings, we have representations of the group $\mathrm{Diff}^+(J_p)$ as well.

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