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Symbolic Parametrization of Algebraic Curves

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Introduction

- Algebraic curves and surfaces have been studied extensively in algebraic geometry.
- Algebraic curves and surfaces play an important and ever increasing role in computer aided geometric design, computer vision, and computer aided manufacturing. (But also in number theory, differential equations, and other areas.)
- Consequently, theoretical results need to be adapted to practical needs. We need efficient algorithms for generating, representing, manipulating, analyzing, rendering algebraic curves and surfaces.

Here we report on symbolic methods for the parametrization of algebraic curves.

Algebraic curves

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0.

By $\mathbb{A}^2(\mathbb{K})$ we denote the **affine plane** over the field $\mathbb{K}$, i.e.

$$\mathbb{A}^2(\mathbb{K}) = \{\ (a, b) \mid a, b \in \mathbb{K}\}.$$

By $\mathbb{P}^2(\mathbb{K})$ we denote the **projective** plane over the field $\mathbb{K}$, i.e.

$$\mathbb{P}^2(\mathbb{K}) = \{\ (a : b : c) \mid (a, b, c) \in \mathbb{K} \setminus \{(0, 0, 0)\}\ \}.$$

For a square-free non-zero polynomial $f \in \mathbb{K}[x, y]$, the subset $\mathcal{C}$ of $\mathbb{A}^2(\mathbb{K})$ defined by

$$\mathcal{C} = \{(a, b) \mid f(a, b) = 0\ \}$$

is an **affine plane algebraic curve**.

For a square-free non-zero form $F \in \mathbb{K}[x, y, z]$, the subset $\mathcal{C}$ of $\mathbb{P}^2(\mathbb{K})$ defined by

$$\mathcal{C} = \{(a : b : c) \mid F(a, b, c) = 0\ \}$$

is a **projective plane algebraic curve**.

Examples of algebraic curves:

Fig. 1.1: $f(x,y) = y^2 - x^3 - x^2$
(rational cubic)

Fig. 1.2: $f(x,y) = y^2 - x^3 + x$
(elliptic curve)

Fig. 1.3: $f(x,y) = 2x^4 - 3x^2y + y^4 - 2y^3 + y^2$
(tacnode)

Fig. 1.4: $f(x,y) = (x^2 + 4y + y^2)^2 - 16(x^2 + y^2)$
(cardioid)

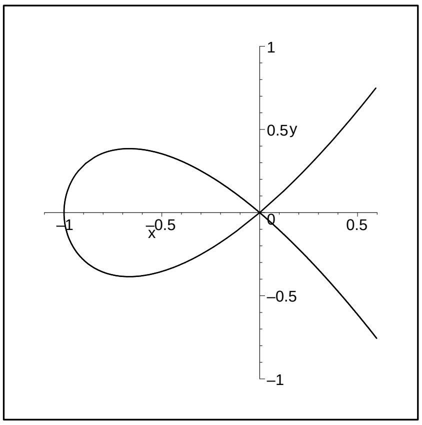


Fig. 1.1

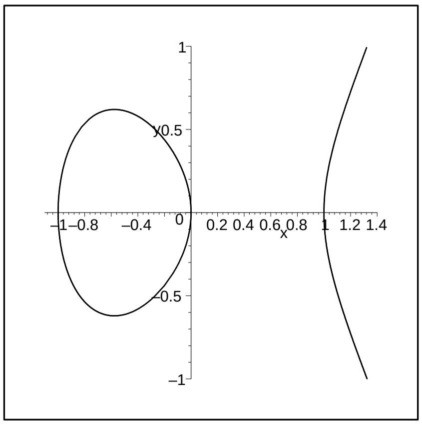


Fig. 1.2

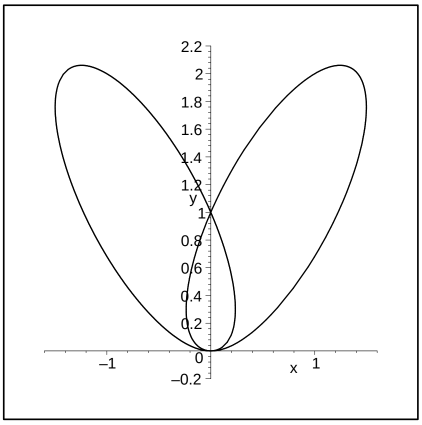


Fig. 1.3

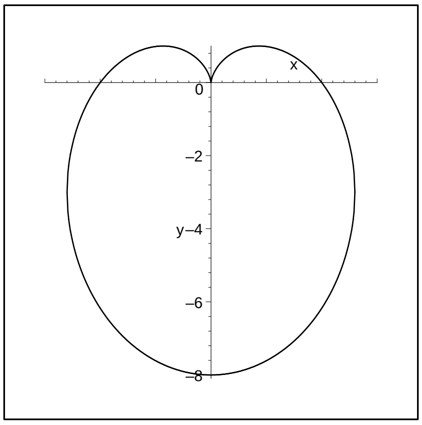


Fig. 1.4

Rational parametrization of curves

A **rational parametrization** of an algebraic curve $\mathcal{C}$ is a pair or rational functions

$$\mathcal{P}(t) = (x(t), y(t)) \in \mathbb{K}(t)$$

such that the Zariski closure of $\{\mathcal{P}(t) | t \in \mathbb{K}\}$ is $\mathcal{C}$.

A curve having a rational parametrization is called **rational**.

We say that $\mathcal{P}$ is a **proper** parametrization iff almost every point (in the sense of the Zariski topology) on $\mathcal{C}$ is generated exactly once by $\mathcal{P}$.

Every rational curve has in fact a proper parametrization. (this is a consequence of Lüroth's Theorem)

Advantages of implicit representation:

- easy to check whether a point lies on the curve
- every algebraic curve has such a representation

Advantages of parametric representation:

- easy to generate points over certain coordinate fields
- tracing of objects
- solution of diophantine equations

An application example:

in analyzing Painleve's 6th equation (a non-linear differential equation), one finds that solutions are plane curves $f(y,t) = 0$, defining $y(t)$ implicitly. Our parametrization method (and the CASA system, which has an implementation of it) have been used to parametrize candidate curves $f(y,t)$, which makes it possible to plug them into the original equation and verify them.

This has been successfully carried out for curves of degree 7 in y, related to Klein's simple group of order 168. Currently we are working on a solution curve of order 12 in y, the largest genus 0 icosahedral solution of Painleve's 6th equation. (Icosahedral solutions with 10 branches were found by Dubrovin-Mazzocco, Invent.Math. 141, 55–147 (2000).)

Criterion for rationality

the **parabola**

$$y = x^2$$

can be parametrized as

$$\{(t, t^2) \,|\, t \in K\}$$

all affine points on the parabola are given by the parametrization (t, t^2).

the **tacnode** curve defined by the polynomial

$$f(x, y) = 2x^4 - 3x^2y + y^2 - 2y^3 + y^4$$

can be represented parametrically as

$$\begin{aligned} x(t) &= \frac{t^3-6t^2+9t-2}{2t^4-16t^3+40t^2-32t+9}, \\ y(t) &= \frac{t^2-4t+4}{2t^4-16t^3+40t^2-32t+9}. \end{aligned}$$

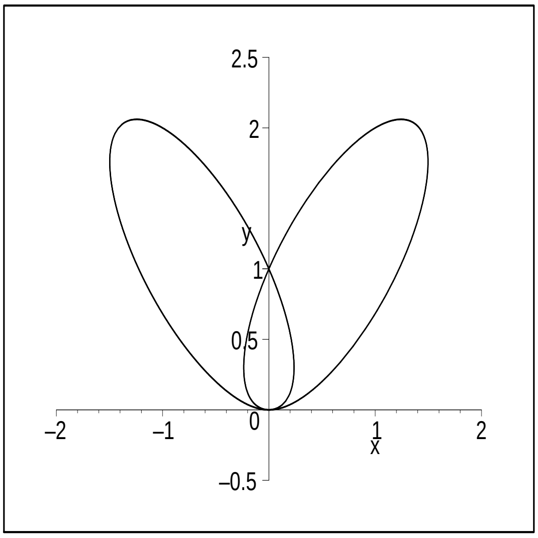


Figure 1: Tacnode curve

However, not all plane algebraic curves can be rationally parametrized.

The rationally parametrizable curves are the curves of genus 0:

$$\text{genus}(\mathcal{C}) = \frac{1}{2}[(d-1)(d-2) - \sum m_P(m_P - 1)]$$

where $d = deg(\mathcal{C})$ and m_P is the multiplicity of P on $\mathcal{C}$.

So, for instance, elliptic curves (of degree 3, no singularity) are not rational.

Some facts:

Theorem *Let* $\chi_1(t), \chi_2(t) \in K(t)$ *be rational functions in reduced form, not both of them constant. Then,*

$$\mathcal{P}(t) = (\chi_1(t), \chi_2(t))$$

parametrizes an irreducible plane curve $\mathcal{C}$ *over* K*. Moreover, if none of the two rational functions is constant and* $f(x, y)$ *is the defining polynomial of* $\mathcal{C}$*, there exists* $r \in \mathbb{N}$ *such that*

$$\mathrm{res}_t(H_1^{\mathcal{P}}(t, x), H_2^{\mathcal{P}}(t, y)) = (f(x, y))^r.$$

Theorem *An irreducible affine curve* $\mathcal{C}$ *is rational if and only if the field of rational functions on* $\mathcal{C}$*, i.e.* $K(\mathcal{C})$*, is isomorphic to* $K(t)$*.*

Theorem *An affine algebraic curve* $\mathcal{C}$ *is rational if and only if it is birationally equivalent to* $\mathbb{A}^1(K)$*.*

Theorem *An algebraic curve* $\mathcal{C}$ *is rational if and only if* $\mathrm{genus}(\mathcal{C}) = 0$*.*

Parametrization by Lines

case of rational conics (i.e. irreducible conics):

Conic through the origin

$$f(x,y) = f_2(x,y) + f_1(x,y)$$

where $\deg(f_i) = i$ (homogeneous components)

Consider the linear system $\mathcal{H}$ of lines through the origin

$$h(x,y,t) = y - tx.$$

compute the intersection points of a generic element of $\mathcal{H}(t)$ and $\mathcal{C}$, i.e. solve for x, y:

$$\begin{cases} y = tx \\ f(x,y) = 0 \end{cases}$$

the solutions are:

$$P = (0,0) \quad \text{and} \quad Q = \left(\frac{-f_1(1,t)}{f_2(1,t)}, \frac{-t \cdot f_1(1,t)}{f_2(1,t)} \right).$$

Theorem *The irreducible conic $\mathcal{C}$ defined by $f(x,y) = f_2(x,y) + f_1(x,y)$ ($\deg(f_i) = i$), has the rational parametrization*

$$\mathcal{P}(t) = \left(\frac{-f_1(1,t)}{f_2(1,t)}, \frac{-t \cdot f_1(1,t)}{f_2(1,t)} \right).$$

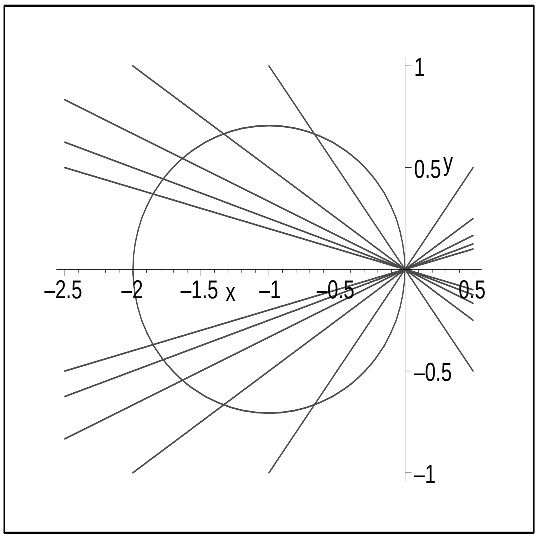


Figure 2: Ellipse $x^2 + 2x + 2y^2 = 0$ and pencil $\mathcal{H}(t)$

Algorithm CONIC-PARAMETRIZATION. Given the defining polynomial $f(x,y)$ of an irreducible conic $\mathcal{C}$, the algorithm computes a rational parametrization.

1. determine a point $(a,b) \in \mathcal{C}$ (choose $(a,b) = (0,0)$ if possible)
2. set $g(x,y) := f(x+a, y+b)$
3. let g_2, g_1 be the homogeneous components of $g(x,y)$
4. Return

$$\mathcal{P}(t) = \left(\frac{-g_1(1,t) + a g_2(1,t)}{g_2(1,t)}, \frac{-t \cdot g_1(1,t) + b g_2(1,t)}{g_2(1,t)} \right).$$

Example Let $\mathcal{C}$ be the ellipse defined by

$$f(x,y) = x^2 + 2y^2 - 1.$$

We apply algorithm CONIC–PARAMETRIZATION.

It is clear that $\mathcal{C}$ does not pass through $(0,0)$. Thus, we take a point on $\mathcal{C}$, for instance $(1,0)$ (Step 1).

In Step 2 we set $g(x,y) = f(x+1,y) = x^2+2x+2y^2$.

So, a parametrization of $\mathcal{C}$ is

$$\mathcal{P}(t) = \left(\frac{-1+2\,t^2}{1+2\,t^2}\,,\ \frac{-2\,t}{1+2\,t^2} \right).$$

case of $\deg(\mathcal{C}) = d$ with a $(d-1)$–fold point:

W.l.o.g. let $P = (0, 0)$ be the $(d-1)$–fold point on $\mathcal{C}$ so the defining polynomial of $\mathcal{C}$ is

$$f(x, y) = f_d(x, y) + f_{d-1}(x, y)$$

(f_i a form of degree i, resp.).

As above, we consider the linear system of lines $\mathcal{H}(t)$ through $(0 : 0 : 1)$. Intersecting $\mathcal{C}$ with an element of $\mathcal{H}$ we get the origin as an intersection point of multiplicity at least $d - 1$. Thus, by Bézout's Theorem, we must get exactly one more intersection point Q depending rationally on the value of t.
So the coordinates of Q are polynomials in t, in fact

$$Q = \left(\frac{-f_{d-1}(1, t)}{f_d(1, t)} \, , \, \frac{-t \cdot f_{d-1}(1, t)}{f_d(1, t)} \right) .$$

This is the rational parametrization of $\mathcal{C}$.

Algorithm PARAMETRIZATION-BY-LINES.
Given the defining polynomial $f(x, y)$ of an irreducible curve $\mathcal{C}$ of degree $d > 1$, having a $(d-1)$–fold point P in $\mathbb{A}^2(\mathbb{K})$, the algorithm computes a rational parametrization of $\mathcal{C}$.

1. Determine the $(d-1)$–fold point $P = (a, b)$ of $\mathcal{C}$.
2. compute $g(x, y) = f(x + a, y + b)$
3. let g_d, g_{d-1} be the homogeneous components of $g(x, y)$
4. Return

$$\mathcal{P}(t) = \left(\frac{-g_{d-1}(1,t)+ag_d(1,t)}{g_d(1,t)}, \frac{-t\cdot g_{d-1}(1,t)+bg_d(1,t)}{g_d(1,t)} \right).$$

Example Let $\mathcal{C}$ be the affine quartic curve defined by

$$f(x,y) = 1 + x - 15\,x^2 - 29\,y^2 + 30\,y^3 - 25\,xy^2 \\ +x^3y + 35\,xy + x^4 - 6\,y^4 + 6\,x^2y$$

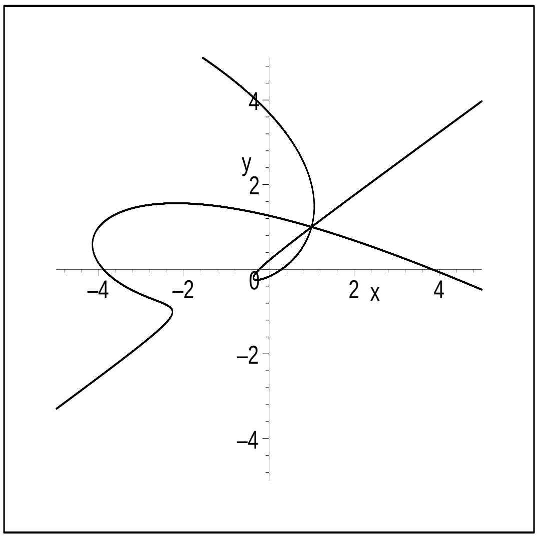


Figure 3: Quartic $\mathcal{C}$

$\mathcal{C}$ has an affine triple point $P = (1,1)$. We apply algorithm PARAMETRIZATION-BY-LINES to parametrize $\mathcal{C}$.

In Step 2 we compute the polynomial

$$g(x,y) = 5\,x^3 + 6\,y^3 - 25\,xy^2 + x^3y + x^4 - 6\,y^4 + 9\,x^2y$$

determining the homogeneous forms of $g(x,y)$, we get the rational parametrization of $\mathcal{C}$

$$\mathcal{P}(t) = \left(\frac{4 + 6\,t^3 - 25\,t^2 + 8\,t + 6\,t^4}{-1 + 6\,t^4 - t}, \frac{4\,t + 12\,t^4 - 25\,t^3 + 9\,t^2 - 1}{-1 + 6\,t^4 - t}\right).$$

Parametrization by Adjoint Curves

Definition A curve $\mathcal{C}'$ is an *adjoint curve* of the irreducible curve $\mathcal{C}$ iff for every singular point P (or in fact, just any point) of $\mathcal{C}$ of multiplicity r, P is a point of multiplicity at least $r-1$ on $\mathcal{C}'$. (This should include also all points "at infinity" and also all "neighboring singularities"). □

The algebraic conditions required in the definition of adjoint curves are linear. Therefore if one fixes the degree, the set of all adjoint curves of $\mathcal{C}$ is a linear system of curves. In fact, if $\mathcal{C}$ has only ordinary singularities, then the set of adjoint curves of degree k of $\mathcal{C}$, denoted by

$$\mathcal{A}_k(\mathcal{C}),$$

is the linear system of curves of degree k generated by the effective divisor

$$\sum_{P \in \mathrm{Sing}(\mathcal{C})} (\mathrm{mult}_P(\mathcal{C}) - 1)P.$$

Notation: let D be an effective divisor on $\mathcal{C}$, i.e. a formal sum of points on $\mathcal{C}$ with finitely many non-zero positive integral coefficients

$$D = \sum_{i=1}^{n} m_n P_n.$$

Then by $\mathcal{H}(k, D)$ we denote the linear system of curves of degree k having the points P_i as points of multiplicity at least m_i.

Theorem *Let* $\deg(\mathcal{C}) \geq d - 2$ *and let* $\mathcal{S}$ *be a set of* $d - 3$ *regular points on* $\mathcal{C}$. *Then*

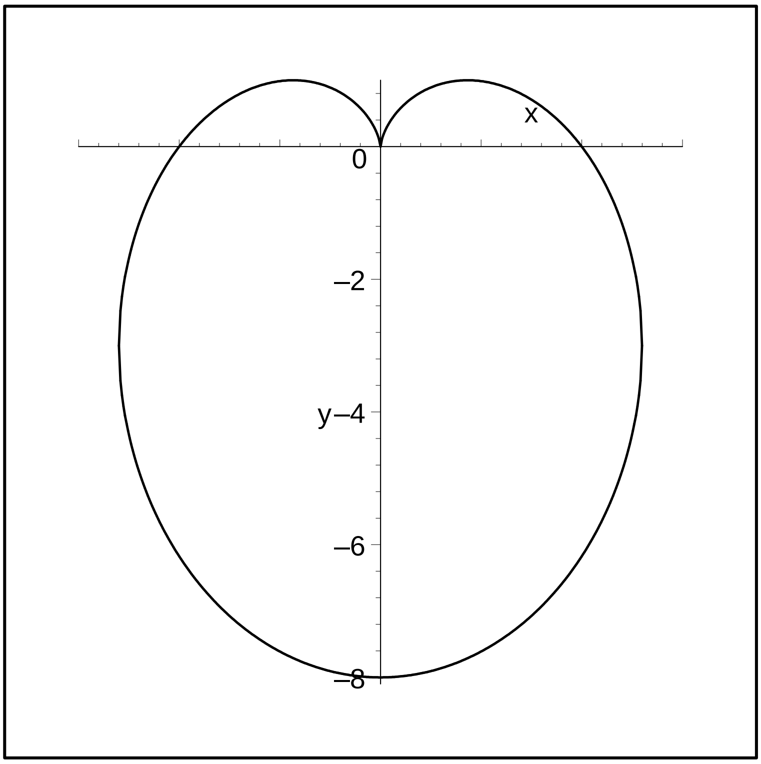
$$\mathcal{A}_{d-2}(\mathcal{C}) \ \cap \ \mathcal{H}(d-2, \sum_{P \in \mathcal{S}} P)$$

parametrizes $\mathcal{C}$. (I.e., if we intersect this system with $\mathcal{C}$, then only one intersection point is "free", and its coordinates give the rational parametrization.)

Algorithm PARAMETRIZATION-BY-ADJOINTS. Given the defining polynomial $f(x, y)$ of a rational irreducible curve $\mathcal{C}$ of degree d, the algorithm computes a rational parametrization of $\mathcal{C}$.

1. If $d \leq 3$ or $\mathrm{Sing}(\mathcal{C})$ contains only one point of multiplicity $d - 1$ apply algorithm PARAMETRIZATION–BY–LINES.
2. compute the defining polynomial of $\mathcal{A}_{d-2}(\mathcal{C})$.
3. Choose a set $\mathcal{S}$ of $d - 3$ regular points on $\mathcal{C}$.
4. Compute the defining polynomial H of
$$\mathcal{H} = \mathcal{A}_{d-2}(\mathcal{C}) \cap \mathcal{H}(d - 2, \sum_{P \in \mathcal{S}} P).$$
5. Return the solution in $K(t)^2$ of the system
$$\{\mathrm{pp}_t(\mathrm{res}_y(F, H)) = 0, \quad \mathrm{pp}_t(\mathrm{res}_x(F, H)) = 0\}.$$

In our presentation we have only considered the case of adjoints of degree $d-2$. But, whereas $d-2$ is the lowest degree for which this process works, we could choose any higher degree. For adjoints of degree higher than or at least d, we have to make sure that the system of adjoints does not contain the curve $\mathcal{C}$, but this can be achieved by requiring a point outside of $\mathcal{C}$ to be a base point of the system of adjoints.



Example: We consider the cardioid curve $\mathcal{C}$ defined by

$$f_1(x, y) = (x^2 + 4y + y^2)^2 - 16(x^2 + y^2) = 0.$$

We apply the algorithm PARAMETRIZATION-BY-ADJOINTS.

$\mathcal{C}$ has 3 double points in the projective plane, namely

$$(0:0:1) \quad \text{and} \quad (1:\pm i:0).$$

So genus($\mathcal{C}$) = 0, which means that $\mathcal{C}$ is rational and must have a parametrization over $\mathbb{C}$ (the picture actually suggests that it is a real curve, and therefore must have a parametrization over $\mathbb{R}$).

System $\tilde{\mathcal{H}}$ of conics passing through all three of these double points:

$$h(x, y, z, s, t) = x^2 + sxz + y^2 + tyz.$$

$\tilde{\mathcal{H}}$ is a system of dimension 2.

Now we need $d - 3 = 1$ regular points on $\mathcal{C}$. From the picture of the cardioid it is obvious that $P = (0, 8)$ is a regular point, and it has coordinates in $\mathbb{Q}$, i.e. we will get a parametrization with coefficients in $\mathbb{Q}$.

We restrict $\tilde{\mathcal{H}}$ to the curves through P:

$$h^\star(x, y, z, t) = x^2 + txz + y^2 + 8yz,$$

reducing the dimension of the system of adjoints by 1, i.e. to a system of dimension 1.

Intersecting the cardioid curve with this system of adjoints of dimension 1 we get the following parametrization $\mathcal{P}(t)$ of $\mathcal{C}$:

$$x(t) = \frac{-1024t^3}{256t^4 + 32t^2 + 1}, \qquad y(t) = \frac{-2048t^4 + 128t^2}{256t^4 + 32t^2 + 1}.$$

Parametrization over $\mathbb{R}$

Real Curves

Definition: The curve $\mathcal{C}$ is a called a **real** curve, if and only if $\mathcal{C}$ has infinitely many points in $\mathbb{A}^2(\mathbb{R})$.

Lemma: ([RS97a]) *If the curve $\mathcal{C}$ is real, then it can be defined by a real polynomial.*

Lemma: ([Wi96]) *Let $\mathcal{C}$ be a real curve. $\mathcal{C}$ is irreducible over $\mathbb{R}$ if and only if it is irreducible over $\mathbb{C}$.*

[SW91]: every parametrizable plane curve over an algebraically closed field $\mathcal{K}$ of characteristic zero can be parametrized over any subfield of $\mathcal{K}$ that contains the coefficients of the irreducible polynomial defining the curve, and the coordinates of one simple point of the curve.

Thus: every real parametrizable plane curve can be parametrized over the reals (this result is also known as the algebraic version of the real Lüroth's theorem), and a parametrizable plane curve is real if and only if it has at least one real simple point.

Example: the cardioid curve $\mathcal{C}_1$ defined by

$$f_1(x,y) = (x^2+4y+y^2)^2 - 16(x^2+y^2) = 0$$

(same curve as above) is a real curve.

In fact $\mathcal{C}_1$ is a parametrizable real curve, and a parametrization over the reals is

$$x(t) = \frac{-1024t^3}{256t^4+32t^2+1}, \qquad y(t) = \frac{-2048t^4+128t^2}{256t^4+32t^2+1}.$$

On the other hand, the curve $\mathcal{C}_2$ defined by

$$f_2(x,y) = 2y^2 + x^2 + 2x^2y^2 = 0$$

is NOT a real curve.

The only point of $\mathcal{C}_2$ in the affine plane over the reals is the double point $(0,0)$. The complex curve $\mathcal{C}_2$ is parametrizable, and a parametrization is

$$x(t) = \frac{-t^2-2t+1}{t^2-2t-1}, \qquad y(t) = \frac{i\,t^2+2i\,t-i}{2t^2+2}.$$

A Real Parametrization Algorithm

$\mathcal{C}$ defined by $f(x, y) \in \mathbb{R}[x, y]$

The corresponding projective curve, also denoted by $\mathcal{C}$, is defined by the homogeneous polynomial $F(x, y, z) \in \mathbb{R}[x, y, z]$, where F is the homogenization of f.

The property of parametrizability is independent of whether we view $\mathcal{C}$ in the affine or the projective plane, and parametrizations can be easily converted [SW91].

In order to parametrize the rational plane curve $\mathcal{C}$ over $\mathbb{R}$, we need to find ONE simple point on $\mathcal{C}$ with real coordinates. We do this by transforming $\mathcal{C}$ birationally to a low degree (1 or 2) curve, for which we can find such points (and transform them back to $\mathcal{C}$).

The following theorem is a refinement of a theorem by Hilbert and Hurwitz.

Theorem 1: (Sendra/W. 1997, adapted to $\mathbb{R}$)
Let $\mathcal{C}$ be a projective rational plane curve of degree d defined by a polynomial over $\mathbb{R}$, $\mathcal{H}_{d-2}$ the linear system of adjoint curves to $\mathcal{C}$ of degree $d-2$ and $\tilde{\mathcal{H}}_{d-2}^{s}$ a linear subsystem of $\mathcal{H}_{d-2}$ of dimension s with all its base points on $\mathcal{C}$. Then we have the following:

(i) If $\Phi_1, \Phi_2, \Phi_3 \in \tilde{\mathcal{H}}_{d-2}^{s}$ are such that the common intersections of the three curves Φ_i and $\mathcal{C}$ are the set of base points of $\tilde{\mathcal{H}}_{d-2}^{s}$, and such that

$$\mathcal{T} = \{y_1 : y_2 : y_3 = \Phi_1 : \Phi_2 : \Phi_3\}$$

is a birational transformation, then the birationally equivalent curve to $\mathcal{C}$, obtained by $\mathcal{T}$, is irreducible of degree s.

(ii) Those values of the parameters for which the rational transformation $\mathcal{T}$ is not birational satisfy some algebraic conditions.

Theorem 2: (Sendra/W. 97)
Let $\mathcal{C}$ be a rational plane curve of degree d defined by a polynomial over $\mathbb{R}$, and $\mathcal{H}_{d-2}$ the linear system of adjoint curves to $\mathcal{C}$ of degree $d-2$. Then every rational linear subsystem of $\mathcal{H}_{d-2}$ of dimension s with all its base points on $\mathcal{C}$ provides curves that generate families of s conjugate simple points over $\mathbb{R}$ by intersection with $\mathcal{C}$.

Theorem 3: (Sendra/W. 97) *Let $\mathcal{C}$ be a rational plane curve of degree d, defined by a polynomial $f(x, y) \in \mathbb{R}[x, y]$.*

(i) $\mathcal{C}$ has families of $d-2, 2d-2$, and $3d-2$ conjugate simple points over $\mathbb{R}$.

(ii) $\mathcal{C}$ has families of 2 conjugate simple points over $\mathbb{R}$.

(iii) If d is odd, then $\mathcal{C}$ has a simple point over $\mathbb{R}$.

(iv) If d is even, then $\mathcal{C}$ has simple points over an algebraic extension of $\mathbb{R}$ of degree 2.

<u>Algorithm</u> REAL-PARAM(f)

Input: $F(x_1, x_2, x_3) \in \mathbb{R}[x_1, x_2, x_3]$ is an irreducible homogeneous polynomial of degree d, that defines a rational plane curve $\mathcal{C}$.

Output: a real parametrization of $\mathcal{C}$, or "`no-real-parametrization`" if no real parametrization exists.

(1) Compute the linear system H of adjoint curves to $\mathcal{C}$ of degree $(d-2)$.

(2) If d is odd, apply Theorem 3 (iii) to find $(d-3)$ simple points of F over $\mathbb{R}$.

(3) If d is even, apply Theorem 3 (ii) to find $(d-3)/2$ families of 2 simple points of F over $\mathbb{R}$.

(4) Determine the linear rational subsystem $\tilde{H}$ obtained by forcing the points computed in steps (2) and (3) to be simple base points on H.

(5) Take $\tilde{\Phi}_1, \tilde{\Phi}_2, \tilde{\Phi}_3 \in \tilde{H}$ such that the common intersections of the three curves $\tilde{\Phi}_i$ and F are the set of base points of $\tilde{H}$, and such that

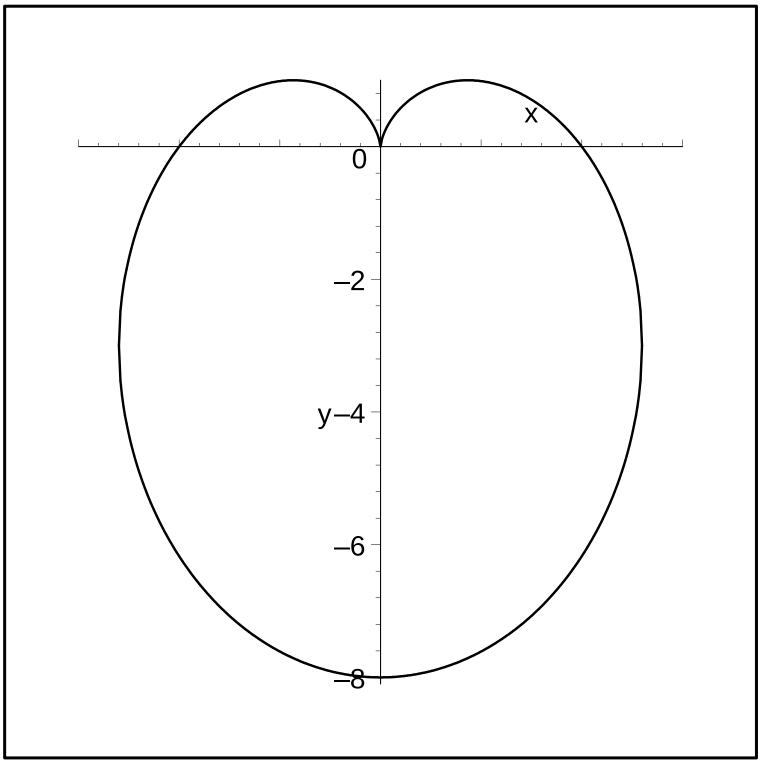
$$\mathcal{T} = \{y_1 : y_2 : y_3 = \tilde{\Phi}_1 : \tilde{\Phi}_2 : \tilde{\Phi}_3\}$$

is a birational transformation (Theorem 1).

(6) Determine the transformed curve $\mathcal{D}$ to $\mathcal{C}$ obtained by $\mathcal{T}$. Note that applying Theorem 1 one has that $\mathcal{D}$ is either a conic or a line depending on whether d is even or odd, respectively. $\mathcal{D}$ can be easily determined by sending a few points from $\mathcal{C}$ to $\mathcal{D}$ and then interpolating.

(7) If d is odd, parametrize the line $\mathcal{D}$ over $\mathbb{R}$. Apply the inverse transformation $\mathcal{T}^{-1}$ to find a parametrization of $\mathcal{C}$ over $\mathbb{R}$. (Or, alternatively, determine as many points on $\mathcal{D}$ over $\mathbb{R}$ as necessary, transfer them back to $\mathcal{C}$ by $\mathcal{T}^{-1}$, and use them for computing a parametrization of $\mathcal{C}$ over $\mathbb{R}$.)

(8) If d is even, decide whether the conic $\mathcal{D}$ can be parametrized over $\mathbb{R}$. If so, parametrize $\mathcal{D}$ over $\mathbb{R}$. Apply the inverse transformation $\mathcal{T}^{-1}$ to find a real parametrization of $\mathcal{C}$ over $\mathbb{R}$. (Or, alternatively, determine as many points on $\mathcal{D}$ over $\mathbb{R}$ as necessary, transfer them back to $\mathcal{C}$ by $\mathcal{T}^{-1}$, and use them for computing a parametrization of $\mathcal{C}$ over $\mathbb{R}$.)
If not, report "`no-real-parametrization`".



Example:

We consider the cardioid curve $\mathcal{C}$ defined by

$$f(x, y) = (x^2 + 4y + y^2)^2 - 16(x^2 + y^2) = 0.$$

We already know that $\mathcal{C}$ is parametrizable, in fact we have already computed a parametrization over $\mathbb{R}$. But now we want to use $\mathcal{C}$ for demonstrating the general algorithm.

Application of REAL-PARAM:

$\mathcal{C}$ has 3 double points in the projective plane, namely

$$(0:0:1) \quad \text{and} \quad (1:\pm i:0).$$

System $\tilde{\mathcal{H}}$ of conics (curve of degree 2) passing through all three of these double points:

$$h(x,y,z,s,t) = x^2 + sxz + y^2 + tyz.$$

$\tilde{\mathcal{H}}$ is a system of dimension 2.

Birational transformation

$$\begin{aligned}\mathcal{T} &= (\Phi_1 : \Phi_2 : \Phi_3) \\ &= (h(x,y,z,0,1) : h(x,y,z,1,0) : h(x,y,z,1,1)),\end{aligned}$$

i.e.

$$\begin{aligned}\Phi_1 &= x^2 + y^2 + yz, \\ \Phi_2 &= x^2 + xz + y^2, \\ \Phi_3 &= x^2 + xz + y^2 + yz.\end{aligned}$$

We determine the birationally equivalent conic $\mathcal{D}$ to $\mathcal{C}$ by sending the 6 points in the families

$$\mathcal{F}_1 = \{(t : -t+2 : 1) \mid 4t^4 - 32t^3 + 80t^2 - 128t + 80\},$$
$$\mathcal{F}_2 = \{(t : 1-2t : 1) \mid t^2 - 4t + 1\}$$

onto $\mathcal{D}$ by $\mathcal{T}$. This gives us the conic defined by

$$15x^2 + 7y^2 + 6xy - 38x - 14y + 23.$$

$\mathcal{D}$ has the real (in fact, rational) point $(1, 8/7)$, which (by $\mathcal{T}^{-1}$) corresponds to the point $P = (0, -8)$ on $\mathcal{C}$.

Now we restrict $\tilde{\mathcal{H}}$ to the curves through P:

$$h^\star(x, y, z, t) = x^2 + txz + y^2 + 8yz.$$

We get the following real parametrization $\mathcal{P}(t)$ of $\mathcal{C}$:

$$x(t) = \frac{-1024t^3}{256t^4 + 32t^2 + 1}, \qquad y(t) = \frac{-2048t^4 + 128t^2}{256t^4 + 32t^2 + 1}.$$

Some of our work on parametrization:

J.R. Sendra, F. Winkler, *Symbolic Parametrization of Curves. J. of Symbolic Computation*, 12/6, 607–631 (1991)

M. Mňuk, J.R. Sendra, F. Winkler, *On the Complexity of Parametrizing Curves. Beiträge zur Algebra und Geometrie*, 37/2, 309–328 (1996)

J.R. Sendra, F. Winkler, *Parametrization of Algebraic Curves over Optimal Field Extensions. J. of Symbolic Computation*, 23/2& 3, 191–207 (1997)

E. Hillgarter, F. Winkler, *Points on Algebraic Curves and the Parametrization Problem.* In: *Automated Deduction in Geometry*, D. Wang (ed.), Springer-Verlag Berlin Heidelberg, LNCS 1360, 189–207 (1998)

J.R. Sendra, F. Winkler, *Algorithms for Rational Real Algebraic Curves*, in special issue on "Symbolic Computation and Artificial Intelligence", *Fundamenta Informaticae*, 39/1-2, 211–228 (1999)

G. Landsmann, E. Hillgarter, J. Schicho, F. Winkler, *Symbolic Parametrization of Pipe and Canal Surfaces*, in *Proc. International Symposium on Symbolic and Algebraic Computation (ISSAC'00)*, St. Andrews, Scotland, C. Traverso (ed.), 202–208, ACM-Press (2000)

G. Landsmann, J. Schicho, F. Winkler, *The Parametrization of Canal Surfaces and the Decomposition of Polynomials into a Sum of Two Squares*, *J. of Symbolic Computation*, 32/1&2, 119–132 (2001)

J.R. Sendra, F. Winkler, *Tracing Index of Rational Curve Parametrizations*, *Computer Aided Geometric Design* 18, 771–795 (2001)

R. Hemmecke, E. Hillgarter, F. Winkler, *CASA*, in: *Handbook of Computer Algebra: Foundations, Applications, Systems*, J.Grabmeier, E.Kaltofen, V.Weispfenning (eds.), 356–359, Springer-Verlag (2003)

J.R. Sendra, F. Winkler, S. Perez-Diaz, *Rational Algebraic Curves — A Computer Algebra Approach*, book project, to be published by Springer-Verlag (2007)