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ON COMBINATORIAL ANALOGS OF THE GROUP OF DIFFEOMORPHISMS OF THE CIRCLE

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YU. A. NERETIN

ABSTRACT. The goal of this article is to construct and study groups which, from the point of view of the theory of representations, should resemble the group of diffeomorphisms of the circle. The first type of such groups are the diffeomorphism groups of p-adic projective lines. The second type are groups consisting of diffeomorphisms (satisfying certain conditions) of the absolutes of Bruhat-Tits trees; they can be regarded as precisely the diffeomorphism groups of Cantor perfect sets. Several series of unitary representations of these groups are constructed, including the analogs of highest-weight representations.

From the point of view of the theory of representations, the group Diff of diffeomorphisms of the circle is an object that is very important and very unusual. Moreover, Diff is an object that is highly complex. (For example, at present it remains practically the only large (=infinite-dimensional) group for which mantles and trains [18] still have not been constructed.) The desire to generalize it is completely natural (if only to obtain an additional way of looking at the group itself), and this desire is evidently shared by the majority of people who have dealt with large groups. However, although the group itself (or its Lie algebra) is included in various series, the theory of representations of Diff turns out to be unique in its own way. This statement is not exactly precise: there are several series of groups with a similar theory of representations, but these groups are more likely different manifestations of Diff than different essences. This was first studied a lot (see [17], [21], and [13]) in semidirect products of Diff and loop groups, as well as the combinatorial analog of Diff discussed here and the group of almost periodic diffeomorphisms of the line recently investigated by Ismagilov [5].

The combinatorial analogs $\mathrm{Diff}(A_p)$ of the group of diffeomorphisms of the circle were constructed by the author in 1983 (see [10]). In the same place it was shown that the constructions of the representations of Diff connected with almost invariant structures (see [8], [9], [12], [13], and [19]) can be partially carried over to $\mathrm{Diff}(A_p)$.

Evidently, our groups are somehow connected with "non-Archimedean field theory" (references can be found in [24]).

I thank G. I. Ol'shanskiĭ for discussing this subject.

§1. CLASSICAL GROUPS

This section contains a summary of the necessary results on infinite-dimensional classical groups. For more details on representations of (G, K)-pairs see [15] and [20], and on the spinor representation of $(O(2\infty, \mathbf{C}), \mathrm{GL}(\infty, \mathbf{C}))$ see [12].

1.1. **(G, K)-pairs.** We denote by $U(\infty)$ the full unitary group of Hilbert space, by

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$O(\infty))$, we obtain a series of unitary representations of $(U(\infty), O(\infty))$ that depends on s.

There also exists a series of imbeddings of $(U(\infty), O(\infty))$ into $(O(4\infty), U(2\infty))$ (see [20]), but its construction is somewhat more complicated.

1.5 **The groups** $(U(2\infty), U(\infty)\times U(\infty))$ **and** $(\mathrm{GL}(2\infty, \mathbf{C}), \mathrm{GL}(\infty, \mathbf{C})\times \mathrm{GL}(\infty, \mathbf{C}))$. Let H be a Hilbert space. The group

$$\mathrm{GL}_\infty = (\mathrm{GL}(2\infty, \mathbf{C}),\ \mathrm{GL}(\infty, \mathbf{C}) \times \mathrm{GL}(\infty, \mathbf{C}))$$

consists of bounded invertible operators in $H \oplus H$ representable in the form

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + T\right), \quad \text{where } A, B \in \mathrm{GL}(\infty, \mathbf{C})$$

(i.e., A and B are bounded operators in H), and T is a Hilbert-Schmidt operator. Its subgroup $(U(2\infty), U(\infty) \times U(\infty))$ consists of unitary operators that belong to GL_∞.

We construct an imbedding of GL_∞ into $(O(4\infty, \mathbf{C}), \mathrm{GL}(2\infty, \mathbf{C}))$ by the formula

$$\nu\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right] = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \\ & \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}.$$

Restricting the spinor representation of $(O(4\infty, \mathbf{C}), \mathrm{GL}(2\infty, \mathbf{C}))$ to GL_∞, we obtain a holomorphic representation of GL_∞ (it splits into a countable sum of irreducible ones).

The same formula (1.1) defines an imbedding of $(U(2\infty), U(\infty) \times U(\infty)$ into $(O(4\infty), U(2\infty))$. Restricting the spinor representation of $(O(4\infty), U(2\infty))$ to $(U(2\infty), U(\infty) \times U(\infty))$, we obtain a unitary representation of $(U(2\infty, U(\infty) \times U(\infty))$.

§2. The p-adic analog of the group of diffeomorphisms of the circle

Let $\mathbf{Q}_p$ be the p-adic number field, $\mathbf{Q}_p^*$ its multiplicative group, $\mathbf{Z}_p$ the ring of p-adic integers, and $\mathbf{F}_p$ the field of p elements. We endow $\mathbf{Q}_p$ with the canonical Haar metric $d\mu(z)$ so that the measure of $\mathbf{Z}_p$ is equal to 1. We denote by $\mathbf{Q}_pP^1$ the p-adic projective line and by An_p the group of analytic diffeomorphisms of $\mathbf{Q}_p$.

2.1. **Complementary series of unitary representations of** $\mathrm{SL}_2(\mathbf{Q}_p)$. Let $0 < s < 1$. Let H_s be the space of real functions on $\mathbf{Q}_p$ with scalar product

$$\langle f, g\rangle = \int_{\mathbf{Q}_p}\int_{\mathbf{Q}_p} |z_1 - z_2|^{s-1} f(z_1) g(z_2)\, dz_1\, dz_2.$$

The unitary representations T_s of the group $\mathrm{SL}_2(\mathbf{Q}_p)$ of the complementary series are realized in the space H_s by the formula (see [4])

$$T_s\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) f(z) = f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) |\gamma z + \delta|^{-s-1}. \tag{2.1}$$

2.2. **Imbeddings of** An_p **in** $(\mathrm{GL}(\infty, \mathbf{R}), O(\infty))$. We extend the representation (2.1) of the group $\mathrm{SL}_2(\mathbf{Q}_p)$ to the group An_p. Let $q \in \mathrm{An}_p$. Then

$$T_s(q) f(z) = f(q(z)) |q'(z)|^{(1+s)/2}.$$

The operators $T_s(q)$ no longer need to be orthogonal. However, the following theorem is valid:

imbedded into the affine symplectic group, and we obtain the possibility of restricting the Weyl representation to An_p. This affine action is defined by the formula

$$f(z) \mapsto f(q(z))|q'(z)| + \lambda(|q'(z)| - 1).$$

2.4. **The even fundamental series of representations of** $\mathrm{SL}_2(\mathbf{Q}_p)$. Let χ be a unitary character of the group $\mathrm{SL}_2(\mathbf{Q}_p)$ (i.e., a homomorphism of $\mathbf{Q}_p^*$ into the group of complex numbers equal to 1 in absolute value). The representations T_χ of the even fundamental series are realized in the space $L_2(\mathbf{Q}_p)$ by the formula

$$T_\chi \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} f(z) = f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) \chi((\gamma z + \delta)^2)|\gamma z + \delta|^{-1}.$$

The representation T_χ is equivalent to $T_{\chi^{-1}}$. The operator that intertwines T_χ and $T_{\chi^{-1}}$ is defined by

$$A_\chi f(z) = \int_{\mathbf{Q}_p} \frac{f(z)\,dz}{|z-u|\chi^2(z-u)} \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \int_{\mathbf{Q}_p} \frac{f(z)\,dz}{|z-u|^{1-\varepsilon}\chi^2(z-u)}.$$

But the representation $T_{\chi^{-1}}$ is complex-conjugate to T_χ; that is, T_χ is equivalent to its conjugate. Hence, T_χ has either real or quaternionic type ([6], §7). Consider the real-linear operator I_χ that intertwines T_χ with itself:

$$I_\chi f(z) = A_\chi \overline{f(z)}.$$

A direct calculation shows that $I_\chi^2 = \lambda E$, where $\lambda > 0$. (For the calculation it is useful to carry out a Fourier transform; all the necessary calculations are contained in [4], II.3.3.) It follows that T_χ has real type (if $\lambda < 0$, then we would have quaternionic type). Thus, $L^2(\mathbf{Q}_p, \mathbf{C})$ contains two real $\mathrm{SL}_2(\mathbf{Q}_p)$-invariant spaces V_+ and V_-:

$$V_\pm = \{v \in L^2 \colon I_\chi v = \pm\sqrt{\lambda} v\}.$$

Multiplication by i interchanges these subspaces.

In particular, L^2 is the complexification of V_+, and so we can define the subgroup $(U(\infty), O(\infty))$ in $U(\infty)$ (see §1.4).

Suppose that the group An_p acts in $L^2(\mathbf{Q}_p)$ by unitary operators according to the formula

$$T_\chi(q)f(z) = f(q(z))\chi(q'(z))|q'(z)|^{1/2}.$$

Theorem 2.2. $T_\chi(q) \in (U(\infty), O(\infty))$.

The theorem is a consequence of the following lemma.

Lemma 2.2. *The operator* $A_\chi(q) = I_\chi T_\chi(q) - T_{\chi^{-1}}(q) I_\chi$ *has finite rank.*

Proof. We have

$$A_\chi(q)f(u) = \int \frac{\overline{f(q(z))}|q'(z)|^{1/2}\chi(p'(z))\,dz}{|z-u|\chi^2(z-u)} - \int \frac{\overline{f(z)}\,dz \cdot \chi^{-1}(p'(u))|p'(u)|^{1/2}}{|z-p(u)|\chi^2(z-p(u))}.$$

Making the change of variable $z = p(w)$ in the second integral, we obtain

$$A_\chi(q)f(u) = \int \overline{f(p(u))}|p'(w)|\chi(p'(w)) \times \left[\frac{1}{|w-u|\chi^{-2}(w-u)} - \frac{|p'(w)|^{1/2}|p'(u)|^{1/2}\chi^{-1}(p'(w))\chi^{-1}(p'(u))}{|p(w)-p(u)|\chi^{-2}(p(w)-p(u))}\right] du.$$

The expression is square brackets is locally constant and equal to 0 in a neighborhood of the diagonal. This proves the lemma.

(c) Each sphere B can be canonically represented as a union of pairwise disjoint spheres $B_1, \dots, B_n$ (we shall say that the B_j are a *canonical partition of* B).

(d) If $B_1 \supset B_2 \supset \cdots$ is a sequence of imbedded spheres ($B_{j+1} \neq B_j$), then $\bigcap B_j$ consists of exactly one point.

We call a homeomorphism q of a sphere B into a sphere C *proper* if q carries subspheres into subspheres and canonical partitions into canonical partitions.

We call a homeomorphism r of a spheroid M into a spheroid N a *spheromorphism* if there exists a partition of N into subspheres $N = \bigcup R_j$ such that $r(R_j)$ is a sphere for all R_j and r is a proper sphere homeomorphism $R_j \to r(R_j)$.

Remark [3]. Let M be a spheroid and $M = P_1 \cup \cdots \cup P_N$ a covering of M by pairwise disjoint spheres. Let d be the remainder of the division of N by $n-1$. Then d does not depend on the partition and is the (unique) invariant of the spheroid under spheromorphisms.

Example. The Cantor set is endowed with a spheroid structure in the obvious way.

Another example of a spheroid is the absolute A_n of the Bruhat-Tits tree J_n (spheres are what were called cells above). This example is universal; to wit, any spheroid can be spheromorphically imbedded into A_n.

Proposition 3.1. *Any analytic transformation* $q \in \mathrm{An}_p$ *is a spheromorphism* $A_p \simeq \mathbf{Q}_p P^1$.

Proof. The assertion is local and, by virtue of the action of $\mathrm{SL}_2(\mathbf{Q}_p)$, without loss of generality we can restrict ourselves to a mapping of a sphere of the form $|z-a| \leq p^k$ into a sphere of the form $|z-b| \leq p^n$. Thus, suppose that in a neighborhood of the point a the mapping has the form

$$q(z) = c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots .$$

We take a neighborhood $B = \{z \colon |z-a| < 1/p^N\}$ so small that the series converges in it and $|q'(z) - c_1| < c_1$. Then q is a proper homeomorphism of the sphere B onto the sphere $\{z \colon |z - c_0| < |c_1|/p^N\}$. This proves the assertion.

3.5. **The group** $\mathrm{Diff}(A_n)$. We define the group $\mathrm{Diff}(A_n)$ as the spheromorphism group of the absolute A_n of the tree J_n. Let us define this group without using the word "spheromorphism".

We take some edge of the tree J_n and cut it in the middle. Then the tree splits into two sets, which we shall call *branches*. To each branch L there naturally corresponds a subset A_L of the absolute, namely, those points to which one can go by moving along paths that lie in this branch (more accurately: A_L consists of equivalence classes of the paths that lie in this branch). We call a set of branches $L_1, \dots, L_k$ such that the L_j are pairwise disjoint and the sets A_{L_j} cover the entire absolute a *broom*.

Let $L_1, \dots, L_k$ and $L'_1, \dots, L'_k$ be two brooms in J_p. Let σ be a permutation of the set $\{1, \dots, k\}$. We map each branch L_j isomorphically onto the branch $L'_{\sigma(j)}$. This set of mappings induces a homeomorphism of the absolute. The group $\mathrm{Diff}(A_n)$ consists of all of the homeomorphisms absolute that can be obtained in this way.

3.6. **Canonical measure on the absolute.** We fix some point ∞ of the absolute A_n. In the set of vertices of the tree J_p we introduce a function h with values in $\mathbf{Z}$ that satisfies the following condition: if $a_1, a_2, \dots$ is a path that leads to ∞, then $h(a_{j+1}) = h(a_j) + 1$. Naturally, this function is unique up to the addition of a constant.

4.1. **The p-adic Hilbert transform.** Let $z \in \mathbf{Q}_p^*$, $z = a_k p^k + a_{k+1}p^{k+1} + \cdots$, where $a_k \neq 0$. We set

$$\operatorname{sgn}(z) = (a_k/p).$$

We define the Hilbert transform in $L^2(\mathbf{Q}_p)$ by

$$If(z) = \frac{1}{\sqrt{p}} \int_{\mathbf{Q}_p} \frac{\operatorname{sgn}(z-u)f(u)\,du}{|z-u|}.$$

If f is a finite function that takes only a finite number of values, then this integral is well defined in the sense of principal value:

$$\text{p.v.} \int_{\mathbf{Q}_p} q(z)\,dz \overset{\text{def}}{=} \lim_{N\to\infty} \int_{|z|\geq 1/p^N} f(z)\,dz.$$

In addition, a direct calculation shows that $\langle If, Ig\rangle = \langle f, g\rangle$ for any compactly supported functions f and g taking only a finite number of values. Hence, I can be uniquely extended to a unitary operator in $L^2(\mathbf{Q}_p)$.

It is not complicated to check that $I^2 = -1$. This can be checked directly, but it is more elegant to carry out a Fourier transform $\mathscr{F}$ in $L^2(\mathbf{Q}_p)$:

$$(\mathscr{F} I \mathscr{F}^{-1})f(u) = i\operatorname{sgn}(u)f(u).$$

In particular, we see that the operator I has two proper subspaces V_+ and V_-, where $V_\pm$ consists of functions whose Fourier transform has support in the set

$$\mathbf{Q}_p^\pm = \{z\mathbf{Q}_p^* \colon \operatorname{sgn} z = \pm 1\}.$$

4.2. **The group** An_p^+. This group consists of analytic transformations of $\mathbf{Q}_p P^1$ such that $\operatorname{sgn} q'(x) = 1$ for all x. If desired, we can interpret An_p^+ as the group of orientation-preserving diffeomorphisms.

We note that $\mathrm{PSL}_2(\mathbf{Q}_p) \subset \mathrm{An}_p^+$.

4.3. **Imbeddings of** An_p^+ **in** GL_∞ **and in** $(U(2\infty), U(\infty)\times U(\infty))$. Let χ be a homeomorphism of $\mathbf{Q}_p^*$ into $\mathbf{C}^*$. We define the representation $T_\chi(q)$ of the group An_p^+ in $L^2(\mathbf{Q}_p)$:

$$T_\chi(q)f(x) = f(q(x))\chi(q'(x))|q'(x)|.$$

In $L^2(\mathbf{Q}_p)$ we distinguished the two subspaces V_+ and V_-. The group $\mathrm{GL}_\infty = (\mathrm{GL}(2\infty, \mathbf{C}), \mathrm{GL}(\infty, \mathbf{C}) \times \mathrm{GL}(\infty, \mathbf{C}))$ consists of operators that "almost preserve $V_\pm$" (see §1.5).

Theorem 4.1. (a) $T_\chi(q) \in \mathrm{GL}_\infty$.

(b) *If* $|\chi| = 1$, *then* $T_\chi(q) \in (U(2\infty), U(\infty)\times U(\infty))$.

Proof. Assertion (b) follows from (a), and (a) is a consequence of the following lemma.

Lemma 4.1. $[T_\chi(q), I]$ *has finite rank.*

Proof. We have

$$(IT_\chi(q) - T_\chi(q)I)f(u) = \int_{\mathbf{Q}_p} \frac{f(q(z))|q'(z)|^{1/2}\chi(q'(z))\,dz}{|z-u|\operatorname{sgn}(z-u)} - \int_{\mathbf{Q}_p} \frac{f(z)|q'(u)|^{1/2}\chi(q'(u))\,dz}{|z-q(u)|\operatorname{sgn}(z-q(u))}.$$

We require that this mapping lie in the group $\mathrm{PSL}_2(\mathbf{F}_q)$ (to emphasize the point, it must lie in $\mathrm{PSL}_2(\mathbf{F}_q)$, not just in $\mathrm{PGL}_2(\mathbf{F}_q)$(!)).

4.6. **The combinatorial Hilbert transform.** We fix the point ∞ on the absolute A_p of the tree J_p. Let v be a vertex of the tree. Then among the $p+1$ edges that go to v the edge l_∞ is selected, namely, the one that is directed toward the side of the point $\infty \in A_p$. Let $l_0, \dots, l_{p-1}$ be the remaining edges that go to v. The elements of the set $l_0, l_1, \dots, l_{p-1}, l_\infty$ are in bijective correspondence with the points of the projective line $\mathbf{F}_pP^1$. Without loss of generality we can assume that l_j corresponds to a point $j \in \mathbf{F}_p$. Then the remaining edges $l_0, \dots, l_{p-1}$ are in bijective correspondence with the points of the affine projective line $\mathbf{F}_p^1$. Without loss of generality we can assume that l_j corresponds to a point $j \in \mathbf{F}_p$. Let $i \neq j$. We set

$$\mathrm{sgn}(l_i, l_j) = ((i-j)/p).$$

Remark. It is important to emphasize that the right-hand side of the equality is invariant with respect to the subgroup $B \subset \mathrm{SL}_2(\mathbf{F}_q)$—the stabilizer of the point ∞ in $\mathbf{F}_pP^1$. Indeed, the group B consists of transformations of the projective line of the form $j \mapsto \alpha^2 j + c$.

Let $h(v)$, $\eta(z_1, z_2)$, and $\rho(z_1, z_2)$ be the same as in §3.6. For two distinct points z_1 and z_2 $(z_j \neq \infty)$ of the absolute we also define the quantity $\mathrm{sgn}(z_1, z_2) = \pm 1$. To do so, we join z_1 and z_2 by a path $\dots, a_{-1}, a_0, a_1, \dots$ leading from z_1 to z_2. Let a_s be the vertex at which the maximum of the function $h(a_j)$ is attained. Let l_1 be the edge $[a_j, a_{j-1}]$ and l_2 the edge $[a_j, a_{j+1}]$. Then

$$\mathrm{sgn}(z_1, z_2) := \mathrm{sgn}(l_1, l_2).$$

We define the Hilbert transform in $L^2(A_p)$ by the formula

$$If(z) = \lambda \int_{A_p} \frac{\mathrm{sgn}(z, u)}{\rho(z, u)} f(u)\, du,$$

where λ is chosen from the condition $I^2 = -1$.

Remark. Here we need to use all of the words that we used in §4.1. The integral in the sense of principal value is understood as

$$\text{p.v.} \int_{A_p} f(u)\, du = \lim_{k\to\infty} \int_{A_p \setminus B_k} f(u)\, du,$$

where B_k is a sequence of spheres, containing u_0, such that $\bigcap B_k = u_0$.

If we identify A_p with $\mathbf{Q}_pP^1$, then our Hilbert transform coincides with the Hilbert transform in §4.1.

4.7. **Imbeddings of** $\mathrm{Diff}^+(\widetilde{J}_p)$ **in** GL_∞ **and in** $(U(2\infty, U(\infty)\times U(\infty))$. Let $\alpha \in \mathbf{C}$. We define the action of $\mathrm{Diff}^+(J_p)$ in $L^2(A_p)$ by

$$T_\alpha(q)f(z) = f(q(z))|q'(z)|^{1/2+i\alpha}.$$

Theorem 4.2. (a) $T_\alpha(q) \in \mathrm{GL}_\infty$.

(b) If $\alpha \in \mathbf{R}$, then $T_\alpha(q) \in (U(2\infty), U(\infty) \times U(\infty))$.

The proof coincides with that of Theorem 4.1.

Naturally, having such imbeddings, we have representations of the group $\mathrm{Diff}^+(J_p)$ as well.

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Moscow Institute of Electronic Machine Building

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