Boundary values of holomorphic functions and spectra of some unitary repesentations

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These notes are based on my lectures on discrete spectra given in Tambov in August 1996 (school "Analysis on homogeneous spaces")

Now spectra in various problems of noncommutative harmonic analysis are completly or partialy evaluated. It is well-known that sometimes such spectra contain discrete increments. Quite often such discrete increments are singular ("exotic") unitary representations and it is very difficult to construct these unitary representations by other way, see [Puk], [Nai], [Boy], [Ism], [Mol1-3], [Str], [Far], [F-J], [Sch], [Kob1], [Kob2], [RSW], [Tsu], [How], [Ada], [Li], [Pat], [BO]

It was observed in [Ner1], [Ols2], [Ols3], [NO], [Ner2] that very often discrete increments to spectra in various problems of noncommutative harmonic analysis (decomposition of tensor products, decomposition of restrictions, decomposition of induced representation) are related to some functional-theoretical phenomena, namely to so-called "trace theorems " (i.e theorems about existence of restrictions of discontinuous functions to submanifolds, for this type theorems see [RS],IX.3,IX.9 and references to this sections, [Bar],chapter 5, [NR] , [Rud], 11.2).

The simplest case of this phenomenon is the tensor product of two representations of complementary series of $SL_2(\mathbb{R})$.Recall the definition of representation T_s of the complementary series. The space $\mathcal{H}_s$ of the repesentation T_s (where $0 < s < 1$) is the space of functions on the circle $z_1 = e^{i\phi}$ provided with the scalar product

$$< f_1, f_2 >= \int_0^{2\pi} \int_0^{2\pi} \frac{f_1(\phi_1)\overline{f_2(\phi_2)} d\phi_1 d\phi_2}{|\sin(\phi_1-\phi_2)/2|^{(1+s)/2}}$$

The representation T_s is defined by the formula

$$T\begin{pmatrix} a & b \\ \overline{b} & \overline{a}\end{pmatrix} f(\phi) = f(\frac{az+b}{\overline{b}z+\overline{a}})|\overline{b}z+\overline{a}|^{s-1}$$

The representation $T_{s_1}\otimes T_{s_2}$ acts in the space $\mathcal{H}_{s_1}\otimes\mathcal{H}_{s_2}$ of functions on two-dimensional torus $z_1 = e^{i\phi_1}, z_2 = e^{i\phi_2}$ equipped with the scalar product

$$< f_1, f_2 >= \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{f_1(\phi_1,\phi_2)\overline{f_2(\psi_1,\psi_2)} d\phi_1 d\phi_2 d\psi_1 d\psi_2}{|\sin(\phi_1-\psi_1)/2|^{(1+s_1)/2}|\sin(\phi_2-\psi_2)/2|^{(1+s_2)/2}}$$

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The group $SL_2(\mathbb{R})$ acts in this space by the formula

$$(T_{s_1} \otimes T_{s_2}) \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} f(z_1, z_2) = f(\frac{az_1 + b}{\overline{b}z_1 + \overline{a}}, \frac{az_2 + b}{\overline{b}z_2 + \overline{a}})|\overline{b}z_1 + a|^{s_1 - 1}|\overline{b}z_2 + a|^{s_2 - 1}$$

If $s_1 + s_2 > 1$ then there exists well-defined operator R of restriction of a function $f \in \mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2}$ to the diagonal $\Delta : \phi_1 = \phi_2$. We emphasis that functions $f \in \mathcal{H}_{s_1} \otimes \mathcal{H}_{s_2}$ are discontinuous and hence f have no values in a individual point of the torus. Neverless the operator R of the restriction of a function f to the diagonal is well defined . Observe that R is interwinning operator from $T_{s_1} \otimes T_{s_2}$ to $T_{s_1+s_2-1}$. Hence $T_{s_1+s_2-1}$ is a subrepresentation in $T_{s_1} \otimes T_{s_2}$.

Existense of the embedding $T_{s_1+s_2-1}$ to $T_{s_1} \otimes T_{s_2}$ was obtained in [Puk] (see also [Nai]). The construction with restriction to the diagonal was observed in [Ner1] (see also [NO]).

Various consructions of the same type are contained in [NO],[Ner1]-[Ner2],[Ols3]. In [NO] we used this approach for constructions of singular unitary representations of the groups $U(p, q), O(p, q), Sp(p, q)$.

These notes is some kind of addendum to the papers [NO],[Ner2]. The aim of these notes is to formulate several open problems on discrete increments to spectra and trace theorems and to discuss a relationship between some spectral problems.

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1 Boundary values of holomorphic functions.

1.1. Let $\Omega \subset \mathbb{C}^n$ be a open domain, let $\partial\Omega$ be its boundary, let $\overline{\Omega}$ be the closure of Ω. We say that Ω is a *regular circle domain* if

a) for all $z \in \Omega$ and $\lambda \in \mathbb{C}$ such that $|\lambda| \leq 1$ we have $\lambda z \in \Omega$

b) for all $z \in \partial\Omega$ and $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$ we have $\lambda z \in \Omega$

Let $K(z, u)$ be a reproducing kernel (see for instance [NO]) in Ω satisfying the condition

$$K(e^{i\phi}z, e^{i\phi}u) = K(z, u)$$

Let H be the hilbert space of holomorphic functions associated to this kernel

Theorem 1.1.*(see [NO]) Let $M \subset \partial\Omega$ be a compact subset. Let μ be a measure supported on M. Let*

a)$K^*(z, u) := \lim_{c \to 1-0} K(cz, cu)$ exist almost sure on $M \times M$ with respect to the measure $\mu \times \mu$.

b)$K^* \in L^1(M \times M, \mu \times \mu)$ and $\lim_{c \to 1-0} K(cz, cu)$ is dominated, i.e. there exists a function $S(z, u) \in L^1(M \times M, \mu \times \mu)$ such that $|K(z, u)| < S(z, u)$ almost sure on $M \times M$.

Then the operator of restriction of a function $f \in H$ on the set M is well-defined operator

$$H \to L^1(M, \mu)$$

The following natural problem arises.

Question. Let $\Omega \in \mathbb{C}^n$ be a a open domain. Let $K(z,u)$ be a reproducing kernel and let H be the associated hilbert space. Let M be a submanifold in the Shilov boundary of Ω. Find conditions for existence of restriction operator from H to some functional hilbert space on M.

Remark. A case which is interesting for harmonic analysis is the following case . Let $\Omega = G/K$ be a homogeneous Cartan domain. Let Q be a subgroup in G and let M be a orbit of Q in the Shilov boundary of Ω. Then operator R of restriction to Ω is a interwinning operator from H to some hilbert space of funtions on M.See discussion of such restrictions in [NO], [Ner2] and below sections 2 and 3

Following subsection show that Theorem 1.1 doesn't cover all cases then a restriction operator exists.

1.2. Denote by B_q the unit ball

$$|z_1|^2 + \cdots + |z_q|^2 < 1$$

in $\mathbb{C}^q$. Let $\gamma(t)$ be a C^1-curve in the $\partial B^q = S^{2q-1}$.

Theorem 1.2_(see [NR],[Rud]) Let $\gamma(t)$ satisfies the condition_

$$\forall t : Im < \gamma(t), \gamma'(t) > \neq 0 \qquad (1)$$

Then for each $f \in H^\infty(B^q)$ the nontangent limit $f(z)$ as $z \to \gamma(t)$ exists almost sure on $\gamma(t)$.

Denote by D' the space of all holomorphic functions of polynomial grouth in B^q

$$f \in D' \Leftrightarrow \exists N : \sup |f(z)|(1 - |z|^2)^N < \infty$$

It is well known that each function $f \in D'$ has limit on the boundary in the sence of distributions (see [RS],IX.3 for discussion of such type theorems and references).

Theorem 1.3._(see [Ner2]) Let $\gamma(t)$ is C^∞-smooth curve in ∂B^q satisfying the condition (1). Then the operator R of restriction of holomorphic function to $\gamma(t)$ extends to a bounded operator from D' to the space of distributions on $\gamma(t)$._

Denote by P^n the polydisk $|z_1| < 1, \ldots, |z_n| < 1$. Let T^n be the torus $z = e^{i\phi_1}, \ldots, z = e^{i\phi_n}$. Let $\gamma(T) = (\phi_1(t), \ldots, \phi_n(t))$ be a C^∞-curve in T^n such that

$$\forall t : \phi_1'(t) > 0, \ldots, \phi_n'(t) > 0$$

Denote by D' the space of holomorhic functions of polynomial grouth in P^n.

Theorem 1.4._(see [Ner2]) Operator R of restriction of holomorphic function f on P^n to the curve $\gamma(t)$ extends to the bounded operator from the space D' to the space of distributions on the curve $\gamma(t)$_

Let $\Omega \subset \mathbb{C}^N$ be a open domain. Let M be a submanifold in the Shilov boundary of Ω. Denote by T_m the tangent space to M in the point $N \in M$.

We identify T_m with a linear submanifold in $\mathbb{C}^n$. Denote by S_m the linear submanifold which consists of vectors

$$i \cdot (v - m) + m$$

where $v \in T_m$. Assume that for each point $m \in M$ there exists a open cone $C_m \subset T_m$ with the vertex m and ϵ-neighbourhood $O_\epsilon(m)$ of m such that

$$C_m \cap \Omega \supset O_\epsilon(m) \cap C_m$$

Conjecture. Each holomorphic function of polynimial grouth in Ω has restriction to M in the sense of distributions.

Theorems 1.3-1.4 are partial cases of this conjecture. It is also similiar to the standard facts on limits of functions of polynomial grouth on the *whole* Shilov boundary mentioned above (see [RS],IX.3). Neverless I couldn't find this fact in the literature.

2 Positive defined kernels on riemann noncompact symmetric spaces

2.1. Matrix balls $B_{p,q}$. Let $p \le q$. Denote by $B_{p,q}$ the space of all complex $p \times q$-matrices z such that $\|z\| < 1$. The group $U(p,q)$ consists of $(p+q)\times(p+q)$-matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying the condition

$$g \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot g^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The group $U(p,q)$ acts on $B_{p,q}$ by the transformations

$$z \mapsto z^{[g]} := (a + zc)^{-1}(b + zd) \tag{2}$$

The stabilizer of the point $z = 0$ consists of matrices having the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ where $a \in U(p), d \in U(q)$ .Hence $B_{p,q}$ is the symmetric space

$$B_{p,q} = U(p,q)/(U(p) \times U(q))$$

Consider the function

$$L_s(z,u) = |\det(1 - zu^*)|^{-2s}$$

where $z, u \in B_{p,q}$.

Theorem 2.1. *Let $s = 0, 1, 2, \ldots, p-1$ or $s > p-1$. Then the function $L_s(z,u)$ is a positive defined kernel on $B_{p,q}$.*

(this theorem is a consequence of the theorem 2.2 below)

Consider the hilbert space defined by the positive defined kernel $L_s(z,u)$. This space contains the total system of vectors $\Psi_z, z \in B_{p,q}$ such that

$$< \Psi_z, \Psi_u >= L_s(z,u)$$

We associate to each vector $h \in H_s$ the function f_h on $B_{p,q}$ by the rule

$$f_h(z) =< h, \psi_z >$$

It is easy to prove that f_h is a real analytic function on $B_{p,q}$. We will identify the space H_s with its image in the space of real analytic functions on $B_{p,q}$.

The group $U(p,q)$ acts in H_s by the unitary operators

$$A_s(g)f(z) = f(z^{[g]})|\det(a+zc)|^{-2s}$$

Problem *Decompose the repersentation* A_s

We will name representations T_s by *spherical kern-representations* of the group $U(p,q)$.

Some partial cases of this question were discussed in the end of 70-ies (See [Ber2],[Rep],[Gut], in fact there was discussed only the case when s is large. In this case the most interesting phenomena don't appear). Then such problems were more or less forgotten. In last several years some this problem attracted interest again(see [NO], [OO], [OZ],[Dij]).

I would like to try to explain why this problem is interesting and also to discuss some approaches to this problem.

2.2. Another formulation of the problem.

Theorem 2.2 *(see [Ber1]) Let Let $s = 0, 1, 2, \ldots, p-1$ or $s > p-1$. Then the kernel*

$$K_s(z,u) = \det{}^{-s}(1 - zu^*)$$

is positive defined on $B_{p,q}$.

Denote by V_s the hilbert space of holomorphic functions on $B_{p,q}$ defined by the kernel $K_s(z,u)$ (see for instance [Ber1], [NO]). The group $U(p,q)$ acts in V_s by the unitary operators

$$T_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = f(z^{[g]})\det{}^{-s}(a+zc)$$

where $z^{[g]}$ is given by the formula (2).

Remark. If s is integer then T_s is a representation of the group $U(p,q)$ itself. If s is not integer then T_s is a representation of the universal covering group of $U(p,q)$.

Denote by T_s^* the representation contragradient to T_s. Consider the tensor product $T_s \otimes T_s^*$. This representation acts in the space of holomorphic functions on $B_{p,q} \times B_{p,q}$ by the operators

$$(T_s \otimes T_s^*) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = f(z_1, z_2) = f(z_1^{[g]}, z_2^{[\overline{g}]})(a + z_1 c)^{-s}(\overline{a} + z_2 \overline{c})^{-s}$$

The $U(p,q)$-invariant scalar product in the space of holomorphic functions on $B_{p,q} \times B_{p,q}$ is defined by the reproducing kernel

$$M(z_1, z_2; u_1, u_2) = \det{}^{-s}(1 - z_1 u_1^*)\det{}^{-s}(1 - z_2 u_2^*)$$

Consider the operator

$$I : V_s \otimes V_s \to H_s$$

defined by the formula

$$If(z) = f(z, \overline{z})$$

Obviously I is a unitary operator interwining

$$T_s \ \otimes T_s^* \leftrightarrow A_s$$

Hence we can formulate our problem in the form:

Problem. *Decompose the tensor product* $T_s \otimes T_s^*$.

2.3.Orbits of the group $U(p,q)$ on the Shilov boundary.

Denote by $M_{p,q}$ the Shilov boundary of $B_{p,q}$. Elements of $M_{p,q}$ are matrices z satisfying the condition

$$z \cdot z^* = 1$$

In the other words z is a matrix of a isometric embedding $\mathbb{C}^p \to C^q$. Hence $M_{p,q}$ is *complex Stiefel manifold*.

The Shilov boundary of $B_{p,q} \times B_{p,q}$ is $M_{p,q} \times M_{p,q}$. The group $U(p,q)$ has $(p+1)$ orbits on $M_{p,q} \times M_{p,q}$. The unique invariant of a orbit is the number

$$\alpha = rk(z - \overline{u}) \tag{3}$$

We denote by Ξ_α the orbit corresponding to a given invariant α (i.e. Ξ_α is the set of all pairs $z, u \in M_{p,q} \times M_{p,q}$ such that (3) is satisfied)

Orbit Ξ_0 is compact, orbit Ξ_p is open. For all α the closure of Ξ_α is $\cup_{\sigma \leq \alpha} \Xi_\sigma$.

2.4. Restriction to $U(p,q)$-orbits. Fix a orbit Ξ_α of $U(p,q)$ in the Shilov boundary of $B_{p,q} \times B_{p,q}$. It can happens (and it realy happens) that for small s function $f \in H_s = V_s \otimes V_s$ has well defined restriction to the orbit Ξ_α. In this case the restriction operator is a interwinning operator from $V_s \times V_s$ to some hilbert space of functions on Ξ_α.

It can also happened (and it realy happens) that for small s all first partial derivatives of function $f \in H_s = V_s \otimes V_s$ have well defined restriction to the orbit Ξ_α etc.(see discussion of this phenimenon in [NO],section 7).

Fix s. For each $\alpha = 0, 1, \ldots, p-1$ consder the maximal number τ_α such that all partial derivatives of functions $f \in V_s \otimes V_s$ have well-defined restrictions to Ξ_α. (this numbers aren't known, but Theorem 1.1 give possibility to estimate them. I don't know are such estimates strict or not).If restrictions of functions $f \in H_s = V_s \otimes V_s$ to Ξ_α don't exist we suppose $\tau_\alpha = -1$.

Remark. For large s restriction operators don't exist, i.e. we have $\tau_\alpha = -1$ for all α.

For each α consider some $i = 1, \cdots, \tau_\alpha$. Denote by $Q[\alpha, \tau_\alpha]$ the space of functions $f \in V_s \otimes V_s$ such that all partial derivatives of f of orders $\leq i$ equals zero on Ξ_α.

We obtain a filtration

$$\begin{gathered} 0 \subset Q[p-1, \tau_{p-1}] \subset Q[p-1, \tau_{p-1} - 1] \subset \cdots \subset Q[p-1, 0] \subset \\ \subset Q[p-2, \tau_{p-2}] \subset \cdots \subset Q[p-2, 1] \subset Q[p-2, 0] \subset \ldots \\ \cdots \subset Q[0, \tau_0] \subset \ldots Q[0, 1] \subset Q[0, 0] \subset V_s \otimes V_s \end{gathered} \tag{4}$$

Remark . For large s this filtration is trivial, neverless for small s it is quite long.

Consider representations of $U(p, q)$ in subquotients of this filtration. Obviously $A_s = T_s \otimes T_s^*$ is equivalent to the direct sum of the subquotients.

Remark. The representations of $U(p, q)$ in the subquotients have simple interpretations. For instance $V_s \otimes V_s / Q[0, 0]$ is a subspace in the space of functions on the orbit Ξ_0 . The space $Q[0, 1]/Q[0, 0]$ a subspace in space of sections of normal bundle to the orbit Ξ_0 . The space $Q[0, 1]/Q[0, 2]$ is a subspace in space of sections of symmetric square of normal bundle etc., see discussion in [NO],$ 7.

It is natural to hope that spectrum in each subquotient is more or less "uniform", i.e. orbit structure give separation of quite complicated spectrum of A_s to the different types (compaire with [GG]-project)

2.5. Large s. If s is large enough then the restriction operators don't exist. In this case the representation A_s is equivalent to standart representation of the group $U(p, q)$ in L^2 on riemann symmetric space $U(p, q)/(U(p) \times U(q))$ see [Ber2],[Rep],[Gut],[OO]. Sufficient (not nessessary) condition for this is $s > p + q - 1$(i.e. T_s is a element of Harish-Chandra discrete seri es).

2.6. Limit as $s \to \infty$. Concider the system of vectors $\Psi_z \in H_s$. Let χ be a distribution in $B_{p,q}$ with a compact support. Consider the vector $\Theta(\chi) \in H_s$ defined by the equality

$$\Theta(\chi) = \int_{B_{p,q}} \det^{-s}(1 - zz^*) \chi(z) \Psi_z dz d\overline{z}$$

Consider a scalar product $\{\cdot, \cdot\}_s$ in the spase of distributions on $B_{p,q}$ with compact support given by the formula

$$\begin{aligned} \{\chi_1, \chi_2\}_s :=< \Theta(\chi_1), \Theta(\chi_2) >= \\ = \int_{B_{p,q}} \int_{B_{p,q}} \left| \frac{\det(1 - zz^*)(\det(1 - uu^*)}{\det^2(1 - zu^*)} \right|^s \chi_1(z) \chi_2(u) dz d\overline{z} du d\overline{u} \end{aligned}$$

We can identify the space H_s with the completion of the space of distributions with respect to scalar produt $\{\cdot, \cdot\}_s$. The group $U(p, q)$ acts in this space of distributions by the formula

$$B_s(g) f(z) = f(z^{[g]})$$

(the formula doesn't depend of s, neverless the scalar product and spectra of representation depend of s essentially)

Denote by $\omega(s)$ the integral

$$\omega(s) = \int_{B_{p,q}} \det{}^s(1 - zz^*)$$

Then for all continuous functions on $B_{p,q}$ with compact support

$$\lim_{s\to+\infty} \frac{\omega(0)}{\omega(s)} \{\phi_1, \phi_2\}_s = \int_{B_{p,q}} \phi_1(z)\phi_2(z) dz d\overline{z}$$

It is natural to think that the limit of kern-representations as $s \to +\infty$ is the canonical representation of $U(p,q)$ in the space L^2 on riemann symmetric space $U(p,q)/(U(p) \times U(q))$.

2.7. Restriction to the compact orbit. The part of spectrum which corresponds to the compact orbit Ξ_0 is purely discrete and it consists of quite exotic representations of $U(p,q)$ and this is relatively simple way for constructing singular unitary representations of $U(p,q)$, see [NO].

2.8. Spherical kern-representations of other classical groups. All classical riemann noncompact symmetric spaces G/K (up to centrum of G) can be realized as matrix balls (see [Ner3]). Namely the space G/K is the space of matrices z over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (see below) satisfying the additional condition (see below) such that $\|z\| < 1$.

Now we are enumerate symmetric spaces G/K, fields and additional conditions.

1^*. $U(p,q)/(U(p) \times U(q))$ is the space of $p \times q$ matrices over $\mathbb{C}$.

2^* . $Sp(2n,\mathbb{R})/U(n)$ is the space of symmetric $n \times n$-matrices over $\mathbb{C}$.

3^*. $SO^*(2n)/U(n)$ is the space of skew symmetric $n \times n$- matrices over $\mathbb{C}$.

4^*. $O(p,q)/(O(p) \times O(q))$ is the space of $p \times q$-matrices over $\mathbb{R}$.

5^*. $GL(n,\mathbb{R})/O(n)$ is the space of symmetric $n \times n$-matrices over $\mathbb{R}$.

6^*. $O(n,\mathbb{C})/O(n)$ is the space of skew-symmetric $n \times n$-matrices over $\mathbb{R}$.

7^*, $GL(n,\mathbb{C})/U(n)$ is the space of hermitian $n \times n$-matrices over $\mathbb{C}$.

8^*. $Sp(p,q)/(Sp(p) \times Sp(q))$ is the space of $p \times q$-matrices over $\mathbb{H}$.

9^*. $GL(n,\mathbb{H})/Sp(p,q)$ is the space of hermitian $n \times n$-matrices over $\mathbb{H}$.

10^*. $Sp(2n,\mathbb{C})/Sp(n)$ is the space of skew hermitian (i.e. $z = -z^*$) $n \times n$-matrices over $\mathbb{H}$.

In all cases enumerated above the group G acts on the matrix ball G/K by fractional-linear transformations

$$z \mapsto z^{[g]} := (a + zc)^{-1}(b + zd) \tag{5}$$

Now let us consider positive definite kernels on G/K having a form

$$L_s(z,u) = |\det(1 - zu^*)|^{-2s}$$

(conditions for positive-definiteness are different for different spaces) Consider the hilbert space defined by the positive defined kernel $L_s(z,u)$. We identify this space with some space of real-analytic functions on G/K by the same way as in 2.1 . The group G acts in H_s by the unitary operators

$$A_s(g)f(z) = f(z^{[g]})|\det(a+zc)|^{-2s}$$

We say A_s is a *spherical kern-representation of* G.

2.9. Nonspherical kern-representations. Fix a matrix ball G/K and a finite dimensional euclidian space $\mathcal{Y}$. Denote by $GL(\mathcal{Y})$ the group of invertible linear operators in $\mathcal{Y}$. We say that a function

$$L : G/K \times G/K \to GL(\mathcal{Y})$$

is a *matrix-valued positive definite kernel* if the function

$$\widetilde{L}((z,\xi);(u,\eta)) := < K(z,u)\xi, \eta > ; (z,\xi),(u,\eta) \in G/K \times \mathcal{Y}$$

is a positive defined kernel on $G/K \times \mathcal{Y}$.

Let we have a matrix-valued positive defined kernel on G/K. Then there exist a hilbert space H and a map $\Psi : G/K \times \mathcal{Y} \to H$ such that

a) The map Ψ is linear on each fibre $z \times \mathcal{Y} \subset G/K \times \mathcal{Y}$.

b) $< \Psi(z,\xi), \Psi(u,\eta) >_H = < K(z,u)\xi, \eta >_{\mathcal{Y}}$

c) The image of map Ψ is dence in H.

For each $h \in H$ we define a function $f_h : G/K \to \mathcal{Y}$ by the rule $< f_h, \xi >_{\mathcal{Y}} = < h, \Psi(z,\xi) >_H$. Consider a symmetric spaces having the form $2^*, 3^*, 5^*, 6^*, 7^*, 9^*, 10^*$. Consider a finite dimensional irreducible representation ρ of the group $GL(n,\mathbb{K})$ or of its universal covering. Assume that the function

$$L(z,u) = \rho(1 - zu^*)$$

is a matrix valued positive definite kernel. Then we consider the associated hilbert space H_ρ of real-analytic functions $G/K \to \mathcal{Y}$ and the unitary kern-representation of G in H_ρ given by the formula

$$T_\rho(g)f(z) = \rho(a+zc)f(z^{[g]}) \qquad (6)$$

Consider the cases $1^*, 4^*, 9^*$. Consider a finite-dimensional irreducible representation $\rho = \rho_1 \otimes \rho_2$ of the group $GL(p,\mathbb{K}) \times GL(q,\mathbb{K})$ or of its universal covering. Assume that the function

$$L_\rho(z,u) := \rho_1(1 - zu^*) \otimes \rho_2(1 - u^*z)$$

is a positive defined matrix-valued kernel on G/K. Then the group G acts in the associated space of real-analytic functions on G/K by the formula

$$T_\rho(g)f(z) = \big(\rho_1(a+zc) \otimes \rho_2(d - cz^{[g]})\big) f(z^{[g]})$$

Remark. Our arguments from subsections 2.5 are valid for general kern-representations.

2.10. Another description of kern-representations (see [OO]). For $G = Sp(2n,\mathbb{R}), U(p,q), SO^*(2n)$ a kern representation is a tensor product of irreducible highest weight representation of G and irreducible lowest weight representation of G.

In other cases a kern representation of G is a restriction of a highest weight representation of the group G^* to the **symmetric** subgroup G:

4^*.	$G = O(p,q)$	$G^* = U(p,q)$	5^*.	$G = GL(n,\mathbb{R})$	$G^* = Sp(2n,\mathbb{R})$
6^*.	$G = O(n,\mathbb{C})$	$G^* = SO^*(2n)$	7^*	$G = GL(n,\mathbb{C})$	$G^* = U(n,n)$
8^*	$G = Sp(p,q)$	$G^* = U(2p,2q)$	9^*	$G = GL(n,\mathbb{H})$	$G^* = SO^*(2n)$
10^*	$G = Sp(2n,\mathbb{C})$	$G^* = Sp(4n,\mathbb{R})$			

Remark. The cases $1^* - 3^*$ can be described by the same way . We have $G^* = G \times G$ and the embedding $G \to G^*$ is given by the formula $g \mapsto (g, g^\theta)$ where θ is the outer automorphism of G.

Remark. There are some additional possibilities related to highest weight representations of $O(p,2)$ and two exeptional groups (see [OO]).

2.11. Action of Olshanskii semigroup. (see [Ols1]) For each matrix ball G/K consider the set of matrices $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that the mapping (6) maps the matrix ball to itself. Obviously Γ is a semigroup and the group G is the group of invertible elements of Γ. The formula (6) defines representation of semigroup Γ. This representation is irreducible and all irreducible representations of Γ can be obtaned by this way(see[Ols1]). See [Ols1], [Ner3],[Ner4],appendix A, for the explicit desvription of semigroups Γ.

Moreover kern-representations extends to representations of some categories (see[Ner3],[Ner4],appendix A).

I don't know any applications of these phenomena to harmonic analysis of kern-representations of groups.

2.12. Bibliographical comments.

a) Let T be a highest weight representation of $G = Sp(2n,\mathbb{R})$, $SO^*(2n)$, $U(p,q)$,... and S be a lowest weight representation of G. Assume that T, S be elements of Harish-Chandra discrete series. Then $T \otimes S$ is equivalent to a representation of G induced from irreducible representation of K (see [Ber2],[Rep],[Gut]).

b) Restriction of a spherical highest weight representation of G^* to the symmetric subgroup G (i.e spherical kern-representation, see notations of 2.10) is equivalent to canonical representation of G in $L^2(G/K)$,see [OO].

c) Discrete spectra associated to the compact orbit in Shilov boundary were investigated in [NO] for the case $G = O(p,q)$. Analogical results are valid for $G = U(p,q), Sp(p,q)$([Ols2],[NO],7.12.). One method of separation of discrete spectrum is discussed in [Ner1], [NO],7.1-7.8. I think that the restriction operator to the compact orbit doesn't exist for $G \neq U(p,q), Sp(p,q), O(p,q)$.

d) The spectrum for spherical kern-representation of $U(2,2)$ was obtained in [OZ2] . For $1 < s < 3/2$ the spectrum consist of two different pieces. One of pieces coincides with the spectrum of $L^2(U(2,2)/(U(2) \times U(2))$. Another piece is a integral of notrivial representations. It is natural to think (it is not proved) that this piece of spectrum is associated to noncompact $U(2,2)$-orbit in the Shilov boundary of $B_{2,2} \times B_{2,2}$.

If it is so it is the unique known case when spectrum associated to noncompact orbit is observed. It is natural to think that such spectra exist in various spectral problems(not only for kern-representations).

e)Spectrum of spherical kern-representations of $U(p,1)$ is obtained in [Dij].

f) Plancherel formula for tensor product of highest weight and lowest weight representations of $SL(2,\mathbb{R})$, see [Mol4]. I don't know analogical results for other kern-representations.

g) Nonspherical kern-representations have discrete spectrum which is not associated to compact orbit. Some possibilities to observe it are contained in the following two sections. A way to observe Harish-Chandra discrete series increments using trace theorems is proposed in [Ner2].

h) Let ρ be the same as in 2.10. Let $\rho_s(\gamma) = \det^{-s}(\gamma)\rho(\gamma)$. Then limit of T_{ρ_s} as $s \to \infty$ is the representation of G induced from finite dimensional representation of K (i.e the representation in sections of vector bundle on G/K). The theory of such representations is more or less equivalent to Harish-Chandra theory of $L^2(G)$.

3 Dual pairs

3.1. Spectrum of dual pairs. Consider the harmonic representation W_{2N} (= Weil representation = Segal-Shale-Weil representation = Friedrichs-Segal-Berezin-Shale-Weil representation = oscilator representation) of the group $Sp(2n,\mathbb{R})$ (see [KV],[Ner4]) for discussion of this representation).

Consider the following subgroups in the simplectic group it (noncompact Howe dual pairs):

$$Sp(2k(p+q),\mathbb{R}) \supset Sp(2k,\mathbb{R}) \times O(p,q)$$
$$Sp(2(k+l)(p+q),\mathbb{R}) \supset U(k,l) \times U(p,q)$$
$$Sp(4k(p+q),\mathbb{R}) \supset SO^*(2k) \times Sp(p,q)$$

Let us restrict W_{2N} to these subgroups and then let us restrict to

$$O(p,q), Sp(2k,\mathbb{R}), U(k,l), U(p,q), SO^*(2k), Sp(p,q)$$

It was proved in [Ada],[Li] that the spectra of these restrictions have discrete increments. This construction is one of standard way to obtain singular unitary representations of groups $U(p,q), O(p,q), Sp(p,q)$.

Proposition 3.1. *(see [NO]) Each representation of* $G = O(p,q)$, $Sp(2k,\mathbb{R})$, $U(k,l)$, $U(p,q)$, $SO^*(2k)$, $Sp(p,q)$ *which occurs in spectra of dual pair discretly (resp.weakly) occurs in spectra of some kern-representation discretly (resp.weakly).*

This proposition is more or less obvious. Concider for instance the case $Sp(2k,\mathbb{R}) \times O(p,q)$. The restriction of $W_{2k(p+q)}$ to the subgroup

$$Sp(2k,\mathbb{R}) \subset Sp(2k,\mathbb{R}) \times O(p,q) \subset Sp(2k(p+q),\mathbb{R})$$

is equivalent to the representation

$$W_{2k}^{\otimes p} \otimes (W_{2k}^*)^{\otimes q} \tag{7}$$

First tensor factor is a direct sum of highest weight representations and second tensor factor is a direct sum of lowest weight representations . Hence (6) is a direct sum of kern-representations.

Consider the following subgroups in $Sp(2k(p+q),\mathbb{R})$:

$$\begin{array}{ccccccc} Sp(2k(p+q),\mathbb{R}) & \supset & O(p,q) & \times & Sp(2k,\mathbb{R}) & \supset & O(p,q) \\ \| & & \cap & & \cup & & \cap \\ Sp(2k(p+q),\mathbb{R}) & \supset & U(p,q) & \times & U(k) & \supset & U(p,q) \end{array}$$

The restriction of $W_{2k(p+q)}$ to $U(p,q)$ is a direct sum of highest weight representations and hence the restriction of $W_{2k(p+q)}$ to $O(p,q)$ is a direct sum of kern-representations.

3.2. Restriction to orbits. We realize the group $Sp(2N,\mathbb{R})$ as the group of $(N+N)\times(N+N)$-matrices with complex coefficients having the form $g = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}$ and satisfying the condition

$$g \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot g^t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Denote by C_N the space of complex symmetric $N\times N$-matrices z . The group $Sp(2n,\mathbb{R})$ acts on C_N by the fractional linear transformations and we have $C_N = Sp(2N,\mathbb{R})/U(N)$. Consider a reproducing kernel

$$K(z,u) = \det{}^{-1/2}(1 - zu^*)$$

on C_N. Denote by H the associated hilbert space.. The group $Sp(2N,\mathbb{R})$ acts in H by the unitary operators

$$W_{2N}^+(g) = f(z^{[g]})\det{}^{-1/2}(\Phi + z\overline{\Psi})$$

The representation W_{2n}^+ is one of two irreducible components of the representation W_{2N}.

Again we have question about restrictions of holomorphic functions to $O(p,q)\times Sp(2k,\mathbb{R})$-orbits in the Shilov boundary of C_N. It seems that a orbit structure of Shilov boundary in this case is very complicated. In any case there exists a orbit

$$\mathcal{F} = Sp(2k,\mathbb{R})/U(n) \times O(p,q)/Q$$

where Q is stabilizer of maximal isotropic subspace in pseudoeuclidean space $\mathbb{R}^{p+q}$. I can show, that operator of restriction to orbit $\mathcal{F}$ exists and this observation give a way to observe a part of discrete spectra for dual pair. It is interesting to calculate this part of spectrum.

Another question which seems interesting to me: is it possible to obtain by such way some handble realizations of some Harish-Chandra discrete series representations?

4 Space L^2 on Stiefel manifolds

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4.1. Stiefel manifolds. We name by *Stiefel manifolds* the following 10 types of homogeneous spaces G/Q:

1°.	$O(p,q)/O(p-t,q-s)$	2°.	$U(p,q)/U(p-t,q-s)$
3°.	$Sp(p,q)/Sp(p-t,q-s)$	4°.	$Sp(2n,\mathbb{R})/Sp(2(n-t),\mathbb{R})$
5°.	$Sp(2n,\mathbb{C})/Sp(2(n-t),\mathbb{C})$	6°.	$O(n,\mathbb{C})/O(n-t,\mathbb{C})$
7°.	$SO^*(2n)/SO^*(2(n-t))$		

$8^\circ - 10^\circ$. The spaces of all linear embeddings

$$\mathbb{R}^{n-t} \to \mathbb{R}^n \quad \mathbb{C}^{n-t} \to \mathbb{C}^n \quad \mathbb{H}^{n-t} \to \mathbb{H}^n$$

In the last 3 cases the group G is $GL(n,\mathbb{R}), GL(n,\mathbb{C}), GL(n,\mathbb{H})$ respectively and Q is the group of matrices having the form

$$\begin{pmatrix} 1_s & * \\ 0 & * \end{pmatrix}$$

Remark The Stiefel manifold $Sp(2n,\mathbb{R})/Sp(2(n-t),\mathbb{R})$ is the space of isometric embeddings of the space $\mathbb{R}^{2t}$ equipped with a nondegenerate skew symmetric bilinear form to the space $\mathbb{R}^{2n}$ equipped with nondegenerated skew symmetric bilinear form. Other Stiefel manifolds $1^\circ - 7^\circ$ have the analogical description.

4.2. Additional symmetries. Consider the case $G/Q = Sp(2n,\mathbb{R})/Sp(2(n-t),\mathbb{R})$. Then the group $Sp(2t,\mathbb{R})$ acts by the obvious way on the space of symplectic-isometric embeddings $\mathbb{R}^{2t} \to \mathbb{R}^{2n}$ (sinse it acts on the space $\mathbb{R}^{2t}$) . Hence the manifold $Sp(2n,\mathbb{R})/Sp(2(n-t),\mathbb{R})$ is a $Sp(2t,\mathbb{R}) \times Sp(2n,\mathbb{R})$-homogeneous space:

$$Sp(2n,\mathbb{R})/Sp(2(n-t),\mathbb{R}) = (Sp(2t,\mathbb{R}) \times Sp(2n,\mathbb{R}))/(Sp(2t,\mathbb{R}) \times Sp(2(n-t),\mathbb{R}))$$

Analogical additional group of symmetries exists in all cases $1^\circ - 10^\circ$. These additional symmetries are useful since the spaces $L^2(G/Q)$ have G-spectrum of infinite multiplicity.

4.3. Spectra of L^2 on Stiefel manifolds. A few is known about spectral decomposition of $L^2(G/Q)$. Neverless it is known that this problem is interesting. See [Sch] and [Kob] for Flensted-Jensen type constructions of discrete spectra in L^2 on

$$O(p,q)/O(p-r,q) \quad U(p,q)/U(p-r,q) \quad Sp(p,q)/Sp(p-r,q)$$

For the case $r=1$ the Plancerel formula is obtained in [OZ1]. Some consrtructions for discrete increments to spectra of

$$L^2(U(p,q)/(U(p-t,q-t) \times U(p) \times U(q))) \subset L^2(U(p,q)/(U(p) \times U(q)))$$

are contained in [RSW].

The cases G/Q where $G = GL(n, \mathbb{R}), GL(n, \mathbb{C}), GL(n, \mathbb{H})$ are very simple.

Proposition 4.1. *Each representation of G which is contained in spectra $L^2(G/Q)$ discretly(resp. weakly) is contained in specrtrum of some kern-representation of G discretly (resp. weakly).*

Proof. We use arguments from [How],[NO]. Concider for instance the case $G = Sp(2n, \mathbb{R})$. The representation

$$W_{2n} \otimes W_{2n}^*$$

of $Sp(2n, \mathbb{R})$ is equivalent to representation of $Sp(2n, \mathbb{R})$ in $L^2(\mathbb{R}^{2n})$. Hence the representation

$$(W_{2n} \otimes W_{2n}^*)^{\otimes 2k} = W_{2n}^{\otimes 2k} \otimes (W_{2n}^*)^{\otimes 2k}$$

is equivalent to representation of $Sp(2n, \mathbb{R})$ in L^2 on the space $Mat_{2k,2n}$ of all $2k \times 2n$-matrices. A generic orbit of $Sp(2n, \mathbb{R})$ in $Mat_{2k,2n}$ is a Stiefel manifold $Sp(2n, \mathbb{R})/Sp(2(n-k), \mathbb{R})$.

Remark. For other groups G Proposition 6.1 can be proved by the same arguments. The basic observation is

$$W_{2n}\Big|_{GL(n,\mathbb{R})} \simeq L^2(\mathbb{R}^n)$$

(see real model of harmonic representation in [KV]).

4.3. Some pseudoriemann symmetric spaces. By the obvious way we have

$$\begin{array}{rcl} L^2\big(O(p,q)/(O(p) \times O(p-r,q))\big) & \subset & L^2\big(O(p,q)/O(p-r,q)\big) \\ L^2\big(U(p,q)/(U(r) \times U(p-r,q))\big) & \subset & L^2\big(U(p,q)/U(p-r,q)\big) \\ L^2\big(Sp(p,q)/(Sp(r) \times Sp(p-r,q))\big) & \subset & L^2\big(Sp(p,q)/Sp(p-r,q)\big) \end{array}$$

and spectra of these spaces are contained in spectra of Stifel manifolds. I don't know such embeddings of spectra for other pseudoriemannian symmetric spaces.

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