Introduction to coherent spaces

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arXiv:1804.01402

April 4, 2018

Abstract. The notion of a coherent space is a nonlinear version of the notion of a complex Euclidean space: The vector space axioms are dropped while the notion of inner product is kept.

Coherent spaces provide a setting for the study of geometry in a different direction than traditional metric, topological, and differential geometry. Just as it pays to study the properties of manifolds independently of their embedding into a Euclidean space, so it appears fruitful to study the properties of coherent spaces independent of their embedding into a Hilbert space.

Coherent spaces have close relations to reproducing kernel Hilbert spaces, Fock spaces, and unitary group representations, and to many other fields of mathematics, statistics, and physics.

This paper is the first of a series of papers and defines concepts and basic theorems about coherent spaces, associated vector spaces, and their topology. Later papers in the series discuss symmetries of coherent spaces, relations to homogeneous spaces, the theory of group representations, C^*-algebras, hypergroups, finite geometry, and applications to quantum physics. While the applications to quantum physics were the main motiviation for developing the theory, many more applications exist in complex analysis, group theory, probability theory, statistics, physics, and engineering.

For the discussion of questions concerning coherent spaces, please use the discussion forum `https://www.physicsoverflow.org`.

MSC2010 classification: 46E22 (primary), 46C50, 43A35

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1 Introduction

In this paper, the first one of a series of papers on coherent spaces, the length and angular properties of vectors in a Euclidean space (embodied in the inner product) are generalized in a similar way as, in the past, metric properties of Euclidean spaces were generalized to metric spaces, differential properties of Euclidean spaces were generalized to manifolds, and topological properties of Euclidean spaces were generalized to topological spaces.

The notion of a coherent space is a nonlinear version of the notion of a complex Euclidean space: The vector space axioms are dropped while the inner product – now called a coherent product – is kept. Thus coherent spaces provide a setting for the study of geometry in a different direction than traditional metric, topological, and differential geometry. Just as it pays to study the properties of manifolds independently of their embedding into a Euclidean space, so it appears fruitful to study the properties of coherent spaces independent of their embedding into a Hilbert space.

However, coherent spaces (in the particular form of coherent manifolds) may also be viewed as a new, geometric way of working with concrete Hilbert spaces in which they are embeddable. In place of measures and integration dominating traditional techniques based on Hilbert spaces of functions, differentiation turns out to be the basic tool for evaluating inner products and matrix elements of linear operators – though this will become apparent only in later papers of the present series, beginning with NEUMAIER & GHAANI FARASHAHI [58]. This makes many calculations easy that are difficult in Hilbert spaces whose inner product is defined through a measure.

One of the strengths of the coherent space approach is that it makes many different things look alike. Coherent spaces have close relations to many important fields of mathematics, statistics, physics, and engineering. A theory of coherent spaces will provide a unified geometric view of these applications. Coherent spaces combine the rich, often highly characteristic variety of symmetries of traditional geometric structures with the computational tractability of traditional tools from numerical analysis and statistics.

Coherent spaces give a natural geometric setting to the concept of coherent states. In particular, the compact symmetric spaces (and many noncompact ones) appear naturally as coherent spaces when equipped with a coherent product derived from the coherent states on semisimple Lie groups (cf. PERELOMOV [67]). In these cases, the coherent product is naturally related to the differential, metric, symplectic, and Kähler structure of the associated symmetric spaces (cf. ZHANG et al. [81, Sections IIIC1 and VI], and Subsection 5.5 below).

Certain coherent spaces are closely related to quantum field theory (BAEZ et al. [12], GLIMM & JAFFE [31]) and the theory of Hida distributions in the white noise calculus for classical stochastic processes (HIDA & SI [34], HIDA & STREIT [35], OBATA [64]).

As we shall see in this paper, coherent spaces abstract the essential geometric properties needed to define a reproducing kernel Hilbert space. Examples of reproducing kernels (i.e., what in the present context are coherent products) were first discussed by ZAREMBA [80] in the context of boundary value problems and by MERCER [46] in the context of integral equations. The theory was systematically developed by Aronszajn [10, 11], Krein [43, 44], and others. For a history see Berg et al. [16] and Stewart [76].

Coherent spaces and reproducing kernel Hilbert spaces are mathematically almost equivalent concepts, and there is a vast literature related to the latter. Most relevant for the present sequence of papers are the books by Perelomov [67], Neeb [48], and Neretin [49]. However, the emphasis in these books is quite different from the present exposition, as they are primarily interested in properties of the associated function spaces and group representations, while we are primarily interested in the geometry and symmetry properties and in computational tractability.

Of particular importance is the use of reproducing kernels in complex analysis (see, e.g., Faraut & Korányi, [25], Upmeier [78], and de Branges [22]) and group theory (see, e.g., Neeb [48]), where they are the basis of many important theorems.

One of the important uses of coherent spaces is that many Euclidean spaces are described most simply and naturally in terms of a nice, small subset of coherent states, and all their properties can be investigated in terms of the associated coherent space. Already Glauber [30, p.2771], who coined the notion of a coherent state, mentioned that "The scalar product may, in fact, be calculated more simply than by using wave functions", and the same can be said for almost everything one wants to calculate in the applications of coherent states.

In particular, while the study of most problems in traditional function spaces for applications rely heavily on measures and integration, the quantum spaces of coherent spaces with an easily computable coherent product can be studied efficiently in terms of the explicit coherent product and differentiation only. This will be substantiated in Neumaier & Farashahi [59] and other papers of this series.

Coherent states are most often discussed as being parameterized by points on a connected manifold. But the concept of a coherent space also makes sense in a nontrivial way for finite spaces. There are strong relations between finite coherent spaces, finite metric spaces, graphs, and combinatorial designs. See Bekka & de la Harpe [14], Brouwer et al. [20], Godsil [32], Neumaier [50, 51, 52, 53]. This shows that the concept of coherent spaces provides a nontrivial extension of the theory of coherent states, in this respect similar to that of the measure-free coherent states of Horzela & Szafraniec [37].

Reproducing kernel Hilbert spaces and the associated coherent states also have applications in many other fields of mathematics (see, e.g., [2, 3, 4, 5, 6, 7, 16, 21, 41, 66, 67, 69, 70]), statistics and stochastic processes (see, e.g., [18, 34, 35, 64, 66, 68, 70]), physics (see, e.g., [2, 6, 7, 21, 29, 38, 41, 67]), and engineering (see, e.g., [4, 29].

In particular, there are relations to
(i) Christoffel–Darboux kernels for orthogonal polynomials,
(ii) Euclidean representations of finite geometries,
(iii) zonal spherical functions on symmetric spaces,
(iv) coherent states for Lie groups acting on homogeneous spaces,
(v) unitary representations of groups,
(vi) abstract harmonic analysis,

(vii) states of C^*-algebras in functional analysis,
(viii) reproducing kernel Hilbert spaces in complex analysis,
(ix) Pick–Nevanlinna interpolation theory,
(x) transfer functions in control theory,
(xi) positive definite kernels for radial basis functions,
(xii) positive definite kernels in data mining,
(xiii) positive definite functions in probability theory,
(xvi) exponential families in probability theory and statistics,
(xv) the theory of random matrices,
(xvi) Hida distributions for white noise analysis,
(xvii) Kähler manifolds and geometric quantization,
(xviii) coherent states in quantum mechanics,
(xix) squeezed states in quantum optics,
(xx) inverse scattering in quantum mechanics,
(xxi) Hartree–Fock equations in quantum chemistry,
(xxii) mean field calculations in statistical mechanics,
(xxiii) path integrals in quantum mechanics,
(xxiv) functional integrals in quantum field theory,
(xxv) integrable quantum systems.

These relations will be established in a series of papers of which the present one is the first, laying the foundations. The web site [54] will display at any time the most recent state of affairs. Other, in some draft form already existing, papers of this series will discuss

• the quantization of coherent spaces with a compatible manifold (or stratification) structure (NEUMAIER & GHAANI FARASHAHI [57, 58, 59]) and related quantum dynamics,

• relations between coherent spaces and C^*-algebras (NEUMAIER & GHAANI FARASHAHI [61]),

• finite coherent spaces related to finite geometries (NEUMAIER [55]) and hypergroups (NEUMAIER & GHAANI FARASHAHI [62]),and

• the classical limit in coherent spaces (NEUMAIER [56]) and related semiclassical expansions.

In this paper we only treat the most basic aspects of coherent spaces, starting from first principles. We define the concept, give some basic examples, and define the corresponding quantum spaces. We then discuss functions of positive type and the construction of the quantum space, and give a long list of constructions of such functions, useful for constructing new coherent spaces from old ones. More advanced topics and applications will be given in later papers of the series. Due to the introductory character of this paper all proofs are carried out in detail.

To illustrate some of the connections to physics and complex analysis, we give in Subsection 3.3 a long but still very incomplete list of examples of coherent spaces. Some of these examples are very elementary and can be understood informally before reading the systematic exposition of the theory. Many more coherent spaces can be constructed by modifying given ones using the recipes from Subsection 2.3.

One of the most important concepts for coherent spaces is that of their symmetries. Indeed, most of the applications of coherent spaces in quantum mechanics and quantum field theory rely on the presence of a large symmetry group. In Neumaier & Ghaani Farashahi [57], the second paper of this series, we define the notion of a coherent map of a coherent space, which will be exploited in depth in later papers of this series. Invertible coherent maps form the symmetry group of the coherent space, while all coherent maps often form a bigger semigroup. In many concrete examples, these are related to so-called Olshanski semigroups (Ol'shanskii [65]). It is shown that one obtains a vast generalization of the theory of normally ordered operator expressions in Fock spaces.

In Neumaier & Ghaani Farashahi [58, 59] we define coherent manifolds and coherent vector fields, the infinitesimal analogue of coherent maps, and their quantization. In Neumaier & Ghaani Farashahi [60], coherent spaces with an additional compatible involution structure are introduced and shown to lead to a natural generalization of the geometric quantization of Kähler manifolds.

Acknowledgments. Thanks to Arash Ghaani Farashahi, Waltraud Huyer, Rahel Knöpfel, David Bar Moshe, Hermann Schichl and Eric Wofsey for useful discussions related to the subject.

2 Euclidean spaces

There is a notational discrepancy in how mathematicians and physicists treat Hilbert spaces. In physics, one often works with finite-dimensional Hilbert spaces treated as $\mathbb{C}^n$ and hence wants to write the Hermitian inner product as $\langle x, y\rangle = x^*y$. This definition dictates the use of a Hermitian inner product that is antilinear in the first argument. This is also the choice adopted in Dirac's bra-ket notation whose usage in quantum mechanics is very widespread.

The notation in this paper was chosen to extend the traditional notation of standard finite-dimensional matrix algebra as closely as possible to arbitrary complex inner product spaces and associated linear operators. In matrix algebra, column vectors and the corresponding matrices with one column are identical objects, row vectors are the linear functionals, and the adjoint is the conjugate transpose. For example, $\mathbb{H} = \mathbb{H}^\times = \mathbb{C}^n$ is the space of column vectors of size n, the dual space $\mathbb{H}^*$ is the space of row vectors of size n, and the operator product $\phi^*\psi$ of a row vector ϕ^* and a column vector ψ is the standard Hermitian inner product of the column vectors ϕ and ψ. We use Greek lower case letters to write vectors, thus emphasizing their intended use as quantum state vectors in quantum mechanics.

On the other hand, mathematicians working on reproducing kernel Hilbert spaces use an inner product (x, y) antilinear in the second argument, related to the physicist's inner product $\langle x, y\rangle$ by $(x, y) = \overline{\langle x, y\rangle}$. Although the two ways of defining the inner product lead to fully equivalent theories, all details look a bit different, a fact that has to be taken into account when reading the literature on the subject. For example, in the description based on the physical tradition it is preferable to work with the antidual space in place of the dual space used in the mathematical tradition.

To avoid possible confusion caused by the different traditions we give in Subsection 2.1 a self-contained introduction to complex inner product spaces (here called Euclidean spaces) and their associate PIP spaces and Hilbert spaces. Moreover, in the final Section 5, we give statements of the relevant main results from the vast literature on functions of positive type, rewritten in the present notation and with detailed proofs.

2.1 Euclidean spaces and their associated PIP space

A **Euclidean space** is a complex vector space $\mathbb{H}$ with a binary operation that assigns to $\phi, \psi \in \mathbb{H}$ the **Hermitian inner product** $\langle \phi, \psi \rangle \in \mathbb{C}$, antilinear in the first and linear in the second argument, such that

$$\overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle, \tag{1}$$

$$\langle \psi, \psi \rangle > 0 \quad \text{for all } \psi \in \mathbb{H} \setminus \{0\}. \tag{2}$$

Here $\alpha > 0$ says that the complex number α is real and positive.

Since every Euclidean space can be completed to a Hilbert space (cf. Theorem 2.6 below), the Euclidean spaces are in fact just the subspaces of Hilbert spaces, with the induced inner product. However, it is of interest to develop the theory of Euclidean spaces independently since some additional topological structure is present that has no simple counterpart in the Hilbert space setting.

We give $\mathbb{H}$ and its subspaces (in particular $\mathbb{H}$) the structure of a locally convex space with the **weak topology** induced by the family of seminorms $|\phi|_\psi := |\phi(\psi)|$ with $\psi \in \mathbb{H}$. Thus $U \subseteq \mathbb{H}$ is a neighborhood of $\phi \in \mathbb{H}$ iff there are finitely many $\psi_k \in \mathbb{H}$ such that U contains all $\phi' \in \mathbb{H}$ with $|\phi'(\psi_k) - \phi(\psi_k)| \leq 1$ for all k. (Note that the 1 can be replaced by any positive constant since the ψ_k can be arbitrarily scaled.) As a consequence, $\phi_\ell \in \mathbb{H}$ **converges** to $\phi \in \mathbb{H}$ in the weak topology iff $\phi_\ell(\psi) \to \phi(\psi)$ for all $\psi \in \mathbb{H}$. We then define $\mathbb{H}^\times$ as the **antidual** of $\mathbb{H}$ to be the vector space of all (weakly) continuous antilinear functionals $\phi : \mathbb{H} \to \mathbb{C}$ satisfying

$$|\phi(\psi)| \leq \sum_{j=1}^{d} |\langle \varphi_j, \psi \rangle| \quad \text{for all } \psi \in \mathbb{H}, \tag{3}$$

for a suitable finite sequence of vectors $\varphi_1, ..., \varphi_d \in \mathbb{H}$. Functionals with this property are called (weakly) **continuous**. Because of (2), we may identify $\psi \in \mathbb{H}$ with the antilinear functional on $\mathbb{H}$ defined by

$$\psi(\phi) := \langle \phi, \psi \rangle \quad \text{for } \phi \in \mathbb{H}.$$

This definition turns $\mathbb{H}$ canonically into a subspace of $\mathbb{H}^\times$.

We may turn the antidual $\mathbb{H}^\times$ into a locally convex space with the **weak-* topology** induced by the family of seminorms $|\phi|_\psi := |\phi(\psi)|$ with $\psi \in \mathbb{H}$ and $\phi \in \mathbb{H}^\times$. It is not difficult to check that the restriction of the weak-* topology to $\mathbb{H}$ is equivalent with the weak topology of $\mathbb{H}$. From now on, the terms **continuous** and **limit** shall mean weakly continuous and weak-* limit, respectively, unless we explicitly mention continuity or limit with respect to another topology.

By definition, the inner product on $\mathbb{H}$ is continuous (in the weak topology). We call two vectors $\phi, \psi \in \mathbb{H}^\times$ **compatible** if

$$\phi^*\psi := \lim_\ell \langle \phi_\ell, \psi_\ell \rangle \tag{4}$$

exists for some pair of nets[1] ϕ_ℓ, ψ_ℓ of vectors in $\mathbb{H}$ indexed by the same directed set converging to ϕ and ψ, respectively, and is independent of the choice of these nets. Clearly, the compatibility relation is symmetric.

2.1 Proposition.
(i) Every $\psi \in \mathbb{H}^\times$ is the limit of a net of vectors from $\mathbb{H}$.

(ii) Every vector $\psi \in \mathbb{H}$ is compatible with every vector from $\phi \in \mathbb{H}^\times$, and we have

$$\phi(\psi) = \psi^*\phi,$$

In particular,

$$\langle \psi, \phi \rangle = \psi^*\phi \quad \textit{for } \phi, \psi \in \mathbb{H}.$$

(iii) If ϕ, ψ are compatible and ϕ, ψ' are compatible then ϕ is compatible with $\lambda\psi + \lambda'\psi'$ for all $\lambda, \lambda' \in \mathbb{C}$.

(iv) Each continuous antilinear functional on $\mathbb{H}$ has a unique extension to a continuous antilinear functional on $\mathbb{H}^\times$.

Proof. (i) For any finite-dimensional subspace V of $\mathbb{H}$ there is a unique $\psi_V \in V$ such that $\psi(\phi) = \langle \phi, \psi_V \rangle$ for all $\phi \in V$. The collection of finite-dimensional subspaces form a directed set under inclusion, hence the ψ_V form a net. The net converges to ψ since for all $\phi \in \mathbb{H}$,

$$(\psi - \psi_V)(\phi) = \langle \phi, \psi - \psi_V \rangle = \langle \phi, \psi \rangle - \langle \phi, \psi_V \rangle \to 0.$$

(ii) Let $\psi \in \mathbb{H}$ and $\phi \in \mathbb{H}^\times$. By part (i), there exists a net ϕ_ℓ converging to ϕ. Also, let $\psi_\ell := \psi$ be the constant net converging to ψ. Then

$$\phi(\psi) = \lim_\ell \phi_\ell(\psi) = \lim_\ell \phi_\ell(\psi_\ell) = \lim_\ell \langle \psi_\ell, \phi_\ell \rangle.$$

Now let $\phi_\ell, \psi_\ell \in \mathbb{H}$ be arbitrary nets converging to ϕ and ψ, respectively. Using the continuity of ϕ we find

$$\phi(\psi) = \lim_k \phi(\psi_k),$$

hence

$$\lim_k \lim_j \langle \psi_k, \phi_\ell \rangle = \lim_k \lim_j \phi_j(\psi_k) = \lim_k \phi(\psi_k) = \phi(\psi).$$

[1] All limits are formulated in terms of nets indexed by a directed set rather than sequences indexed by nonnegative integers, to cover the possibility of nonseparable quantum spaces. If a Hilbert space is separable, net convergence and sequence convergence are equivalent. If not, there is a difference and nets are needed to obtain the correct topology.
For those not familiar with nets – they are generalizations of sequences defining the appropriate form of the limit in the nonseparable case. In the separable case, nets can always be replaced by sequences. Thus readers will grasp the main content if, on first reading, they simply think of nets as being sequences.

Restriction to the diagonal subnet of the product net now gives $\lim_{\ell}\langle\psi_\ell,\phi_\ell\rangle = \phi(\psi)$. This shows that ψ is compatible with ϕ and $\psi^*\phi = \phi(\psi)$.

(iii) Suppose that $\{\psi_i\}_{i\in I}$ converges to ψ and $\{\psi'_j\}_{j\in J}$ converges to ψ'. Then, $I \times J$ is also a direct set and $\varphi_{(i,j)} := \lambda\psi_i + \lambda'\psi'_j$ is a net in $\mathbb{H}$ that converges to $\lambda\psi + \lambda'\psi'$.

(iv) By (i) $\mathbb{H}$ is dense in $\mathbb{H}^\times$. Hence, each continuous antilinear functional on $\mathbb{H}$ has a unique extension to a continuous antilinear functional on $\mathbb{H}^\times$. □

(4) defines a partial binary operation * on $\mathbb{H}^\times$ satisfying

$$\overline{\phi^*\psi} = \psi^*\phi. \tag{5}$$

We then define the **adjoint** ψ^* of $\psi \in \mathbb{H}^\times$ to be the linear functional on $\mathbb{H}$ that maps $\phi \in \mathbb{H}$ to $\psi^*\phi$. This gives * a second meaning notationally compatible with the first one.

The operation * is a Hermitian **partial inner product** (**PIP**) on $\mathbb{H}^\times$. Unless $\mathbb{H}$ is already a Hilbert space, the partial inner product is not everywhere defined. (For example, in the antidual of the Schwartz space over $\mathbb{R}$, the inner product of two delta functions at the same point is not defined.) Proposition 2.1(i) implies that $\mathbb{H}$ is dense in $\mathbb{H}^\times$. In particular, $\mathbb{H}^\times$ is a **PIP space** in the sense of Antoine & Trapani [8].

2.2 Proposition.
*(i) Every continuous linear mapping $f : \mathbb{H} \to \mathbb{C}$ can be written in the form $f = \phi^*$ for some $\phi \in \mathbb{H}^\times$. Thus * is an antiisomorphism from $\mathbb{H}^\times$ to the space of all continuous linear functionals on $\mathbb{H}$, the **dual** of $\mathbb{H}$.*

(ii) For every continuous antilinear functional Ψ on $\mathbb{H}^\times$ there is a vector $\psi \in \mathbb{H}$ such that

$$\Psi(\phi) = \phi^*\psi \quad \textit{for all } \phi \in \mathbb{H}^\times. \tag{6}$$

(iii) If $\psi \in \mathbb{H}^\times$ is compatible with all vectors from $\mathbb{H}^\times$ then $\psi \in \mathbb{H}$.

Proof. (i) The mapping $\phi : \mathbb{H} \to \mathbb{C}$ defined by $\phi(\psi) := \overline{f\psi}$ is antilinear, hence $\phi \in \mathbb{H}^\times$. Since $f\psi = \overline{\phi(\psi)} = \overline{\psi^*\phi} = \phi^*\psi$ we conclude that $f = \phi^*$.

(ii) Fix $\varepsilon > 0$. The set U consisting of all $\phi \in \mathbb{H}^\times$ such that

$$|\Psi(\phi)| \le \varepsilon \tag{7}$$

is a neighborhood of 0, by the definition of continuity. By definition of the weak-* topology, this implies that there are finitely many $\psi_k \in \mathbb{H}$ such that U contains all $\phi \in \mathbb{H}^\times$ with $|\phi(\psi_k)| \le 1$ for all k. In particular, U contains all ϕ in the kernel of the linear mapping $X : \mathbb{H}^\times \to \mathbb{C}^n$ defined by

$$(X\phi)_k := \psi_k^*\phi = \phi(\psi_k).$$

Therefore (7) holds for any such ϕ. With ϕ, also $\delta^{-1}\varepsilon\phi$ belongs to the kernel for any $\delta > 0$, hence $|\Psi(\phi)| \le \delta$, and in the limit $\delta \to 0$ we find $\Psi(\phi) = 0$. Thus Ψ vanishes on the kernel of X. Therefore $X\phi = X\phi'$ implies $\Psi(\phi) = \Psi(\phi')$. This implies that $\Psi = f \circ X$ is the composition of some antilinear $f : \mathbb{C}^n \to \mathbb{C}$ with X, and we have $\Psi(\phi) = f(X\phi)$. Since the argument of f is finite-dimensional, we conclude that $f(x) = x^*u$ for some $u \in \mathbb{C}^n$. Therefore

$$\Psi(\phi) = (X\phi)^*u = \sum \overline{(X\phi)_k} u_k = \sum \overline{\psi_k^*\phi} u_k = \sum \phi^*\psi_k u_k = \phi^*\psi,$$

where $\psi := \sum u_k\psi_k \in \mathbb{H}$.

(iii) In this case, (6) defines a continuous antilinear functional Ψ on $\mathbb{H}^\times$, and (ii) shows that $\psi \in \mathbb{H}$. $\square$

If U and V are (complex) topological vector spaces we write $\mathrm{Lin}\,(U, V)$ for the vector space of all continuous linear mappings from U to V, and $\mathrm{Lin}\,U$ for $\mathrm{Lin}\,(U, U)$. We identify V with the space $\mathrm{Lin}\,(\mathbb{C}, V)$ via

$$\psi\alpha := \alpha\psi \quad \text{for } \alpha \in \mathbb{C},\ \psi \in V.$$

If U, V are Euclidean spaces and $A \in \mathrm{Lin}\,(U, V^\times)$,

$$(A^*\phi)(\psi) := (A\psi)^*\phi \quad \text{for } \phi \in V, \psi \in U$$

defines an antilinear functional $A^*\phi \in U^\times$.

The dependence on ϕ is linear; therefore this defines a linear operator $A^* \in \mathrm{Lin}\,(V, U^\times)$, called the **adjoint** of A. Clearly,

$$(A\psi)^*\phi = \psi^*(A^*\phi).$$

2.3 Proposition. *The mapping* * *that maps* A *to* A^* *is an antilinear mapping from* $\mathrm{Lin}\,(U, V^\times)$ *to* $\mathrm{Lin}\,(V, U^\times)$ *and satisfies*

$$A^{**} = A.$$

Proof. Proposition 2.2(ii) gives $V^{\times\times} = V$ and $U^{\times\times} = U$. Hence $A^* : V \to U^\times$ is given by $A^*\psi(\phi) = (A\phi)^*\psi$ for all $\psi \in V$ and $\phi \in U$. Thus we have for all $\phi \in U$ and $\psi \in V$,

$$A^{**}\phi(\psi) = (A^*\psi)^*\phi = A\phi(\psi),$$

which implies that $A^{**} = A$. $\square$

Since $V \subseteq V^\times$, the adjoint is also defined for $A \in \mathrm{Lin}\,(U, V)$ and then makes sense as a mapping $A^* \in \mathrm{Lin}\,(V^\times, U^\times)$, and we have

$$A^*B^* = (BA)^* \quad \text{if } A \in \mathrm{Lin}\,(U, V),\ B \in \mathrm{Lin}\,(V, W^\times).$$

We write

$$\mathrm{Lin}^\times \mathbb{H} := \mathrm{Lin}\,(\mathbb{H}, \mathbb{H}^\times)$$

for the vector space of continuous linear operators from a Euclidean space $\mathbb{H}$ to its antidual.

2.4 Proposition. *Let U and V be Euclidean spaces and let $A : U \to V^\times$ be a linear map. Then the following are equivalent:*

(i) $A \in \mathrm{Lin}\,(U, V^\times)$.

(ii) For every vector $\phi \in V$ there exists a finite subset S of vectors in U such that

$$|A\psi(\phi)| \leq \sum_{\chi \in S} |\chi^* \psi|, \quad \textit{for all } \psi \in U. \tag{8}$$

(iii) The linear map A^* that maps any vector $\phi \in V$ to the antlinear functional $A^*\phi : U \to \mathbb{C}$ which is given by $A^*\phi(\psi) := \overline{A\psi(\phi)}$ for all $\psi \in U$, belongs to $\mathrm{Lin}\,(V, U^\times)$.

Proof. (i)$\leftrightarrow$(ii) holds by definition of (weak) continuity for the linear map $A : U \to V^\times$ and the structure of the topology on U and $V^\times$. Indeed, let $A : U \to V^\times$ be linear and continuous. Then for any open set $O \subseteq V^\times$ containing 0, the set $A^{-1}(O)$ is open in U. For any $\phi \in V$, the set $O_\phi := \{\varphi \in V^\times : |\varphi(\phi)| \leq 1\}$ is open in $V^\times$. Hence $A^{-1}(O_\phi)$ is open in U. Thus there is a finite subset S of U such that

$$\{\psi \in U : |\chi^*\psi| \leq 1 \quad \text{for all } \chi \in S\} = A^{-1}(O_\phi).$$

Thus, if $\psi' \in U$ with $|\chi^*\psi'| \leq 1$ for all $\chi \in S$ then w$|A\psi'(\phi)| \leq 1$. Now let $\psi \in U$ be arbitrary. Thus

$$\psi' := \Big(\sum_{\chi \in S} |\chi^*\psi|\Big)^{-1} \psi \in A^{-1}(O_\phi).$$

Therefore, $|A\psi'(\phi)| \leq 1$. Since A is linear, we get

$$\Big(\sum_{\chi \in S} |\chi^*\psi|\Big)^{-1} |A\psi(\phi)| \leq 1,$$

thus we have

$$|A\psi(\phi)| \leq \sum_{\chi \in S} |\chi^*\psi|,$$

which implies (8). A similar argument works for the converse.

(ii)$\rightarrow$(iii): First, we shall show that $A^*\phi \in U^\times$ for all $\phi \in V$. To this end, let $\phi \in V$. Since $A : V \to U^\times$ satisfies (ii), there exists a finite subset S of U such that

$$|A^*\phi(\psi)| = |\overline{A\psi(\phi)}| = |A\psi(\phi)| \leq \sum_{\chi \in S} |\chi^*\psi|$$

for all $\psi \in U$. This implies that the antilinear map $A^*\phi : U \to \mathbb{C}$ is continuous, hence $A^*\phi \in U^\times$. The linearity of the map $A^* : V \to U^\times$ is obvious. Thus to deduce that $A^* \in \mathrm{Lin}\,(V, U^\times)$ it suffices to prove that $A^* : V \to U^\times$ is continuous. Suppose that $\phi \in U$. Since $A\phi : V \to \mathbb{C}$ is continuous, (3) implies that there is a finite subset R of V such that

$$|A\phi(\psi)| \leq \sum_{\chi \in R} |\chi^*\psi| \quad \text{for all } \psi \in V.$$

Thus by definition of A^*,

$$|A^*\psi(\phi)| = |A\phi(\psi)| \le \sum_{\chi\in S} |\chi^*\psi| \quad \text{for all } \psi \in V.$$

From (ii), applied with A^*, R in place of A, S, we conclude that $A^* \in \mathrm{Lin}\,(V, U^\times)$.

(iii)→(i): Suppose that $A^* \in \mathrm{Lin}\,(V, U^\times)$. Applying (ii)→(iii) with A^* in place of A gives $A^{**} \in \mathrm{Lin}\,(U, V^\times)$. Since $A = A^{**}$ by Proposition 2.3 we conclude that $A \in \mathrm{Lin}\,\mathrm{Lin}\,(U, V^\times)$. □

2.5 Corollary. *If $A \in \mathrm{Lin}^\times \mathbb{H}$ then $A^* \in \mathrm{Lin}^\times \mathbb{H}$ and we have*

$$\phi^*A\psi = (\phi^*A)\psi = \phi^*(A\psi) = (A^*\phi)^*\psi \quad \textit{for } \phi, \psi \in \mathbb{H}. \tag{9}$$

Thus $\phi^*A\psi$ defines a sesquilinear form on $\mathbb{H}$.

Here ϕ^* is treated as the adjoint $\phi^* : \mathbb{H}^\times \to \mathbb{C}$ of $\phi : \mathbb{C} \to \mathbb{H}$ under the identification $V = \mathrm{Lin}\,(\mathbb{C}, V)$. We call $A \in \mathrm{Lin}^\times \mathbb{H}$ **Hermitian** if $A^* = A$; then the associated sesquilinear form is Hermitian, too.

2.6 Theorem. *The set $\overline{\mathbb{H}}$ of all $\psi \in \mathbb{H}^\times$ compatible with themselves is a Hilbert space with **Euclidean norm** $\|\psi\|$ defined by*

$$\|\psi\| := \sqrt{\psi^*\psi}.$$

Any two vectors in $\overline{\mathbb{H}}$ are compatible, and we have

$$\mathbb{H} \subseteq \overline{\mathbb{H}} \subseteq \mathbb{H}^\times. \tag{10}$$

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Proof. (i) We first note that the **Cauchy–Schwarz inequality**

$$|\phi^*\psi| \le \|\phi\|\,\|\psi\| \tag{11}$$

and the **triangle inequality**

$$\|\phi - \psi\| \le \|\phi\| + \|\psi\| \tag{12}$$

for vectors from $\mathbb{H}$ are proved as usual.

(ii) If $\psi \in \overline{\mathbb{H}}$ then by the independence requirement in the definition of compatibility, all nets ψ_j converging to ψ satisfy $\psi^*\psi = \lim \psi_j^*\psi_j$ and hence $\|\psi_j\| \to \|\psi\|$. Conversely, this implies $\psi \in \overline{\mathbb{H}}$. This immediately shows that $\alpha\psi \in \overline{\mathbb{H}}$ whenever $\alpha \in \mathbb{C}$ and $\psi \in \overline{\mathbb{H}}$. Now suppose that $\phi, \psi \in \overline{\mathbb{H}}$ and that $\phi = \lim \phi_j$, $\psi = \lim \psi_j$ for nets with $\phi_j, \psi_j \in \mathbb{H}$. Then

$$|\phi_j^*\psi_j - \phi_k^*\psi_k| = |(\phi_j - \phi_k)^*\psi_j + \phi_k^*(\psi_j - \psi_k)| \le \|\phi_j - \phi_k\|\,\|\psi_j\| + \|\phi_k\|\,\|\psi_j - \psi_k\|$$

converges to zero as $j,k \to \infty$. Therefore the $\phi_j^*\psi_j$ form a Cauchy net and the limit (4) exists. A similar argument shows that the limit is independent of the choice of the nets. Therefore any two vectors in $\overline{\overline{\mathbb{H}}}$ are compatible.

(iii) This implies that the sum of vectors $\phi, \psi \in \overline{\overline{\mathbb{H}}}$ also belongs to $\overline{\overline{\mathbb{H}}}$. Indeed, we may represent ϕ, ψ as limits $\phi = \lim \phi_j$, $\psi = \lim \psi_j$ and find

$$\begin{aligned} \lim_(\phi + \psi)_j^*(\phi + \psi)_j &= \lim(\phi_j^*\phi_j) + \lim_j(\phi_j^*\psi_j) + \lim(\psi_j^*\phi_j) + \lim_j(\psi_j^*\psi_j) \\ &= \phi^*\phi + \phi^*\psi + \psi^*\phi + \psi^*\psi = (\phi+\psi)^*(\phi+\psi), \end{aligned}$$

Thus $\overline{\overline{\mathbb{H}}}$ is a vector space with a Hermitian inner product. Its definiteness follows by taking limits. Thus $\overline{\overline{\mathbb{H}}}$ is a Euclidean space. In particular, the Cauchy–Schwarz inequality and the triangle inequality are valid for vectors from $\overline{\overline{\mathbb{H}}}$.

(iv) It remains to show completeness. Let ϕ_ℓ be a Cauchy net in $\overline{\overline{\mathbb{H}}}$. Then for every $\psi \in \mathbb{H}$,

$$|\phi_\ell^*\psi - \phi_k^*\psi| = |(\phi_\ell - \phi_k)^*\psi| \leq \|\phi_\ell - \phi_k\| \, \|\psi\| \to 0 \quad \text{for } k, \ell \to \infty,$$

hence the $\phi_\ell^*\psi$ form a Cauchy net in $\mathbb{C}$ and converge. Thus

$$f\psi := \lim_{\ell \to \infty} \phi_\ell^*\psi$$

defines a map $f : \mathbb{H} \to \mathbb{C}$. Since for $\mu, \mu' \in \mathbb{C}$ and $\psi, \psi' \in \mathbb{H}$,

$$f(\mu\psi + \mu'\psi') - \mu f\psi - \mu' f\psi' = \lim_{\ell \to \infty} \Big(\phi_\ell^*(\mu\psi + \mu'\psi') - \mu\phi_\ell^*\psi - \mu'\phi_\ell^*\psi'\Big) = 0,$$

f is linear, and by Proposition 2.2(i), $f = \phi^*$ for some $\phi \in \mathbb{H}^\times$. Clearly $\phi_\ell \to \phi$. Now if also $\psi_\ell \to \psi$ then

$$\phi_\ell^*\psi_\ell - \phi_k^*\psi_k = (\phi_\ell - \phi_k)^*\psi_\ell - \phi_k^*(\psi_k - \psi_\ell) \to 0 \quad \text{for } k, \ell \to \infty,$$

hence the $\phi_\ell^*\psi_\ell$ form a Cauchy net in $\mathbb{C}$ and converge. By interlacing several nets converging to ϕ it is seen that the limit is independent of the net chosen, proving that ϕ is compatible with itself. Thus $\phi \in \overline{\overline{\mathbb{H}}}$. Therefore $\overline{\overline{\mathbb{H}}}$ is complete, hence a Hilbert space. (10) is obvious. $\square$

We call $\overline{\overline{\mathbb{H}}}$ the **completion** of $\mathbb{H}$. It is a subspace of the antidual $\mathbb{H}^\times$. If $\mathbb{H}$ is finite-dimensional then $\mathbb{H} = \overline{\overline{\mathbb{H}}} = \mathbb{H}^\times$ by standard arguments. If $\mathbb{H}$ is infinite-dimensional then $\mathbb{H} = \overline{\overline{\mathbb{H}}} = \mathbb{H}^\times$ iff $\mathbb{H}$ is already a Hilbert space, and $\mathbb{H} \neq \overline{\overline{\mathbb{H}}} \neq \mathbb{H}^\times$ otherwise. For example, the space $\mathbb{H} := C([-1,1])$ of continuous functions on $[-1,1]$ has as additional antilinear functionals not only all elements of the Hilbert space $\overline{\overline{\mathbb{H}}} = L^2([-1,1])$ of square integrable functions on $[-1,1]$ but also all function evaluation maps, corresponding to distributions. All these are elements of the antidual $\mathbb{H}^\times$.

In general, the Hilbert space topology in $\overline{\overline{\mathbb{H}}}$ is typically weaker than the topology in $\mathbb{H}$ but stronger than the topology in $\mathbb{H}^\times$. E.g., a Cauchy net in $\mathbb{H}$ has a weak limit in $\overline{\overline{\mathbb{H}}}$; but it converges in $\mathbb{H}$ only if the weak limit is in $\mathbb{H}$.

2.7 Theorem. ***(Riesz representation theorem)***
(i) $\psi \in \mathbb{H}^\times$ is in $\overline{\mathbb{H}}$ iff ψ is compatible with all $\phi \in \overline{\mathbb{H}}$.

(ii) For every continuous antilinear functional Ψ on $\overline{\mathbb{H}}$ there is a vector $\psi \in \overline{\mathbb{H}}$ such that

$$\Psi(\phi) = \phi^*\psi \quad \textit{for all } \phi \in \overline{\mathbb{H}}. \tag{13}$$

Proof. (i) $\Rightarrow$: By the above if $\psi \in \overline{\mathbb{H}}$ there is no $\phi \in \overline{\mathbb{H}}$ that is not compatible with it. $\Leftarrow$: By definition of $\overline{\mathbb{H}}$, if ψ is compatible with all $\phi \in \overline{\mathbb{H}}$ it is in $\overline{\mathbb{H}}$, too.

(ii) Let $\Psi : \overline{\mathbb{H}} \to \mathbb{C}$ be a continuous antilinear functional. By Proposition 2.1(iv), the antilinear functional Ψ has a unique extension to an antilinear functional $\widetilde{\Psi} : \mathbb{H}^\times \to \mathbb{C}$. Now Proposition 2.2(ii) implies that there exists a unique vector $\psi \in \mathbb{H}$ such that $\widetilde{\Psi}(\phi) = \phi^*\psi$, for all $\phi \in \mathbb{H}^\times$. Therefore $\Psi(\phi) = \phi^*\psi$ for all $\phi \in \overline{\mathbb{H}}$. $\square$

An **isometry** from a Euclidean space U to a Euclidean space V is a linear map $A \in \mathrm{Lin}\,(U, V)$ such that

$$\|A\psi\| = \|\psi\| \quad \text{for all } \psi \in U.$$

Isometries are injective since $A\psi = 0$ implies $\|\psi\| = 0$ and hence $\psi = 0$. An **isomorphism** from U to V is a surjective isometry $A : U \to V$. Its inverse is an isomorphism from V to U. if such an isomorphism exists the Euclidean spaces U and V are called **isometric** or **isomorphic**.

2.2 Functions of positive type

A complex $n \times n$ matrix G is **Hermitian** if $\overline{G}_{jk} = G_{kj}$ for $j, k = 1, \dots, n$, **positive semidefinite** if $u^*Gu \geq 0$ for all $u \in \mathbb{C}^n$, and **conditionally semidefinite** if $u^*Gu \geq 0$ for all $u \in \mathbb{C}^n$ with $\displaystyle\sum_k u_k = 0$.

Let Z be a nonempty set. We call a function $F : Z \times Z \to \mathbb{C}$ **of positive type** (resp. **conditionally positive**) over Z if, for every finite sequence $z_1, \dots, z_n$ in Z, the **Gram matrix** of $z_1, \dots, z_n$, i.e., the $n \times n$-matrix G with entries

$$G_{jk} = F(z_j, z_k), \tag{14}$$

is Hermitian and positive semidefinite (resp. conditionally semidefinite). In particular, every function of positive type is conditionally positive.

The basic intuition for the above definition comes from the following examples. (Note that z and z' are unrelated points.)

2.8 Proposition. *Let Z be a subset of a Euclidean space $\mathbb{H}$. Then the functions $F, F', F'' : Z \times Z \to \mathbb{C}$ defined by*

$$F(z, z') := z^*z', \quad F'(z, z') := z'^*z, \quad F''(z, z') := \mathrm{Re}\, z^*z'$$

are of positive type.

Proof. Let G, G', G'' be the Gram matrices computed with F, F', F'', respectively. Clearly, G is Hermitian; it is positive semidefinite since

$$u^*Gu = \sum_{j,k} \overline{u}_j z_j^* z_k u_k = \Big\| \sum_k z_k u_k \Big\|^2 \geq 0.$$

$G' = \overline{G}$ and $G'' = \frac{1}{2}(G + \overline{G})$ are easily seen to be Hermitian and positive semidefinite, too. $\square$

The Moore–Aronszejn theorem (Theorem 5.1 below) provides a converse of Proposition 2.8.

2.9 Proposition. *If $F : Z \times Z \to \mathbb{C}$ is conditionally positive then, for any function $f : Z \to \mathbb{C}$ and any $\gamma \geq 0$, the function $\widetilde{F} : Z \times Z \to \mathbb{C}$ defined by*

$$\widetilde{F}(z, z') := \overline{f(z)} + f(z') + \gamma F(z, z') \quad \text{for } z, z' \in Z \tag{15}$$

is conditionally positive.

Proof. Let $G, \widetilde{G}$ be the Gram matrices computed with F and $\widetilde{F}$, respectively. Clearly, $\widetilde{G}$ is Hermitian, and

$$\widetilde{G}_{jk} = \overline{f(z_j)} + f(z_k) + \gamma G_{jk},$$

hence $\sum_\ell u_\ell = 0$ implies

$$u^*\widetilde{G}u = \sum_{j,k} \overline{u}_j(\overline{f(z_j)} + f(z_k) + \gamma G_{jk})u_k = \gamma \sum_{j,k} \overline{u}_j G_{jk} u_k = \gamma u^*Gu \geq 0.$$

Thus $\widetilde{G}$ is conditionally semidefinite. $\square$

2.10 Proposition. *Let Z be a subset of a Euclidean space $\mathbb{H}$. Then for any function $g : Z \to \mathbb{C}$, the function $\widetilde{F} : Z \times Z \to \mathbb{C}$ defined by*

$$\widetilde{F}(z, z') := \overline{g(z)} + g(z') - \|\overline{z} - z'\|^2 \quad \text{for } z, z' \in Z \tag{16}$$

is conditionally positive.

Proof. This follows from Propositions 2.8 and 2.9 since $\widetilde{F}(z, z') = \overline{f(z)} + f(z') + F''(z, z')$, where $f(z) = g(z) - \|z\|^2$. $\square$

For appropriate converses of Propositions 2.9 and 2.10 see the theorems bySchoenberg ansd Menger in Section 5.4 below.

2.3 Constructing functions of positive type

In this subsection we discuss a toolkit for the construction of such explicit functions of positive type from simpler ingredients. We provide a number of constructions that allow one to verify positivity properties. For further constructions and numerous examples in the form of exercises see Berg et al. [16].

2.11 Proposition. *For every family ϕ_z $(z \in Z)$ of vectors ϕ_z in a Euclidean vector space $\mathbb{H}$, the function F defined by*

$$F(z, z') := \langle \phi_z, \phi_{z'} \rangle$$

is of positive type.

Proof. The corresponding matrix G from (14) is clearly Hermitian, and

$$x^* G x = \sum_{j,k} \overline{x}_j G_{jk} x_k = \sum_{j,k} \overline{x}_j \langle \phi_{z_j}, \phi_{z_k} \rangle x_k = \Big| \sum_k x_k \phi_{z_k} \Big|^2 \geq 0.$$

□

Basic examples of functions of positive type arise from the above constructions by choosing the family of ϕ_z in such a way that their inner products can be expressed in closed form. Others come from a number of constructions which modify or combine functions of positive type.

2.12 Proposition.
(i) Every positive semidefinite Hermitian form on a complex vector space Z is of positive type.

(ii) If F is of positive type over Z and $Y \subseteq Z$ then the ***restriction*** *$F|_Y$ of F to $Y \times Y$ is of positive type.*

(iii) If F_0 is of positive type over Z_0 and $u : Z \to Z_0$ then

$$F(z, z') := F_0(u(z), u(z'))$$

is of positive type.

(iv) If F is of positive type over Z, $\gamma > 0$, and $\nu : Z \to \mathbb{C}$ then

$$F'(z, z') := \gamma \overline{\nu(z)} F(z, z') \nu(z')$$

is of positive type. In particular, if $F(z, z) > 0$ for all z then the ***normalization*** *$F_{\rm norm}$ of F, defined by*

$$F_{\rm norm}(z, z') := \frac{F(z, z')}{\sqrt{F(z, z) F(z', z')}},$$

is of positive type, and satisfies $F_{\rm norm}(z, z) = 1$ for all z.

(v) If L is a countable set and each F_ℓ ($\ell \in L$) is of positive type over Z then, for arbitrary positive weights w_ℓ for which

$$F(z,z') := \sum_{\ell\in L} w_\ell F_\ell(z,z')$$

is everywhere defined, F is of positive type.

(vi) Let Z be the disjoint union of a family of sets Z_ℓ indexed by $\ell \in L$. If $F_\ell : Z_\ell \times Z_\ell \to \mathbb{C}$ is of positive type for all $\ell \in L$ then the function $F : Z \times Z \to \mathbb{C}$ defined by

$$F(z,z') := \begin{cases} F_\ell(z,z') & \text{if } z,z' \in Z_\ell, \\ 0 & \text{otherwise,} \end{cases}$$

is of positive type.

(vii) If the F_ℓ ($\ell = 0,1,2\ldots$) are of positive type over Z and the limit

$$F(z,z') := \lim_{\ell\to\infty} F_\ell(z,z')$$

exists for $z,z' \in Z$ then F is of positive type.

(viii) If μ is a positive measure on a set L and each F_ℓ ($\ell \in L$) is of positive type over Z then

$$F(z,z') := \int_L d\mu(\ell) F_\ell(z,z'),$$

if everywhere defined, is of positive type.

(ix) Let $F_0 : Z_0 \times Z_0 \to \mathbb{C}$ be of positive type, let $d\mu$ be a positive measure on a set L. If $u : Z \times L \to Z_0$ is such that the integral

$$F(z,z')(\ell) := \int_L d\mu(\ell') F_0(u(z,\ell), u(z',\ell'))$$

exists for all $\ell \in L$ and $z,z' \in Z$ then F is of positive type on Z.

Proof. (i)–(viii) are straightforward, and (ix) follows from (iii) and (viii). □

Note that many examples of interest are analytic in the second argument. Unfortunately, this property does not persist under normalization as in Proposition 2.12(iv).

It is easily checked that all constructions of Proposition 2.12 produce conditionally positive functions when the ingredients are only required to be conditionally positive rather than of positive type.

2.13 Theorem. *(Schur [74])*
(i) If F_1 is of positive type on Z_1 and F_2 is of positive type on Z_2 then

$$F((z_1,z_2),(z_1',z_2')) := F_1(z_1,z_1')F_2(z_2,z_2')$$

is of positive type on $Z = Z_1 \times Z_2$.

(ii) If F_1 and F_2 are of positive type then the pointwise product

$$F(z,z') := F_1(z,z')F_2(z,z')$$

is of positive type.

Proof. (i) For $t = 1, 2$, the Gram matrix G_t of $z_{t1}, \dots, z_{tn}$ computed with respect to F_t is positive semidefinite, hence has a Cholesky factorization $G_t = R_t^* R_t$. The Gram matrix of $(z_{11}, z_{21}), \dots, (z_{11}, z_{21})$ computed with respect to F has entries

$$\begin{aligned} G_{jk} &= G_{1jk}G_{2jk} = \Big(\sum_\ell \overline{R_{1\ell j}} R_{1\ell k}\Big)\Big(\sum_m \overline{R_{2mj}} R_{2mk}\Big) \\ &= \sum_{\ell,m} \overline{R_{1\ell j}R_{2mj}} R_{1\ell k}R_{2mk}, \end{aligned}$$

so that

$$u^*Gu = \sum_{j,k} \overline{u}_j G_{jk} u_k = \sum_{\ell,m} \Big|\sum_j u_j R_{1\ell j} R_{2mj}\Big|^2 \geq 0.$$

Thus G is positive definite, proving that F is of positive type.

(ii) follows from (i) and Proposition 2.12(iii) by mapping to the diagonal. □

2.14 Theorem.
(i) All pointwise powers

$$F^n(z,z') := F(z,z')^n \quad (n = 1, 2, \dots)$$

of a function F of positive type are of positive type.

(ii) If F is of positive type then for any $\beta \geq 0$, the function F_β defined by

$$F_\beta(z,z') := e^{\beta F(z,z')}$$

is of positive type, too.

(iii) Write $B(0;1) := \{x \in \mathbb{C} \mid |x| < 1\}$ for the open complex unit disk. If F is of positive type and $|F(z,z')| < c < \infty$ for all $z, z' \in Z$ then

$$F_{\mathrm{inv}}(z,z') := \frac{1}{c - F(z,z')}$$

is of positive type, too. (This is related to Nevanlinna–Pick interpolation theory; cf. Agler *&* McCarthy *[1].)*

Proof. (i) follows from Theorem 2.13(ii) by induction. (ii) and (iii) then follow from Proposition 2.12(v) since $e^x = \sum_0^\infty \frac{x^n}{n!}$ for $x \in \mathbb{C}$ and $\frac{1}{c-x} = \sum_0^\infty \frac{x^n}{c^{n+1}}$ for $|x| < c$, and constant functions with positive values are of positive type. □

This theorem is related to the Berezin–Wallach set discussed in Section 5.5.

3 Coherent spaces and their quantum spaces

3.1 Coherent spaces

Let Z be a nonempty set. A **coherent product** on Z is a function $K : Z \times Z \to \mathbb{C}$ of positive type.[2] A **coherent space** is a nonempty set Z with a distinguished coherent product $K : Z \times Z \to \mathbb{C}$. We regard the same set with different coherent products as different coherent spaces.

3.1 Examples.
(i) Any subset Z of a Euclidean space is a coherent space with coherent product

$$K(z, z') := z^* z'.$$

(ii) Any subset Z' of a coherent space Z is again a coherent space, with the coherent product inherited from Z by restriction.

(iii) For practical applications, it is often important that the coherent products are given as explicit expressions $K(z, z')$ with which one can work analytically, or at least expressions which can be efficeintly approximated numerically. The easiest way to construct such expressions is by using one of the many constructions from Subsection 2.3.

Many interesting examples will appear in other papers on this subject, starting with Neumaier & Ghaani Farashahi [57]. We just give one particularly important example. As we shall see in [57], the corresponding quantum spaces are the Fock spaces upon which quantum field theory is based.

3.2 Example. Let V be a Euclidean space. In a notation where pairs are denoted by square brackets, we write

$$z := [z_0, \mathbf{z}] \in \mathbb{C} \times V.$$

for the elements of $Z = \mathbb{C} \times V$. Since

$$F(z, z') := \overline{z}_0 + z'_0 + \mathbf{z}^* \mathbf{z}' \tag{17}$$

is conditionally positive by Proposition 2.9. Hence Theorem 5.8 below implies that

$$K(z, z') := e^{\overline{z}_0 + z'_0 + \mathbf{z}^* \mathbf{z}'} \tag{18}$$

is a coherent product, with respect to which Z is a coherent space. We call this coherent space the **Klauder space** over V and denote it by $Kl[V]$. (For $V = \mathbb{C}$, the associated coherent states were first discussed in Klauder [39, p.1062].) We shall discuss Klauder spaces in more detail in Neumaier & Ghaani Farashahi [57].

[2] One obtains the more general concepts of **semicoherent products** and **semicoherent spaces** by weakening the requirement of having positive type to the requirement that the supremum $\mathrm{ns}(Z)$ of the number of negative eigenvalues of Gram matrices constructed from K is finite. Much of the subsequent theory remains valid, but the inner products need no longer be positive semidefinite and the quantum spaces discussed below become Pontryagin spaces with $\mathrm{ns}(Z)$ negative squares; cf. Alpay et al. [6]. In the present paper, this generalization is not considered further.

In particular, coherent spaces generalize Euclidean spaces, and the coherent product $K(z, z')$ generalizes the Hermitian inner product z^*z', but in general no linear structure is assumed on Z. This is similar to the way how metric spaces generalize the distance in Euclidean spaces without keeping their linear structure.

We draw some simple but useful general consequences. The Hermiticity of the Gram matrix of z, z' gives

$$\overline{K(z, z')} = K(z', z). \tag{19}$$

Since the diagonal elements of a Hermitian positive semidefinite matrix are real and nonnegative,

$$K(z, z) \geq 0 \quad \text{for all } z \in Z. \tag{20}$$

In particular, we may define the **length** of $z \in Z$ to be

$$n(z) := \sqrt{K(z, z)} \geq 0. \tag{21}$$

Since every principal submatrix of a Hermitian positive semidefinite matrix has real nonnegative determinants, the determinants of size 2 lead to

$$|K(z, z')|^2 \leq K(z, z)K(z', z'). \tag{22}$$

Taking square roots gives the **coherent Cauchy–Schwarz inequality**

$$|K(z, z')| \leq n(z)n(z'). \tag{23}$$

This allows us to define the **angle** between two points $z, z' \in Z$ of positive length by

$$\angle(z, z') := \arccos \frac{|K(z, z')|}{n(z)n(z')} \in [0, \pi[. \tag{24}$$

3.2 Quantum spaces

Let Z be a coherent space. A **quantum space** of Z is a Euclidean space $\mathbb{Q}(Z)$ spanned by (i.e., consisting of all finite linear combinations of) a distinguished set of vectors $|z\rangle$ $(z \in Z)$ satisfying

$$\langle z|z'\rangle := \langle z|\,|z'\rangle = K(z, z') \quad \text{for } z, z' \in Z,$$

with the linear functionals[3]

$$\langle z| := |z\rangle^*$$

acting on $\mathbb{Q}^\times(Z)$. Thus there is a distinguished map from Z to $\mathbb{Q}(Z)$ mapping z to the vectors $|z\rangle$ $(z \in Z)$; these are called the **coherent states** of Z in $\mathbb{Q}(Z)$. In this paper, we use this Dirac bra/ket notation *only* for coherent states and their adjoints.

We call the completion $\overline{\mathbb{Q}}(Z) := \overline{\mathbb{Q}(Z)}$ of a quantum space the corresponding **completed quantum space** of Z. The corresponding **augmented quantum space** is the antidual $\mathbb{Q}^\times(Z) := \mathbb{Q}(Z)^\times$. We have

$$\mathbb{Q}(Z) \subseteq \overline{\mathbb{Q}}(Z) \subseteq \mathbb{Q}^\times(Z).$$

[3] With this convention, $\langle z|$ is a linear functional mapping $\psi \in \mathbb{Q}^\times(Z)$ to $\langle z|psi$, while $|z\rangle \in \mathbb{Q}(Z)$ is an antilinear functional mapping $\psi \in \mathbb{Q}^\times(Z)$ to $\psi^*|z\rangle$.

If the quantum space is infinite-dimensional, $\mathbb{Q}(Z)$ is usually a proper subspace of the Hilbert space $\overline{\mathbb{Q}}(Z)$. By definition of the weak-* topology of $\mathbb{Q}^\times(Z)$, $\psi_\ell \in \mathbb{Q}^\times(Z)$ converges to $\psi \in \mathbb{Q}^\times(Z)$ iff $\langle z|\psi_\ell \to \langle z|\psi$ for all $z \in Z$.

3.3 Proposition.
(i) Let $\mathbb{H}$ be a Euclidean space. Then for any set Z and any mapping $c : Z \to \mathbb{H}$,

$$K(z, z') := c(z)^* c(z') \tag{25}$$

defines a coherent product on Z that turns Z into a coherent space whose quantum space $\mathbb{Q}(Z)$ is the space consisting of the finite linear combinations of coherent states $|z\rangle := c(z)$. ($\mathbb{Q}(Z)$ is usually a proper subspace of $\mathbb{H}$.)

(ii) Conversely, every coherent product can be written in the form (25) *such that the coherent states are given as $|z\rangle = c(z)$.*

Proof. (i) follows by combining Example 3.1(i) with the definition of the quantum space. To see (ii), take $\mathbb{H} = \mathbb{Q}(Z)$ and define $c(z) := |z\rangle$. □

3.4 Theorem. *Every coherent space Z has a quantum space $\mathbb{Q}(Z)$. It is unique up to an isomorphism that maps coherent states with the same label to each other.*

Proof. By definition of a coherent space, the coherent product K is of positive type. Hence the Moore–Aronszejn theorem (Theorem 5.1 below) applies and provides a Hilbert space $\overline{\mathbb{Q}}$. If we define the coherent states $|z\rangle := q_z$ and their adjoints $\langle z| := q_z^*$, we find from (50) below that

$$\langle z|z'\rangle = \langle q_z, q_{z'}\rangle = K(z, z').$$

Thus the space $\mathbb{Q}$ consisting of the finite linear combinations of coherent states is a quantum space. If $\mathbb{Q}$ and $\mathbb{Q}'$ are quantum spaces for Z with coherent states $|z\rangle$ and $|z\rangle'$, respectively, then

$$I(\phi) := \sum a_k |z_k\rangle' \quad \text{if } \phi = \sum a_k |z_k\rangle$$

defines a map $I : \mathbb{Q} \to \mathbb{Q}'$. Indeed, if $\phi = \sum b_k |z_k\rangle$ is another representation of ϕ then

$$\sum a_k K(z, z_k) = \sum a_k \langle z|z_k\rangle = \langle z|\phi = \sum b_k \langle z|z_k\rangle = \sum b_k K(z, z_k).$$

Thus $\phi' := \sum b_k |z_k\rangle'$ satisfies

$${}'\langle z|\phi' = \sum b_k{}' \langle z|z_k\rangle' = \sum b_k K(z, z_k) = \sum a_k K(z, z_k) = \sum a_k{}' \langle z|z_k\rangle' = {}'\langle z|I(\phi)$$

for all $z \in Z$, whence $\phi' = I(\phi)$. This map is easily seen to be an isomorphism. □

3.5 Proposition. *Let Z be a coherent space and $\mathbb{Q}(Z)$ be a quantum space of Z. Also, let $\phi : \mathbb{Q}(Z) \to \mathbb{C}$ be an antilinear map. Then, $\phi \in \mathbb{Q}(Z)^\times$ (i.e., ϕ is continuous) iff for every $z \in Z$ there exists a constant $M > 0$ and a finite subset W of Z such that*

$$|\phi(\psi)| \le M \sum_{w \in W} |\langle w|\psi\rangle| \quad \textit{for all } \psi \in \mathbb{Q}(Z). \tag{26}$$

Proof. Suppose ϕ satisfies (26) for some constant $M > 0$ and finite subset W of Z. Applying (3) for the vectors $\varphi := M|w\rangle$ with $w \in W$ we find that ϕ is continuous. Conversely, let $\phi \in \mathbb{Q}(Z)^\times$. Then $\phi : \mathbb{Q}(Z) \to \mathbb{C}$ is continuous. Then there is a finite set of vectors $\varphi_1, ..., \varphi_d \in \mathbb{Q}(Z)$ such that (3) holds. Each φ_j $(1 \le j \le d)$ can be written as a finite sum

$$\varphi_j = \sum_\ell c_\ell^j |w_\ell^j\rangle$$

with $c_\ell^j \in \mathbb{C}$ and $w_\ell^j \in Z$. With $W := \{w_\ell^j \mid 1 \le j \le d\}$ and $M := \max\{|c_\ell^j| \mid 1 \le j \le d\}$, we now have for all $\psi \in \mathbb{Q}(Z)$,

$$\begin{aligned} |\phi(\psi)| &= \le \sum_{j=1}^d |\langle \varphi_j|\psi\rangle| = \sum_{j=1}^d \Big| \sum_\ell \overline{c_\ell^j} \langle w_\ell^j|\psi\rangle \Big| \\ &\le \sum_{j=1}^d \sum_\ell |\overline{c_\ell^j}| |\langle w_\ell^j|\psi\rangle| \le M \sum_{j=1}^d \sum_\ell |\langle w_\ell^j|\psi\rangle| = M \sum_{w\in W} |\langle w|\psi\rangle|, \end{aligned}$$

which implies (26). $\square$

The next theorem presents a characterization for continuous linear maps from a quantum space of a coherent space into its antidual.

3.6 Theorem. *Let Z be a coherent space and $\mathbb{Q}(Z)$ be a quantum space of Z. Also, let $\mathbf{X} : \mathbb{Q}(Z) \to \mathbb{Q}(Z)^\times$ be a linear map. Then $\mathbf{X} \in \mathrm{Lin}^\times \mathbb{Q}(Z)$ (i.e., $\mathbf{X}$ is continuous) iff for every $z \in Z$ there exists a constant $M > 0$ and a finite subset W of Z such that*

$$\Big| \langle z|\mathbf{X}\psi \Big| \le M \sum_{w\in W} |\langle w|\psi\rangle| \quad \text{for all } \psi \in \mathbb{Q}(Z). \tag{27}$$

Proof. (i) Let $\mathbf{X} \in \mathrm{Lin}^\times \mathbb{Q}(Z)$ and $z \in Z$. Applying Proposition 2.4 for $\phi = |z\rangle \in \mathbb{Q}(Z)$, there exists a finite subset U of $\mathbb{Q}(Z)$ such that

$$|\mathbf{X}\psi(\phi)| \le \sum_{\chi\in U} |\chi^*\psi|, \quad \text{for all } \psi \in \mathbb{Q}(Z).$$

Each $\chi \in U$ can be written as a finite sum

$$\chi = \sum_\ell c_\ell^\chi |w_\ell^\chi\rangle$$

with $c_\ell^\chi \in \mathbb{C}$ and $w_\ell^\chi \in Z$. With $W := \{w_\ell^\chi \mid \chi \in U\}$ and $M := \max\{|c_\ell^\chi| \mid \chi \in U\}$, we now have for all $\psi \in \mathbb{Q}(Z)$,

$$\begin{aligned} |\langle z|\mathbf{X}\psi| &= |\mathbf{X}\psi(|z\rangle)| \le \sum_{\chi\in U} |\langle \chi|\psi\rangle| = \sum_{\chi\in U} \Big| \sum_\ell \overline{c_\ell^\chi} \langle w_\ell^\chi|\psi\rangle \Big| \\ &\le \sum_{\chi\in U} \sum_\ell |\overline{c_\ell^\chi}| |\langle w_\ell^\chi|\psi\rangle| \le M \sum_{\chi\in U} \sum_\ell |\langle w_\ell^\chi|\psi\rangle| = M \sum_{w\in W} |\langle w|\psi\rangle|, \end{aligned}$$

which implies (27).

(ii) Conversely, assume that (27) holds, for all $z \in Z$. Fix $\phi = \sum_{k \in J} c_k |z_k\rangle \in \mathbb{Q}(Z)$. For each $k \in J$, the assumption for z_k implies that there exists a positive constant M_k and a finite subset W_k of Z such that

$$\Big|\langle z_k\Big|\mathbf{X}\psi| \le M_k \sum_{w \in W_k} |\langle w|\psi\rangle| \quad \text{for all } \psi \in \mathbb{Q}(Z).$$

Let $M := \max_{k \in J} |c_k| M_k$ and $W := \bigcup_{k \in J} W_k$. Then, for all $\psi \in \mathbb{Q}(Z)$,

$$\begin{aligned} |\mathbf{X}\psi(\phi)| &= \Big|\mathbf{X}\psi\Big(\sum_k c_k|z_k\rangle\Big)\Big| = \Big|\sum_k c_k\langle z_k|\mathbf{X}\psi\Big| \le \sum_k |c_k||\langle z_k|\mathbf{X}\psi| \\ &\le \sum_k |c_k| M_k \sum_{w \in W_k} |\langle w|\psi\rangle| \le M \sum_k \sum_{w \in W_k} |\langle w|\psi\rangle| = M \sum_{w \in W} |\langle w|\psi\rangle|. \end{aligned}$$

Now Proposition 2.4 shows that $\mathbf{X} \in \mathrm{Lin}^{\times}\, \mathbb{Q}(Z)$. $\square$

Let Z, Z' be coherent spaces. A **morphism** from Z to Z' is a map $\rho : Z \to Z'$ such that

$$K'(\rho(z), \rho(z)) = K(z, w) \quad \text{for } z, w \in Z; \tag{28}$$

if $Z' = Z$, ρ is called an **endomorphism**. Two coherent spaces Z and Z' with coherent products K and K', respectively, are called **isomorphic** if there is a bijective morphism $\rho : Z \to Z'$. In this case we write $Z \cong Z'$ and we call the map $\rho : Z \to Z'$ an **isomorphism** of the coherent spaces. Clearly, $\rho^{-1} : Z' \to Z$ is then also an isomorphism. If $Z' = Z$ and $K' = K$ we call ρ an **automorphism** of Z. Automorphisms are closely related to the more general concept of coherent maps, introduced in Neumaier & Ghaani Farashahi [57].

3.7 Proposition. *Let Z, Z' be coherent spaces and $\rho : Z \to Z'$ be an isomorphism. Then,*

(i) $K(\rho^{-1}(z'), \rho^{-1}(w')) = K'(z', w')$ for all $z', w' \in Z'$.

(ii) $K'(z', \rho(z)) = K(\rho^{-1}(z'), z)$ for all $z \in Z$ and $z' \in Z'$.

Proof. (i) is straightforward.

(ii) Let $z \in Z$ and $z' \in Z'$. Then $K'(z', \rho(z)) = K'(\rho(\rho^{-1}(z')), \rho(z)) = K(\rho^{-1}(z'), z)$. $\square$

3.8 Proposition. *Let Z be a coherent space with coherent product K and Z' be an arbitrary set. Then for any map $A : Z' \to Z$,*

$$K'(z, z') := K(Az, Az') \quad \textit{for } z, z' \in Z'$$

defines a coherent product on Z'. This turns Z' into a coherent space with respect to which A is a morphism.

3.9 Proposition. *Let Z, Z' be isomorphic coherent spaces. Then any two quantum spaces $\mathbb{Q}(Z)$ and $\mathbb{Q}(Z')$ are isometric Euclidean spaces.*

Proof. Let Z, Z' be isomorphic coherent spaces. Let $\mathbb{Q}(Z)$ and $\mathbb{Q}(Z')$ be quantum spaces of Z and Z', respectively. Let $\rho : Z \to Z'$ be an isomorphism of coherent spaces. We define the map $T_\rho : \mathbb{Q}(Z) \to \mathbb{Q}(Z')$ given by

$$T_\rho\Big(\sum_k c_k|z_k\rangle\Big) := \sum_k c_k|\rho(z_k)\rangle' \quad \text{for all } \sum_k c_k|z_k\rangle \in \mathbb{Q}(Z).$$

Now

$$\begin{aligned}\Big\|T_\rho\Big(\sum_k c_k|z_k\rangle\Big)\Big\|^2_{\mathbb{Q}(Z')} &= \Big\|\sum_k c_k|\rho(z_k)\rangle'\Big\|^2_{\mathbb{Q}(Z')} = \sum_k\sum_j \overline{c_k}c_j K'(\rho(z_k),\rho(z_j)) \\ &= \sum_k\sum_j \overline{c_k}c_j K(z_k,z_j) = \Big\|\sum_k c_k|z_k\rangle\Big\|^2_{\mathbb{Q}(Z)}.\end{aligned}$$

This implies that T_ρ is a well-defined isometry. Since ρ is surjective, T_ρ is surjective as well. Thus, T_ρ is an isomorphism. □

3.3 Some examples

We now give a long list of basic examples of coherent spaces exhibiting the flavor of the relations to other fields of mathematics and science. As indicated in the introduction, this is just the tip of an iceberg; many other coherent spaces will be discussed in subsequent papers of this series.

The first group of examples arises in applications to quantum mechanics. For the physical background see, e.g., Neumaier & Westra [63].

3.10 Examples. The simplest instances of coherent spaces are the spaces formed by the subsets Z of $\mathbb{C}^n$ which are closed under conjugation and scalar multiplication, with one of the coherent products

$$K(z,z') := \begin{cases} 1 & \text{if } z' = \overline{z}, \\ 0 & \text{otherwise,}\end{cases} \tag{29}$$

$$K(z,z') := z^*z', \tag{30}$$

$$K(z,z') := (z^*z')^{2j} \qquad (j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots), \tag{31}$$

$$K(z,z') := e^{(z^*z' - \frac{1}{2}\|z\|^2 - \frac{1}{2}\|z'\|^2)/\hbar}, \tag{32}$$

where $\hbar$ is a positive real number. In the applications to quantum mechanics, $\hbar$ is the **Planck constant**. The axioms are easily verified using the constructions of Proposition 2.12 and Theorem 2.14.

(i) $Z = \mathbb{C}^n$ with the coherent product (29) corresponds to the phase space of a classical system of n oscillators, with n position and n momentum degrees of freedom, via the identification

$$z = q + ip, \quad q = \operatorname{Re} z, \quad p = \operatorname{Im} z.$$

In the corresponding quantum space, the associated coherent states are orthonormal basis vectors, indexed by the phase space points.

(ii) The unit sphere Z in $\mathbb{C}^2$ with the coherent product (29) corresponds to the phase space of a **classical spin**, such as a polarized light beam or a spinning top fixed at its point.

(iii) The unit ball Z in $\mathbb{C}^2$ with the coherent product (29) corresponds to the classical phase space of (monochromatic) **partially polarized light**.

(iv) $Z = \mathbb{C}^n$ with the coherent product (30) has as quantum space the Hilbert space $\mathbb{C}^n$ of an n-level quantum system. The associated coherent states are all state vectors.

(v) The unit sphere Z in $\mathbb{C}^2$ with the coherent product (31) corresponds to the Poincaré sphere (or Bloch sphere) representing a single quantum mode of an atom with spin j, or for $j = 1$ the polarization of a single photon mode. The corresponding quantum space has dimension $2j + 1$. The associated coherent states are the so-called **spin coherent states**. (This example shows that a given set Z may carry more than one interesting coherent product, resulting in different coherent spaces with nonisomorphic quantum spaces.) For $j \to \infty$, the space degenerates into the coherent space of a classical spin.

(vi) $Z = \mathbb{C}^n$ with the coherent product (32) has as quantum space the bosonic Fock space with n degrees of freedom, corresponding to n independent harmonic oscillators. The associated coherent states are the so-called **Glauber coherent states**. In the so-called classical limit $\hbar \to 0$ (which can be taken mathematically, though not in reality), the space degenerates into the coherent space of a classical system with n spatial degrees of freedom.

We note that for (32), the power construction from Theorem 2.14(i) just amounts to a replacement of $\hbar$ by $\hbar/n$. Therefore the classical limit amounts here to applying the power construction for arbitrary n and considering the limit $n \to \infty$. Generalizing this to arbitrary coherent spaces provides a general definition of the **classical limit**, even when $\hbar$ does not appear in the coherent product. For example, the power construction applied to (30) produces (31) with $2j = n$; thus the classical limit amounts here to the limit of infinite spin. The classical limit and related semiclassical expansions are investigated in general in a later paper of this series (Neumaier [56]).

We also note that in order that a coherent product results, $\hbar$ can take in (32) any positive value, while in (31), $2j$ must be a nonnegative integer. (The latter is already needed in order that the power is unambiguously defined.) This phenomenon is captured through the concept of a Berezin–Wallach set (see Subsection 5.5 below).

3.11 Example. The set $Z = \mathbb{R}_+$ of positive real numbers is a real coherent space with trivial conjugation for any of the coherent products

$$K(z, z') = \min(z, z'),$$

$$K(z,z') = (z+z')^{-1}.$$

(i) In the first case, a completed quantum space is $L^2(\mathbb{R}_+)$ with coherent states

$$k_z(z') = \begin{cases} 1 & \text{if } z' \le z, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) In the second case, a completed quantum space is $L^2(\mathbb{R}_+)$ with coherent states

$$k_z(z') = e^{-zz'}$$

since

$$\langle k_z, k_{z'} \rangle = \int_{0+}^{\infty} dy\, \overline{k_z(y)}\, k_{z'}(y) = \int_0^{\infty} dy\, e^{-zy} e^{-z'y} = \int_0^{\infty} dy\, e^{-(z+z')y} = \frac{1}{z+z'}.$$

The following spaces are important not only in complex analysis but are also relevant in quantum physics, for the analysis of quantum mechanical scattering problems (de Branges & Rovnyak [23, Theorem 4]). Example 3.12(ii) below is relevant in signal processing.

For a function $f : Z \subset \mathbb{C} \to \mathbb{C}$ we define its conjugate $\overline{f} : Z \to \mathbb{C}$ by

$$\overline{f}(z) := \overline{f(\overline{z})}. \tag{33}$$

3.12 Examples.
(i) A **de Branges function** is an entire analytic function $E : \mathbb{C} \to \mathbb{C}$ satisfying

$$|E(\overline{z})| < |E(z)| \quad \text{if } \operatorname{Im} z > 0. \tag{34}$$

With the coherent product

$$K(z,z') := \begin{cases} \overline{E}'(\overline{z})E(z') - E'(\overline{z})\overline{E}(z') & \text{if } z' = \overline{z}, \\[2ex] \dfrac{\overline{E}(\overline{z})E(z') - E(\overline{z})\overline{E}(z')}{2i(\overline{z} - z')} & \text{otherwise,} \end{cases}$$

$Z = \mathbb{C}$ is a coherent space. A corresponding quantum space is the subspace of $L^2(\mathbb{R})$ spanned by the coherent states q_z, denoted by $\mathcal{H}(E)$, defined by

$$q_z(t) = \frac{K(\overline{z}, t)}{E(t)} := \lim_{\varepsilon \downarrow 0} \frac{K(\overline{z}, t + i\varepsilon)}{E(t + i\varepsilon)} \quad \text{for } t \in \mathbb{R}.$$

(The denominator on the right is nonzero by (34). The limit exists and is continuous as a function of t since at an n-fold zero t of E, the function $K(\overline{z}, \cdot)$ has t as a zero of multiplicity at least r.) Indeed, the formula $q_z^* q_{z'} = K(z,z')$ follows by evaluating the integral expression for $q_z^* q_{z'}$ using the residue theorem. For details see de Branges [22, Theorem 19, p.50], where the quantum space is more fully characterized.

(ii) $Z = \mathbb{C}$ is a coherent space with the coherent product

$$K(z,z') := \operatorname{sinc}(\overline{z} - z'), \quad \operatorname{sinc}(z) := \begin{cases} 1 & \text{if } z = 0, \\ \sin(z)/z & \text{otherwise.} \end{cases}$$

This is the special case $E(z) = e^{-iz}$ of (i).

(iii) A **Schur function** (SCHUR [75]) is an analytic function s from the open unit disk $B(0;1)$ in $\mathbb{C}$ into its closure. With the coherent product

$$K(z,z') := \frac{1 - \overline{s(z)}s(z')}{1 - \overline{z}z'}.$$

$Z := B(0;1)$ is a coherent space. Note that the inverse is defined since $|\overline{z}z'| < 1$. Coherence follows from results by DE BRANGES & ROVNYAK [24]. The corresponding quantum spaces are the sub-Hardy spaces discussed by SARASON [71], also called **de Branges–Rovnyak spaces**; see the recent survey by BALL & BOLOTNIKOV [13].

(iv) The **Szegö space** is the coherent space defined on the open unit disk in $\mathbb{C}$,

$$D(0,1) := \{z \in \mathbb{C} \mid |z| < 1\},$$

by the coherent product

$$K(z,z') := (1 - \overline{z}z')^{-1},$$

cf. SZEGÖ [77]. This example from 1911 is probably the earliest nontrival explicit, nontrivial coherent product in the literature. It is the special case $s = 0$ of (iii); its quantum space is the Hardy space on the unit disk. Coherence also follows directly from Theorem 2.14(iii).

In general, unlike in these (and other simple) examples, there need not be a simple realization of a quantum space in terms of an L^2 space with respect to a suitable measure. Fortunately, such a description is usually not needed in applications to physics since one can work comfortably in the quantum space using only its defining properties. This is one of the strengths of the concept of coherent spaces, as it allows one to avoid the often cumbersome evaluation of integrals in the computation of inner products.

4 Nondegenerate and projective coherent spaces

In this section we consider some desirable conditions a coherent space may or may not have. Some of these conditions are satisfied in many coherent spaces relevant for the applications.

4.1 The distance

Theorem 3.4 implies that, in a sense, coherent spaces are just the subsets of Euclidean spaces. However, separating the structure of a coherent space Z from the notion of a Euclidean space allows many geometric features to be expressed in terms of Z and the coherent product alone, without direct references to the quantum space. The latter only serves as a convenient tool for proving assertions of interest. For example, the study of symmetry in NEUMAIER & GHAANI FARASHAHI [57] benefits from this separation. Another example is the distance function induced on Z by the Euclidean distance, as in the proof of Proposition 4.1 below. It will play an important role in the study of coherent manifolds in NEUMAIER & GHAANI FARASHAHI [59].

4.1 Proposition. *(*PARTHASARATHY & SCHMIDT *[66, Corollary 1.3/4])*
The ***distance***

$$d(z,z') := \sqrt{K(z,z)+K(z',z')-2\operatorname{Re}K(z,z')}, \tag{35}$$

of two points $z,z' \in Z$ is nonnegative and satisfies the triangle inequality. With (21) *we have*

$$|n(z)-n(z')| \le d(z,z') \le n(z)+n(z'), \tag{36}$$

$$|K(y,z)-K(y',z')| \le d(y,y')n(z')+n(y)d(z,z'). \tag{37}$$

Proof. The expression under the square root of (35) is

$$\langle z|z\rangle + \langle z'|z'\rangle - \langle z|z'\rangle - \langle z'|z\rangle = \Big\| |z\rangle - |z'\rangle \Big\|^2, \tag{38}$$

whence $d(z,z')$ is just the Euclidean distance between $|z\rangle$ and $|z'\rangle$. This implies non-negativity and the triangle inequality. $n(z)$ is the length of $|z\rangle$, and (36) follows. The Cauchy–Schwarz inequality gives

$$|K(y,z)-K(y,z')| = \Big|\langle y|\Big(|z\rangle - |z'\rangle\Big)\Big| \le n(y)d(z,z'),$$

hence

$$\begin{aligned} |K(y,z)-K(y',z')| &= |K(y,z)-K(y,z')+K(y,z')-K(y',z')| \\ &\le |K(y,z')-K(y',z')| + |K(y,z)-K(y,z')| \\ &\le d(y,y')n(z') + d(z,z')n(y). \end{aligned}$$

This proves (37). □

We call a coherent space **nondegenerate** if $K(z'',z') = K(z,z')$ for all $z' \in Z$ implies $z'' = z$. Clearly, this is the case iff the mapping from Z to $\mathbb{Q}(Z)$ that maps each $z \in Z$ to the corresponding coherent state $|z\rangle$ is injective.

4.2 Proposition. *The distance map d is a metric on Z iff K is nondegenerate on Z.*

Proof. (38) implies that $d(z,z') = 0$ iff $|z\rangle = |z'\rangle$. Hence d is a metric on Z iff K is nondegenerate on Z. □

The distance map d is a quasimetric, hence it induces in the standard way a topology on Z called the **metric topology** and denoted by τ_m. There is a second topology on nondegenerate coherent spaces Z, the **coherent topology** denoted by τ_c, defined by calling a net z_ℓ **coherently convergent** to z iff $K(z_\ell,z') \to K(z,z')$ for all $z' \in Z$. It can be readily checked that the coherent topology τ_c is at least as fine as the metric topology τ_m, because if $z_n \to z$ in the metric topology then $z_n \to z$ in the coherent topology, too.

4.3 Theorem. *In any coherent space, the metric topology is the weakest (coarsest) topology in which K is continuous.*

Proof. We equip $Z \times Z$ with the product topology induced by the metric topology on Z. Let (z_n, z_n') be a convergent sequence to $(z, z') \in Z \times Z$. Then $z_n \to z$ and $z_n' \to z'$ in the metric topology. Hence the sequence of $n(z_n)$ is bounded. Thus

$$|K(z_n, z_n') - K(z, z')| \leq d(z_n, z)n(z') + d(z_n', z')n(z_n),$$

which implies that $\lim_n K(z_n, z_n') = K(z, z')$.

Now let τ be any topology on the coherent space Z such that $K : Z \times Z \to \mathbb{C}$ is continuous. To prove that τ is at least as fine as τ_m we assume that $w_n \to w$ in Z with respect to τ. Since K is continuous with respect to τ and $K(w, w) = n(w)^2$, we find

$$\begin{aligned} \lim_n d(w_n, w) &= \lim_n \sqrt{K(w_n, w_n) + n(w)^2 - 2\,\mathrm{Re}\,K(w_n, w)} \\ &= \sqrt{K(w, w) + n(w)^2 - 2\,\mathrm{Re}\,K(w, w)} \\ &= \sqrt{n(w)^2 + n(w)^2 - 2\,\mathrm{Re}\,n(w)^2} = 0, \end{aligned}$$

which implies that $w_n \to w$ in Z with respect to the metric topology as well. Thus τ is at least as fine as τ_m. This implies that the metric topology is the weakest (coarsest) topology in which K is continuous. ☐

4.2 Normal coherent spaces

We call a coherent space **normal** if

$$\begin{cases} K(z, z') = 1 & \text{if } z' = z, \\ |K(z, z')| < 1 & \text{otherwise.} \end{cases}$$

In a normal coherent space, coherent states have norm 1, hence the distance simplifies to

$$d(z, z') := c\sqrt{1 - \mathrm{Re}\,K(z, z')}, \quad c = \sqrt{2}. \tag{39}$$

This distance was studied by ARCOZZI et al. [9] with $c = 1$ rather than the above value. (39) implies that a normal coherent space Z is nondegenerate.

4.4 Proposition. *Let Z be a coherent space with coherent product K. Then, for any function $\gamma : Z \to \mathbb{C}$, the set Z with* ***scaled coherent product***

$$K_\gamma(z, z') := \overline{\gamma(z)}K(z, z')\gamma(z')$$

is also a coherent space.

Proof. The Gram matrix G' of the scaled coherent product has entries

$$G_{jk} := K_\gamma(z_j, z_k) = \overline{\gamma(z_j)}K(z_j, z_k)\gamma(z_k)$$

and is clearly Hermitian. For any vector u, we define the vector v with components $v_k := \gamma(z_k)u_k$ and find

$$u^*G'u = \sum_{j,k} \overline{u}_j\overline{\gamma(z_j)}K(z_j,z_k)\gamma(z_k)u_k = \sum_{j,k} \overline{v}_jK(z_j,z_k)v_k \geq 0.$$

Thus G' is positive semidefinite. $\square$

4.5 Proposition. *Let Z be a coherent space. If the coherent product is not identically zero then there is a normal, coherent space Z' such that there is an isomorphism $\alpha : \mathbb{Q}(Z) \to \mathbb{Q}(Z')$ with*

$$\{\alpha|z\rangle \mid z \in Z\} \subseteq \{\lambda|z'\rangle \mid \lambda \in \mathbb{C},\ z' \in Z'\}.$$

Thus any image of a coherent state of Z is a multiple of some coherent state of Z'.

Proof. If $K(z,z) = 0$, the coherent state $|z\rangle$ vanishes by the Cauchy–Schwarz inequality (11). Thus we can delete such points from Z. By scaling using Proposition 4.4, we may assume that $K(z,z) = 1$ without changing the Hilbert space. Now the proof of the Cauchy–Schwarz inequality (11) shows that if $|K(z,z')| = 1$ then the coherent states $|z\rangle$ and $|z'\rangle$ differ by a phase only; so we may delete one of them without changing the Hilbert space. The new coherent space is normal. $\square$

4.3 Projective coherent spaces

We call a coherent space Z **projective** if there is a **scalar multiplication** that assigns to each $\lambda \in \mathbb{C}^\times$ and each $z \in Z$ a point $\lambda z \in Z$ such that

$$K(z,\lambda z') \;=\; \lambda^e K(z,z') \quad \text{for all } z,z' \in Z, \tag{40}$$

for some $e \in \mathbb{Z} \setminus \{0\}$ called the **degree**. Note that a coherent space cannot be both normal and projective. Example 3.10(v) is projective of degree $e = 2j$, Example 3.11(i) and (ii) are projective of degree 1 and -1, respectively.

There are important degenerate projective spaces where the scalar multiplication is not associative because it is not canonically defined. An example are the Klauder spaces from Example 3.2, which are projective of degree 1 with the scalar multiplication

$$\lambda[z_0,\mathbf{z}] := [z_0 + \log\lambda, \mathbf{z}],$$

using an arbitrary but fixed branch of log. The need to restrict to a fixed branch causes the associative law to be not valid universally. On the other hand, we have:

4.6 Proposition. *Let Z be a nondegenerate and projective space. Then the scalar multiplication is associative:*

$$\lambda(\mu z) = (\lambda\mu)z \quad \textit{for } \lambda,\mu \in \mathbb{C}^\times,\ z \in Z, \tag{41}$$

Proof. Let $z \in Z$ and $\lambda, \mu \in \mathbb{C}^\times$. For all $z' \in Z$, we have

$$K(\lambda(\mu z), z') = \overline{\lambda}^e K(\mu z, z') = \overline{\lambda}^e \overline{\mu}^e K(z, z') = (\overline{\lambda\mu})^e K(z, z') = K((\lambda\mu)z, z').$$

Now nondegeneracy of K implies (41). $\square$

For projective coherent spaces the continuity conditions of Theorem 3.5 and Theorem 3.6 simplify:

4.7 Proposition. *Let Z be a projective coherent space and let $\mathbb{Q}(Z)$ be a quantum space of Z. Then:*

(i) An antilinear functional $\psi : \mathbb{Q}(Z) \to \mathbb{C}$ belongs to $\mathbb{Q}(Z)^\times$ iff

$$|\psi(\phi)| \le \sum_{w \in W'} |\langle w|\phi| \quad \textit{for all } \phi \in \mathbb{Q}(Z) \tag{42}$$

for some finite subset W' of Z.

(ii) A linear map $A : \mathbb{Q}(Z) \to \mathbb{Q}(Z)^\times$ belongs to $\mathrm{Lin}^\times \mathbb{Q}(Z)$ iff for every $z \in Z$ there exists a finite subset W' of Z such that

$$\Big|\langle z|A\psi\Big| \le \sum_{w \in W'} |\langle w|\psi\rangle| \quad \textit{for all } \psi \in \mathbb{Q}(Z). \tag{43}$$

Proof. (i) First assume that (42) holds for W'. Proposition 3.5 with W' in place of W and $M = 1$ guarantees that $\psi \in \mathbb{Q}(Z)^\times$. Conversely, let $\psi \in \mathbb{Q}(Z)^\times$. Hence, ψ satisfies (26) for some constant M and finite subset W of Z. Then $W' := MW = \{Mw : w \in W\}$ holds for ψ.

(ii) This follows using the argument of (i) and applying Theorem 3.6 on the projective space Z. $\square$

4.8 Proposition. *Let Z be a projective coherent space of degree e. Then*

$$K(\lambda z, z') \;=\; \overline{\lambda}^e K(z, z') \quad \textit{for all } z, z' \in Z, \tag{44}$$

$$|\lambda z\rangle = \lambda^e |z\rangle \quad \textit{for } \lambda \in \mathbb{C}^\times,\ z \in Z, \tag{45}$$

$$K(z, \lambda z') = K(\overline{\lambda} z, z') \quad \textit{for } \lambda \in \mathbb{C}^\times,\ z \in Z. \tag{46}$$

Proof. (44) follows from the definition and (19). To prove (45), let $z \in Z$ and $\lambda \in \mathbb{C}$. Then, for all $z' \in Z$,

$$\langle z'|\lambda z\rangle = K(z', \lambda z) = \lambda^e K(z', z) = \lambda^e \langle z'|z\rangle.$$

Finally, using (40) and (44), we get

$$K(z, \lambda z') = \lambda^e K(z, z') = K(\overline{\lambda} z, z').$$

$\square$

Formula (46) suggests that it might be fruitful to consider more general maps $A : Z \to Z$ satisfying

$$K(z, \lambda z') = K(\overline{\lambda} z, z') \quad \text{for } \lambda \in \mathbb{C}^\times,\ z \in Z.$$

Such maps are called **coherent maps** and are studied in detail in NEUMAIER & GHAANI FARASHAHI [57] and many later papers of this series. Invertible coherent maps are of fundamental importance as they describe the symmetry group of a coherent space.

Any coherent space can be extended to a projective coherent space without changing the quantum space. The idea of a projective extension can be traced back to KLAUDER [40].

4.9 Proposition.
Let Z be a coherent space and e be a nonzero integer. Then the ***projective extension*** *$PZ := \mathbb{C}^\times \times Z$ of degree e is a projective coherent space with coherent product*

$$K_{\mathrm{pe}}((\lambda, z), (\lambda', z')) \ := \ \overline{\lambda}^e\, K(z, z')\, \lambda'^e \tag{47}$$

and scalar multiplication $\lambda'(\lambda, z) := (\lambda'\lambda, z)$. The corresponding quantum spaces $\mathbb{Q}(Z)$ and $\mathbb{Q}(PZ)$ are canonically isomorphic.

Proof. It is straightforward to check that PZ with respect to the projective extension kernel K_{pe} is a projective coherent space of degree e. The map $T : \mathbb{Q}(PZ) \to \mathbb{Q}(Z)$ given by

$$T \sum_\ell c_\ell |(\lambda_\ell, z_\ell)\rangle := \sum_\ell c_\ell \lambda_\ell^e |z_\ell\rangle \quad \text{for all } \sum_\ell c_\ell |(\lambda_\ell, z_\ell)\rangle \in \mathbb{Q}(PZ),$$

is well-defined and linear. Also, we have

$$\begin{aligned} \Big\| T \sum_\ell c_\ell |(\lambda_\ell, z_\ell)\rangle \Big\|^2_{\mathbb{Q}(Z)} &= \Big\| \sum_\ell c_\ell \lambda_\ell^e |z_\ell\rangle \Big\|^2_{\mathbb{Q}(Z)} = \sum_j \sum_k \overline{c_j}\, \overline{\lambda_j^e} c_k \lambda_k^e K(z_j, z_k) \\ &= \sum_j \sum_k \overline{c_j} c_k K_{\mathrm{pe}}((\lambda_j, z_j), (\lambda_k, z_k)) = \Big\| \sum_\ell c_\ell |(\lambda_\ell, z_\ell)\rangle \Big\|^2_{\mathbb{Q}(PZ)}, \end{aligned}$$

which implies that T is an isometric linear operator. Thus, T is injective as well. Let $\psi = \sum_\ell c_\ell |z_\ell\rangle \in \mathbb{Q}(Z)$ with $c_\ell \neq 0$ for all ℓ. Then $\phi := \sum_\ell |(c_\ell^{-e}, z_\ell)\rangle \in \mathbb{Q}(PZ)$ with $T\phi = \psi$. Thus, T is an isomorphism. $\square$

4.10 Corollary. *Let Z be a coherent space and e be a nonzero integer. Then Z is projective of degree e iff $P_e Z \cong Z$.*

Proof. Let Z be a projective space of degree $e \in \mathbb{Z}$. We then define $\rho : P_e Z \to Z$ by $\rho(\lambda, z) := \lambda z$, for all $(\lambda, z) \in P_e Z$. It is easy to check that $\rho : P_e Z \to Z$ is an isomorphism. Hence $P_e Z \cong Z$. Conversely, suppose that $P_e Z \cong Z$ and let $\rho : P_e Z \to Z$ be an isomorphism of coherent spaces. Then, with multiplication defined by $\lambda z := \rho\lambda\rho^{-1}z$, Z is projective of degree e. Indeed,using Proposition 3.7(ii) for $z, z' \in Z$, we have

$$K(z, \lambda z') = K(z, \rho\lambda\rho^{-1}z') = K_e(\rho^{-1}z, \lambda\rho^{-1}z') = \lambda^e K_e(\rho^{-1}z, \rho^{-1}z') = \lambda^e K(z, z').$$

□

4.4 Nondegenerate coherent spaces

4.11 Proposition.
Let Z be a coherent space. Define on Z an equivalence relation $\equiv$ by

$$z \equiv z' \quad \Leftrightarrow \quad K(z, z'') = K(z', z'') \quad \text{for all } z'' \in Z.$$

Then the set $[Z]$ of equivalence classes

$$[z] := \{z' \in Z | z' \equiv z\} \quad (z \in Z)$$

is a nondegenerate coherent space with the coherent product

$$K([z], [z']) \;:= K(z, z') \quad \text{for all } z, z' \in Z. \tag{48}$$

The corresponding quantum spaces $\mathbb{Q}(Z)$ and $\mathbb{Q}([Z])$ are canonically isomorphic. In particular, if Z is projective then $[Z]$ is projective, with scalar multiplication $\lambda[z] := [\lambda z]$.

Proof. Let Z be a coherent space and $z, z', w, w' \in Z$ with $[z] = [w]$ and $[z'] = [w']$. Then

$$K([z], [z']) = K(z, z') = K(w, z') = K(w, w') = K([w], [w']).$$

Thus, $K : [Z] \times [Z] \to \mathbb{C}$ is well-defined. It is straightforward to check that $([Z], K)$ is a coherent space. Now let $z, w \in Z$ with $K([z], [z']) = K([w], [z'])$ for all $z' \in Z$. Hence

$$K(z, z') = K([z], [z']) = K([w], [z']) = K(w, z'),$$

for all $z' \in Z$, giving $[z] = [w]$. Thus $[Z]$ is nondegenerate. Let $T : \mathbb{Q}(Z) \to \mathbb{Q}([Z])$ be given by $\psi \to T\psi$, where $T\psi := \sum c_\ell |[z_\ell]\rangle$ for $\psi = \sum c_\ell |z_\ell\rangle \in \mathbb{Q}(Z)$. If $\psi = \sum c_\ell |z_\ell\rangle = 0$ then, for all $w \in Z$,

$$\langle [w]| T\psi = \sum c_\ell \langle [w]|[z_\ell]\rangle = \sum c_\ell K([w], [z_\ell]) = \sum c_\ell K(w, z_\ell) = 0.$$

Thus $T\psi = 0$. Hence $T : \mathbb{Q}(Z) \to \mathbb{Q}([Z])$ is a well-defined linear operator. Also, for $\psi \in \mathbb{Q}(Z)$, we have

$$\|T\psi\|^2 = \textstyle\sum_j \sum_k \overline{c_j} c_k K([z_j], [z_k]) = \sum_j \sum_k \overline{c_j} c_k K(z_j, z_k) = \|\psi\|^2,$$

which implies that T is an isometry, hence injective. It is straightforward to see that T is surjective as well. Hence T is an isomrphism.

If Z is projective then $[Z]$ is projective with the same degree, with scalar multiplication $\lambda[z] := [\lambda z]$. Indeed, if Z is projective of degree e, we have

$$K([z], \lambda[z']) = K([z], [\lambda z']) = K(z, \lambda z') = \lambda^e K(z, z') = \lambda^e K([z], [z']),$$

for all $z, z' \in Z$ and $\lambda \in \mathbb{C}^\times$. $\square$

4.12 Corollary. *Let Z be a projective coherent space. The canonical scalar multiplication on the nondegeneration space $[Z]$ is associative.*

4.13 Theorem. *Let Z be a coherent space and $A : Z \to Z$ be a coherent map with adjoint A^*. Then the class map $[A] : [Z] \to [Z]$ defined by*

$$[A][z] := [Az] \quad \text{for all } z \in Z,$$

is a well-defined and coherent map with the unique adjoint $[A]^* = [A^*]$.

Proof. Let $z, z' \in Z$ with $[z] = [z']$. Using coherence of A, we have

$$K(Az, z'') = K(z, A^*z'') = K(z', A^*z'') = K(Az', z''),$$

for all $z'' \in Z$. Thus, $[Az] = [Az']$ and hence $[A] : [Z] \to [Z]$ is well-defined. Then, using coherence of A and applying the definition of the class map for the coherent maps A and A^*, we get

$$\begin{aligned} K([A][z], [z'']) &= K([Az], [z'']) = K(Az, z'') = K(z, A^*z'') \\ &= K([z], [A^*z'']) = K([z], [A^*][z'']) \end{aligned}$$

for all $z, z'' \in Z$. This guarantees that the class map $[A]$ is a coherent map with the unique adjoint $[A]^* = [A^*]$. $\square$

4.14 Theorem. *Let Z be a coherent space. Then $[PZ] \cong P[Z]$, using a canonical identification. In particular,*

(i) if Z is projective then we have $[PZ] \cong [Z]$.

(ii) if Z is nondegenerate then PZ is nondegenerate.

Proof. The canonical map $\rho : [PZ] \to P[Z]$ given by $[(\lambda, z)] \to (\lambda, [z])$ is well-defined. It is also straightforward to check that ρ is a bijection. Let $[(\lambda, z)], [(\lambda', z')] \in [PZ]$. Then, we have

$$\begin{aligned} K_{\mathrm{pe}}(\rho[(\lambda, z)], \rho[(\lambda', z')]) &= K_{\mathrm{pe}}((\lambda, [z]), (\lambda', [z'])) = \overline{\lambda} K([z], [z'])\lambda' \\ &= \overline{\lambda} K(z, z')\lambda' = K_{\mathrm{pe}}((\lambda, z), (\lambda', z')) = K_{\mathrm{pe}}([(\lambda, z)], [(\lambda', z')]), \end{aligned}$$

implying that $\rho : [PZ] \to P[Z]$ is an isomorphism of coherent spaces. If Z is projective then $PZ \cong Z$. Thus, we get $[PZ] \cong [Z]$. If Z is nondegenerate then $[Z] \cong Z$. Hence, we have $[PZ] \cong P[Z] \cong PZ$, which implies that PZ is nondegenerate as well. $\square$

5 Classical theory of functions of positive type

This section is independent of the remainder. It provides, in the present physics-oriented terminology (cf. the introduction to Section 2) and with full proofs, a self-contained synopsis (and sometimes slight generalization) of a number of classical results from the literature about functions of positive type and reproducing kernel Hilbert spaces. In particular, the Moore–Aronszajn Theorem 5.1 provides the existence of the quantum space of a coherent space, hence is of fundamental importance. However, on first reading, this theorem can be taken for granted, and the study of the remainder of the section can be postponed until the material is needed in later papers on coherent spaces.

5.1 The Moore–Aronszajn theorem

This section discusses how to reconstruct a Hilbert space from a spanning set of vectors whose inner product is known, and the properties that must be satisfied for arbitrarily assigned formal inner products to produce a Hilbert space.

The following theorem is due to Aronszajn [11] (1943), who attributed[4] it to Moore (1935).

5.1 Theorem. *(Moore, Aronszajn)*
Let $K : Z \times Z \to \mathbb{C}$ be of positive type. Then there is a unique Hilbert space $\overline{\mathbb{Q}}$ of complex-valued functions on Z with the Hermitian inner product $\langle \cdot, \cdot \rangle$ (antilinear in the first component) such that the following properties hold.

(i) $\overline{\mathbb{Q}}$ contains the functions $q_z : Z \to \mathbb{C}$ defined for $z \in Z$ by

$$q_z(x) := K(x, z) = \overline{K(z, x)}. \tag{49}$$

(ii) The space $\mathbb{Q}$ of finite linear combinations of the q_z is dense in $\overline{\mathbb{Q}}$.

(iii) The following relations hold:

$$\langle q_z, q_x \rangle = K(z, x), \tag{50}$$

$$\psi(z) = \langle q_z, \psi \rangle \quad \text{for all } \psi \in \overline{\mathbb{Q}}. \tag{51}$$

(iv) For each $z \in Z$, the linear functional ι_z defined by

$$\iota_z \psi := \psi(z) \tag{52}$$

is continuous.

[4] Aronszajn [10, Théorème 2] states the theorem and gives a detailed proof (in French), but his later English paper [11] states the theorem on p.344 and attributes it to Moore. He cites Moore [47] (and a very short notice from 1916) on p.338, but the theorem does not seem to be in one of these references. (Moore discusses in Chapter III functions of positive type under the name *positive Hermitian matrices* – cf. the statement at the top of p.182 – but does not construct a Hilbert space from them.) Faraut & Korányi [25, p. 170] ascribes the theorem to Bergman [17] (1933), but the theorem does not seem to be there either. Kolmogorov [42, Lemma 2] (1941) contains the result for the special case where Z is countable.

Proof. The vector space $\mathbb{Q}$ spanned by the q_z consists of all linear combinations

$$\widehat{f} := \sum_z f(z) q_z \tag{53}$$

with f in the space $\mathbb{F}$ of all maps $f : Z \to \mathbb{C}$ for which all but finitely many values $f(z)$ vanish. Thus the sum is finite, and by (49), function values can be calculated by

$$\widehat{f}(x) = \sum_z f(z) q_z(x) = \sum_z K(x,z) f(z). \tag{54}$$

Since it might be possible that a function $\psi \in \mathbb{Q}$ can be written in several ways in the form (53), the definition of an inner product on $\mathbb{Q}$ requires some care. The mapping defined on $\mathbb{F} \times \mathbb{F}$ by

$$(g,f) := \sum_x \overline{g(x)} \widehat{f}(x) = \sum_{x,z} \overline{g(x)} K(x,z) f(z) \tag{55}$$

is a Hermitian form since

$$\overline{(g,f)} = \sum_{x,z} g(x) \overline{K(x,z) f(z)} = \sum_{x,z} \overline{f(z)} K(z,x) g(x) = (f,g).$$

Now

$$(g,f) = \overline{(f,g)} = \sum_x f(x) \overline{\widehat{g}(x)} = \sum_z \overline{\widehat{g}(z)} f(z). \tag{56}$$

If $\widehat{g} = \widehat{u}$ and $\widehat{f} = \widehat{v}$ then

$$\begin{aligned} (g,f) &= \sum_z \overline{\widehat{g}(z)} f(z) = \sum_z \overline{\widehat{u}(z)} f(z) = (u,f) \\ &= \sum_x \overline{u(x)} \widehat{f}(x) = \sum_x \overline{u(x)} \widehat{v}(x) = (u,v). \end{aligned}$$

Therefore (g,f) depends only on the functions $\widehat{g}$ and $\widehat{f}$. Thus

$$\langle \psi, \psi' \rangle := (g,f) \quad \text{if } \psi = \widehat{g},\ \psi' = \widehat{f}$$

defines a Hermitian form $\langle \cdot, \cdot \rangle$ on $\mathbb{Q}$ satisfying

$$\langle \widehat{g}, \widehat{f} \rangle = (g,f). \tag{57}$$

The function $g_z \in \mathbb{F}$ defined (for arbitrary but fixed $z \in Z$) by $g_z(x) = 1$ if $x = z$ and $g_z(x) = 0$ otherwise, satisfies

$$\widehat{g}_z = q_z \tag{58}$$

by (53), hence by (56),

$$\langle q_z, \widehat{f} \rangle = \langle \widehat{g}_z, \widehat{f} \rangle = (g,f) = \sum_x \overline{g_z(x)} \widehat{f}(x) = \widehat{f}(z).$$

Since by definition of $\mathbb{Q}$, any $\psi \in \mathbb{Q}$ can be written as $\psi = \widehat{f}$, we conclude (51). Specialization to $\psi = q_x$ and using (49) (with z and x interchanged) yields (50).

Since K is of positive type, $(f, f) \geq 0$ for all $f \in \mathbb{F}$. Thus the form is positive semidefinite on $\mathbb{F}$. In particular, the Cauchy–Schwarz inequality $|(f, f')|^2 \leq (f, f)(f', f')$ holds. It implies that $(f, f) = 0$ only if $(f, f') = 0 = (f', f)$ for all f', and (53) then shows that $\widehat{f}(z) = 0$ for all z. Hence $\widehat{f} = 0$. Therefore the Hermitian form $\langle \cdot, \cdot \rangle$ is positive definite, hence defines a Hermitian inner product on $\mathbb{Q}$. Thus $\mathbb{Q}$ is a Euclidean space. The completion with respect to the norm

$$\|\psi\| := \sqrt{\langle \psi, \psi \rangle}$$

(which can be done constructively using Theorem 2.6) gives the desired Hilbert space, and a limiting argument shows that (51) holds in general: If $\psi \in \overline{\mathbb{Q}}$, there is a net of $\psi_j \in \mathbb{Q}$ converging to ψ in the norm, and

$$|\langle q_z, \psi \rangle - \psi_j(z)| = |\langle q_z, \psi \rangle - \langle q_z, \psi_j \rangle| = |\langle q_z, \psi - \psi_j \rangle| \leq \|q_z\| \, \|\psi - \psi_j\| \to 0,$$

hence $\psi(z) = \lim\limits_j \psi_j(z) \to \langle q_z, \psi \rangle$.

(iv) Since $\iota_z \psi = \psi(z) = \langle q_z, \psi \rangle$, we have $\|\iota_z\| = \|q_z\|$. Thus ι_z is bounded and hence continuous.

The uniqueness of $\overline{\mathbb{Q}}$ is clear from the construction. □

5.2 Reproducing kernel Hilbert spaces and Mercer's theorem

5.2 Proposition. *Let ψ_α ($\alpha \in I$) be an orthonormal basis for $\overline{\mathbb{Q}}$. Then*

$$K(z, w) = \sum_{\alpha \in I} \psi_\alpha(z) \overline{\psi_\alpha(w)}. \tag{59}$$

Proof. By the polarized version of the Parseval identity, Theorem 5.27 of Folland [28], we have

$$q_w = \sum_{\alpha \in I} \langle \psi_\alpha, q_w \rangle \psi_\alpha = \sum_{\alpha \in I} \overline{\psi_\alpha(w)} \psi_\alpha$$

for all $w \in Z$. Hence for all $z, w \in Z$,

$$K(z, w) = \langle q_z, q_w \rangle = \Big\langle q_z, \sum_{\alpha \in I} \overline{\psi_\alpha(w)} \psi_\alpha \Big\rangle = \sum_{\alpha \in I} \langle q_z, \psi_\alpha \rangle \overline{\psi_\alpha(w)} = \sum_{\alpha \in I} \psi_\alpha(z) \overline{\psi_\alpha(w)},$$

which implies (59). □

A **reproducing kernel Hilbert space** is a Hilbert space $\mathbb{K}$ of functions on a set Z together with a **reproducing kernel** $K : Z \times Z \to \mathbb{C}$ such that the functions[5] k_z $(z \in Z)$ defined by

$$k_z(x) := K(x, z) \tag{60}$$

span a space dense in $\mathbb{K}$ and satisfy

$$\psi(z) = k_z^* \psi \quad \text{for all } \psi \in \mathbb{K},\ z \in Z. \tag{61}$$

If we define for $\psi \in \overline{\mathbb{Q}}$ the function $\widetilde{\psi} : Z \to \mathbb{C}$ by

$$\widetilde{\psi}(z) := \psi(\widetilde{z}),$$

(51) says that $\widetilde{\mathbb{Q}} := \{\psi \mid \widetilde{\psi} \in \overline{\mathbb{Q}}\}$ is a reproducing kernel Hilbert space with reproducing kernel K and $k_z = \widetilde{q}_z$.

Proposition 5.2 is related to Mercer's theorem (Mercer [46]), which represents certain reproducing kernels by an infinite sum of the form

$$K(z, w) = \sum_{\alpha \in I} \lambda_\alpha \phi_\alpha(z) \overline{\phi_\alpha(w)}$$

with positive real numbers λ_α and functions ϕ_α satisfying additional properties. Precise statements of Mercer's theorem and its generalizations (e.g., Ferreira & Menegatto [26]) require additional structure on Z and K concerning measurability and continuity, hence are not valid in the generality discussed here. We therefore refrain here from giving details, and refer to a future paper in this series for the discussion of measure theoretic properties of coherent states and associated overcompleteness relations.

5.3 Theorems by Bochner and Kreĭn

Bochner [19, Satz 4] proved the following optimality result for q_x.

5.3 Theorem. *(Bochner)*
Let $K : Z \times Z \to \mathbb{C}$ be of positive type, and let $\mathbb{Q}$ be the space constructed in the Moore–Aronszejn theorem (Theorem 5.1). If $x \in Z$ satisfies $K(x, x) \neq 0$ then

$$\min\{\psi^* \psi \mid \psi \in \mathbb{Q},\ \psi(x) = \alpha\} = \frac{|\alpha|^2}{K(x, x)}.$$

The minimum is attained just for $\psi = \dfrac{\alpha}{K(x, x)} q_x$. In particular, if $\alpha = K(x, x)$, the minimum is attained precisely at q_x.

[5] Slightly more generally, a reproducing kernel Hilbert space may be defined as a Hilbert space $\mathbb{K}$ of functions on a set Z with an involution $^-$ together with a reproducing kernel $K : Z \times Z \to \mathbb{C}$ such that the functions k_z $(z \in Z)$ defined by $k_z(x) := K(\overline{x}, z)$ span a space dense in $\mathbb{K}$ and satisfy $\psi(z) = k_{\overline{z}}^* \psi$ for all $\psi \in \mathbb{K},\ z \in Z$. If we define, with an arbitrary choice of an involution $\sim$ on Z, for $\psi \in \overline{\mathbb{Q}}$ the function $\widetilde{\psi} : Z \to \mathbb{C}$ by $\widetilde{\psi}(z) := \psi(\widetilde{z})$, (51) says that $\widetilde{\mathbb{Q}} := \{\psi \mid \widetilde{\psi} \in \overline{\mathbb{Q}}\}$ is a reproducing kernel Hilbert space with reproducing kernel K and $k_z = \widetilde{q}_z$. This generalization (which just amounts to a relabeling of the arguments of the functions k_z) is useful when considering sets Z with the structure of a complex manifold, and wants the functions k_z to be analytic rather than antianalytic.

Proof. This is trivial for $\alpha = 0$. For $\alpha \neq 0$ we may rescale the assertion; thus it is enough to prove the case $\alpha = K(x,x)$. In this case

$$\begin{aligned}\psi^*\psi &= \langle \psi - q_x, \psi - q_x\rangle + 2\,\mathrm{Re}\langle q_x, \psi\rangle - \langle q_x, q_x\rangle \\ &= \langle \psi - q_x, \psi - q_x\rangle + 2\,\mathrm{Re}\,\psi(x) - K(x,x) = \|\psi - q_x\|^2 + K(x,x) \geq K(x,x) = \alpha,\end{aligned}$$

with equality iff $\psi - q_x = 0$. ☐

Our next result, a variant of KREĬN [43], characterizes which functions $\psi \in \mathbb{Q}^\times$ belong already to the Hilbert space $\overline{\mathbb{Q}}$.

5.4 Theorem. *(Kreĭn)*
Let $K : Z \times Z \to \mathbb{C}$ be of positive type and $\psi : Z \to \mathbb{C}$. Define the function $K_\varepsilon : Z \times Z \to \mathbb{C}$ by

$$K_\varepsilon(z,z') := K(z,z') - \varepsilon\psi(z)\overline{\psi(z')}.$$

(i) If $\psi \in \overline{\mathbb{Q}}$ and $0 < \varepsilon \leq \|\psi\|^{-2}$ then K_ε is of positive type.

(ii) If K_ε is of positive type for some $\varepsilon > 0$ then $\psi \in \overline{\mathbb{Q}}$.

Proof. (i) Hermiticity is obvious. To show that K_ε is of positive type we need to show for any finite sequence of complex numbers u_k and points $z_k \in Z$ the nonnegativity of the sum

$$\sigma := \sum_{j,k} \overline{u}_j K_\varepsilon(z_j, z_k) u_k = \sum_{j,k} \overline{u}_j \langle q_{z_j}, q_{z_k}\rangle u_k - \varepsilon \sum_{j,k} \overline{u}_j \psi(z_j)\overline{\psi(z_k)} u_k,$$

where we used (50). Writing

$$q := \sum_k q_{z_k} u_k,$$

we find that

$$\langle \psi, q\rangle = \sum_k \langle \psi, q_{z_k}\rangle u_k = \sum_k \overline{\langle q_{z_k}, \psi\rangle} u_k = \sum_k \overline{\psi(z_k)} u_k,$$

hence

$$\sigma = \|q\|^2 - \varepsilon|\langle \psi, q\rangle|^2 \geq \|q\|^2 - \varepsilon\|\psi\|^2\|q\|^2 \geq 0.$$

(ii) In this case, with $\mathbb{F}$ and $\widehat{f}$ as in the proof of the Moore–Aronszejn theorem (Theorem 5.1), we consider the antilinear mapping $\Psi : \mathbb{F} \to \mathbb{C}$ defined by

$$\Psi(f) := \sum_{z \in Z} \overline{f(z)}\psi(z).$$

Since K_ε is of positive type, we have

$$\begin{aligned}0 &\leq \sum_{z,z'} \overline{f(z)} K_\varepsilon(z,z') f(z') = \sum_{z,z'} \overline{f(z)} K(z,z') f(z') - \varepsilon \sum_{z,z'} \overline{f(z)}\psi(z)\overline{\psi(z')} f(z') \\ &= (f,f) - \varepsilon|\Psi(f)|^2 = \|\widehat{f}\|^2 - \varepsilon|\Psi(f)|^2\end{aligned}$$

by definition of K_ε, (55), (57), and the definition of Ψ. Therefore

$$|\Psi(f)| \le \varepsilon^{-1/2}\|\widehat{f}\|.$$

In particular, $\widehat{f} = 0$ implies $\Psi(f) = 0$. Therefore Ψ defines a unique antilinear mapping $\widehat{\Psi} : \mathbb{Q} \to \mathbb{C}$ with $\widehat{\Psi}(\widehat{f}) = \Psi(f)$ for all $f \in \mathbb{F}$. By the above, $|\widehat{\Psi}(\widehat{f})| \le \varepsilon^{-1/2}\|\widehat{f}\|$. Thus $\widehat{\Psi}$ is bounded, hence can be completed to a continuous antilinear functional (denoted the same) on $\overline{\mathbb{Q}}$. Now the Riesz representation theorem (Theorem 2.7) implies the existence of a unique $\psi' \in \overline{\mathbb{Q}}$ such that

$$\widehat{\Psi}(\phi) = \langle \phi, \psi' \rangle \quad \text{for all } \psi \in \overline{\mathbb{Q}}.$$

Since by (58) and (51),

$$\psi(z) = \Psi(g_z) = \widehat{\Psi}(\widehat{g}_z) = \widehat{\Psi}(q_z) = \langle q_z, \psi' \rangle = \psi'(z) \tag{62}$$

for all $z \in Z$, we conclude that $\psi = \psi' \in \overline{\mathbb{Q}}$. □

5.4 Theorems by Schoenberg and Menger

In this subsection we prove the promised converse of Propositions 2.9–2.10.

5.5 Theorem. *(Schoenberg [73, p.49])*
If F is conditionally positive then the function P_a, defined for any $a \in Z$ by

$$P_a(z, z') := F(z, z') - F(z, a) - F(a, z') + F(a, a), \tag{63}$$

is of positive type. Conversely, if a map $F : Z \times Z \to \mathbb{C}$ is such that if P_a is of positive type for some *$a \in Z$ then F is conditionally positive.*

Proof. Let $G, \widetilde{G}$ be the Gram matrices of $z_1, \dots, z_n$ computed with F and P_a, respectively. Then

$$\widetilde{G} = G - g\mathbf{1}^* - \mathbf{1}g^* + \gamma\mathbf{1}\mathbf{1}^*,$$

where $\mathbf{1}$ is the all-one column vector, g the column vector with components $g_j := F(z_j, a)$, and $\gamma := F(a, a)$. The Gram matrix of $z_1, \dots, z_n, a$ computed with F is therefore

$$G' := \begin{pmatrix} G & g \\ g^* & \gamma \end{pmatrix}.$$

Now $v \in \mathbb{C}^{n+1}$ satisfies $\displaystyle\sum_j v_j = 0$ iff, for some $u \in \mathbb{C}^n$,

$$v = \begin{pmatrix} u \\ -s \end{pmatrix}, \quad s = \mathbf{1}^* u,$$

and then

$$\begin{aligned} v^*G'v &= \begin{pmatrix} u \\ -s \end{pmatrix}^* \begin{pmatrix} G & g \\ g^* & \gamma \end{pmatrix} \begin{pmatrix} u \\ -s \end{pmatrix} = u^*Gu - u^*gs - \overline{s}g^*u + \gamma\overline{s}s \\ &= u^*(G - g\mathbf{1}^* - \mathbf{1}g^* + \gamma\mathbf{1}\mathbf{1}^*)u = u^*\widetilde{G}u. \end{aligned}$$

This shows that F is conditionally positive iff all $\widetilde{G}$ are positive semidefinite, i.e., iff P_a is of positive type for some a and hence for all a. $\square$

5.6 Theorem. *A map $F : Z \times Z \to \mathbb{C}$ is conditionally positive iff there is an embedding $z \to q_z$ of Z into a Euclidean space $\mathbb{H}$ such that*

$$F(z, z') = \overline{f(z)} + f(z') + q_z^* q_{z'} \tag{64}$$

holds for some $f : Z \to \mathbb{C}$.

Proof. (i) Suppose that F is conditionally positive. Fix a and define P_a by (63). By Theorem 5.5, P_a is of positive type. Hence the Moore–Aronszejn theorem (Theorem 5.1) gives an embedding $z \to q_z$ into a Hilbert space such that

$$P_a(z, z') = q_z^* q_{z'}, \tag{65}$$

applying (50) of the theorem to P_a in place of K. The definition of P_a then implies

$$F(z, z') - F(z, a) - F(a, z') + F(a, a) = q_z^* q_{z'}.$$

Putting $z = z' = a$ gives $q_a^* q_a = 0$, hence $q_a = 0$. One now easily verifies that

$$D(z, z') := F(z, z') - q_z^* q_{z'}$$

satisfies $\overline{D(z, z')} = D(z', z)$ and

$$D(z, z') - D(z, a) - D(a, z') + D(a, a) = 0.$$

This implies that $D(z, z') = \overline{f(z)} + f(z')$ with

$$f(z) := D(a, z) - \frac{1}{2}D(a, a).$$

Therefore (64) holds.

(ii) Conversely, if (64) holds then (65) and (63) imply that $P_a(z, z') = \langle q_z - q_a, q_{z'} - q_a \rangle$, hence P_a is of positive type. By Schoenberg's Theorem 5.5, F is conditionally positive. $\square$

The following converse of Proposition 2.10 is related to results by Menger [45] in the context of characterizing metric spaces embeddable into a finite-dimensional real vector space.

5.7 Corollary. *A map $F : Z \times Z \to \mathbb{C}$ satisfying $F(z, z') = F(z', z)$ for $z, z' \in Z$ is conditionally positive iff there is an embedding $z \to q_z$ of Z into a real Euclidean space such that*

$$F(z, z') = g(z) + g(z') - \|q_z - q_{z'}\|^2 \tag{66}$$

holds for some $g : Z \to \mathbb{R}$.

Proof. If there is such an embedding then F is conditionally positive by Proposition 2.10. Conversely, suppose that F is conditionally positive. Then $\frac{1}{2}F$ is also conditionally positive. By Theorem 5.6, there is an embedding $z \to q_z$ of Z into a complex Euclidean space $\mathbb{H}$ such that

$$\frac{1}{2}F(z, z') = \overline{f(z)} + f(z') + q_z^* q_{z'}$$

holds for some $f : Z \to \mathbb{C}$. Since we assumed $F(z, z') = F(z', z)$ for $z, z' \in Z$, $f(z)$ is real. Thus the inner products $K(z, z') = q_z^* q_{z'}$ are also real, and the q_z span a real Euclidean space. Substitution of $g(z) = 2f(z) + \|q_z\|^2$ now shows that (66) holds. $\square$

5.5 The Berezin–Wallach set

Many coherent products of interest have the exponential form discussed in the following theorem. It is due to Schoenberg [72] in the case where F takes only finite values and is zero on the diagonal, to Herz [33, Proposition 6] in the case where F takes only finite values, and to Horn [36] in the general case. The present proof is much shorter than Horn's.

To be able to formulate the results, we put

$$e^{-\infty} := 0$$

and call a function $F : Z \times Z \to \mathbb{C} \cup \{-\infty\}$ **conditionally positive** if either (i) there is an equivalence relation $\equiv$ on Z such that F is conditionally positive on each equivalence class, and $F(z, z') = -\infty$ whenever $z \not\equiv z'$, or (ii) F takes only infinite values. This reduces to the original definition if the value $-\infty$ is not attained, which holds iff there is only one equivalence classe.

5.8 Theorem.
(i) If $F : Z \times Z \to \mathbb{C} \cup \{-\infty\}$ is conditionally positive then, for all $\beta > 0$,

$$K(z, z') := e^{\beta F(z,z')} \tag{67}$$

is of positive type.

(ii) Let $F : Z \times Z \to \mathbb{C} \cup \{-\infty\}$. If there is a sequence of positive numbers β_k converging to 0 such that

$$K_k(z, z') := e^{\beta_k F(z,z')}$$

is of positive type for all k then F is conditionally positive.

Proof. (i) If F takes only finite values then Theorem 5.5 shows that (for any $z_0 \in Z$), the function $\widetilde{F}$ defined by

$$\widetilde{F}(z,z') := \beta(F(z,z') - F(z,z_0) - F(z_0,z') + F(z_0,z_0))$$

is of positive type. Theorem 2.14 therefore implies that $K(z,z') := e^{\widetilde{F}(z,z')}$ defines a function K of positive type. Rescaling this by Proposition 2.12(iv), we see that (67) is of positive type, too. If F takes infinite values only, K is identically zero and hence of positive type. If F takes finite and infinite values, the previous argument may be applied to the restriction of K to each equivalence class, and shows that this restriction is of positive type. Then Proposition 2.12(vi) implies that K itself is of positive type.

(ii) We may assume w.l.o.g. that Z cannot be decomposed as in Proposition 2.12(vi).

Case 1: $K(z,z') \neq 0$ for all $z,z' \in Z$. We fix $a \in Z$ and use Proposition 2.12(iv) to rescale $K := e^{\beta F}$ (consistently for all β) such that all $K(a,z) = 1$, hence all $F(a,z)$ vanish. Theorem 5.5, applied with K_k in place of F, implies that the map $P_a : Z \times Z \to \mathbb{C}$ defined by

$$P_a(z,z') = K_k(z,z') - K_k(z,a) - K_k(a,z') + K_k(a,a) = K_k(z,z') - 1$$

is of positive type. Therefore the functions F_k defined by

$$F_k(z,z') := \frac{K_k(z,z') - 1}{\beta_k} = \frac{e^{\beta_k F(z,z')} - 1}{\beta_k} = F(z,z') + \beta_k F(z,z')^2 + O(\beta_k)^2$$

are also of positive type. Since $\beta_k \to 0$, $F_k(z,z') \to F(z,z')$ for $k \to \infty$. By Proposition 2.12(vii), F is of positive type. Undoing the scaling and using Theorem 5.5 now proves that F is conditionally positive.

Case 2: $K(z,z) = 0$ for some $z \in Z$. Then the positivity of the Gram matrix (14) for $n = 2$ implies that $K(z,z') = K(z',z) = 0$ for all $z' \in Z$, and the indecomposability assumed at the beginning of (ii) implies that $Z = \{z\}$ and K is identically zero. Thus F takes only infinite values and is therefore conditionally positive.

Case 3: $K(z,z) \neq 0$ for all $z \in Z$ but $K(x,y) = 0$ for some $x,y \in Z$. By Proposition 2.12(iv) we may normalize K such that $K(z,z) = 1$ for all $z \in Z$. The Gram matrix

$$G = \begin{pmatrix} K(x,x) & K(x,y) & K(x,z) \\ K(y,x) & K(y,y) & K(y,z) \\ K(z,x) & K(z,y) & K(z,z) \end{pmatrix} = \begin{pmatrix} 1 & K(x,y) & K(x,z) \\ K(y,x) & 1 & K(y,z) \\ K(z,x) & K(z,y) & 1 \end{pmatrix}$$

of $x,y,z \in Z$ is Hermitian and poitive semidefinite, hence its determinant is nonnegative,

$$0 \le 1 - |K_k(x,z)|^2 - |K_k(y,z)|^2 = 1 - |K(x,z)|^{2\beta_k} - |K(y,z)|^{2\beta_k}.$$

Unless at least one of $K(x,z)$ or $K(y,z)$ vanishes, the two negative terms tend for $k \to \infty$ both to 1, hence the right hand side converges to -1. This holds for any z, whence Z can be split into two subsets X and Y such that $K(x,z) = 0$ for $z \in Y$ and $K(y,z) = 0$ for $z \in X$. By Hermiticity, $K(z,x) = 0$ for $z \in Y$ and $K(z,y) = 0$ for $z \in X$. Repeating the argument for all zeros constructible this way shows that Z decomposes as in Proposition 2.12(iii), contradiction. $\square$

If we want to discuss a possible generalization of Theorem 2.14 to other exponents we need to assume that the power exists, which suggests to assume for $K(z, z')$ an exponential form.

The **Berezin–Wallach set** of a mapping $F : Z \times Z \to \mathbb{C} \cup \{-\infty\}$ is the set $W(F)$ of nonnegative real numbers β for which

$$K(z, z') := e^{\beta F(z,z')} \tag{68}$$

is of positive type. The **Berezin–Wallach set** of a coherent space is the set $W(F)$ where

$$F(z, z') := \log K(z, z'),$$

using the principal value of the logarithm and $\log 0 = -\infty$. (Thus the Berezin–Wallach set of a coherent space always contains 1.)

This set was introduced by WALLACH [79] in the context of representations of Lie groups. But already earlier, BEREZIN [15] computed the Berezin–Wallach set for the case when F is the Kähler potential of a Siegel domain. Indeed, in many cases of interest, Z is a so-called Kähler manifold and F the associated Kähler potential; see, e.g., ZHANG et al. [81, Section VI]. For the Berezin–Wallach sets corrsponding to Hermitian symmetric spaces see, e.g., FARAUT & KORÁNYI [25, Section XIII.2].

5.9 Theorem.
(i) The Berezin–Wallach set $W(F)$ is a closed set containing 0.

(ii) $W(F)$ contains with β and β' their sum and hence all linear combinations with nonnegative integral coefficients.

(iii) If $W(F)$ contains an open set it contains all sufficiently large positive real numbers.

(iv) If F is conditionally positive then $W(F)$ contains all nonnegative real numbers.

(v) If 0 *is a limit point of $W(F)$ then F is conditionally positive.*

Proof. (i)–(iv) follow easily from Proposition 2.13(ii). (v) follows from Theorem 5.8. $\square$

In the most interesting cases, the Berezin–Wallach set is of the form $\alpha\mathbb{N}_0 \cup [\beta, \infty]$ or $\alpha\mathbb{N}_0$, where $\alpha, \beta > 0$ and $\mathbb{N}_0$ denotes the set of nonnegative integers. In general, the Berezin–Wallach set may have a very complicated structure, already for Z with three elements only. FITZGERALD & HORN [27] show that the Berezin–Wallach set of every finite coherent space Z with real, nonnegative coherent product contains the interval $[|Z| - 2, \infty[$.

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