Binary categories

Arnold Neumaier

Fakultät für Mathematik, Universität Wien
Nordbergstr. 15 , A-1090 Wien, Austria
email: Arnold.Neumaier@univie.ac.at
WWW: `http://www.mat.univie.ac.at/~neum`

Version 0.9, October 4, 2009

1 Introduction

These notes (still in draft form) arose from a challenge arising in the discussion of FMathL on the n-Category Café. I had claimed: " For example, one can express with this naturally (and every mathematician immediately understands without the need for explanations) that ordered monoids are the objects in the intersection of Order and Monoid satisfying the compatibility relation R (suitably defined), and many other similar constructs."

Mike Shulman had replied: "As far as I can tell, this is not true even in FMathL (and I don't see how it could ever be true)." And Toby Bartels had replied: "I would like to see you formalise this!"

This notes answer the challence by giving a formalization in FMathL. For simplicity, I restrict to binary algebraic structures.

This formalization also shows how specifications in FMathL work on the informal level. We simply state which objects we want to have, which properties they are supposed to have, and we make sure that, in as far as we define something (below: an intersection) that has already syntactical meaning before our definition, the properties required by this meaning are satisfied. (Below, this check is needed for intersection.) In addition, we may require whatever we like (including inconsistent things, such as assuming the existence of a smallest nonabelian finite simple group of odd order).

2 Categories and sets

Categories and sets are already defined in the main document [1], but for the sake of completeness, I repeat the relevant definitions.

Axiom A21.
An object f is called an **arrow** from $\mathrm{DOM}(f)$, the **domain** of f, to $\mathrm{COD}(f)$, the **codomain** of f, if $\mathrm{DOM}(f)$ and $\mathrm{COD}(f)$ exist. The **arrow product** $g \diamond f$ of two arrows g and f exists iff $\mathrm{DOM}(g) = \mathrm{COD}(f)$ and is then an arrow from $\mathrm{DOM}(f)$ to $\mathrm{COD}(g)$. For any object A, the **identity** Id_A on A exists. For arrows f, g, h and objects A, B, C, D,

(P1) $f : A \to B$ iff $\mathrm{DOM}(f) = A$ and $\mathrm{COD}(f) = B$;
(P2) if $f : A \to B$, $g : B \to C$, $h : C \to D$ then $h \diamond (g \diamond f) = (h \diamond g) \diamond f$;
(P3) $\mathrm{Id}_A : A \to A$.
(P4) $f : A \to B$ implies $\mathrm{Id}_B \diamond f = f$ and $f \diamond \mathrm{Id}_A = f$.

It is customary in mathematics to talk about the arrow $f : A \to B$, meaning the arrow f with $f : A \to B$.

Axiom A22.
An object C is called a **category** if $\mathrm{HOM}(C)$ exists. The elements of $\mathrm{HOM}(C)$ are called **homomorphisms** (or C-**morphisms**) of the category C. An object A is called a **structure** of the category C if $C(A) = A$. For every structure A of the category C, the object Id_A is a homomorphisms of C. For every homomorphism f of the category C, the objects $\mathrm{DOM}(f)$ and $\mathrm{COD}(f)$ are structures of C. For any category C, any object A, and any two arrows f, g,

(P5) $A \in C$ iff $C(A) = A$;
(P6) $C(C(A)) = A$ if $C(A)$ exists;
(P7) $f, g \in \mathrm{HOM}(C)$ implies $g \diamond f \in \mathrm{HOM}(C)$ if $g \diamond f$ exists.

If $C(A)$ exists, it is the structure of category C associated with the object A. The nature of this association depends on the category and must be postulated in each case.

Note that, in FMathL, a category contains all its objects. This is possible without difficulties since $\in$ is not extensive. Thus FMathL does not need a separate *Ob* operator.

Axiom A24.
An object A is called a **set** if $\mathrm{SET}(A) = A$. We say that A is a **subset** of the

object Z if A is a set and $A \subseteq Z$. The object SET is a category satisfying

(P8) $\text{SET}(A) = (A \mid \text{CONST}(\mathbf{1}))$;

(P9) If A and B are sets then $A = B$ iff $x \in A \Leftrightarrow x \in B$.

The **empty set** $\emptyset$ is a set. If the objects x and y exist then $\{x, y\}$ is a set. If A is a set and every $B \in A$ is a set then $\text{UNION}(A)$ is a set. If the object f is defined on the set A then the image $f[A]$ of A under f and the set restriction $A \mid f$ of A by f are sets.

Axiom A25.
An object f is called a **map** (or **mapping**) from the set A to the object Z if $f : A \to Z$. In this case, $f[A]$ is called the **range** of the map f. For any two maps $f, g : A \to Z$,

(P10) $f = g$ iff $f(x) = g(x)$ for all $x \in A$;

(P11) $f(x)$ exists iff $x \in A$;

(P12) $x \in A$ implies $f(x) \in Z$;

(P13) $h \diamond f = h \circ f$ if Z is a set and $h : Z \to Y$;

(P14) $\text{Id}_A = \text{Id}\Big|_A$.

3 The new part

A **subcategory** of a category C is a category C' such that $A \in C'$ implies $A \in C$, and $f \in \text{HOM}(C')$ implies $f \in \text{HOM}(C)$. In this case, we write $C' \le C$. Clearly, $\le$ is a partial order on categories. The **intersection** of two categories S and S' is the category $S \cap S'$ with $A \in S \cap S'$ iff $A \in S$ and $A \in S'$, and $f \in \text{HOM}(S \cap S')$ iff $f \in \text{HOM}(S)$ and $f \in \text{HOM}(S')$. (Clearly, this is a category, and the notion of intersection is consistent with the general FMathL properties of intersections.) It is easily seen that $S \cap S'$ is the infimum of S and S' with respect to the partial order on categories.

To define binary algebraic structures we introduce two new distinguished constants OPNAMES and ACTION.

A **binary structure** is an object S for which the objects $X := \text{SET}(S)$, $O := \text{OPNAMES}(S)$, and $c := \text{ACTION}(S)$ exist, such that O is a finite set of symbols, c maps O to $(X \times X \to X) \cup (X \times X \to \{\mathbf{0}, \mathbf{1}\})$, where $\mathbf{0}$ and $\mathbf{1}$ stand for true and false, respectively. If $x, y \in X$ and $\omega \in O$, we write

$$x \, \omega \, y := c(\omega)(x, y).$$

We call $\omega \in O$ an **operation** if $x\,\omega\,y \in X$ for all $x, y \in X$, and a **relation** if $x\,\omega\,y \in \{\mathbf{0}, \mathbf{1}\}$ for all $x, y \in X$. (Occasionally but not usually, $\omega \in O$ might be both an operation and a relation.)

By Axiom A24, $x \in S$ is equivalent with $x \in X$; note that $\in$ is not extensive in FMathL.

[define O-homomorphism]

Call a structure S an order if $<\in O$ and the usual laws for a strict partial order are satisfied, and a monoid if $* \in O$ and the usual laws for a monoid are satisfied. [state laws for both]

We introduce a new object $\textsc{Bin}$ such that, for every finite set O (whose elements will be called **symbols**), such that $S \in \textsc{Bin}(O)$ iff S is a binary structure with $O \subseteq \textsc{Opnames}(S)$, and $\textsc{Hom}(\textsc{Bin}(O))$ contains precisely the O-homomorphisms between such structures. (To show that the object $\textsc{Bin}$ actually exists possibly needs some more reflection that has not been made explicit so far. Thus, for the moment, we take its existence as additional axiom.)

Clearly, $\textsc{Bin}(O)$ is a category, called the **full binary category** with **symbol set** O. A **binary category** is a subcategory of some full binary category.

It is easily seen that

$$\textsc{Cat}(O) \cap \textsc{Cat}(O') = \textsc{Cat}(O \cup O').$$

Thus the intersection of categories is a useful, unifying structural concept. It captures natural intuition about relationships present between binary algebraic structures with different operations and/or relations.

We now apply it to the original query. We define the category *Order* to be the subcategory of $\textsc{Bin}(\{<\})$ consisting of all binary structures that have a relation $<$ which is a strict partial order. We define the category *Monoid* to be the subcategory of $\textsc{Bin}(\{*\})$ consisting of all binary structures that have an operation $*$ which is associative and has a neutral element. We define the category *OrderedMonoid* to be the subcategory of $\textsc{Bin}(\{<, *\})$ consisting of all binary structures S that have a relation $<$ which is a strict partial order and an operation $*$ that turns S into a strictly ordered monoid in the standard sense. An example is the strictly ordered monoid of natural numbers (without 0) with the standard strict order $<$ and the standard multiplication $*$.

Clearly, $OrderedMonoid$ is a subcategory of both $Order$ and $Monoid$, and hence a subcategory of $Order \cap Monoid$. it contains precisely the binary structure $S \in Order \cap Monoid$ that satisfy

$$x < y \quad \Rightarrow \quad xz < yz,\ zx < zy \quad \text{for all } x, y, z \tag{1}$$

This completes our task, but a few related things follow.

The category $AMonoid$ of **additive monoids** is a subcategory of $\textsc{Bin}(\{+\})$ canonically isomorphic to $Monoid$, but different as an object in FMathL. In general, there are canonical functors between any two binary categories C and C' that differ only in the labeling of the operations and relations. [It seems we need the category of categories to get the natural concept of isomorphism of categories. Can this be defined in FMathL?]

Let S be a binary structure with $X = \textsc{Set}(S)$, $O = \textsc{Opnames}(S)$, and $c = \textsc{Action}(S)$, and let S' be a binary structure with $X' = \textsc{Set}(S')$, $O' = \textsc{Opnames}(S')$, and $c' = \textsc{Action}(S')$. We call S and S' **conforming** if

$$c(\omega)(x, y) = c'(\omega)(x, y)$$

for all $\omega \in O \cap O'$ and all $x, y \in X \cap X'$. In this case, we define, consistent with the general FMathL properties of intersections and the above definition of a binary structure, the **intersection** $S \cap T$ to be the binary structure with $\textsc{Set}(S \cap S') := X \cap X'$, $\textsc{Opnames}(S \cap S') := O \cup O'$, and, for $\omega \in O \cup O'$, $\textsc{Action}(S \cap S')(\omega)$ maps (x, y) to $x\,\omega\,y$ if $x, y \in X \cap X'$. Note that this gives a well-defined map $\textsc{Action}(S \cap S')$ since S and S' are conforming. (For non-conforming structures, we do not specify anything, so statements involving these are usually undecidable.)

4 Conclusion

The above gives a complete formalization of ordered monoids as the objects in the intersection of Order and Monoid satisfying the compatibility relation (1) is complete.

It was achieved without using any abuse of language. It provides a unified formal account of what mathematicians use intuitively when they combine algebraic structures.

Surely this is part of the essence of mathematics. Therefore I challenge the owners and visitors of the n-Category Café to describe the same information and relationships according to the categorial moral code propagated there, also without using any abuse of language.

Then we can compare readability, complexity, and elegance.

References

[1] A. Neumaier, The FMathL mathematical framework, Manuscript (2009).
http://www.mat.univie.ac.at/~neum/ms/fmathl.pdf