Classical Models for Quantum Light II

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Lecture given on April 8, 2016 at the

Zentrum für Oberflächen- und Nanoanalytik

Johannes Kepler Universität Linz, Österreich

http://www.mat.univie.ac.at/~neum/papers/physpapers.html#CQlightslides2

In this lecture the results of the historical review given in my lecture

Classical models for quantum light

http://www.mat.univie.ac.at/~neum/papers/physpapers.html#CQlightslides

are utilized to reassess the meaning of observables and stochastic processes for the classical and quantum description of light.

In particular we discuss the description of partially coherent, fluctuating light through classical stochastic Maxwell equations (with uncertainty in the initial conditions only), and look at a generalization that works for all quantum aspects of arbitrary quantum systems.

1. Can we learn from polarization about quantum mechanics?
2. Macroscopic observables
3. Polarization and random fields
4. The stochastic Maxwell equations
5. Beyond the stochastic Maxwell equations
6. Quantum mechanics of single objects

Can we learn from polarization about quantum mechanics?

Yesterday we saw that polarized light is the simplest quantum phenomenon.

It is the only quantum phenomenon
that was understood quantitatively already in 1852,
long before the birth of quantum mechanics in 1900.

In modern terminology, the behavior is identical to that
of a qubit in a pure state (for fully polarized light)
or in a mixed state (for partially polarized light).

Polarization is a macroscopic phenomenon.
The counterintuitive features of quantum mechanics
irritating the untrained intuition are still absent.
Experiments with polarization filters are easy to perform;
probably they are already known from school.

Thus polarization experiments show the basic principles
of quantum mechanics in a clean and transparent way.

Is it perhaps more than a toy example?
We also saw that the stochastic Maxwell equations fully describe
the modes of single photons and of coherent light.

In quantum mechanics, observables are conventionally described by Hermitian operators, whose observation reveals one of their eigenvalues as the observed value.

Predictable is not the single case but only the mean of many observations of A
$\langle A \rangle := \text{tr } \rho A$,
where the density matrix ρ describes the state, and
$\langle 1 \rangle = \text{tr } \rho = 1$.

In Stokes' classical description, observables are **intensities**
$\rho' = \text{tr } \rho T^* T = \langle T^* T \rangle$
of beams after passing some instrument with Jones matrix T.

In particular, $\langle 1 \rangle = \text{tr } \rho$ is the intensity of the original beam itself.

Why was there no problem for Stokes to interpret the qubit in a classical way? Because he had a different conception of observable and measurement.

In Stokes' classical view, the observables were four real numbers, the components of the Stokes vector; equivalently, four linearly dependent complex numbers, the components of the polarization matrix.

In the traditional quantum view, the observables are operators (2×2 matrices). The observable values are their eigenvalues – discrete numbers, related to the density matrix (normalized polarization matrix) only in an indirect, statistical way.

What is classical about the quantum mechanics of polarization?

The behavior of

- sources (preparation)
- thermal matter (irreversibility)
- detectors (irreversible amplification of microscopic events)
- fields (what is in between sources and detectors)

GLAUBER's dictum *"photons are what photodetectors count"* gets its quantitative expression as

$$\text{intensity} = \text{rate of clicks}$$

That we have a rate rather than a probability accounts for the unnormalized nature of the coherence matrix.

Since any Hermitian $A = T^*T$ can be written in the form
$A = a_0 + \mathbf{a} \cdot \sigma$,
and $S_0 = 1$, $S_j = \langle \sigma_j \rangle$ $(J = 1, 2, 3)$, we have
$\langle A \rangle = a_0 S_0 + \mathbf{a} \cdot \mathbf{S}$.

Thus the components of the Stokes vector form a complete basis for the classical observables (or the quantum rates).

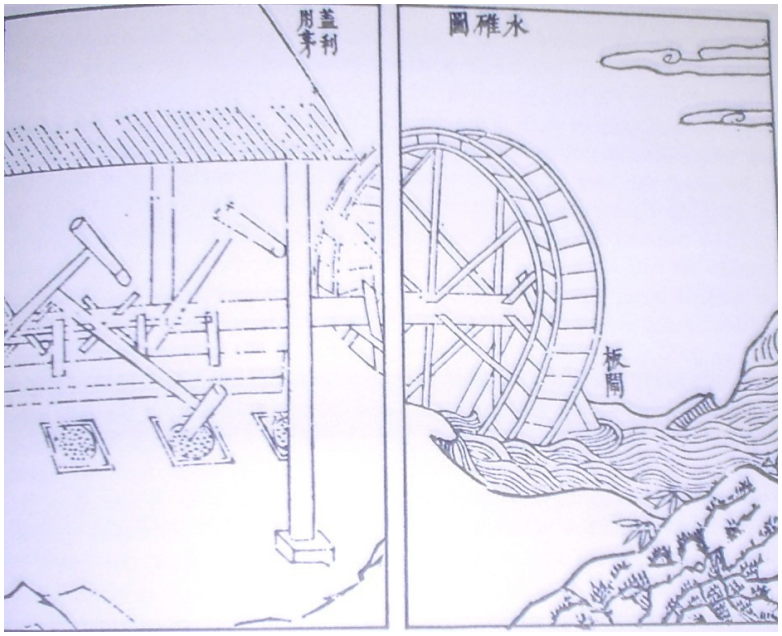
What are the true observables of a quantum system?
The intensities or the discrete detection events?

Our discussion of the photoelectric effect showed that the statistics of photodetector clicks is predicted correctly even for classical, unquantized light which have no inherently discrete properties.

This points to the possibility that the rates of clicks, i.e., **various intensities are the most sensible observables** of a quantum system.

At least they might turn out to have a classical interpretation, as in the case of polarization.

Measuring the intensity of light with a photodetector
is a bit like measuring the water supply of a trip hammer



powered by a water wheel by the number of the hammers' impacts:
They become few and erratic when the water supply
is not rich enough.

One needs to take long-time averages to get accurate results.

image from
`https://en.wikipedia.org/wiki/File:Hydraulic-Powdered_Trip_Hammers.jpg`

Macroscopic observables

What are the true observables of a quantum system?
For a microscopic quantum system this is difficult to answer
since we observe only macroscopic responses
(e.g., detector clicks, current peaks, etc.).

Therefore we first look at what quantum mechanics says
about the observation of macroscopic variables.

This is the domain of statistical mechanics.

Statistical mechanics derives from quantum mechanics
the behavior of ordinary macroscopic matter.

Macroscopic means that the so-called **theormdynamic limit**
(where the system size tends to infinity)
is an excellent approximation of the real situation.

For example, one can derive
the Navier–Stokes equations for moving air
or the thermodynamic behavior of homogeneous substances.

Since these can be directly observed, there is no question about
what the observables are.

The only question is their microscopic form.

We now look at which quantum expressions encode the macroscopic observables.

We find that in equilibrium, the thermodynamics of an arbitrary substance composed of C kinds of molecules is completely characterized by the volume V, the internal energy $\langle H \rangle$ and the total masses $\langle M_j \rangle$ of molecules of kind j.

Here H is the Hamiltonian operator of the system and M_j the mass operator for molecules of kind k, the product of the corresponding number operator with the molecular mass.

Out of equilibrium, derivations are more complicated and typically restricted to sufficiently simple systems.

But for argon gas, say, the observable extensive fields appearing in the Navier–Stokes equations are microscopically again expectations of the form $\langle \phi(x) \rangle$
with appropriate quantum field expression $\phi(x)$.

For very dilute gases, where the local equilibrium approximation is inadequate, one can derive similarly the Boltzmann equation.

Its observable fields are again given by expectations – this time expectations of Wigner transforms of 1-particle operators.

Linear response theory lets us predict the observable responses of a macroscopic system to small external fields.

If one checks the microscopic expressions used for their prediction, one finds again that they are expectations
of products of operators with different space-time arguments.

Thus on the macroscopic level it is always expectations of suitable operators that define the observable variables:

(T1) All macroscopically observable variables are expectations of selected operators A,

$$\bar{A} := \langle A \rangle = \text{tr } \rho A.$$

If we reduce the size of a macroscopic system,
the thermodynamic limit becomes less and less accurate,
and at some point quantum-mechanical fluctuations start playing a role.

In statistical mechanics, these are expressed quantitatively by means of fluctuation-dissipation theorems.

In the simplest case, in equilibrium, this takes the form of a computable **intrinsic uncertainty**

$$\sigma_A := \sqrt{\langle (A - \bar{A})^2 \rangle}.$$

The following additional rules summarizes the practice of statistical mechanics:

(T2) Upon measuring a Hermitian operator A,
the measured result will be approximately $\bar{A}$,
with an uncertainty at least of the order of σ_A.

(T3) If the measurement can be sufficiently often repeated
(on objects with the same or sufficiently similar state)
then σ_A will be a lower bound on the standard deviation
of the measurement results.

Polarization and random fields

In the first 464 pages of their well-known, comprehensive book
Optical Coherence and Quantum Optics,
Mandel & Wolf 1995 deduce from the Maxwell equations
the properties of classical light, including polarization
and the (semi)classical photo effect.

Normal polarized light in general (not only for quasimonochromatic beams) behaves exactly according to the free Maxwell equations.

Unpolarized light cannot be modelled
directly by the Maxwell equations.

Instead, within Maxwell's theory, the description
of unpolarized light is an effective description
of observable degrees of freedom of very complex
"fluctuating" solutions of the Maxwell equations.

This effective description is obtained by short-time averaging,
over a time short with respect to the time resolution
of the observations but long with respect to the time
light needs to complete one fluctuation.

In this effective description, the short-time average
of the fields themselves is essentially zero, while
the short-time average of quadratic expressions in the field
are nonzero, varying slowly enough to be observable.

The short-time average introduces naturally
a fuzzy element into the description.

This leads to the effective observables $\langle f \rangle$, defined as
the short-time average of a microscopic quantity f
described in terms of an exact solution
of the Maxwell equations.

Thus the ensemble (if one can still speak of one)
is formed by the uncertainty in the fuzzy nature
of the time of preparation or measurement, rather than
by repeated independent preparation of the field.

Formally, one can take a limit where the fluctuations
are infinitely fast and the short-time average extends
over infinitely short times.

Then one arrives at a stochastic version of the Maxwell equations,
where the fluctuating fields are described as random fields
satisfying the Maxwell equations, and observables are the
expectations of functionals of the fields.
The uncertainty is in the initial conditions (the state) only.

Note that in this limit, the actual observables
are not the (classical) random fields f themselves
but their (classical) expectations $\langle f \rangle$!

This very much resembles the situation in statistical mechanics.

The stochastic Maxwell equations

Random space-time fields which are stationary in time
give models for beams of partially polarized light.

Their coherence matrices provide the microscopic basis
for the macroscopic description of beams of light
in terms of the density matrix or the Stokes vector.

On the basis of the Maxwell equations and their stochastic
interpretation, the second order coherence theory
fully explains all partial polarization effects
in a classical way.
(The photoeffect ist already explained classically
by the deterministic Maxwell equations.)

The **second order coherence theory** of the
stochastic Maxwell equations gives a valid description
not only for classical beams of light
but for general classical electromagnetic radiation in the vacuum
(or, with appropriate modifications, in other media).

The relevant effective observables for classical e/m fields
are the momentum-dependent 3×3 coherence matrices

$$C(t, \mathbf{p}_1, \mathbf{p}_2) := \langle \mathbf{A}(t, \mathbf{p}_1) \mathbf{A}(t, \mathbf{p}_2)^* \rangle,$$

where $\mathbf{A}(t, \mathbf{p})$ is the spatial Fourier transform of $\mathbf{A}(t, \mathbf{x})$.

Here one assumes for radiation that $\langle \mathbf{A}(t, \mathbf{p}) \rangle = 0$, i.e.,
macroscopically observable field averages are subtracted.

They satisfy the transversality conditions

$$\mathbf{p}_1^T C(t, \mathbf{p}_1, \mathbf{p}_2) = 0, \quad C(t, \mathbf{p}_1, \mathbf{p}_2)\mathbf{p}_2 = 0,$$

and completely determine all short-time averages
of second order expressions in the fields.

After passing an instrument with transition matrix
$T_\psi(t) := \int d\mathbf{p}\psi(\mathbf{p}) \cdot \mathbf{A}(t, \mathbf{p})$
one measures the intensity

$$\langle T_\psi(t)^* T_\psi(t) \rangle = \int d\mathbf{p} d\mathbf{p}' \psi(\mathbf{p})^* C(t, \mathbf{p}_1, \mathbf{p}_2) \psi(\mathbf{p}') = \psi^* C(t) \psi.$$

By taking linear combinations and limits, this gives an
operational meaning to the observables $C(t, \mathbf{p}_1, \mathbf{p}_2)$.

Note that compared to the deterministic setting,
the stochastic setting has many more observables!

We may interpret $C(t, \mathbf{p}_1, \mathbf{p}_2)$ as the kernel of the time-dependent operator $C(t)$ defined by

$$(C(t)\psi)(\mathbf{p}) := \int d\mathbf{p}' C(t, \mathbf{p}, \mathbf{p}')\psi(\mathbf{p}').$$

$C(t)$ maps transversal wave functions ψ with $\mathbf{p} \cdot \psi(\mathbf{p}) = 0$ into transversal wave functions $C(t)\psi$. Thus it defines a linear operator on the Hilbert space

$$\mathbb{H} = \{\psi \in L^2(\mathbb{R}^3, \mathbb{R}^3) \mid p^T \psi(p) = 0\}.$$

The $C(t, \mathbf{p}_1, \mathbf{p}_2)$ describe Fourier modes of local intensities with frequencies $\omega = (|\mathbf{p}_2| - |\mathbf{p}_1|)c/\hbar$, with a dynamics given by

$$i\hbar \frac{d}{dt} C(t, \mathbf{p}_1, \mathbf{p}_2) = c(|\mathbf{p}_2| - |\mathbf{p}_1|) C(t, \mathbf{p}_1, \mathbf{p}_2).$$

This classical dynamics for the kernel of observable intensities can be written as an operator equation

$$i\hbar \frac{d}{dt} C(t) = [H, C(t)],$$

where $H = c|\mathbf{p}|$ is the kinetic energy of a massless relativistic particle with momentum $\mathbf{p}$.

This is the **Heisenberg equation** with the free Hamiltonian $H = c|\mathbf{p}|$, a dynamical equation otherwise only known from quantum theory.

Thus the classical second order coherence theory of the Maxwell equations is fully equivalent with

the quantum theory of a transverse vector particle
with kinetic energy $H = c|\mathbf{p}|$ in the momentum representation.

It is therefore capable of faithfully representing
all quantum properties of a **single** photon.

Beyond the stochastic Maxwell equations

The stochastic Maxwell equations fail to describe both antibunching and nonlocal effects observed in time correlations in the presence of nonclassical multi-photon states.

We now indicate how to construct more general classical models that also account for the properties of entangled photon pairs such as those produced by positron annihilation or parametric down-conversion.

Indeed, it is possible to generalize the approach of Stokes – to use intensities as the true variables of a quantum system – to arbitrary quantum systems, and treat the expectation values as the correct observables.

In this way one obtains a deterministic dynamics for the "new" observables of any quantum system.

Whle this was first observed by Strocchi 1966, and repeatedly rediscovered later, it was never turned into a fundamental approach.

Because of its origin in thermodynamics, I call this the **thermal interpretation of quantum mechanics**.

A systematic exposition is given in Chapters 8–10
of the online book Neumaier & Westra 2008, 2011.
A revised version will be published in 2017 by de Gruyter.

Quantum mechanics of single objects

Thermodynamics applies to single objects such as a particular piece of metal.

It is on first sight paradoxical that
the results of statistical mechanics apply to single objects
although their derivation is based on statistical ensembles.

Gibbs (who coined the notion of an ensemble)
justified the application of statistical techniques to single systems
by treating ensembles as fictitous imagined repetitions
of the single system under consideration.

How can arguments involving fictitious copies lead to correct physics?

It is the mathematical rules that make the theory,
not the intuition associated with them.

Just as mathematical vectors are not necessarily vectors in the original spatial sense, an expectation value is not necessarily the mean value of a series of statistical measurements.

Thus one has to free the intuition from regarding expectations as implying a fictitious repetition.

The mathematics of expectations (and hence of the physics where it is applied) is completely independent of this intuition.

Indeed, statistical techniques are often applied in practice to unique situations.

Even perfectly well-defined nonrandom objects such as
prime numbers are investigated with stochastic methods.

Rounding errors in deterministic floating-point computations
are analyzed statistically to get realistic error distributions.

Unique climate data such as the El Niño Southern Oscillation
are handled statistically.

Everyday weather forecasts are unthinkable
without statistical techniques although
weather is a unique, non-repeatable dynamical process.

Where then is the influence of statistics in statistical mechanics?
The single system consists there of myriads of identical parts,
so that the law of large numbers applies.

Weak law of large numbers

For a family of quantities A_l $(l = 1, \ldots, N)$ satisfying
$\langle A_k A_l \rangle = \langle A_k \rangle \langle A_l \rangle \quad \text{for } k \neq l,$
$\langle A_l \rangle = \bar{A} \quad \text{for } l = 1, \ldots, N,$
the mean quantity $\widehat{A} := \dfrac{1}{N} \displaystyle\sum_{l=1}^{N} A_l$ satisfies

$$\langle \widehat{A} \rangle = \bar{A}, \quad \sigma_{\widehat{A}} = \sigma(A)/\sqrt{N}.$$

Macroscopic variables are simply those $\bar{A}$ where,
due to the law of large numbers, $\sigma_A \ll |\bar{A}|$, so that
the expectation is highly significant.

Quantum field theory poses a special problem
for the foundations of quantum mechanics.

In a full QED (i.e., space-time) treatment of light,
one needs to make an operational definition of how to interpret
quantum expectations when only **one** space-time field exists,
rather than an ensemble of independently prepared fields.

Thus, as in statistical mechanics, the ensemble consists of
fictitious, imagined copies of the one, existing field.

However, quantum field theory also describes
the scattering of a few particles only.

Thus we need to find out what happens for small systems
where the law of large number no longer applies.
If the system size is reduced to the level of single molecules,
the law of large numbers, and hence statistical mechanics,
is no longer applicable.

However, quantum chemists routinely use the
Born-Oppenheimer approximation
to compute molecular potential energy surfaces,
subsequently used in classical molecular mechanics simulations.

(Related, improved matrix-valued approximations can
also account for conical intersections where
the Born-Oppenheimer approximation itself breaks down.)

In the Born-Oppenheimer approximation one
replaces the position operators of the nuclei
(which are much heavier than the electrons) by their expectation,
and solves the resulting Schrödinger equation for electrons alone.

Since for well-behaved functions f,
$f(A) = f(\overline{A}) + O(\sigma_A)$, this is justified
as long as the intrinsic uncertainty is relatively small.

This is typically the case for nuclei; there
$\sigma_A = O(m_{\text{electron}}/m_{\text{nucleus}}) = O(10^{-3})$.

Thus even at the size of the nuclei,
the expectations $\overline{A}$ figure as good classical observables.

Since there seems no limit to the applicability of rules (T1)–(T3) originating in statistical mechanics, it is legitimate to propose to make these three rules the basic interpretation principle for all of quantum mechanics.

Applied to single quantum systems, the expectation is not interpreted statistically, except in the limited sense given in (T3).

This is the **thermal interpretation of quantum mechanics**.

Some variables of microscopic systems may have zero uncertainty: In a pure state ψ, $\sigma_A = 0$ holds iff $A\psi = \overline{A}\psi$.

Thus arbitrarily sharp measurements of A are possible precisely in eigenstates of A whose eigenvalue is the measured value.

However in typical physical states most variables have a positive intrinsic uncertainty and cannot be measured sharply.

In such cases, it makes no sense to assign a meaning to the exact value $\bar{A}$ – only the range $\bar{A} \pm \sigma_A$ has an observable meaning.

Uncertain values naturally go hand in hand with extendedness
with a somewhat fuzzy boundary – in the same way as
we can locate the position of a city like Vienna on an atlas,
but not very accurately due to its extendedness.

To give more accuracy than a few km is simply meaningless.

In the thermal interpretation, all quantum objects have an
uncertain position at each time, and their trajectory forms a fuzzy
tube within which the object is located.

This fuzzy tube is fully determined by the state –
but it is an uncertain position only, not specifiable
to more accuracy than the diameter of the tube.

In the thermal interpretation, the observables
(i.e., expectations of operators) have a **deterministic dynamics** given by the **Ehrenfest theorem**

$$\frac{d}{dt}\langle A\rangle_t = \langle H \angle A\rangle_t,$$

where $\langle\cdot\rangle_t$ encodes the state at time t and

$$H \angle A = \frac{i}{\hbar}[H, A]$$

is the quantum Poisson bracket. It turns out (Strocchi 1966) that

$$\langle A\rangle \angle \langle B\rangle := \langle A \angle B\rangle$$

defines a corresponding classical Poission bracket.

The Ehrenfest dynamics is a Hamiltonian dynamics with respect to this bracket.

Traditionally, the Ehrenfest theorem is interpreted as being a link between quantum mechanics and classical mechanics.

However, in the traditional statistical interpretation of quantum mechanics, the theorem does not make much sense.

For example, the Ehrenfest theorem describes the dynamics of the "mean position" and "mean momentum" of a single particle,

$$\langle \dot{q} \rangle = \langle H_p(p,q) \rangle, \quad \langle \dot{p} \rangle = -\langle H_q(p,q) \rangle.$$

But tradition gives no idea **of what** $\langle q \rangle$ and $\langle p \rangle$ should be the mean:

The single realization of a quantum particle is not supposed to have simultaneously position and momentum – so how can their mean exist?

The thermal interpretation therefore restores common sense
to the interpretation of the dynamics of single particles:

They are extended objects that have – like a car –
an uncertain but within some accuracy definite
position, momentum, and trajectory.

However, it is meaningless to inquire about
this position, momentum, or trajectory beyond an accuracy
specified by the intrinsic uncertainty.

In particular, one has the limits given by
Heisenberg's uncertainty relation.

Related material

Recent publications and preprints
`http://www.mat.univie.ac.at/~neum/papers/physpapers.html`

Arnold Neumaier and Dennis Westra,
Classical and Quantum Mechanics via Lie algebras, 2008, 2011.
`http://lanl.arxiv.org/abs/0810.1019`

A **theoretical physics FAQ**
`http://www.mat.univie.ac.at/~neum/physfaq/physics-faq.html`
with topics such as:
What is the meaning of probabilities?
Postulates for the formal core of quantum mechanics
What is a photon?

Physics Overflow
A question and answer site for graduate+ level physics
`http://www.physicsoverflow.org`

Thank you for your attention!

Appendix

1. The thermal interpretation of quantum mechanics
2. Probability via expectation
3. What exists in a beam of light?
4. The position of quantum objects

1. The thermal interpretation of quantum mechanics

Following the exposition of statistical thermodynamics
in the book Neumaier & Westra 2008,
we introduce formal statistical mathematics, phrased
in a way not using probabilistic terminology.

In thermodynamics, all we ever measure are macroscopic values
of the form $\langle g \rangle := \operatorname{tr} \rho g$, where g is a microscopic quantity
and ρ is the density matrix encoding the state of the system.

With some care, this **thermal interpretation**
can be extended to the microscopic regime in a way
that it remains sensible even for single, small quantum objects.

Instead of calling $\langle g \rangle := \text{tr } \rho g$ the expectation value,
we continue to call it the **value** of g,
as in the thermal case.

For a Hermitian quantity g with nonzero value $\langle g \rangle$,
we define the **limit resolution**

$$\text{Res}\,(g) := \sqrt{\langle g^2 \rangle / \langle g \rangle^2 - 1}.$$

The expression under the square root is always nonnegative,
and vanishes in a pure state iff it is an eigenstate of g.

Therefore, the limit resolution is an uncertainty measure
specifying how accurately one can treat g as a sharp number,
given by this value.

In experimental practice, the limit resolution is a lower bound on the relative accuracy with which one can expect $\langle g \rangle$ to be determinable reliably from (ideal) measurements of a single object at a single time.

In particular, a quantity
g is considered to be **significant** if $\mathrm{Res}\,(g) \ll 1$,
while g is considered to be **noise** if $\mathrm{Res}\,(g) \gg 1$.

If g is a quantity and $\widetilde{g}$ is a good approximation of its value then $\Delta g := g - \widetilde{g}$ is noise.

For a single system at a single time,
the values $\langle g \rangle$ have an experimentally testable meaning
precisely when $\mathrm{Res}\,(g)$ is significantly smaller than 1.

However, time averages of time-dependent values $\langle g(t_l) \rangle$
at different times t_l may have a testable meaning:
They are the values of the quantity $\tilde{g}$ defined as the average
of the $g(t_l)$. By the law of large numbers, this quantity
may be significant in the above precise sense
even when no single $g(t_l)$ is.

This terminology captures correctly the experimental practice,
without imposing any statistical or probabilistic connotations.

On the contrary, it determines the precise conditions
under which statistical reasoning is necessary:
namely precisely when the limit resolution of a quantity
is larger than the desired accuracy.

2. Probability via expectation

Kolmogorov 1933 gave the modern axiomatic foundations of (classical, commutative) probability theory, impeccably basing the concept of random variables on measure theory.

There is a nice book by Peter Whittle 1992 with the title "Probability via Expectation", who proceeds in a different way.

He starts with expectation, and **derives**
the measure theoretic approach rather than postulating it.

This has important advantages since the measure theoretic approach is limited to classical physics, where all quantities commute.

In quantum mechanics, noncommuting pairs of quantities,
such as position and momentum, defy a probabilistic description
in measure theoretic terms.

The probability interpretation is restricted to families
of pairwise commuting quantites X_j determined
by an experimental set-up.

Probabilistic foundations of quantum mechanics are therefore
intrinsically muddled up with the problem of measurement.

But measurement cannot be fundamental since what constitutes
a good measurement is itself a complicated question
dependent on quantum theory and statistical mechanics.

This might explain the unsatisfactory state (Schlosshauer 2005)
of the foundations of quantum mechanics even over 80 years
after its mathematical basis was established.

The thermal interpretation together with Whittle's approach gives a convenient alternative to the probabilistic foundations.

Indeed, if the X_j are pairwise commuting quantum observables, Whittle's (classical) development implies that there is a way consistent with Kolmogorov to define for any quantum state random variables H and X_j such that the expectation of all sufficiently regular functions $f(H, X)$ defined on the joint spectrum of (H, X) agrees with the value of f.

Thus, in the pairwise commuting case, it is always possible to construct a probability interpretation for the quantities, *no matter whether or not some sort of microscopic or submicroscopic structure is assumed to exist.*

Even without a postulated probabilistic structure,
quantum states can be given a frequentist interpretation
in a context where many repeated experiments are feasible.

Then $\langle g \rangle$ is the **expectation value** of g,
empirically defined as an average over many realizations.

In this case, the limit resolution $\mathrm{Res}\,(g)$ becomes
the standard deviation of g, divided by the absolute value
of the expectation.

Therefore, when the frequentist interpretation applies,
the limit resolution measures the relative accuracy
of the realizations.

Thus the thermal interpretation of the
quantum mechanical formalism in terms
of the value $\langle g \rangle$ of a quantity g
and its limit resolution Res g
gives both a nonstatistical fuzzy view
applicable to single quantum objects,
and an additional statistical view applicable
when frequent repetition is feasible.

New aspects appear when we consider quantum fields,
needed for the understanding of photons
in terms of quantum electrodynamics.

3. What exists in a beam of light?

Many quantum experiments start with a source producing a
well-collimated, monochromatic, unpolarized beam of light.
Such a beam is always stationary – nothing happens in the beam.
It is simply constant in time, apart from an unobservable phase.
(Only relative phases are observable.)

A sun ray shining on a sunny day through a small
(but not too small) hole in a dark room
with a specially colored window (acting as a chromatic filter),
filled with dusty air, gives an intuitive picture of such a beam.

The dusty air only serves to make the beam visible.
If one clears the air, one can only see the hole
and the colored spot on the wall.

From a knowledge of geometric optics and the location
of hole and spot, one can reconstruct the direction of the beam
by drawing a line between the hole and the spot.
The beam is still there but cannot be seen.

But is it really there?

According to the Copenhagen interpretation,
apparently not, since it is not observed.

To observe the beam, one needs the dust,
and a person in the room to notice the reflected light.

"No elementary phenomenon is a phenomenon
until it is an observed phenomenon" (Wheeler)
is a frequently repeated quantum credo.

But light had been studied centuries before the
Copenhagen interpretation.

No one had had any difficulty with the idea that the beam
is objectively there, with the definite properties studied
in classical optics, namely the complex amplitude
or the two real quadratures of the beam.

These properties could always be reproducably tested by putting
objects (mirrors, filters, prisms, etc.) into the path of the beam
and observing the results of the interaction.

Although this of course destroys the observed part of the beam
(which is analogous to an alleged 'collapse of the wave function'),
it had never been interpreted as a proof that the beam
has objective properties only when we observe it,
or as an indicator that the state of the beam is
not objectively determined (though initially unknown).

4. The position of quantum objects

The somewhat spooky view of the Copenhagen interpretation
stems basically from Heisenberg's uncertainty relation.

This relation implies that it is impossible
to assign consistently a sharp position
and a sharp momentum to a quantum object.

To account for this, the Copenhagen interpretation
simply denies quantum objects to have definite properties
– except in the special case where the state
is an eigenstate of the corresponding operator.

This may have been reasonable in the old days
where particles could only be observed collectively.

But nowadays there are many experiments
done routinely with single quantum objects.

Quantum mechanics is very successfully applied to the analysis of single quantum objects, whether they are electrons in a quantum dot, atoms in an ion trap, or big objects like a buckyball or the sun.

On the other hand, the interpretation of quantum mechanics
is conventionally based on the assumption that
quantum measurements are done only on an
ensemble of similarly prepared systems. Strictly speaking,
the traditional statistical interpretation even requires that
these similarly prepared systems are stochastically independent.

Thus the traditional foundations appear to be
inadequate for modern quantum mechanics.
There seems to be a discrepancy between the practice
of quantum mechanics and its traditional foundations.

We need a language for quantum mechanics that allows us
to speak of the position (and other properties)
of single quantum objects at least in an approximate way.

There is no doubt that to some limited accuracy,
quantum objects have a definite position – they are located
under this atom microscope, in that quantum dot, etc.,
and this implies knowledge of an approximate position.

It also implies knowledge of an approximate momentum
– they are roughly at rest in the laboratory frame.

In which sense is this compatible with a quantum description?

Clearly, the uncertainty relation is not violated.

In some sense, this approximate position and momentum
are a mean position and a mean momentum.

But the mean of what?

Traditional quantum mechanics does not answer this.
But it provides formulas for the **computation**
of the mean position $\langle q \rangle$ and the mean momentum $\langle p \rangle$
of each quantum object which can be prepared
as an **individual** ...

... provided that one assigns a state to each individual object.

Those strictly adhering to a statistical interpretation
may find this a forbidden use.

But how else shall we encode into quantum mechanics the
knowledge that, at a particular time, a particular object
is at a particular place in the experimental setup?