

MATHEMATICAL RESEARCHES OF D. P. ZHELOBENKO

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ABSTRACT. This is a brief overview of researches of Dmitry Petrovich Zhelobenko (1934–2006). He is the best known for his book "Compact Lie groups and their representations" and for the classification of all irreducible representations of complex semisimple Lie groups. We tell also on other his works, especially on the spectral analysis of representations.

1. Brief personal data. The mathematical researches of Dmitrii Petrovich Zhelobenko (Ul'yanovsk, 1934 - Moscow, 2006) are mainly devoted to the representation theory of semisimple Lie groups and Lie algebras and to the noncommutative harmonic analysis.

He graduated from the Physical Department of the Moscow State University (1958) and took his postgraduate program at the Steklov Mathematical Institute under the supervision of S. V.Fomin and of M. A.Naimark (1961). He has defended his PhD thesis "*Harmonic Analysis on the Lorentz group and some questions of the theory of linear representations*" in 1962. The doctoral thesis "*Harmonic analysis of functions on semisimple Lie groups and its applications to the representation theory*" was defended in the Steklov Institute in 1972 (official opponents: I. M. Gel'fand, I. R. Shafarevich, and A. A. Kirillov). In 1974 he was an invited speaker at the Mathematical Congress in Vancouver¹

Since 1961 and for the rest of his life he has worked in P. Lumumba People's Friendship University².

It appears that this job gave him an opportunity to use his teaching abilities, but there were not many not graduate students³.

In 1963–1978 M. A. Naimark, D. P. Zhelobenko, and A. I. Stern held a seminar on representation theory in the Steklov Mathematical Institute. The lecturers were, among others: F. A. Berezin, N. Ya. Vilenkin, R. S. Ismagilov, A. A. Kirillov, E. V. Kissin, G. L. Litvinov, V. I. Man'ko, M. B. Menskii, V. F. Molchanov, M. G. Krein, G. I. Olshansky, I. Sigal, V. N. Tolstoy, C. Foiaş, Dj. Khadzhev, P. Halmos, A. Ya. Khelemsky, E. Hewitt. Dmitrii Petrovich gave lectures for physicists many times; they took place in Dubna at 1965, in the College for Mathematical

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¹In fact he was not let abroad at that time. By the way, Naimark's report at Nice Congress at 1970 was the joint with Zhelobenko by the contents.

²PFU, later renamed into People's Friendship University of Russia (PFUR).

³besides he happened face certain administrative impediments when trying to 'pick up' a gifted student. Here is the list of postgraduates, who had defended PhD under the supervision of Zhelobenko: M. S. Al-Nator, A. S. Kazarov, A. G. Knyazev, A. V. Lutsyuk. We took borrowed it from his Curriculum Vitae. Partially, Vijay Jha (he later worked in number theory) is also a student of Zhelobenko

Physics of the Independent University of Moscow in early 90-ths, in the Sofia University in 1973–1974, and on some physical conferences.

Since 1959 and up to the last days of his life Dmitrii Petrovich was seriously ill (as a result of unlucky mountain hike)⁴.

Dmitrii Petrovich is the author of 9 books⁵ and of more than 70 papers (according to MathSciNet and Zentralblatt). The goal of our short survey is to discuss the most significant, in our regard, works of Zhelobenko. Clearly, this survey is not complete. Besides, having set such a goal we then cannot deviate from the subject of a purely professional text. MCCME⁶ publishers and Yu. N. Torkhov are preparing now edition of a new book by Zhelobenko ”*Gauss algebras*”. It is assumed to include reminiscences of his colleges and friends, G. L. Litvinov and V. F. Molchanov, and a paper of his students, M. S. Al-Nator, I. V. Goldes, A. S. Kazarov, V. R. Nigmatullin. The bibliography of his papers should be also present there.

2. The book ”Compact Lie groups and their representations” (1970).

This is the classical book, which after 37 years remains one of the best books on representation theory in both genres: as a textbook (or an introductory book), and as a monograph. This text, peculiar in its structure as well as in its style⁷, finally became a specimen of ”mathematics with a human face”. It was designed for a reader with a background of 2-3 grades of a mathematical (or physical) department of that time; it can serve as a textbook for the primary acquaintance or studying, as well as a book, interesting to specialists, or a handbook in a certain field.

We are not going to discuss the content (the book is worth reading itself), but note that its ”complicated chapters” (problems of spectral analysis) are in many respects based on the original results of D. P. Zhelobenko, in particular [39], [40]⁸.

The original invention of Zhelobenko was the consistent exposition of a big number of classical problems of the representation theory in the framework of so-called ”method of Z -invariants”. The method is based on a realization of representations of semisimple groups in functions on the maximal unipotent (strictly triangular) subgroup; these functions must satisfy a certain system of partial differential equations [39], [40]. Virtually this method anticipates the theory of D -modules on flag manifolds⁹, see A. Beilinson, I. Bernstein [4], J.-L. Brylinsky, M. Kashiwara [10].

3. The book ”Harmonic Analysis on complex semisimple Lie groups” (1974).

Essentially, this is an original work written upon Zhelobenko’s papers¹⁰ [41]–[46], [48]–[49], [58]–[59] (1963–1973), partially joint with M. A. Naimark. The main result is the Zhelobenko-Naimark theorem on classification of all irreducible

⁴Zhelobenko rescued a tourist group in Altai mountains in emergency conditions. Here follows a fragment of a letter of V. I. Chilin. ”I learned from the conversations with Dmitrii Petrovich, that he got ill after a hard mountain hike. After an agreement with the leader of the group he went forward, searching the path. Then a heavy snowfall began and he could not return back to the camp in the approaching dusk. He was was bound to circle around throughout the whole night in order not to freeze. As a result he got cold developed later into a serious illness”.

⁵Two of them are now in press, [56], [57].

⁶The Moscow Center for Continuous Mathematical Education.

⁷It seems that the evident popularity of his ”wrongly” written (”for physicists”) book surprised Dmitrii Petrovich up to the end of his life.

⁸It is worth to note that the paper by R. Godement [21] influenced much on these works.

⁹Strictly triangular subgroup is a ”big cell” of the flag variety

¹⁰Some of these works have overlaps with parallel researches of M. Duflo of that time, see references in [14], [50].

representations¹¹ of complex semisimple Lie groups. Let us recall this statement for the group $G = \text{GL}(n, \mathbb{C})$.¹²

Denote by $U(n)$ the subgroup of all unitary matrices. Denote by $T \subset G$ the subgroup of upper-triangular matrices. For $A \in T$ denote by a_{jj} the diagonal elements of A . Fix a collection of complex numbers $p_1, \dots, p_n, q_1, \dots, q_n$. Consider the character $T \rightarrow \mathbb{C}^*$, given by the relation

$$\chi_{p,q}(A) = \prod_{j=1}^n a_{jj}^{p_j} \bar{a}_{jj}^{q_j}.$$

All $(p_j - q_j)$ must be integers; otherwise complex powers make no sense.

Consider the representation $\rho_{p,q}$ of the group G induced from the representation $\chi_{p,q}$ of the group T .¹³ What we get is called the *principal series representation*. For generic p, q ¹⁴ principal series representations are irreducible. For exceptional values of parameter the representation $\rho_{p,q}$ has a finite Jordan–Hölder composition series.

Next, for generic p, q a simultaneous permutation of the tuples (p_1, \dots, p_n) and (q_1, \dots, q_n) get an equivalent representation.¹⁵ For exceptional values of p, q such representations are not equivalent, but their composition series have the same factors (we use the term "subfactor" below).

Now we formulate the Zhelobenko-Naimark theorem: *All irreducible representations of $\text{GL}(n, \mathbb{C})$ are in one-to-one correspondence with orbits of the symmetric group S_n on the set of collections*

$$(p_1, \dots, p_n; q_1, \dots, q_n)$$

Namely, for each orbit one of the subfactors of the corresponded $\rho_{p,q}$ is taken.

Let us describe the subfactor more precisely. Without loss of generality we can set

$$p_1 - q_1 \geq p_2 - q_2 \geq \dots \geq p_n - q_n.$$

Now one should take the subfactor (it is actually a subrepresentation), whose restriction to the subgroup $K = U(n)$ contains a representation with the highest weight $(p_1 - q_1, \dots, p_n - q_n)$.

The theorem (1966) by itself was a part of thirty-year-long story around Harish-Chandra subfactor theorem. Principal series and the parabolic induction were introduced in the book by M. I. Gel'fand and M. A. Naimark [18]. In 1954 Harish-Chandra [22] proved that any irreducible representation of a semisimple Lie group

¹¹in Fréchet spaces up to infinitesimal equivalence of modules of K -finite vectors.

¹²Below G is arbitrary semisimple group, K is its maximal compact subgroup, $\mathfrak{g} \supset \mathfrak{k}$ are their Lie algebras. Now $G = \text{GL}(n, \mathbb{C})$, $K = U(n)$.

¹³For the reader accustomed to another language. We consider representations of $\text{GL}(n, \mathbb{C})$ in smooth sections of complex linear bundles over a flag manifold.

¹⁴The exceptional condition is $p_i - p_j \in \mathbb{Z}$, $q_i - q_j \in \mathbb{Z}$ for at least one pair i, j

¹⁵For the reader slightly familiar with the Lorentz group $\text{SL}(2, \mathbb{C})$ and with the representation theory. This is almost evident. First, for $\text{GL}(2, \mathbb{C})$ this is a well-known and easily checked statement. Next, denote by T_i the generalized upper-triangular subgroup, where the element $a_{(i+1)i}$, which stands below the diagonal, is allowed to be nonzero. The Lie group T_i has $\text{SL}(2, \mathbb{C})$ as a semisimple factor. Now we can construct our representation taking a successive induction, first from T to T_i , and second to G . But now for $\text{SL}(n, \mathbb{C})$ we can permute the parameters (p_i, q_i) and (p_{i+1}, q_{i+1}) .

can be realized as a subfactor of the principal series¹⁶. F. A. Berezin (1956), [5], [6], has immediately made an attempt to turn this result into an explicit classification in the case of complex groups.¹⁷ Ten years later (1966) the Zhelobenko–Naimark Theorem has appeared, then in 1973 R. Langlands [27] provided a classification of all irreducible representations of semisimple groups, and Casselman [11] in 1975 proved the subrepresentation theorem.^{18,19}

Zhelobenko [44] also introduces the following structure.²⁰ Denote by $U(\mathfrak{g})$ the enveloping algebra of the Lie algebra \mathfrak{g} , and by $U(\mathfrak{k})$ the enveloping algebra of the subalgebra \mathfrak{k} . Denote by π_λ the irreducible representation of the group K with the highest weight λ . By $\ker \pi_\lambda \subset U(\mathfrak{k})$ we denote the two sided ideal, which consists of elements v , such that $\pi_\lambda(v) = 0$. Next, by $U_{\lambda\mu}$ we denote the set of all $u \in U(\mathfrak{g})$ such that

$$(\ker \pi_\lambda)u \subset U(\mathfrak{g})(\ker \pi_\mu).$$

It is easy to check that $U_{\lambda\mu}U_{\mu\nu} \subset U_{\lambda\nu}$. Therefore we get a category whose objects are dominant weights λ , and morphisms are elements $u \in U_{\lambda\mu}$.

For a representation of the group G in the space H consider its restriction to the subgroup K ; it decomposes into a direct sum $H = \bigoplus H_\lambda$, where in H_λ the group K acts by means of representation π_λ with certain multiplicity. It is easy to check that $u \in U_{\lambda\mu}$ sends H_μ to H_λ , i.e, we get a representation of our category.

This subject, which is discussed in the book in detail, can be naturally regarded as an analog of the classical operational calculus. The enveloping algebra is simultaneously the algebra of left-invariant differential operators on the Lie group, and the action of $U(\mathfrak{g})$ in representations is an analog of the Fourier transform of polynomial differential operators.

Integral formulas for intertwining operators for the principal nonunitary series together with a detailed description of the adjoint action of the Lie algebra \mathfrak{k} in $U(\mathfrak{g})$ enable to describe the algebra $U_{\lambda\lambda}/\ker \pi_\lambda U(\mathfrak{g})$ and classify all irreducible representations with the lowest weight λ . This result is then used in the classification of all irreducible representations of a complex semisimple Lie group.

¹⁶see also [13], 9.4

¹⁷Literally the paper yet fell short of the classification; to be more precise the final theorem was given without proof. Serious gaps were found in this paper, and a discussion between Harish-Chandra and Berezin on the correctness of proofs followed. In the next paper ("A Letter to Editors") Berezin removed these gaps. Berezin's approach (bringing up to the classification) is expounded in Appendix A of the discussed book of Zhelobenko (however we are bound to note that Zhelobenko applied additional arguments).

By the way, this work of Berezin contains a calculation of radial parts of Laplace operators on semisimple groups, which was no less important than the incipient "struggle" for the classification of representations.

¹⁸Any irreducible representation is a subrepresentation in a principal series representation.

¹⁹The proofs were published in the book by A. Borel and–N. Wallach in 1980 [8] and in the paper by B. Casselman and D. Milicic in 1982 [12]; see also the book by A. Knapp [25]. The real case in essence is more complicated compared to complex, but the results are less transparent. The classification of all representations is based on the classification of discrete series. Representations of discrete series themselves are rather complicated objects; a few descriptions of them are known, but they all are still far from transparent. After the work of M. Flensted-Jensen[17] had appeared it seemed that the discrete series were just about to be quite understood. Thirty years afterwards the work of Flensted-Jensen remains unclear.

²⁰This structure is also considered in the book by Dixmier [13] 1974, Chapter 9, with a reference to the work of J. Lepowsky and G. W. MacCollum, this structure is quite natural and probably was introduced before.

From the "sports" point of view, the Zhelobenko-Naimark Theorem is left behind long ago. On the other hand, the book still remains an instructive text on complex semisimple groups (which are generally understood better than real) and together with accompanying papers could become a "basis" for progressive movement in the future. Unfortunately, the book is written in the style used in the modern mathematics; it happens to be legally precise text for experts.²¹ As a result, the "operational calculus" is known and understood less than it deserves. An important text on this subject is M. Duflou's notes [14] 1974.²²

Now, the unfinished Zhelobenko's book "Operational calculus on complex semisimple Lie groups" is in press (we did not see it yet). Judging by the title, it continues the subjects discussed above. Hopefully it can help entering of these topics to a common knowledge.

4. The book "Representations of reductive Lie algebras". 1994. This work and the related series of papers probably represent now the most interesting part of the original scientific heritage of D. P. Zhelobenko. The main part of the book is devoted to the exposition of the so-called reduction algebras, or Mickelsson algebras [30].

Let $\mathfrak{g} \supset \mathfrak{k}$ be complex Lie algebras, and \mathfrak{k} reductive.²³ We are interested in the problem of restriction of finite-dimensional representations of \mathfrak{g} to \mathfrak{k} .²⁴ Decompose \mathfrak{k} into a sum of lowering, diagonal (Cartan) and raising (step up) subalgebras, $\mathfrak{k} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$. Consider the ideal $U(\mathfrak{g})\mathfrak{n}_+$ and the quotient $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}_+$ over this ideal. The reduction algebra (or *Mickelsson algebra* $S(\mathfrak{g}, \mathfrak{k})$) is defined as a subspace of \mathfrak{n}_+ -highest vectors in $U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{n}_+$.

It is easy to make sure that $S(\mathfrak{g}, \mathfrak{k})$ is an algebra indeed and that it acts on the space $V^{\mathfrak{n}_+}$ of \mathfrak{n}_+ -highest vectors of any \mathfrak{g} -module V . Next²⁵, \mathfrak{g} -module V can be reconstructed from $S(\mathfrak{g}, \mathfrak{k})$ -module $V^{\mathfrak{n}_+}$.

The important tool for the study of reduction algebras is the "extremal projector" of R. M. Asherova–Yu. F. Smirnov–V. N. Tolstoy (1971). For a Lie algebra \mathfrak{sl}_2 with a standard basis $e, h,$ and f it was obtained by P. O. Löwdin, [29] in 1964,

$$(1) \quad P = \sum_{n=0}^{\infty} \frac{f^n e^n}{\prod_{j=1}^n (h + j + 1)} = \prod_{n=1}^{\infty} \left(1 - \frac{fe}{n(h + n + 1)} \right)$$

This expression defines an operator in any finite-dimensional \mathfrak{sl}_2 -module²⁶, the operator P satisfies the relations

$$eP = Pf = 0, \quad P^2 = P$$

In other words, P is the projector to the subspace of highest weight vectors.

²¹Afterwards Dmitrii Petrovich regretted to have submitted to the influence of "bourbakism", just coming into a fashion that time.

²²Under the influence of these notes, Zhelobenko's technique was used in the work of P. Torasso [35], it was also used in one of the variants of the proofs of Baum-Connes conjecture by Vincent Lafforgue [26].

²³It is worth to note the unexpected generality: subalgebra \mathfrak{k} does not need not to be symmetric.

²⁴It is reasonable to take in mind a slightly more generality: we consider modules with highest weights over \mathfrak{g} , whose restriction to \mathfrak{k} is the sum (not necessary direct) of modules with highest weight.

²⁵assuming the conditions of the previous footnote are satisfied;

²⁶It is defined in a more general setting, but this requires an additional discussion

R. M. Asherova, Yu. F. Smirnov and V. N. Tolstoy found an analogous expression for arbitrary semisimple algebra \mathfrak{k} . Namely for arbitrary positive root α we consider the related \mathfrak{sl}_2 subalgebra with the basis $e_\alpha, h_\alpha, f_\alpha$, then write down the corresponding projector P_α , and take the product $\prod P_\alpha$ over all positive roots in a correct²⁷ order.

The formula for P contains a division by elements of the enveloping algebra $U(\mathfrak{k})$, i.e., the element P does not belong to $U(\mathfrak{k})$.

In this connection, Zhelobenko introduces a certain extension of the algebra $U(\mathfrak{k})$ (with partially allowed division) and identifies it with locally-finite endomorphisms of some universal module; after this an existence and uniqueness of the projection operator becomes a tautology.

The involvement of the projector force to localize²⁸ the algebra $S(\mathfrak{g}, \mathfrak{k})$ over expressions $h_\gamma + k$, where h_γ is a coroot, and k is an integer. In the localized algebra $Z(\mathfrak{g}, \mathfrak{k})$, there appear natural generators $z_g = Pg$, where g runs the orthogonal complement to \mathfrak{k} in \mathfrak{g} . Generators z_g satisfy quadratic-linear relations with coefficients being rational expressions in Cartan elements. The complete or a partial knowledge of these relations allows to obtain various information about representations of the Lie algebra \mathfrak{g} . For instance, a description of the Mickelsson algebra for the pair $\mathfrak{gl}_{n+1} \supset \mathfrak{gl}_n$ gives eventually a possibility to obtain explicit formulas for the action of generators of \mathfrak{gl}_n in Gel'fand-Tsetlin basis ([2]), while the investigation of a pair, related to a symmetric space is applied in the classification of certain series of Harish-Chandra modules over real Lie groups.²⁹

One may "transpose" the defining properties of generators z_g and get another generators z'_g , which also satisfy quadratic-linear relations. Zhelobenko constructs an operator Q such that $z'_g = Qg$. The operator Q (known to specialists as the "Zhelobenko cocycle") also admits a factorization over the roots and reminds P in sl_2 case:

$$(2) \quad Q_\alpha(x) = \sum_{n=0}^{\infty} \text{Ad}(e_\alpha)^n x \cdot f_\alpha^n \frac{(-1)^n}{\prod_{j=1}^n j(h_\alpha - j + 1)},$$

where $\text{Ad}(e_\alpha)x := [e_\alpha, x]$ is an operator of the adjoint action. Later operators (2) appeared in mathematical physics under the name of dynamical Weyl group, see G. Felder [16], P. Etingof, A. Varchenko [15], see also [24]. Even so, the general understanding of the very formula (2) is still unsatisfactory.

The discussed book of Zhelobenko contains quite a few inaccuracies; besides, the last chapter is, to our point of view, unsuccessful. But as a whole, the book is an impressive and very original work.

5. The book "Principal structures of the representation theory".
2004. The work is very unusual by the style and structure. This is a text of "patchwork structure", which was not originally presumed for a "systematic study" in the proper sense of this word. The book consists of fragments; each of them, however, is quite readable and interesting. From one side, they are expected (written) for the beginners, on the other hand, the professional mathematician could find there quite a number of interesting and unexpected.

²⁷so-called, normal,

²⁸i.e., to allow a division partially

²⁹For other applications see [32].

6. Indecomposable representations of the Lorentz group (1958–1959).

Infinite-dimensional nonunitary representations of the Lorentz group $SL(2, \mathbb{C})$ (just as of any other semisimple group) are not completely reducible (i.e., are not decomposed into a direct sum of irreducible representations). In particular, for exceptional values of the parameters (see above) the representation of principal series of $SL(2, \mathbb{C})$ splits into a finite-dimensional representation and an infinite-dimensional “tail”³⁰. Two notes [37]–[38] of Zhelobenko in “Doklady AN SSSR” are devoted to the problem of description of all indecomposable representations of the Lorentz group (with a finite composition series). Only subrepresentations with the same infinitesimal character could be linked.

Zhelobenko found a way to reduce this problem to a problem of finite-dimensional linear algebra. Namely, one should classify collections of linear operators $d_+ : V \rightarrow W$, $d_- : W \rightarrow V$, $\delta : W \rightarrow W$ such that the block matrices

$$a := \begin{pmatrix} 0 & d_- \\ d_+ & 0 \end{pmatrix}, \quad b := \begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix}$$

satisfy the relations:

$$ab = ba, \quad a, b \text{ are nilpotent}$$

In its turn, this problem of linear algebra turned out to be quite curious, it was solved by I. M. Gel’fand and V. A. Ponomarev [19]. (1968).

An analogous reduction to finite-dimensional linear algebra problem was done for the groups $SO(n, 1)$ and $SU(n, 1)$, see [23]. For arbitrary real semisimple groups, a general method (being far from effective algorithm) was suggested in [7]. Probably these results are not ultimate.

7. Gel’fand-Tsetlin bases. Finite-dimensional irreducible representations of semisimple groups admit a simple parametrization (Cartan highest weight theorem), however individual representations are complicated objects and the description of their explicit implementations is a rather sophisticated problem. An analogous problem appeared earlier for symmetric groups S_n . In 1931, Alfred Young [36] suggested a way of construction of models of representations with a help of successive restriction of a representation to subgroups $S_n \supset S_{n-1} \supset S_{n-2} \supset \dots$.

This method was applied by I. M. Gel’fand and M. L. Tsetlin [20], 1950, to classical groups $GL(n, \mathbb{C})$ and $O(n, \mathbb{C})$. Recall that restricting an irreducible finite-dimensional representation of $GL(n, \mathbb{C})$ to $GL(n - 1, \mathbb{C})$, we get a multiplicity free direct sum. Then we restrict each of the obtained representations to $GL(n - 2, \mathbb{C})$ and so on. We get finally a decomposition of the initial space into direct sum of one-dimensional spaces. Choosing in each of them a vector (not canonically) we get a basis. This part of arguments is relatively trivial³¹. The deep result is the explicit formulas for the action of generators of the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ in this basis. Gel’fand and Tsetlin announced these formulas in two notes in “Doklady AN SSSR”

Later on different authors published several proofs, the earliest are the work of Zhelobenko [39] and of G. Baird, L. Biedenharn [3]. More important is that Zhelobenko managed to develop here methods applicable for the solutions of other

³⁰which by itself is a point of the principal series

³¹One can refer to the Pieri formula.

spectral problems (and this was one of the starting points for the books [47], [54], see also the survey [52]).³²

8. Holomorphic families of intertwining operators and "operators of discrete symmetry". Finally, we discuss the works of Zhelobenko on intertwining operators. Partially they are included into the book "*Harmonic analysis on complex groups*", but there is a later paper [51] on real groups.

Above (see n.3) we described a construction of intertwining operators for representations of the principal series of $GL(n, \mathbb{C})$; for any permutation $\sigma \in S_n$ of the parameters of a representation it gives a meromorphic operator-valued function of the parameters of the representation³³ with possible poles at the points of reducibility. For instance, for the group $GL(n, \mathbb{C})$ this is a function on p_1, \dots, p_n . The numbers $p_j - q_j$ are integers, i.e., we have \mathbb{Z}^n of meromorphic functions.

The behavior of such functions at the points of reducibility is an important and complicated question, discussed in a plenty of papers. Zhelobenko introduces (for complex groups) unobvious normalization, making this family of operators holomorphic and nonvanishing everywhere. In particular, this gives explicit formulas for intertwining operators in reducibility points as well.

However, these are not all intertwining operators for the principal series. Sometimes (only in reducibility points) there are operators, which connect representations with different collections of $\{p_j - q_j\}$ ³⁴. Zhelobenko realized that an existence of such exceptional symmetries is related to the natural action of the double of symmetric group $S_n \times S_n$ on the set of collections $p_1, \dots, p_n, q_1, \dots, q_n$. He also presents explicit constructions of these operators.

For arbitrary real semisimple Lie algebra \mathfrak{g} this is related to the action of the Weyl group of the complexification of $\mathfrak{g}_{\mathbb{C}}$ on the dual space to the Cartan subalgebra³⁵.

As far as we remember Moscow of 1970-80, D. P. Zhelobenko looked rather detached person. But many people, mathematicians and physicists, students and experts, in the Soviet Union and abroad, read the book "*Compact Lie groups*". To our mind, Zhelobenko's activity seemed to influence upon the formation, interest and tastes of a quite a number of the young at that time mathematicians: A. V. Zelevinsky, G. L. Litvinov, A. I. Molev, M. L. Nazarov, Yu. A. Neretin, G. I. Olshanski, V. N. Tolstoy, S. M. Khoroshkin, I. V. Cherednik,³⁶ though none of them was Zhelobenko's student neither formally nor in fact.

³²Note also that Zhelobenko (1962, see [39], [47]) obtained the spectrum of the reduction of a finite-dimensional representation of $Sp(2n, \mathbb{C})$ to $Sp(2n-2, \mathbb{C})$. It is not multiplicity free, but resembles the spectrum of the restriction of $O(k, \mathbb{C})$ to $O(k-2, \mathbb{C})$; this enforced to think that Gel'fand-Tseitlin bases for $Sp(2n, \mathbb{C})$ exist.

The question was many times discussed. see e.g. an approach suggested by A. A. Kirillov and realized by V. V. Stepin [34], 1986. The problem was finally solved by A. I. Molev [31], 1999.

Notice also that the note of Zhelobenko on this subject in Russian Math. Surveys 42 (1987) is mistaken.

³³or, according to Bruhat [9], a function with values in G -invariant distributions on the product of flag manifolds.

³⁴It seems that Naimark [33] was the first who observed that in the study of degenerations of principal series.

³⁵An heuristic explanation for the reader slightly familiar with the subject: eigenvalues of Laplace operators are constant on such orbits

³⁶Of course, authors confirm this relatively to themselves.

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³⁷May be, there exists an English or French translation (Mir publishers), but we can not find it.