Introduction to coherent quantization

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arXiv:1804.01400

April 4, 2018

Abstract.

This paper is the second in a series of papers on coherent spaces and their applications. It begins the study of coherent quantization – the way operators in a quantum space can be studied in terms of objects defined directly on the coherent space. The results may be viewed as a generalization of geometric quantization to the non-unitary case.

Care has been taken to work with the weakest meaningful topology and to assume as little as possible about the spaces and groups involved. Unlike in geometric quantization, the groups are not assumed to be compact, locally compact, or finite-dimensional. This implies that the setting can be successfully applied to quantum field theory, where the groups involved satisfy none of these properties.

The paper characterizes linear operators acting on the quantum space of a coherent space in terms of their coherent matrix elements. Coherent maps and associated symmetry groups for coherent spaces are introduced, and formulas are derived for the quantization of coherent maps.

The importance of coherent maps for quantum mechanics is due to the fact that there is a quantization operator that associates homomorphically with every coherent map a linear operator from the quantum space into itself. This operator generalizes to general symmetry groups of coherent spaces the second quantization procedure for free classical fields. The latter is obtained by specialization to Klauder spaces, whose quantum spaces are the bosonic Fock spaces. A coordinate-free derivation is given of the basic properties of creation and annihilation operators in Fock spaces.

For the discussion of questions concerning coherent spaces, please use the discussion forum `https://www.physicsoverflow.org`.

MSC2010 classification: 81S10 (primary) 81R15, 47B32, 46C50, 43A35, 46E22

Contents

1 Introduction

This paper is the second in a series of papers [17] on coherent spaces and their applications. It begins the study of coherent quantization – the way operators in a quantum space can be studied in terms of objects defined directly on the coherent space.

Care has been taken to work with the weakest meaningful topology and to assume as little as possible about the groups involved. In particular, unlike in geometric quantization (Bates & Weinstein [3], Woodhouse [28]), we do not assume the groups to be compact, locally compact, or finite-dimensional. This implies that the setting can be successfully applied to quantum field theory, where the groups involved satisfy none of these properties.

More specifically, we characterize linear operators acting on the quantum space of a coherent space in terms of their coherent matrix elements. We discuss coherent maps and associated symmetry groups for the coherent spaces introduced in Neumaier [18], and derives formulas for the quantization of coherent maps.

An early paper by Itô [12] describes unitary group representations in terms of what are now called (generalized) coherent states. Group theoretic work on the subject was greatly extended by Perelomov [24, 25], Gilmore [7], and others; see, e.g., the survey by Zhang et al. [29]. The present setting may be viewed as a generalization of this to the non-unitary case.

The importance of coherent maps for quantum mechanics is due to the fact proved in Theorem 3.12 below that there is a **quantization operator** Γ that associates homomorphically with every coherent map A a linear operator $\Gamma(A)$ on the augmented quantum space $\mathbb{Q}^\times(Z)$ that maps the quantum space $\mathbb{Q}(Z)$ into itself. This operator generalizes the **second quantization** procedure of free classical fields to general symmetry groups of coherent spaces.

In follow-up papers (Neumaier & Ghaani Farashahi [20, 21]) from the present series, we shall introduce additional differentiability structure that turns the present quantization procedure into an even more powerful tool.

Contents. In the present section we review notation, terminology, and some results of Neumaier [18], on which the present paper is based.

Section 2 provides fundamental but abstract necessary and sufficient conditions for recognizing when a kernel, i.e., a map from a coherent space into itself is a shadow (i.e., definable in terms of coherent matrix elements), and hence determines an operator on the corresponding quantum space.

Section 3 discusses symmetries of a coherent space, one of the most important concepts for studying and using coherent spaces. Indeed, most of the applications of coherent spaces in quantum mechanics and quantum field theory rely on the presence of a large symmetry group. The main reason is that there is a quantization map that furnishes a representation of the semigroup of coherent maps on the quantum space, and thus provides easy access to a class of very well-behaved linear operators on the quantum space.

Section 4 looks at self-mappings of coherent spaces satisfying homogeneity or separability properties. These often give simple but important coherent maps.

In Section 5 we prove quantization theorems for a restricted class of coherent spaces for which many operators on a quantum space have a simple description in terms of normal kernels. These generalize the normal ordering of operators familiar from quantum field theory.

In the final Section 6 we discuss in some detail the coherent quantization of Klauder spaces, a class of coherent spaces with a large semigroup of coherent maps, introduced in NEUMAIER [18]. The corresponding coherent states are closely related to those introduced by SCHRÖDINGER [27]) and made prominent in quantum optics by GLAUBER [8]. The quantum spaces of Klauder spaces are the bosonic Fock spaces, which play a very important role in quantum field theory (BAEZ et al. [2], GLIMM & JAFFE [9]), and the theory of Hida distributions in the white noise calculus for classical stochastic processes (HIDA & SI [10], HIDA & STREIT [11], OBATA [22]). In particular, we give a coordinate-free derivation of the basic properties of creation and annihilation operators in Fock spaces.

Acknowledgments. Thanks to Rahel Knöpfel, David Bar Moshe, and Hermann Schichl for useful discussions related to the subject.

1.1 Euclidean spaces

In this paper, we use the notation and terminology of NEUMAIER [18], quickly reviewed here.

We write $\mathbb{C}$ for the field of complex numbers and $\mathbb{C}^\times$ for the multiplicative group of nonzero complex numbers. $\mathbb{C}^X$ denotes the vector space of all maps from a set X to $\mathbb{C}$.

A **Euclidean space** is a complex vector space $\mathbb{H}$ with a Hermitian form that assigns to $\phi, \psi \in \mathbb{H}$ the **Hermitian inner product** $\langle \phi, \psi \rangle \in \mathbb{C}$, antilinear in the first and linear in the second argument, such that

$$\overline{\langle \phi, \psi \rangle} = \langle \psi, \phi \rangle, \tag{1}$$

$$\langle \psi, \psi \rangle > 0 \quad \text{for all } \psi \in \mathbb{H} \setminus \{0\}. \tag{2}$$

Associated with $\mathbb{H}$ is the triple of spaces

$$\mathbb{H} \subseteq \overline{\mathbb{H}} \subseteq \mathbb{H}^\times, \tag{3}$$

where $\overline{\mathbb{H}}$ is a Hilbert space completion of $\mathbb{H}$, and $\mathbb{H}$ is dense in the vector space $\mathbb{H}^\times$ of all continuous antilinear functionals on $\mathbb{H}$, with

$$\psi(\phi) := \langle \phi, \psi \rangle \quad \text{for } \phi \in \mathbb{H}.$$

Here and later, continuity is always understood in the weak sense.

$\mathbb{H}^\times$ carries a Hermitian partial inner product $\phi^*\psi$ with

$$\phi^*\psi = \langle \phi, \psi \rangle \quad \text{for } \phi, \psi \in \overline{\mathbb{H}}.$$

and is a PIP space in the sense of ANTOINE & TRAPANI [1].

Let U and V be (complex) topological vector spaces. We write $\mathrm{Lin}\,(U,V)$ for the vector space of all continuous linear mappings from U to V, $\mathrm{Lin}\,U$ for $\mathrm{Lin}\,(U,U)$, and $U^\times$ for the **antidual** of U, the space of all continuous linear mappings from U to $\mathbb{C}$. We identify V with the space $\mathrm{Lin}\,(\mathbb{C},V)$. We write $\mathrm{Lin}^\times \mathbb{H} := \mathrm{Lin}\,(\mathbb{H},\mathbb{H}^\times)$ for the vector space of continuous linear operators from a Euclidean space $\mathbb{H}$ to its antidual. If $\mathbb{H}_1,\mathbb{H}_2$ are Euclidean spaces and $A \in \mathrm{Lin}\,(\mathbb{H}_1,\mathbb{H}_2)$, the adjoint operator is $A^* \in \mathrm{Lin}\,(\mathbb{H}_2^\times,\mathbb{H}_1^\times)$.

1.2 Coherent spaces

A **coherent space** is a nonempty set Z with a distinguished function $K : Z \times Z \to \mathbb{C}$ of positive type called the **coherent product**. Thus

$$\overline{K(z,z')} = K(z',z), \tag{4}$$

and for all $z_1,\dots,z_n \in Z$, the $n \times n$ matrix G with entries $G_{jk} = K(z_j,z_k)$ is positive semidefinite.

The coherent space Z is called **nondegenerate** if

$$K(z'',z') = K(z,z') \;\; \forall\, z' \in Z \quad \Rightarrow \quad z'' = z.$$

For any coherent space Z, $[Z]$ denotes the nondegenerate coherent space defined in [18, Proposition 4.11], with the same quantum spaces as Z.

The coherent space Z is called **projective** if there is a **scalar multiplication** that assigns to each $\lambda \in \mathbb{C}^\times$ and each $z \in Z$ a point $\lambda z \in Z$ such that

$$K(z',\lambda z) = \lambda^e K(z',z) \tag{5}$$

for some $e \in \mathbb{Z} \setminus \{0\}$ called the **degree**. Equivalently,

$$|\lambda z\rangle = \lambda^e |z\rangle.$$

For any coherent space Z, the **projective extension** of Z of degree e (a nonzero integer) is the coherent space $PZ := \mathbb{C}^\times \times Z$ with coherent product

$$K_{\mathrm{pe}}((\lambda,z),(\lambda',z')) \;:=\; \overline{\lambda}^e\, K(z,z')\, \lambda'^e \tag{6}$$

and scalar multiplication $\lambda'(\lambda,z) := (\lambda'\lambda,z)$, defined in [18, Proposition 4.9], with the same quantum spaces as Z.

Throughout the paper, Z is a fixed coherent space with coherent product K. A **quantum space** $\mathbb{Q}(Z)$ of Z is a Euclidean space spanned (algebraically) by a distinguished set of vectors $|z\rangle$ ($z \in Z$) called **coherent states** satisfying

$$\langle z|z'\rangle \;=\; K(z,z') \quad \text{for } z,z' \in Z, \tag{7}$$

where $\langle z| := |z\rangle^*$. The associated **augmented quantum space** $\mathbb{Q}^\times(Z)$, the antidual of $\mathbb{Q}(Z)$, contains the **completed quantum space** $\overline{\mathbb{Q}}(Z)$, the Hilbert space completion of $\mathbb{Q}(Z)$. The following results were proved in [18] (Proposition 3.5 and Theorem 3.6):

1.1 Proposition. *Let Z be a coherent space, let $\mathbb{Q}(Z)$ be a quantum space of Z, and let $\psi : \mathbb{Q}(Z) \to \mathbb{C}$ be an antilinear functional. Then $\psi \in \mathbb{Q}(Z)^\times$ (i.e., ψ is continuous) iff*

$$|\psi(\phi)| \leq M \sum_{w\in W} \Big|\langle w|\phi\Big| \quad \textit{for all } \phi \in \mathbb{Q}(Z) \tag{8}$$

for some non-negative constant M and some finite subset W of Z.

1.2 Theorem. *Let Z be a coherent space and $\mathbb{Q}(Z)$ be a quantum space of Z. Also, let $\mathbf{X} : \mathbb{Q}(Z) \to \mathbb{Q}(Z)^\times$ be a linear map. Then $\mathbf{X} \in \operatorname{Lin}^\times \mathbb{Q}(Z)$ (i.e., $\mathbf{X}$ is continuous) iff for every $z \in Z$ there exists a constant $M > 0$ and a finite subset W of Z such that*

$$\Big|\langle z|\mathbf{X}\psi\Big| \leq M \sum_{w\in W} |\langle w|\psi\rangle| \quad \textit{for all } \psi \in \mathbb{Q}(Z). \tag{9}$$

2 Quantization through admissibility conditions

We regard the **quantization** of a coherent space Z as the problem of describing interesting classes of linear operators from $\operatorname{Lin}^\times \mathbb{Q}(Z)$ and their properties in terms of objects more tangibly defined on Z. The key to coherent quantization is the observation that one can frequently define and manipulate operators on the quantum space in terms of their coherent matrix elements, without needing a more explicit description in terms of differential or integral operators on a Hilbert space of functions.

A **kernel** is a map $X \in \mathbb{C}^{Z\times Z}$. The **shadow** of a linear operator $\mathbf{X} \in \operatorname{Lin}^\times \mathbb{Q}(Z)$ is the kernel $\operatorname{Sh}\mathbf{X} \in \mathbb{C}^{Z\times Z}$ defined by (cf. Klauder [13])

$$\operatorname{Sh}\mathbf{X}(z,z') := \langle z|\mathbf{X}|z'\rangle \quad \text{for } z,z' \in Z.$$

Thus shadows represent the information in the **coherent matrix elements** $\langle z|\mathbf{X}|z'\rangle$ of an operator $\mathbf{X}$.

This section discusses admissibility conditions. They provide fundamental but abstract necessary and sufficient conditions for recognizing when a kernel is a shadow and hence determines an operator $\mathbf{X} \in \operatorname{Lin}^\times \mathbb{Q}(Z)$. Later sections then provide applications to more concrete situations.

The admissibility conditions are infinite generalizations of the simple situation when Z is finite. In this case we may w.l.o.g. take $Z = \{1, 2, \dots, n\}$ and regard kernels as $n \times n$ matrices. Then the coherent producxt is just a positive semidefinite matrix $K = R^*R$, and the shadow of an operator $\mathbf{X}$ is $X = R^*\mathbf{X}R$. Admissibility of X, here equivalent with strong admissibility, is the condition that for any column vector c, $Rc = 0$ implies $Xc = 0$ and $X^*c = 0$, which forces X to have at most the same rank as K. It is not difficult to see (and follows from the results below) that this condition implies that X has the form $X = R^*\mathbf{X}R$ for some matrix $\mathbf{X}$, so that an admissible X is indeed a shadow.

2.1 Admissibility

Let Z be a coherent space with the coherent product K. We want to characterize the functions $f : Z \to \mathbb{C}$ for which there is a $\psi : \mathbb{Q}(Z) \to \mathbb{C}$ such that

$$f(z) = \langle z|\psi \quad \text{for all } z \in Z. \tag{10}$$

We call a function $f : Z \to \mathbb{C}$ **admissible** if for arbitrary finite sequences of complex numbers c_k and points $z_k \in Z$,

$$\sum c_k K(z_k, z) = 0 \ \ \forall\, z \in Z \quad \Rightarrow \quad \sum c_k f(z_k) = 0, \tag{11}$$

and **strongly admissible** if there are a constant $M \geq 0$ and a finite subset W of Z such that

$$\Big|\sum_k c_k f(z_k)\Big| \leq M \sum_{w\in W} \Big|\sum_k c_k K(z_k, w)\Big|, \tag{12}$$

for arbitrary finite sequences of complex numbers c_k and points $z_k \in Z$. Clearly, (12) implies (11); thus strongly admissible functions are admissible.

For example, for $K(z, z') = 0$ for all z, z' one gets a trivial coherent space whose quantum space is $\{0\}$, and only the zero function is admissible. On the other hand, a condition guaranteeing that every map is strongly admissible is given in Proposition 5.1 below.

2.1 Theorem. *Let Z be a coherent space. For a quantum space $\mathbb{Q}(Z)$ of Z, the following conditions on a function $f : Z \to \mathbb{C}$ are equivalent.*

(i) There is an antilinear functional $\psi : \mathbb{Q}(Z) \to \mathbb{C}$ such that (10) *holds.*

(ii) For arbitrary finite sequences of complex numbers c_k and points $z_k \in Z$,

$$\sum c_k |z_k\rangle = 0 \quad \Rightarrow \quad \sum \overline{c}_k f(z_k) = 0. \tag{13}$$

(iii) f is admissible.

Moreover, in (i), ψ is uniquely determined by f.

Proof. (ii)$\Leftrightarrow$(i): Let $f : Z \to \mathbb{C}$ be a function satisfying (13). We define the antilinear functional $\psi \in \mathbb{Q}(Z) \to \mathbb{C}$ by

$$\psi\Big(\sum c_k |z_k\rangle\Big) := \sum \overline{c_k} f(z_k) \quad \text{for all } \sum c_k |z_k\rangle \in \mathbb{Q}(Z). \tag{14}$$

Because of (13), ψ is well-defined; it is clearly antilinear. Thus, ψ defines an antilinear functional on the quantum space $\mathbb{Q}(Z)$. Specializing (14) to the case of a sum containing a single term only gives

$$\langle z|\psi = \psi(|z\rangle) = f(z) \quad \text{for } z \in Z,$$

so that ψ satisfies (10). If (10) also holds for ψ' in place of ψ then $\psi = \psi'$ since the coherent states span $\mathbb{Q}(Z)$. This shows that ψ is uniquely determined by f and (10).

Conversely, let $f : Z \to \mathbb{C}$ be a function that satisfies (10) for some antilinear functional ψ on $\mathbb{Q}(Z)$. If the left hand side of (11) holds then

$$\sum \bar{c}_k f(z_k) = \sum \bar{c}_k \langle z_k|\psi = \psi\Big(\sum c_k|z_k\rangle\Big) = 0.$$

(iii)⇔(ii): Clearly (11) is equivalent to

$$\sum \bar{c}_k K(z_k, z) = 0 \ \ \forall\, z \in Z \quad\Rightarrow\quad \sum \bar{c}_k f(z_k) = 0,$$

The left hand side of (13) is equivalent to

$$0 = \sum \bar{c}_k \langle z_k|z\rangle = \sum \bar{c}_k K(z_k, z') \quad \text{for } z \in Z.$$

Since

$$\langle z|\sum \bar{c}_k|z_k\rangle = \sum c_k K(z, z_k) = \overline{\sum \bar{c}_k K(z_k, z)},$$

this is equivalent to (13). Thus if f is admissible then (13) follows from (11). Conversely, the left hand side of (11) implies that

$$\Big(\sum c_k\langle z_k|\Big)|z\rangle = \sum c_k\langle z_k|z\rangle = 0 \quad \text{for } z \in Z,$$

hence $\sum c_k\langle z_k| = 0$. Therefore $\sum \bar{c}_k|z_k\rangle = 0$. If (ii) holds, we may substitute in (13) $\bar{c}_k$ for c_k and find that $\sum c_k f(z_k) = 0$. Hence (11) follows and f is admissible. □

2.2 Theorem. *A function $f : Z \to \mathbb{C}$ is strongly admissible iff there is a vector $\psi \in \mathbb{Q}(Z)^\times$ satisfying* (10). *In this case, ψ is uniquely determined by f.*

Proof. It is straightforward to check that any functiohe full classical symmetry group is visible only in the projective completion of the symmetric spaces, i.e., the associated line bundle, since the associated unitary representation of the symmetry groups of symmetric spaces are only projective representations.n satisfying (10) with a vector $\psi \in \mathbb{Q}(Z)^\times$ is strongly admissible. Conversely, let $f : Z \to \mathbb{C}$ be a strongly admissible function. Then f is admissible, and by Theorem 2.1, there exists a unique antilinear functional $\psi : \mathbb{Q}(Z) \to \mathbb{C}$ satisfying (10). We claim that ψ is continuous. To show this, let $\phi = \sum_k c_k|z_k\rangle \in \mathbb{Q}(Z)$.

Then by (12),

$$|\psi(\phi)| = \Big|\psi\Big(\sum_k c_k|z_k\rangle\Big)\Big| = \Big|\sum_k \bar{c}_k\langle z_k|\psi\Big| = \Big|\sum_k \bar{c}_k f(z_k)\Big| \le M \sum_{w\in W} \Big|\sum_k \bar{c}_k K(z_k, w_j)\Big|.$$

Now

$$\sum_k \bar{c}_k K(z_k, w_j) = \sum_k \bar{c}_k\langle z_k|w_j\rangle = \phi^*|w\rangle,$$

and we find

$$|\psi(\phi)| \le M \sum_{w\in W} \Big|\phi^*|w\rangle\Big| = M \sum_{w\in W} \Big|\langle w|\phi\Big| \quad \text{for } \phi \in \mathbb{Q}(Z).$$

This implies that ψ is continuous. Thus $\psi \in \mathbb{Q}(Z)^\times$. □

The **admissibility space** of Z is the set $\mathbb{A}(Z)$ of all strongly admissible functions over the coherent space Z. It is easy to see that $\mathbb{A}(Z)$ is a vector space with respect to pointwise addition of functions and pointwise multiplication by complex numbers.

2.3 Theorem. *Let Z be a coherent space and let $\mathbb{Q}(Z)$ be a quantum space of Z.*

(i) For every strongly admissible function $f : Z \to \mathbb{C}$,

$$\theta_f\Big(\sum c_k|z_k\rangle\Big) := \sum \overline{c_k} f(z_k) \quad \text{for } \sum c_k|z_k\rangle \in \mathbb{Q}(Z), \tag{15}$$

defines a continuous antilinear functional on $\mathbb{Q}(Z)$.

*(ii) The **identification map** $\Theta : \mathbb{A}(Z) \to \mathbb{Q}(Z)^\times$ given by*

$$\Theta(f) := \theta_f \tag{16}$$

is a vector space isomorphism. In particular, the admissibility space $\mathbb{A}(Z)$ can be equipped with a locally convex topology into a locally convex space such that the linear map Θ is a homeomorphism.

Proof. (i) Let $f : Z \to \mathbb{C}$ be an strongly admissible function. By Theorem 2.2, $\theta_f = \psi$ is the unique vector in $\mathbb{Q}(Z)^\times$ satisfying (10).

(ii) By (i), the linear map $\Theta : \mathbb{A}(Z) \to \mathbb{Q}(Z)^\times$ given by $\Theta(f) := \theta_f$ is a vector space homomorphism. Let $\psi \in \mathbb{Q}(Z)^\times$ be a given continuous antilinear functional and define $f : Z \to \mathbb{C}$ via $f(z) := \langle z|\psi$, for all $z \in Z$. Then, it is easy to check that $f \in \mathbb{A}(Z)$ and $\theta_f = \psi$. Thus Θ is an isomorphism of topological vector spaces. □

2.4 Corollary. *Let Z be a coherent space. The admissible spaces $\mathbb{A}(Z)$, $\mathbb{A}(PZ)$, and $\mathbb{A}([Z])$ are canonically isomorphic as topological vector space.*

2.2 Kernels and shadows

For any kernel X we define the related kernels X^T, $\overline{X}$, and X^* by

$$X^T(z,z') := X(z',z), \quad \overline{X}(z,z') := \overline{X(z,z')}, \quad X^*(z,z') := \overline{X(z',z)}.$$

Clearly,

$$X^{TT} = \overline{\overline{X}} = X^{**} = X, \quad X^* = \overline{X}^T = \overline{X^T}.$$

For example, any coherent product is a kernel K; it is Hermitian iff $K^T = K$. Given a kernel X and $z \in Z$, we define the functions $X(z,\cdot), X(\cdot,z) \in \mathbb{C}^Z$ by

$$X(\cdot,z)(z') := X(z',z), \quad X(z,\cdot)(z') := X(z,z') \quad \text{for } z' \in Z.$$

2.5 Proposition.
(i) The shadow of the identity operator 1 *is* $\operatorname{Sh} 1 = K$.

(ii) For $\mathbf{X} \in \operatorname{Lin}^{\times} \mathbb{Q}(Z)$, *we have*

$$\langle z|\mathbf{X}^*|z'\rangle = \overline{\langle z'|\mathbf{X}|z\rangle} \quad \textit{for all } z, z' \in Z,$$

$$(\operatorname{Sh} \mathbf{X})^* = \operatorname{Sh} \mathbf{X}^*.$$

Proof. (i) holds since $\operatorname{Sh} 1(z, z') = \langle z|1|z'\rangle = \langle z|z'\rangle = K(z, z')$ for all $z, z' \in Z$.

(ii) Theorem 1.2 gives $\mathbf{X}^* \in \operatorname{Lin}^{\times} \mathbb{Q}(Z)$. Then, for $z, z' \in Z$,

$$\langle z|\mathbf{X}^*|z'\rangle = \mathbf{X}^*|z'\rangle(|z\rangle) = \overline{\mathbf{X}|z\rangle(|z'\rangle)} = \overline{\langle z'|\mathbf{X}|z\rangle},$$

$$\operatorname{Sh} \mathbf{X}^*(z, z') = \langle z|\mathbf{X}^*|z'\rangle = \overline{\langle z'|\mathbf{X}|z\rangle} = (\operatorname{Sh} \mathbf{X})^*(z, z').$$

□

The following characterization of shadows is the fundamental theorem on which all later quantization results are based.

2.6 Theorem. *Let* Z *be a coherent space and let* $X \in \mathbb{C}^{Z \times Z}$ *be a kernel.*

(i) X *is a shadow iff* $X(z, \cdot)$ *and* $\overline{X}(\cdot, z)$ *are strongly admissible for all* $z \in Z$. *If this holds, there is a unique operator* $\mathbf{X} \in \operatorname{Lin}^{\times} \mathbb{Q}(Z)$ *whose shadow is* X, *i.e.,*

$$\langle z|\mathbf{X}|z'\rangle = X(z, z') \quad \textit{for } z, z' \in Z. \tag{17}$$

(ii) If $X(z, \cdot)$ *and* $\overline{X}(\cdot, z)$ *are admissible for all* $z \in Z$, *there is a unique linear mapping* $\mathbf{X}$ *from* $\mathbb{Q}(Z)$ *to its algebraic dual such that* (17) *holds. (Equivalently,* $\mathbf{X}$ *defines a Hermitian form on* $\mathbb{Q}(Z)$.*)*

Proof. (i) Let $z \in Z$, $\mathbf{X} \in \operatorname{Lin}^{\times} \mathbb{Q}(Z)$, and $X := \operatorname{Sh} \mathbf{X}$. Then $\mathbf{X}|z\rangle \in \mathbb{Q}(Z)^{\times}$, and by Proposition 1.1, there is an $M > 0$ and a finite subset W of Z such that

$$|\psi^* \mathbf{X}|z\rangle| = |\mathbf{X}|z\rangle(\psi)| \le M \sum_{w \in W} |\langle w|\psi| \quad \text{for all } \psi \in \mathbb{Q}(Z).$$

For $z_1, ..., z_n \in Z$ and $c_1, ..., c_n \in \mathbb{C}$ we have

$$\overline{X}(z_\ell, z) = \overline{\operatorname{Sh} \mathbf{X}(z_\ell, z)} = \overline{\langle z_\ell|\mathbf{X}|z\rangle},$$

hence

$$\begin{aligned}
\Big|\sum_\ell c_\ell \overline{X}(z_\ell, z)\Big| &= \Big|\sum_\ell \overline{c_\ell} \langle z_\ell|\mathbf{X}|z\rangle\Big| = \Big|\Big(\sum_\ell c_\ell |z_\ell\rangle\Big)^* \mathbf{X}|z\rangle\Big| \\
&\le M \sum_{w \in W} \Big|\langle w|\Big(\sum_\ell c_\ell |z_\ell\rangle\Big)\Big| \le M \sum_{w \in W} \Big|\sum_\ell \overline{c_\ell} \langle w|z_\ell\rangle\Big| \\
&= M \sum_{w \in W} \Big|\sum_\ell c_\ell \overline{K(w, z_\ell)}\Big| = M \sum_{w \in W} \Big|\sum_\ell c_\ell K(z_\ell, w)\Big|.
\end{aligned}$$

This implies that $\overline{X}(.,z)$ is strongly admissible. We then claim that $X(z,\cdot)$ is strongly admissible. Using Proposition 2.5(ii), for $z,z' \in Z$, we have

$$X(z,z') = \langle z|\mathbf{X}|z'\rangle = \overline{\langle z'|\mathbf{X}^*|z\rangle} = \overline{\mathrm{Sh}\,\mathbf{X}^*}(z',z).$$

Hence, we have $X(z,\cdot) = \overline{\mathrm{Sh}\,\mathbf{X}^*}(\cdot,z)$ for all $z \in Z$. This implies that $X(z,.)$ is strongly admissible as well.

Conversely, let X be a kernel such that $\overline{X}(\cdot,z)$ and $X(z,\cdot)$ are strongly admissible for all $z \in Z$. For given vectors $\phi = \sum_k c'_k|z'_k\rangle$ and $\psi = \sum_\ell c_\ell|z_\ell\rangle \in \mathbb{Q}(Z)$, define the complex number

$$(\psi,\phi)_X := \sum_\ell \sum_k \overline{c_\ell} c'_k X(z_\ell, z'_k).$$

For the moment, fix ϕ. Since $\overline{X}(\cdot,z'_k)$ is strongly admissible, there exist finite subsets W_k of Z and positive constants M_k such that

$$\begin{aligned}\Big|\sum_\ell \overline{c_\ell} X(z_\ell,z'_k)\Big| &= \Big|\sum_\ell c_\ell \overline{X(z_\ell,z'_k)}\Big| = \Big|\sum_\ell c_\ell \overline{X}(z_\ell,z'_k)\Big| \\ &\le M_k \sum_{w\in W_k}\Big|\sum_\ell c_\ell K(z_\ell,w)\Big|.\end{aligned}$$

Thus

$$\begin{aligned}|(\psi,\phi)_X| &= \Big|\sum_k\sum_\ell c'_k\overline{c_\ell}X(z_\ell,z'_k)\Big| \le \sum_k |c'_k|\Big|\sum_\ell \overline{c_\ell}X(z_\ell,z'_k)\Big| \\ &\le \sum_k |c'_k| M_k \sum_{w\in W_k}\Big|\sum_\ell c_\ell K(z_\ell,w)\Big|\end{aligned}$$

for all $\psi = \sum c_\ell|z_\ell\rangle \in \mathbb{Q}(Z)$. Thus

$$\psi \to \psi^*\mathbf{X}\phi := (\psi,\phi)_X, \quad \text{for all } \psi \in \mathbb{Q}(Z)$$

defines a continuous antilinear functional $\mathbf{X}\phi : \mathbb{Q}(Z) \to \mathbb{C}$, that is $\mathbf{X}\phi \in \mathbb{Q}(Z)^\times$. Clearly, $\phi \to \mathbf{X}\phi$ defines a linear map $\mathbf{X} : \mathbb{Q}(Z) \to \mathbb{Q}(Z)^\times$.

Now suppose that $z \in Z$. Since the function $X(z,\cdot)$ is strongly admissible, there is a constant $M > 0$ and a finite subset W of Z satisfying (12). We then have

$$\begin{aligned}|\langle z|\mathbf{X}\phi| &= \Big|\sum_k c'_k X(z,z'_k)\Big| \le M\sum_{w\in W}\Big|\sum_k c'_k K(z'_k,w)\Big| \\ &= M\sum_{w\in W}\Big|\sum_k \overline{c'_k}\overline{K(z'_k,w)}\Big| = M\sum_{w\in W}\Big|\sum_k \overline{c'_k}K(w,z'_k)\Big| \\ &= M\sum_{w\in W}\Big|\sum_k \overline{c'_k}\langle w|z'_k\rangle\Big| = M\sum_{w\in W}\Big|\langle w|\Big(\sum_k c'_k|z'_k\rangle\Big)\Big| = M\sum_{w\in W}\Big|\langle w|\phi\Big|,\end{aligned}$$

for all $\phi \in \mathbb{Q}(Z)$. Thus Theorem 1.2 implies that $\mathbf{X} \in \mathrm{Lin}^\times\,\mathbb{Q}(Z)$. It is easy to check that $\langle z|\mathbf{X}|z'\rangle = X(z,z')$ for all $z,z' \in Z$. Thus $\mathrm{Sh}\,\mathbf{X} = X$. Finally, it can be readily checked that $\mathbf{X}$ is the unique operator which satisfies $X = \mathrm{Sh}\,\mathbf{X}$. This proves (i).

Under the conditions (ii) we cannot prove continuity, hence only get an mapping from $\mathbb{Q}(Z)$ to its algebraic dual. □

3 Coherent maps and their quantization

This section discusses symmetries of a coherent space, one of the most important concepts for studying and using coherent spaces. Indeed, most of the applications of coherent spaces in quantum mechanics and quantum field theory rely on the presence of a large symmetry group. The main reason is that – as we show in Theorem 3.12 below – there is a quantization map that furnishes a representation of the semigroup of coherent maps on the quantum space, and thus provides easy access to a class of very well-behaved linear operators on the quantum space.

Let Z, Z' be coherent spaces. Recall from Neumaier [18] that a **morphism** from Z to Z' is a map $\rho : Z \to Z'$ such that

$$K'(\rho z, \rho w) = K(z, w) \quad \text{for } z, w \in Z; \tag{18}$$

if $Z' = Z$, ρ is called an **endomorphism**. Two coherent spaces Z and Z' are called **isomorphic** if there is a bijective morphism $\rho : Z \to Z'$. In this case we write $Z \cong Z'$ and we call the map $\rho : Z \to Z'$ an **isomorphism** of the coherent spaces. Clearly, $\rho^{-1} : Z' \to Z$ is then also an isomorphism.

In the spirit of category theory one should define the symmetries of a coherent space Z in terms of its **automorphisms**, i.e., isomorphisms from Z to itself. Remarkably, however, coherent spaces allow a significantly more general concept of symmetry, based on the notion of a coherent map.

3.1 Coherent maps

Let Z and Z' be coherent spaces with coherent products K and K', respectively. A map $A : Z' \to Z$ is called **coherent** if there is an **adjoint map** $A^* : Z \to Z'$ such that

$$K(z, Az') = K'(A^*z, z') \quad \text{for } z \in Z,\ z' \in Z' \tag{19}$$

If Z' is nondegenerate, the adjoint is unique, but not in general. A coherent map $A : Z' \to Z$ is called an **isometry** if it has an adjoint satisfying $A^*A = 1$. A **coherent map** on Z is a coherent map from Z to itself.

A **symmetry** of Z is an invertible coherent map on Z with an invertible adjoint. We call a coherent map A **unitary** if it is invertible and $A^* = A^{-1}$. Thus unitary coherent maps are isometries.

3.1 Example. An **orbit** of a group $\mathbb{G}$ acting on a set S is a set consisting of all images Ax ($A \in \mathbb{G}$) of a single vector. The group is **transitive** on S if S is an orbit. The orbits of groups of linear self-mappings of a Euclidean space give coherent spaces with predefined transitive symmetry groups. Indeed, in the coherent space formed by an arbitrary subset Z of a Euclidean space with coherent product $K(z,z') := z^*z'$, all linear operators mapping Z into itself are coherent maps, and all linear operators mapping Z bijectively onto itself are symmetries. This is the reason why coherent spaces are important in the theory of group representations.

For example, the symmetric group Sym(5) acts as a group of Euclidean isometries on the 12 points of the icosahedron in $\mathbb{R}^3$. The coherent space consisting of these 12 points with the induced coherent product therefore has Sym(5) as a group of unitary symmetries. The skeleton of the icosahedron is a distance-regular graph, here a double cover of the complete graph on six vertices. As shown in NEUMAIER [19], many more interesting examples of finite coherent spaces are related to Euclidean representations of distance regular graphs (BROUWER et al. [4]) and other highly symmetric combinatorial objects.

3.2 Proposition.
(i) Every unitary coherent map is a symmetry.

(ii) The map $A : Z \to Z$ is a unitary coherent map iff

$$K(Az, Az') = K(z, z') \quad \text{for all } z, z' \in Z.$$

Proof. (i) A^{-1} exists and is coherent by the preceding since $A^{-1} = A^*$.

(ii) Replace z in (19) by Az. □

3.3 Proposition.
(i) Every morphsim A with right inverse A' is coherent, with adjoint $A^* = A'$.

(ii) Every isometry is a morphism.

(iii) A map $A : Z \to Z$ is an automorphism of Z iff it is a unitary coherent map.

Proof. (i) Put $A^* := A'$. Then $AA^* = 1$, and we have $K(z, Az') = K(AA^*z, Az') = K(A^*z, z')$, proving the claim.

(ii) Let $A : Z \to Z'$ be an isometry. Then, for $z, z' \in Z$,

$$K'(Az, Az') = K(z, A^*Az') = K(z, z').$$

(iii) Let $A : Z \to Z$ be an automorphism of Z. Since A is a morphism and invertible, for $z, z' \in Z$, we get

$$K(z, Az') = K(AA^{-1}z, Az') = K(A^{-1}z, z').$$

This implies that A is coherent with $A^* := A^{-1}$ with $A^*A = AA^* = 1$. Hence A is unitary. Conversely, assume that $A : Z \to Z$ is a unitary coherent map. Then Proposition 3.2(ii) implies that A is a morphism. Since A is bijective, it is an automorphism of Z as well. □

3.4 Proposition. *Let Z be a coherent space and $A : Z \to Z$ be a coherent map. Then for $z, z' \in Z$,*

$$K(Az, z') = K(z, A^*z'), \tag{20}$$

$$\langle z|Az'\rangle = \langle A^*z|z'\rangle, \quad \langle Az|z'\rangle = \langle z|A^*z'\rangle. \tag{21}$$

Proof. For $z, z' \in Z$, (4) implies

$$\langle Az|z'\rangle = K(Az, z') = \overline{K(z', Az)} = \overline{K(A^*z', z)} = K(z, A^*z') = \langle z|A^*z'\rangle.$$

This proves both (20) and the second half of (21). The first half of (21) follows directly from (19). □

3.5 Proposition. *Coherent maps are continuous in the metric topology.*

Proof. $z_\ell \to z$ in the the metric topology says by definition [18] that

$$K(z_\ell, z') \to K(z, z') \quad \text{for all } z' \in Z.$$

If this holds and A is a coherent map then

$$K(Az_\ell, z') = K(z_\ell, A^*z') \to K(z, A^*z') = K(Az, z'),$$

so that $Az_\ell \to Az$. Thus A is continuous in the metric topology. □

3.6 Theorem. *Let Z be a coherent space. Then the set* $\mathrm{Coh}\, Z$ *consisting of all coherent maps is a semigroup with identity. Moreover:*

(i) Any adjoint A^* of $A \in \mathrm{Coh}\, Z$ is coherent.

(ii) For any invertible coherent map $A : Z \to Z$ with an invertible adjoint, the inverse A^{-1} is coherent.

Proof. The identity map $I : Z \to Z$ is trivially coherent. Let $A, B \in \mathrm{Coh}\, Z$. Then, for $z, z' \in Z$,

$$K(z, ABz') = K(A^*z, Bz') = K(B^*A^*z, z'),$$

which implies that AB is coherent with adjoint $(AB)^* = B^*A^*$.

(i) Using Proposition 3.4, we can write

$$K(z, A^*z') = K(Az, z'),$$

which implies that A^* is coherent with $A^{**} = A$.

(ii) Let $A : Z \to Z$ be a coherent map with an adjoint A^* such that A and A^* are invertible with the inverses A^{-1} and $(A^*)^{-1}$. Then, for $z, z' \in Z$,

$$K(A^{-1}z, z') = K(A^{-1}z, A^*(A^*)^{-1}z') = K(AA^{-1}z, (A^*)^{-1}z') = K(z, (A^*)^{-1}z'),$$

which implies that A^{-1} is coherent with $(A^{-1})^* = (A^*)^{-1}$. □

3.7 Corollary. *Let Z be a nondegenerate coherent space. Then* $\operatorname{Coh} Z$ *is a $*$-semigroup with identity, i.e.,*

$$1^* = 1, \quad A^{**} = A, \quad (AB)^* = B^*A^* \quad \text{for } A, B \in \operatorname{Coh} Z.$$

Moreover, the set $\operatorname{sym}(Z)$ *of all invertible coherent maps with invertible adjoint is a $*$-group, and*

$$A^{-*} := (A^{-1})^* = (A^*)^{-1} \quad \text{for } A \in \operatorname{sym}(Z).$$

Proof. If Z is nondegenerate then the adjoint is unique. Therefore the claim follows from the preceding result. □

3.8 Proposition. *Let Z be a coherent space. Then,*

$$\operatorname{Coh}[Z] = \{[A] \mid A \in \operatorname{Coh} Z\}.$$

Proof. Let $A \in \operatorname{Coh} Z$. Then $[A] \in \operatorname{Coh}[Z]$ by [18, Theorem 4.12]. Thus $\{[A] : A \in \operatorname{Coh} Z\} \subseteq \operatorname{Coh}[Z]$. Let $\imath : [Z] \to Z$ be a choice function, that is a function which satisfies $[\imath[z]] = [z]$ for all $z \in Z$. For any coherent map $\mathcal{A} : [Z] \to [Z]$, define $A : Z \to Z$ by $z \to Az := \imath(\mathcal{A}[z])$. Then $A : Z \to Z$ is a well-defined map. Thus, for $z, z' \in Z$,

$$\begin{aligned} K(Az, z') &= K(\imath(\mathcal{A}[z]), z') = K([\imath(\mathcal{A}[z])], [z']) = K(\mathcal{A}[z], [z']) \\ &= K([z], \mathcal{A}^*[z']) = K([z], [\imath(\mathcal{A}^*[z'])]) = K(z, \imath(\mathcal{A}^*[z'])). \end{aligned}$$

This implies that A is a coherent map with an adjoint $A^* : Z \to Z$ given by $A^*z = \imath(\mathcal{A}^*[z'])$. For $z, z' \in Z$, we have

$$K([A][z], [z']) = K([Az], [z']) = K([\imath(\mathcal{A}[z])], [z']) = K(\mathcal{A}[z], [z']),$$

implying that $[A] = \mathcal{A}$. □

3.9 Theorem. *Let PZ be the projective extension of degree 1 of the coherent space Z.*

(i) Let $A : Z \to Z$ be a map with the property

$$K(z, Az')v(z') = \overline{w(z)}K(A^*z, z') \quad \text{for } z, z' \in Z, \tag{22}$$

for suitable $v, w : Z \to \mathbb{C}$ and $A^* : Z \to Z$. Then

$$[\alpha, A](\lambda, z) := (\alpha v(z)\lambda, Az), \quad [\alpha, A]^*(\lambda, z) := (\overline{\alpha}w(z)\lambda, A^*z)$$

define a coherent map $[\alpha, A]$ of PZ and its adjoint $[\alpha, A]^*$.

(ii) For every coherent map $A : Z \to Z$ and every $\alpha \in \mathbb{C}$, the map $[\alpha, A] : PZ \to PZ$ defined by

$$[\alpha, A](\lambda, z) := (\alpha\lambda, Az) \quad \text{for all } (\lambda, z) \in PZ,$$

is coherent.

Proof. Let $(\lambda, z), (\lambda', z') \in PZ$. Then

$$K_{\mathrm{pe}}((\lambda, z), (\lambda', z')) = \overline{\lambda} K(z, z')\lambda'.$$

Therefore,

$$\begin{aligned}
K_{\mathrm{pe}}((\lambda, z), [\alpha, A](\lambda', z')) &= K_{\mathrm{pe}}((\lambda, z), (\alpha v(z')\lambda', Az')) = \overline{\lambda} K(z, Az')\alpha v(z')\lambda' \\
&= \overline{\lambda\overline{\alpha}} K(z, Az')v(z')\lambda' = \overline{\lambda\overline{\alpha}w(z)} K(A^*z, z')\lambda' \\
&= K_{\mathrm{pe}}((\lambda\overline{\alpha}w(z), A^*z), (\lambda', z')) = K_{\mathrm{pe}}([\alpha, A]^*(\lambda, z), (\lambda', z')).
\end{aligned}$$

This proves (i), and (ii) is the special case of (i) where v and w are identically 1. □

Something similar can be shown for projective extensions of any integral degree $e \neq 0$.

In the applications, a group $\mathbb{G}$ of quantum symmetries is typically first defined classically on a symmetric space. In a quantization step, it is then represented by a unitary representation on a Hilbert space. Typically, unitary representations of the symmetry groups of symmetric spaces are only projective representations, defined in terms of a family of multipliers satisfying a cocycle condition. Therefore, in geometric quantization (Woodhouse [28]), the symmetric space (typically a Kähler manifold) needs to be extended to a line bundle on which a central extension of the group acts classically, and this central extension (defined through the respective cocycle) is represented linearly in the Hilbert space defined through the geomentric quantization procedure.

In the coherent space setting, the coherent product defined on an orbit Z of $\mathbb{G}$ on the symmetric space Z via the coherent states available from geometric quantization leads in these cases to a coherent space. However, on this space, most elements of $\mathbb{G}$ are not represented coherently since they only satisfy a relation (22) with multipliers that are not constant. Theorem 3.9 shows that the projective extension PZ of degree 1 represents the central extensiion coherently. This shows that projective coherent spaces are the natural starting point for coherent quantization since they represent all classically visible symmetries in a coherent way. The projective property is therefore typically needed whenever one has a quantum system given in terms of a coherent space and wants to describe all symmetries of the quantum system through coherent maps.

3.2 Some examples

We now give two simple examples demonstrating that related coherent spaces with the same quantum space can have very different symmetry groups, the large groups being associated with projective coherent spaces. Another important example of this situation, though with different details, is treated extensively in Section 6.

3.10 Example. (Szegö [26], 1911) The **Szegö space** (a special case of [18, Example 3.12(i)]) is the coherent space defined on the open unit disk in $\mathbb{C}$,

$$D(0,1) := \{z \in \mathbb{C} \mid |z| < 1\},$$

by the coherent product

$$K(z, z') := (1 - \overline{z}z')^{-1}.$$

the inverse is defined since $|\overline{z}z'| < 1$. A corresponding quantum space is the **Hardy space** of power series

$$f(x) = \sum_{k=0}^{\infty} f_\ell x^\ell$$

such that

$$\|f\| := \sqrt{\sum |f_\ell|^2} < \infty,$$

describing analytic functions on Z that are square integrable over the positively oriented boundary ∂Z of Z, with inner product

$$f^* g := \sum \overline{f}_\ell g_\ell = \int_0^{2\pi} d\phi \overline{f(e^{i\phi})} g(e^{i\phi}) = \int_{\partial Z} |dz| \overline{f(z)} g(z).$$

The associated coherent states are the functions

$$k_z(x) = (1 - zx)^{-1}$$

with $(k_z)_\ell = z^\ell$, since

$$k_z^* k_{z'} = \sum \overline{z}^\ell (z')^\ell = \frac{1}{1 - \overline{z}z'} = K(z, z').$$

The set of coherent maps is easily seen to consist only of the scalar multiplication maps $z \to \lambda z$ for $\lambda \in \mathbb{C}$, with the complex conjugate as adjoint.

3.11 Example. The **Möbius space** $Z = \{z \in \mathbb{C}^2 \mid |z_1| > |z_2|\}$ is a coherent space with coherent product

$$K(z, z') := (\overline{z}_1 z'_1 - \overline{z}_2 z'_2)^{-1}$$

with the same quantum spaces as the Szegö space. Indeed, the functions

$$f_z(x) = (z_1 - z_2 x)^{-1} \tag{23}$$

from the Szegö space from Example 3.10 are associated **Möbius coherent states**. The Möbius space is a projective coherent space of degree -1; indeed, with the scalar multiplication induced from $\mathbb{C}^2$, we have

$$K(z, \lambda z') = (\overline{z}_1 \lambda z'_1 - \overline{z}_2 \lambda z'_2)^{-1} = \lambda^{-1} (\overline{z}_1 z'_1 - \overline{z}_2 z'_2)^{-1} = \lambda^{-1} K(z, z')$$

for all $z, z' \in Z$. It is now easy to see that the projective completion of the Szegö space for this degree is isomorphic to the Möbius space.

Unlike the Szegö space, the Möbius space has a large symmetry group. Indeed, if $A \in \mathbb{C}^{2\times 2}$, put

$$\alpha := |A_{11}|^2 - |A_{21}|^2, \quad \beta := \overline{A}_{11} A_{12} - \overline{A}_{21} A_{22}, \quad \gamma := |A_{22}|^2 - |A_{12}|^2,$$

If the inequalities

$$\alpha > 0, \quad |\beta| \le \alpha, \quad \gamma \le \alpha - 2|\beta| \tag{24}$$

hold and $z \in Z$ then, with $\beta = |\beta|\delta$, where $|\delta| = 1$,

$$\begin{array}{rcl} |(Az)_1|^2 - |(Az)_2|^2 & = & \alpha|z_1|^2 + 2\,\mathrm{Re}(\beta\overline{z}_1 z_2) - \gamma|z_2|^2 \\ & \geq & \alpha|z_1|^2 + 2\,\mathrm{Re}(\beta\overline{z}_1 z_2) + (2|\beta| - \alpha)|z_2|^2 \\ & = & |\beta|\,|z_1 + \delta z_2|^2 + (\alpha - |\beta|)(|z_1|^2 - |z_2|^2) \geq 0. \end{array}$$

Equality in the last step is possible only if $|\beta| = \alpha > 0$ and $z_1 + \delta z_2 = 0$, contradicting $|z_1| > |z_2|$. Hence $Az \in Z$. Thus A maps Z into itself whenever (24) holds. Now

$$\begin{array}{rcl} K(z, Az') & = & \Big(\overline{z}_1(A_{11}z'_1 + A_{12}z'_2) - \overline{z}_2(A_{21}z'_1 + A_{22}z'_2)\Big)^{-1} \\ & = & \Big((A_{11}\overline{z}_1 - A_{21}\overline{z}_2)z'_1 - (-A_{12}\overline{z}_1 + A_{22}\overline{z}_2)z'_2\Big)^{-1} = K(A^\sigma z, z'), \end{array}$$

where

$$A^\sigma = \begin{pmatrix} \overline{A}_{11} & -\overline{A}_{21} \\ -\overline{A}_{12} & \overline{A}_{22} \end{pmatrix}.$$

Thus every linear mapping $A \in \mathbb{C}^{2\times 2}$ satisfying (24) is coherent, with adjoint A^* given by A^σ rather than by the standard matrix adjoint. These mappings form a semigroup, a homogeneous version of an Olshanski semigroup of compressions (Olshanskii [23]). We have $\beta = 0$ iff

$$\overline{A}_{11}A_{12} = \overline{A}_{21}A_{22}, \quad \alpha = |A_{11}|^2 - |A_{21}|^2, \quad \gamma = |A_{22}|^2 - |A_{11}|^2. \tag{25}$$

(To get the last formula, solve the first for A_{12}, substitute it into $\alpha + \gamma \geq 0$, and divide by $1 - |A_{21}/A_{11}|^2$.) The $A \in \mathbb{C}^{2\times 2}$ satisfying (25) and $\gamma = \alpha > 0$ preserve the Hermitian form $|z_1|^2 - |z_2|^2$ up to a positive factor α. Thus the group $GU(1,1)$ of all these matrices is a group of symmetries of Z. This fact is relevant for applications to quantum systems with a dynamical symmetry group $SU(1,1)$ or the closely related groups $SO(2,1)$, $SL(2,\mathbb{R})$.

This example generalizes to central extensions of other semisimple Lie groups and associated line bundles over symmetric spaces. This follows from the material on the corresponding coherent states discussed in detail in Perelomov [25] from a group theoretic point of view, and in Zhang et al. [29] in terms of applications to quantum mechanics. Other related material is in the books by Faraut & Korányi, [6], Neeb [15], and Neretin [16].

3.3 Quantization of coherent maps

3.12 Theorem. *Let Z be a coherent space, $\mathbb{Q}(Z)$ a quantum space of Z, and let A be a coherent map on Z.*

(i) There is a unique linear map $\Gamma(A) \in \mathrm{Lin}\,\mathbb{Q}(Z)$ such that

$$\Gamma(A)|z\rangle = |Az\rangle \quad \textit{for all } z \in Z. \tag{26}$$

(ii) For any adjoint map A^* of A,

$$\langle z|\Gamma(A) = \langle A^*z| \quad \textit{for all } z \in Z, \tag{27}$$

$$\Gamma(A)^*|_{\mathbb{Q}(Z)} = \Gamma(A^*). \tag{28}$$

(iii) $\Gamma(A)$ *can be extended to a linear map* $\Gamma(A) := \Gamma(A^*)^* \in \operatorname{Lin} \mathbb{Q}^x(Z)$, *and this extension maps* $\overline{\mathbb{Q}}(Z)$ *into itself.*

We call $\Gamma(A)$ and its extension the **quantization**[1] of A and Γ the **quantization map**.

Proof. (i) Let $A : Z \to Z$ be a coherent map and $S : Z \times Z \to \mathbb{C}$ be the kernel given by

$$S(z, z') := K(z, Az') \quad \text{for all } z, z' \in Z.$$

Then $\overline{S}(z, \cdot), S(\cdot, z)$ are strongly admissible functions, for all $z \in Z$. To show this, let $z \in Z$ and $c_1, ..., c_n \in \mathbb{C}$ and $z_1, ..., z_n \in Z$. Then

$$\begin{aligned}\Big|\sum c_\ell \overline{S}(z, z_\ell)\Big| &= \Big|\sum c_\ell \overline{S(z, z_\ell)}\Big| = \Big|\sum c_\ell \overline{K(z, Az_\ell)}\Big| \\ &= \Big|\sum c_\ell K(Az_\ell, z)\Big| = \Big|\sum c_\ell K(z_\ell, A^*z)\Big| = \Big|\sum c_\ell K(z_\ell, w)\Big|\end{aligned}$$

Therefore the function $\overline{S}(z, \cdot)$ is strongly admissible with $M = 1$ and $W = \{A^*z\}$. A similar argument guarantees that $S(\cdot, z)$ is strongly admissible with $M = 1$ and $W = \{Az\}$. Invoking Theorem 2.6, let $\Gamma(A) : \mathbb{Q}(Z) \to \mathbb{Q}(Z)^\times$ be the unique continuous linear operator satisfying

$$S(z, z') = \langle z|\Gamma(A)|z'\rangle \quad \text{for all } z, z' \in Z. \tag{29}$$

To prove the theorem we need to show that the images are actually in $\mathbb{Q}(Z)$. Using (26), we have

$$\langle z|Az'\rangle = K(z, Az') = S(z, z') = \langle z|\Gamma(A)|z'\rangle \quad \text{for all } z, z' \in Z. \tag{30}$$

which implies that $\Gamma(A)|z'\rangle = |Az'\rangle$ for all $z' \in Z$. We conclude that $\Gamma(A)$ maps $\mathbb{Q}(Z)$ already into the smaller space $\mathbb{Q}(Z)$. Hence $\Gamma(A) \in \operatorname{Lin} \mathbb{Q}(Z)$.

(ii) Let $z \in Z$ and $\phi = \sum c_k|z_k\rangle \in \mathbb{Q}(Z)$. Then (27) follows from

$$\begin{aligned}\Big(\Gamma(A)^*|z\rangle\Big)(\phi) &= \langle z|\Gamma(A)\phi = \langle z|\Gamma(A)\sum c_k|z_k\rangle = \langle z|\sum c_k\Gamma(A)|z_k\rangle \\ &= \langle z|\sum c_k|Az_k\rangle = \sum c_k\langle z|Az_k\rangle = \sum c_k\langle A^*z|z_k\rangle \\ &= \langle A^*z|\sum c_k|z_k\rangle = \langle A^*z|\phi,\end{aligned}$$

By Theorem 3.6(i), the map A^* is coherent as well. Thus we have

$$\begin{aligned}\langle z|\Gamma(A)^*|z'\rangle &= \overline{\langle z'|\Gamma(A)|z\rangle} = \overline{\langle z'|Az\rangle} = \overline{K(z', Az)} \\ &= K(Az, z') = K(z, A^*z') = \langle z|A^*z'\rangle = \langle z|\Gamma(A^*)|z'\rangle,\end{aligned}$$

which implies that the restriction of $\Gamma(A)^*$ into $\mathbb{Q}(Z)$ is precisely $\Gamma(A^*)$, as claimed.

[1]In the literature (see, e.g., DEREZIŃSKI & GÉRARD [5]), $\Gamma(A)$ is called the **second quantization** of A when applied to the special case (treated in Subsection 6.3) where Z is a Klauder space and $\overline{\mathbb{Q}}(Z)$ is a Fock space.

(iii) is a simple consequence of (i) and (ii). □

We now show that the quantization map Γ furnishes a representation of the semigroup of coherent maps on Z in the quantum space of Z.

3.13 Theorem. *The quantization map Γ has the following properties.*

(i) The identity map 1 on Z is coherent, and $\Gamma(1) = 1$.

(ii) For any two coherent maps A, B on Z,

$$\Gamma(AB) = \Gamma(A)\Gamma(B).$$

(iii) For any invertible coherent map $A : Z \to Z$ with an invertible adjoint, $\Gamma(A)$ is invertible with inverse

$$\Gamma(A)^{-1} = \Gamma(A^{-1}).$$

(iv) For a coherent map $A : Z \to Z$, A is unitary iff $\Gamma(A)$ is unitary.

Proof. (i) is straightforward.

(ii) Let A, B be coherent maps and $z, z' \in Z$. Then we have

$$\langle z|\Gamma(AB)|z'\rangle = K(z, ABz') = \langle z|\Gamma(A)|Bz'\rangle = \langle z|\Gamma(A)\Gamma(B)|z'\rangle,$$

which implies that $\Gamma(AB) = \Gamma(A)\Gamma(B)$.

(iii) follows from $\Gamma(1) = 1$ and the fact that $AA^{-1} = A^{-1}A = 1$. Indeed, using Theorem 3.6(ii), A^{-1} is coherent and we get

$$\Gamma(A)\Gamma(A^{-1}) = \Gamma(AA^{-1}) = \Gamma(1) = \Gamma(A^{-1}A) = \Gamma(A^{-1})\Gamma(A),$$

which implies that $\Gamma(A)$ is invertible with $\Gamma(A)^{-1} = \Gamma(A^{-1})$.

(iv) Let A be a coherent map. Also, suppose that A is unitary as well. Then, A is invertible with the inverse $A^{-1} = A^*$. Thus, A and A^* are invertible. Then, we get

$$\Gamma(A)\Gamma(A)^* = \Gamma(A)\Gamma(A^*) = \Gamma(AA^*) = \Gamma(1) = 1,$$

and also

$$\Gamma(A)^*\Gamma(A) = \Gamma(A^*)\Gamma(A) = \Gamma(A^*A) = \Gamma(1) = 1.$$

Hence, we deduce that $\Gamma(A)$ is a unitary linear operator. Conversely, assume that $\Gamma(A)$ is a unitary linear operator. Then we get $AA^* = 1$ and also $A^*A = 1$, which means that A is unitary. □

The quantization map is important as it reduces many computations with coherent operators in the quantum space of Z to computations in the coherent space Z itself. By Theorem 3.13, large semigroups of coherent maps A produce large semigroups of coherent operators $\Gamma(A)$, which may make complex calculations much more tractable. Coherent spaces with many coherent maps are often associated with symmetric spaces in the sense of differential geometry. In this case, the linear differential operators can be coherently quantized, too, through weak-* limits of suitable linear combinations of operators of the form $\Gamma(A)$. This yields quantization procedures for Lie algebras defined by coherent differential operators on coherent spaces. Details will be given in Neumaier & Ghaani Farashahi [20].

4 Homogeneous and separable maps

In this section we look at self-mappings of coherent spaces satisfying homogeneity or separability properties. These often give simple but important coherent maps.

4.1 Homogeneous maps and multipliers

Let Z be a coherent space. We say that a function $m : Z \to \mathbb{C}$ is a **multiplier** for the map $A : Z \to Z$ if

$$|z'\rangle = \lambda|z\rangle \quad \Rightarrow \quad m(z')|Az'\rangle = \lambda m(z)|Az\rangle, \tag{31}$$

for all $\lambda \in \mathbb{C}$ and $z, z' \in Z$, equivalently if

$$K(w, z') = \lambda K(w, z)\ \forall w \in Z \quad \Rightarrow \quad m(z')K(w, Az') = \lambda m(z)K(w, Az)\ \forall w \in Z.$$

A function $m : Z \to \mathbb{C}$ is called **homogeneous** if

$$|z'\rangle = \lambda|z\rangle,\ \lambda \neq 0 \quad \Rightarrow \quad m(z') = m(z); \tag{32}$$

this is the case iff it is a multiplier for the identity map.

We call a map $A : Z \to Z$ **homogeneous** if

$$|z'\rangle = \lambda|z\rangle \quad \Rightarrow \quad |Az'\rangle = \lambda|Az\rangle; \tag{33}$$

this is the case iff $m = 1$ is a multiplier for A. We write $\operatorname{hom} Z$ for the set of all homogeneous maps $A : Z \to Z$.

4.1 Theorem. *Let Z be a coherent space. Then,*

(i) each coherent map is homogeneous.

(ii) the composition of any two homogeneous maps is homogeneous.

Proof. (i) Let A be coherent map with an adjoint A^*. Suppose that $z, z' \in Z$ and $\lambda \in \mathbb{C}^\times$ with $|z'\rangle = \lambda|z\rangle$. Then, for $z'' \in Z$, we get

$$\langle z''|Az'\rangle = \langle A^*z''|z'\rangle = \lambda\langle A^*z''|z\rangle = \lambda\langle z''|Az\rangle.$$

Thus $|Az'\rangle = \lambda|Az\rangle$. Therefore, $m = 1$ is a multiplier for A and hence A is homogeneous.

(ii) Let $A, B \in \hom(Z)$. Suppose that $z, z' \in Z$ and $\lambda \in \mathbb{C}^\times$ with $|z'\rangle = \lambda|z\rangle$. Since B is homogeneous, we have $|Bz'\rangle = \lambda|Bz\rangle$. Then applying homogeneity of A, we have $|ABz'\rangle = \lambda|ABz\rangle$. Therefore, $m = 1$ is a multiplier for AB and hence AB is homogeneous. □

4.2 Theorem. *Let Z be a projective coherent space. We then have*

$$K(z, \lambda z') = K(\overline{\lambda} z, z'),$$

for all $z, z' \in Z$ and $\lambda \in \mathbb{C}^\times$. In particular, if Z is a nondegenerate and projective coherent space the scalar multiplication map $\lambda : Z \to Z$ is coherent, with unique adjoint $\lambda^* = \overline{\lambda}$.

Proof. Let $\lambda \in \mathbb{C}^\times$ be given. Then, for $z, z' \in Z$, we have

$$K(\overline{\lambda} z, z') = \overline{K(z', \overline{\lambda} z)} = \overline{\overline{\lambda}^e K(z', z)} = \lambda^e \overline{K(z', z)} = \lambda^e K(z, z') = K(\lambda z, z').$$

In particular, if Z is nondegenerate then the multiplication map λ is coherent with the unique adjoint $\overline{\lambda}$. □

4.3 Proposition. *Let Z be a projective and non-degenerate coherent space. Then:*

(i) $m : Z \to \mathbb{C}$ is a multiplier for $A : Z \to Z$ iff

$$m(\mu z)|A\mu z\rangle = m(z)|\mu A z\rangle \quad \textit{for all } \mu \in \mathbb{C}^\times.$$

(ii) A map $A : Z \to Z$ is homogeneous iff $A\mu = \mu A$ for all $\mu \in \mathbb{C}^\times$.

(iii) A map $m : Z \to \mathbb{C}$ is homogeneous iff $m\mu = \mu m$ for all $\mu \in \mathbb{C}^\times$.

Proof. In a projective coherent space, $|\lambda z\rangle = \lambda^e|z\rangle$, so nondegeneracy implies that for any choice of the eth root,

$$|z'\rangle = \lambda|z\rangle \quad \Leftrightarrow \quad z' = \mu z, \; \mu = \lambda^{1/e}.$$

The definition of a multiplier now gives (i), and a straightforward specialization gives (ii) and (iii). □

4.2 Separable maps

Let Z be a coherent space. We call a map $\alpha : Z \to Z$ **separable** if there is a number $\chi(\alpha) \in \mathbb{C}$, called a **separation constant**, such that

$$K(z, \alpha z') = \chi(\alpha) K(z, z') \quad \text{for } z, z' \in Z. \tag{34}$$

4.4 Proposition. *Let Z be a coherent space and $\alpha : Z \to Z$ be a map. Then, α is separable iff there exists a complex constant λ_α, such that for any quantum space $\mathbb{Q}(Z)$ of Z we have*

$$|\alpha z\rangle = \lambda_\alpha |z\rangle \quad \textit{for all } z \in Z. \tag{35}$$

In this case, $\chi(\alpha) = \lambda_\alpha$.

Proof. Let $\mathbb{Q}(Z)$ be a quantum space of Z and $z, z' \in Z$. If α is separable with the separation constant $\chi(\alpha)$, then

$$\langle z'|\alpha z\rangle = K(z', \alpha z) = \chi(\alpha) K(z', z) = \chi(\alpha)\langle z'|z\rangle = \langle z'|\Big(\chi(\alpha)|z\rangle\Big).$$

Hence $|\alpha z\rangle = \chi(\alpha)|z\rangle$ and (35) holds with $\lambda_\alpha := \chi(\alpha)$. Conversely, suppose that (35) holds for some complex number λ_α. Then, for $z, z' \in Z$, we get

$$K(z', \alpha z) = \langle z'|\alpha z\rangle = \langle z'|\Big(\chi(\alpha)|z\rangle\Big) = \chi(\alpha)\langle z'|z\rangle = \chi(\alpha) K(z', z).$$

This implies that α is a separable map with the separation constant $\chi(\alpha) := \lambda_\alpha$. □

4.5 Proposition.
(i) Eveey separable map $\alpha : Z \to Z$ satisfies

$$K(\alpha z, z') = \overline{\chi(\alpha)} K(z, z') \quad \textit{for } z, z' \in Z. \tag{36}$$

(ii) Every separable map α with $\chi_\alpha = 1$ is coherent, with adjoint 1.

(iii) Every separable map $\alpha : Z \to Z$ satisfies

$$K(\alpha z, \alpha z') = |\chi(\alpha)|^2 K(z, z') \quad \textit{for all } z, z' \in Z.$$

(iv) Every separable map is homogeneous.

Proof. (i) and (ii) are straightforward.

(iii) Let $\alpha \in \mathrm{Sep}Z$ and $z, z' \in Z$. Then (36) implies

$$K(\alpha z, \alpha z') = \overline{\chi(\alpha)} K(z, \alpha z') = \chi(\alpha)\overline{\chi(\alpha)} K(z, z') = |\chi(\alpha)|^2 K(z, z')$$

(iv) Let $\alpha : Z \to Z$ be a separable map with the separation constant $\chi(\alpha)$. Suppose that $z, z' \in Z$ and $\lambda \in \mathbb{C}^\times$ with $|z'\rangle = \lambda|z\rangle$. Then, for $z'' \in Z$,

$$\begin{aligned} \langle z''|\alpha z'\rangle &= K(z'', \alpha z') = \chi(\alpha) K(z'', z') \\ &= \chi(\alpha)\langle z''|z'\rangle = \chi(\alpha)\lambda\langle z''|z\rangle = \lambda\langle z''|\alpha z\rangle. \end{aligned}$$

Thus $|\alpha z'\rangle = \lambda|\alpha z\rangle$. Therefore, $m = 1$ is a multiplier for α and hence α is homogeneous. □

We denote the set of all separable maps by $\mathrm{Sep}Z$ and the set of all separable maps with nonzero separation constants by $\mathrm{Sep}_\times Z$. It is easy to check that, any invertible separable map has a nonzero separation constant.

4.6 Proposition. *Let Z be a coherent space. Then:*

(i) The identity 1 *is a separable map with* $\chi(1) = 1$.

(ii) The composition of separable maps is separable.

(iii) Any adjoint α^* *of a coherent and separable map* α *is separable with* $\chi(\alpha^*) = \overline{\chi(\alpha)}$.

(iv) The inverse α^{-1} *of any invertible separable map is separable with* $\chi(\alpha^{-1}) = \chi(\alpha)^{-1}$.

Proof. (i) and (ii) are straightforward.

(iii) Let $\alpha : Z \to Z$ be a coherent and separable map and let α^* be an adjoint for α. Using (36) we find for $z, z' \in Z$,

$$K(\alpha^* z, z') = K(z, \alpha z') = \chi(\alpha) K(z, z').$$

This implies that α^* is separable with $\chi(\alpha^*) := \overline{\chi(\alpha)}$.

(iv) Let $\alpha \in \mathrm{Sep}Z$ be invertible with the inverse α^{-1}. Since $\chi(\alpha) \neq 0$, for $z, z' \in Z$, we have

$$K(\alpha^{-1} z, z') = \overline{\chi(\alpha^{-1})} K(\alpha\alpha^{-1} z, z') = \overline{\chi(\alpha)^{-1}} K(z, z'),$$

which implies that α^{-1} is separable with separation constant $\chi(\alpha^{-1}) := \chi(\alpha)^{-1}$. □

4.7 Proposition. *Let Z be a coherent space. Then,*

(i) $\mathrm{Sep}Z$ *is a semigroup with identity.*

(ii) $\mathrm{Sep}Z \cap \mathrm{Coh}\,Z$ *is* $*$*-semigroup.*

(iii) The separable maps α *with* $\chi_\alpha = 1$ *form a subsemigroup* $\mathrm{Sep}_1(Z)$ *of* $\mathrm{Sep}Z$.

(iv) In the nondegenerate case, χ *is an injective multiplicative homomorphism into* $\mathbb{C}$ *and* $\mathrm{Sep}_1(Z)$ *consists of the identity only.*

(v) Each separable map α *with* $|\chi_\alpha| = 1$ *preserves the coherent product. In particular, elements of* $\mathrm{Sep}_1(Z)$ *preserves the coherent product.*

Proof. Straightforward. □

4.8 Theorem. *Let Z be a coherent space. Then* $Z_\times := (\mathrm{Sep}Z) \times Z$ *with the coherent product*

$$K_\times((\alpha, z); (\alpha', z')) := K(\alpha' z, \alpha z') \quad \text{for all } (\alpha, z), (\alpha', z') \in Z_\times \tag{37}$$

is a coherent space.

Proof. Let $\alpha_1, .., \alpha_n \in \text{Sep}Z$ and $z_1, .., z_n \in Z$. Then, for all $c_1, ..., c_n \in \mathbb{C}$, we have

$$\begin{aligned}\sum_{j,k} \overline{c_j} c_k K_\times((\alpha_j, z_j); (\alpha_k, z_k)) &= \sum_{j,k} \overline{c_j} c_k K(\alpha_k z_j, \alpha_j z_k) \\ &= \sum_{j,k} \overline{c_j} c_k \overline{\chi_{\alpha_j}} \chi_{\alpha_k} K(z_j, z_k) \\ &= \sum_{j,k} \overline{d_j} d_k K(z_j, z_k) \geq 0,\end{aligned}$$

where $d_\ell := c_\ell \chi_{\alpha_\ell}$ for $1 \leq \ell \leq n$. □

4.9 Theorem. *Let Z be a coherent space. Then, for any $A : Z \to Z$ and $f : Z \to \mathbb{C}^\times$, the map $\mathcal{H}_{(\alpha,A)} : PZ \to PZ$ defined via*

$$\mathcal{H}_{(f,A)}(\lambda, z) := (f(z)\lambda, Az) \quad \text{for all } (\lambda, z) \in PZ,$$

is a homogeneous map.

Proof. Let $\lambda, \lambda' \in \mathbb{C}^\times$ and $z \in Z$. Then, we have

$$\mathcal{H}_{(f,A)}(\lambda'\lambda, z) = (f(z)\lambda'\lambda, Az) = (\lambda' f(z)\lambda, Az) = \lambda'(f(z)\lambda, Az) = \lambda'\mathcal{H}_{(f,A)}(\lambda, z).$$

□

4.10 Proposition. *The separable maps on a projective and nondegenerate coherent space of degree $e = \pm 1$ are precisely the multiplication maps.*

Proof. Clearly each multiplication map on a projective and non-degenerate coherent space is separable. Conversely, let Z be such a coherent space and let α be a separable map with separation constant $\chi(\alpha)$. Then, for $z, z' \in Z$,

$$K(z, \alpha z') = \chi(\alpha) K(z, z') = K(z, \chi(\alpha)^e z')$$

since α is separable and Z is projective. Since Z is nondegenerate we conclude $\alpha z = \chi(\alpha)^e z$. □

For any coherent space Z, PZ denotes the projective extension defined in [18, Proposition 4.9], with the same quantum spaces as Z.

4.11 Theorem. *Let Z be a coherent space, $S : Z \to Z$ be a separable map with separation constant $\chi(S) \in \mathbb{C}$. Then, the maps $\mathcal{A}_S : PZ \to PZ$ and $\mathcal{B}_S : PZ \to PZ$ defined via*

$$\mathcal{A}_S(\lambda, z) := (\lambda, Sz) \quad \text{for all } (\lambda, z) \in PZ,$$

$$\mathcal{B}_S(\lambda, z) := (\chi(S)\lambda, z) \quad \text{for all } (\lambda, z) \in PZ,$$

are coherent with $\mathcal{A}_S^* = \mathcal{B}_S$ and $\mathcal{B}_S^* = \mathcal{A}_S$.

Proof. Let $(\lambda, z), (\lambda', z') \in PZ$. Then, we have

$$\begin{aligned} K_{\mathrm{pe}}(\mathcal{A}_S(\lambda, z), (\lambda', z')) &= K_{\mathrm{pe}}((\lambda, Sz), (\lambda', z')) = \overline{\lambda} K(Sz, z')\lambda' \\ &= \overline{\lambda} K(z, z')\chi(S)\lambda' = K_{\mathrm{pe}}((\lambda, z), \mathcal{B}_S(\lambda', z')). \end{aligned}$$

Thus, $\mathcal{A}_S$ is coherent with $\mathcal{A}_S^* = \mathcal{B}_S$. This also implies that $\mathcal{B}_S$ is coherent with $\mathcal{B}_S^* = \mathcal{A}_S$. □

4.12 Proposition. *Let Z be a coherent space. Then:*

(i) The map $\mathcal{P} : \mathbb{C} \times \mathrm{Coh}\, Z \to \mathrm{Coh}\, PZ$ given by $(\alpha, A) \to [\alpha, A]$ is an anti-homomorphism of $*$-semigroups.

(ii) The map $\mathcal{A} : \mathrm{Sep}Z \to \mathrm{Coh}\, PZ$ given by $S \to \mathcal{A}_S$ is a homomorphism of semigroups.

(iii) The map $\mathcal{B} : \mathrm{Sep}Z \to \mathrm{Coh}\, PZ$ given by $S \to \mathcal{B}_S$ is a homomorphism of semigroups.

Proof. (i) Let $(\alpha, A), (\beta, B) \in \mathbb{C}^\times \times \mathrm{Coh}\, Z$. Then, for $(\lambda, z) \in PZ$, we have

$$\begin{aligned} [\alpha, A]\mathcal{P}_{(\beta,B)}(\lambda, z) &= \mathcal{P}_{(\beta,B)}(\alpha\lambda, Az) = (\beta\alpha\lambda, BAz) \\ &= \mathcal{P}_{(\beta\alpha,BA)}(\lambda, z) = \mathcal{P}_{(\beta,B)(\alpha,A)}(\lambda, z). \end{aligned}$$

(ii) Let $S, S' \in \mathrm{Sep}Z$. Then, for $(\lambda, z) \in PZ$, we have

$$\mathcal{A}_{SS'}(\lambda, z) = (\lambda, SS'z) = \mathcal{A}_S(\lambda, S'z) = \mathcal{A}_S\mathcal{A}_{S'}(\lambda, z).$$

(iii) Let $S, S' \in \mathrm{Sep}Z$. Then, for $(\lambda, z) \in PZ$, we have

$$\mathcal{B}_{SS'}(\lambda, z) = (\chi(SS')\lambda, z) = (\chi(S)\chi(S')\lambda, z) = \mathcal{B}_S(\chi(S')\lambda, z) = \mathcal{B}_S\mathcal{B}_{S'}(\lambda, z).$$

□

4.13 Corollary. *Let Z be a coherent space. Then*

$$\mathrm{Sep}\,[PZ] \cong \mathrm{Sep}P[Z] \cong \mathbb{C}^\times.$$

In particular, the map $\chi : \mathrm{Sep}\,[PZ] \cong \mathrm{Sep}P[Z] \to \mathbb{C}^\times$ is a group isomorphism.

A map $A : Z \to Z$ is called **strongly homogeneous** if $A\alpha = \alpha A$ for all separable maps $\alpha \in \mathrm{Sep}Z$. We write $\mathrm{hom}_s\, Z$ for the set of all strongly homogeneous maps over Z. It can be readily checked that $\mathrm{Sep}Z \subseteq \mathrm{hom}_s\, Z$ and $\mathrm{hom}_s\, Z \subseteq \mathrm{hom}(Z)$.

A function $f : Z \to \mathbb{C}$, or a kernel $X : Z \times Z \to \mathbb{C}$ is called **strongly homogeneous** if

$$f(\alpha z) = f(z) \quad \text{for } \alpha \in \mathrm{Sep}Z,\ z \in Z,$$

or

$$X(\alpha z, \alpha' z') = X(z, z') \quad \text{for } \alpha, \alpha' \in \mathrm{Sep}Z,\ z, z' \in Z,$$

respectively.

4.14 Proposition. *Let Z be a coherent space. Then,*

(i) Any adjoint of a coherent and strongly homogeneous map is homogeneous.

(ii) The set $\mathrm{Coh}\, Z \cap \mathrm{Hom}(Z)$ *is $*$-subsemigroup of* $\mathrm{Coh}\, Z$.

Proof. (i) Let $A : Z \to Z$ be a strongly homogeneous coherent map with an adjoint A^*. Then, for all $\alpha \in \mathrm{Sep}Z$, we have

$$K(A^*(\alpha z), z') = K(\alpha z, Az') = \chi(\alpha)K(z, Az') = \chi(\alpha)K(A^*z, z') = K(\alpha A^*z, z'),$$

for all $z, z' \in Z$. Thus, A^* is strongly homogeneous.

(ii) is straightforward. □

4.15 Proposition. *Let Z be a nondegenerate coherent space. Then,*

(i) each coherent map is strongly homogeneous.

(ii) $\mathrm{Sep}Z$ *is in the center of* $\mathrm{Coh}\, Z$.

(iii) For $z \in Z$, $\alpha \in \mathrm{Sep}Z$, and $A \in \mathrm{Coh}\, Z$ we have $|A\alpha z\rangle = \chi(\alpha)|Az\rangle$.

Proof. (i) Let $A : Z \to Z$ be a coherent map. Then, for all $z, z' \in Z$ and $\alpha \in \mathrm{Sep}Z$, we have

$$K(A\alpha z, z') = K(\alpha z, A^*z') = \chi(\alpha)K(z, A^*z') = \chi(\alpha)K(Az, z') = K(\alpha Az, z').$$

Since K is nondegenerate over Z, we get $A \circ \alpha = \alpha \circ A$ for all $\alpha \in \mathrm{Sep}Z$.

(ii) Let $\alpha \in \mathrm{Sep}Z$ with the separation constant $\chi(\alpha)$. Also, let $A \in \mathrm{Coh}\, Z$ be given. Using (i), A is strongly homogeneous as well. Thus, by definition of strongly homogeneous we have $A\alpha = \alpha A$. Hence α belongs to the center of $\mathrm{Coh}\, Z$.

(iii) Using (ii) and Proposition 4.20(i) we have

$$|A\alpha z\rangle = |\alpha Az\rangle = \chi(\alpha)|Az\rangle.$$

□

4.16 Proposition. *Let Z be a coherent space and $z, z' \in Z$. If there exists a separable map $\alpha \in \mathrm{Sep}Z$ such that $\alpha z = z'$ then the coherent states $|z\rangle, |z'\rangle$ are parallel. In this case, we have $|z'\rangle = \overline{\chi(\alpha)}|z\rangle$.*

Proof. Suppose that there exists a separable map $\alpha \in \mathrm{Sep}Z$ such that $\alpha z = z'$. Then, for $w \in Z$, we have

$$\langle w|z'\rangle = K(w, z') = K(w, \alpha z) = \overline{\chi_\alpha}K(w, z) = \overline{\chi_\alpha}\langle w|z\rangle.$$

Thus we get $|z\rangle = \overline{\chi(\alpha)}|z'\rangle$. □

4.17 Remark. If Z is a projective and nondegenerate coherent space then $\hom Z = \hom_s Z$. Indeed, a function $f : Z \to \mathbb{C}$, or a kernel $X : Z \times Z \to \mathbb{C}$ is homogeneous iff

$$f(\alpha z) = f(z) \quad \text{for } \alpha \in \mathbb{C}^\times,\ z \in Z,$$

or

$$X(\alpha z, \alpha' z') = X(z, z') \quad \text{for } \alpha, \alpha' \in \mathbb{C}^\times,\ z, z' \in Z,$$

respectively.

The next result shows that each coherent map over a projective coherent space is automatically homogeneous as well.

4.18 Corollary. *Let Z be a projective and nondegenerate coherent space. Then,*

(i) every coherent map is homogeneous.

(ii) $\mathbb{C}^\times$ is in the center of $\mathrm{Coh}\, Z$.

Proof. The results follow directly from Propositions 4.15 and 4.10. □

4.19 Corollary. *Let Z be a coherent space. Then*

(i) $\mathrm{Coh}\ [PZ] \subseteq \hom\ [PZ]$ *and* $\mathrm{Coh}\, P[Z] \subseteq \hom P[Z]$.

(ii) $\mathrm{Sep}\ [PZ]$ *is in the center of* $\mathrm{Coh}\ [PZ]$.

(iii) $\mathrm{Sep} P[Z]$ *is in the center of* $\mathrm{Coh}\, P[Z]$.

Proof. Apply Corollary 4.18 to the projective and non-degenerate spaces $Z' := [PZ]$ and $Z'' := P[Z]$. □

4.20 Proposition. *Let Z be a coherent space and $z \in Z$. Then*

(i) For $\alpha \in \mathrm{Sep} Z$ and $A \in \hom_s Z$ we have

$$|A\alpha z\rangle = \chi(\alpha)|Az\rangle.$$

(ii) For $A \in \mathrm{Coh}\, Z$ and $\alpha \in \mathrm{Sep} Z$ we have

$$|A\alpha z\rangle = \chi(\alpha)\Gamma(A)|z\rangle = |\alpha A z\rangle.$$

Proof. Straightforward. □

5 Slender coherent spaces

In this section we prove quantization theorems for a restricted class of coherent spaces for which many operators on a quantum space have a simple description in terms of normal kernels. These generalize the normal ordering of operators familiar from quantum field theory.

5.1 Slender coherent spaces

The simplest interesting situation is the following.

5.1 Proposition. *Let Z be a coherent space. Then the admissibility space is $\mathbb{A}(Z) = \mathbb{C}^Z$ iff any finite set of distinct coherent states is linearly independent.*

Proof. If any finite set of distinct coherent states is linearly independent then the hypothesis of (12) implies that all c_k vanish, hence (12) always holds. Thus each function $f : Z \to \mathbb{C}$ is strongly admissible with $M = 1$ and $W = \{w\}$, where $K(z, w) \neq 0$ for all $z \in Z$. Hence $\mathbb{A}(Z) = \mathbb{C}^Z$. Conversely, suppose that $\mathbb{A}(Z) = \mathbb{C}^Z$, and $\sum c_\ell |z_\ell\rangle = 0$ with distinct z_ℓ. Then every $f = \delta_k$ is strongly admissible and we deduce from (12) that

$$0 = \sum \overline{c}_\ell f(\overline{z}_\ell) = \sum \overline{c}_\ell \delta_k(z_\ell) = \overline{c_k},$$

which implies that $c_k = 0$. This holds for all k, whence any finite set of distinct coherent states is linearly independent. □

The most interesting cases are covered by a slightly more general class of coherent spaces. We call a coherent space **slender** if any finite set of linearly dependent, nonzero coherent states in a quantum space $\mathbb{Q}(Z)$ of Z contains two parallel coherent states. Clearly, every subset of a slender coherent space is again a slender coherent space.

5.2 Proposition. *Let S be a subset of the Euclidean space $\mathbb{H}$ such that any two elements of S are linearly independent. Then the set $Z = \mathbb{C}^\times \times S$ with the coherent product*

$$K((\lambda, s); (\lambda', s')) := \overline{\lambda}\lambda' s^* s' \quad \textit{for all } (\lambda, s), (\lambda', s') \in Z$$

and scalar multiplication $\alpha(\lambda, s) := (\alpha\lambda, s)$ is a slender, projective coherent space of degree 1.

Proof. It is easy to see that $\mathbb{Q}(Z) := \operatorname{Span} S$ is a quantum space of Z. Let the $z_k \in Z$ be such that $\sum c_k |z_k\rangle = 0$ with $c_k \neq 0$ for all k. We then have $z_k = (\lambda_k, z_k')$ with $\lambda_k \in \mathbb{C}^\times$ and $z_k' \in S$, hence

$$\sum c_k \lambda_k z_k' = \sum c_k |(\lambda_k, z_k')\rangle = \sum c_k |z_k\rangle = 0.$$

But the z_k' are linearly independent, hence $c_k\lambda_k = 0$ for all k, and since $\lambda_k \neq 0$, all c_k vanish. Thus Z is slender. Projectivity is obvious. $\square$

Thus slender coherent spaces are very abundant. However, proving slenderness for a *given* coherent space is a nontrivial matter once Z contains infinitely many elements.

5.3 Theorem. *The Möbius space defined in Example 3.11 is a slender coherent space.*

Proof. Suppose that the Möbius space Z is not slender. Then there is a nontrivial finite linear dependence $\sum c_k|z_k\rangle = 0$ such that no two $|z_k\rangle$ are parallel. Since Z is projective of degree -1, this implies that the numbers $\mu_k := z_{k2}/z_{k1}$ are distinct, and $|\mu_k| < 1$ by definition of Z. Since $x = \binom{1}{\mu} \in Z$ for $|\mu| < 1$, we have

$$0 = \langle\overline{x}|\sum c_k|z_k\rangle = \sum c_k K(\overline{x}, z_k) = \sum \frac{c_k}{x_1 z_{k1} - x_2 z_{k2}} = \sum \frac{c_k z_{k1}^{-1}}{1 - \mu\mu_k} \quad \text{for } |\mu| < 1.$$

The right hand side is the partial fraction decomposition of a rational function of μ vanishing in an open set. Since the partial fraction decomposition is unique, each term vanishes. Therefore $c_k z_{k1}^{-1} = 0$ for all k, which implies that all c_k vanish, contradiction. Thus Z is slender. $\square$

5.4 Proposition.
(i) A projective coherent space is slender iff $\sum_{k\in I}|z_k\rangle = 0$ *implies that there exist distinct* $j, k \in I$ *such that* $|z_k\rangle = \alpha|z_j\rangle$ *for some* $\alpha \in \mathbb{C}$.

(ii) A nondegenerate projective coherent space is slender iff $\sum_{k\in I}|z_k\rangle = 0$ *implies that there exist distinct* $j, k \in I$ *such that* $z_k = \alpha z_j$ *for some* $\alpha \in \mathbb{C}$.

(iii) A coherent space Z *is slender iff its projective extension* PZ *is slender.*

Proof. In the projective case, $\sum \alpha_k|z_k\rangle = 0$ implies $\sum|\beta_k z_k\rangle = 0$ with $\beta_k := \alpha_k^{1/e}$. Thus we may assume w.l.o.g. that the linear combination in the definition of slender is a sum. Hence (i) holds. (ii) is straightforward.

(iii) Let Z be a slender coherent space with a quantum space $\mathbb{Q}(Z)$, and let PZ be a projective extension of Z of degree e with the same quantum space $\mathbb{Q}(PZ) = \mathbb{Q}(Z)$. Let $\sum_k|(\lambda_k, z_k)\rangle = 0$ in $\mathbb{Q}(PZ)$. Then $\sum_k \lambda_k^e|z_k\rangle = 0$ in $\mathbb{Q}(Z)$, and we may assume that the sum extends only over the nonzero λ_k. Since Z is slender, there exists distinct j, k with $\lambda_k \neq 0$ such that $|z_j\rangle = \alpha|z_k\rangle$ for some $\alpha \in \mathbb{C}$. But then

$$|(\lambda_j, z_j)\rangle = \lambda_j^e|z_j\rangle = \lambda_j^e\alpha|z_k\rangle = \Big(\frac{\lambda_j}{\lambda_k}\Big)^e\alpha|(\lambda_k, z_k)\rangle.$$

Thus PZ is slender. The converse is obvious. $\square$

5.5 Proposition. *Let Z be a slender coherent space and let $\mathbb{Q}(Z)$ be a quantum space of Z. Let I be a finite index set. If the $z_k \in Z$ $(k \in I)$ satisfy $\sum_{k\in I} c_k|z_k\rangle = 0$ then there is a partition of I into nonempty subsets I_t $(t \in T)$ such that $k \in I_t$ implies $|z_k\rangle = \alpha_k|z_t\rangle$ with $\sum_{k\in I_t} c_k\alpha_k = 0$ for all $t \in T$.*

Proof. It is easy to see that it is enough to consider the case where none of the $c_k|z_k\rangle$ vanishes, as the general case can be reduced to this case by removing zero contributions to the sum. Let T be a maximal subset of I with the property that no two coherent states $|z_t\rangle$ are multiples of each other. For each $t \in T$, let I_t be the set of $k \in I$ such that $|z_k\rangle$ is a multiple of $|z_t\rangle$, say, $|z_k\rangle = \alpha_k|z_t\rangle$. Then the I_t $(t \in T)$ form a partition of I. If we define for $t \in T$ the numbers

$$a_t := \sum_{k\in I_t} c_k\alpha_k$$

we have

$$\sum_{t\in T} a_t|z_t\rangle = \sum_{t\in T}\Big(\sum_{k\in I_t} c_k\alpha_k\Big)|z_t\rangle = \sum_{k\in I} c_k|z_k\rangle = 0.$$

Since Z is a slender coherent space and no two of the $|z_t\rangle$ $(t \in T)$ are parallel, the $|z_t\rangle$ $(t \in T)$ are linearly independent. We conclude that all a_t vanish. Therefore $\sum_{k\in I_t} c_k\alpha_k = 0$ for all $t \in T$. $\square$

5.6 Corollary. *Let Z be a slender coherent space, projective of degree e and let $\mathbb{Q}(Z)$ be a quantum space of Z. Let I be a finite index set. If $z_k \in Z$ $(k \in I)$ satisfies $\sum_{k\in I} |z_k\rangle = 0$ then there is a partition of I into nonempty subsets I_t $(t \in T)$ such that $k \in I_t$ implies $z_k = \alpha_k z_t$ with $\sum_{k\in I_t} \alpha_k^e = 0$, for all $t \in T$.*

5.2 Quantization theorems

5.7 Theorem. *Let Z be a slender coherent space and suppose that $m : Z \to \mathbb{C}$ is a multiplier map for the map $A : Z \to Z$. Then there exists a unique linear operator $\Gamma_m(A) : \mathbb{Q}(Z) \to \mathbb{Q}(Z)$, the **quantization** of A relative to m, such that*

$$\Gamma_m(A)|z\rangle = m(z)|Az\rangle \quad \textit{for all } z \in Z.$$

Proof. Let $m : Z \to \mathbb{C}$ be a multiplier for the map $A : Z \to Z$. We then define $\Gamma_m(A) : \mathbb{Q}(Z) \to \mathbb{Q}(Z)$ by

$$\Gamma_m(A)\Big(\sum_k c_k|z_k\rangle\Big) := \sum_k c_k m(z_k)|Az_k\rangle \quad \text{for all } \sum_k c_k|z_k\rangle \in \mathbb{Q}(Z).$$

Let $\sum c_k|z_k\rangle = 0$. Then, we have $\sum |c_k z_k\rangle = 0$. Hence, using Proposition 5.5, there is a partition of $I := \{k : c_k \neq 0\}$ into nonempty subsets I_t $(t \in T)$ such that $k \in I_t$ implies $|z_k\rangle = \alpha_k|z_t\rangle$ with $\sum_{k\in I_t} c_k\alpha_k = 0$, for all $t \in T$. Since α is a multiplier for A, we have for $t \in T$ and $k \in I_t$,

$$m(z_k)|Az_k\rangle = \alpha_k m(z_t)|Az_t\rangle. \tag{38}$$

Thus, using (38), we get

$$\begin{aligned}\sum_k c_k m(z_k)|Az_k\rangle &= \sum_{t\in T}\sum_{k\in I_t} c_k m(z_k)|Az_k\rangle = \sum_{t\in T}\sum_{k\in I_t} c_k\alpha_k m(z_t)|Az_t\rangle \\ &= \sum_{t\in T}\Big(\sum_{k\in I_t} c_k\alpha_k\Big)m(z_t)|Az_t\rangle = 0.\end{aligned}$$

Therefore, $\Gamma_m(A) : \mathbb{Q}(Z) \to \mathbb{Q}(Z)$ is a well-defined linear map. In particular, we have

$$\Gamma_m(A)|z\rangle = m(z)|Az\rangle \quad \text{for } z \in Z.$$

□

5.8 Corollary. *Let Z be a slender, projective, and non-degenerate coherent space, and let $\mathbb{Q}(Z)$ be a quantum space of Z. Then for every homogeneous map $A : Z \to Z$, there is a unique linear operator $\Gamma(A) : \mathbb{Q}(Z) \to \mathbb{Q}(Z)$, the* ***quantization*** *of A, such that*

$$\Gamma(A)|z\rangle = |Az\rangle \quad \text{for } z \in Z. \tag{39}$$

Proof. We define $\Gamma(A) : \mathbb{Q}(Z) \to \mathbb{Q}(Z)$ by $\Gamma(A) := \Gamma_1(A)$. Then, $\Gamma(A)$ satisfies (39). □

Note that when Z is slender, Theorem 3.12 is less general than Theorem 5.8, but Theorem 3.12 holds for a larger class of coherent spaces. Moreover, for coherent maps A, there is a simple relationship (28) between $\Gamma(A)^*$ and $\Gamma(A^*)$, that does not generalize to the situation of Theorem 5.8.

5.9 Proposition. *Let Z be a slender, projective and non-degenerate coherent space. The quantization map $\Gamma : \hom Z \to \operatorname{Lin}\mathbb{Q}(Z)$ is a semigroup homomorphism,*

$$\Gamma(AB) = \Gamma(A)\Gamma(B) \quad \text{for } A, B \in \hom Z. \tag{40}$$

Proof. It is straightforward to check that $AB \in \hom Z$. By Theorem 5.8,

$$\Gamma(AB)|z\rangle = |ABz\rangle = \Gamma(A)|Bz\rangle = \Gamma(A)\Gamma(B)|z\rangle \quad \text{for all } z \in Z.$$

Thus (40) holds. □

5.10 Theorem. *Let Z be a slender coherent space and let $\mathbb{Q}(Z)$ be a quantum space of Z. Then for every homogeneous function $m : Z \to \mathbb{C}$ there is a unique linear operator $\mathbf{a}(m) : \mathbb{Q}(Z) \to \mathbb{Q}(Z)$ such that*

$$\mathbf{a}(m)|z\rangle = m(z)|z\rangle \quad \textit{for } z \in Z. \tag{41}$$

Proof. We define $\mathbf{a}(m) : \mathbb{Q}(Z) \to \mathbb{Q}(Z)$ by $\mathbf{a}(m) := \Gamma_m(1)$. Then (41) follows easily. □

This generalizes the property of traditional coherent states to be eigenstates of annihilator operators. Indeed, in the special case of Klauder spaces treated in Subsection 6.4, the $\mathbf{a}(m)$ are found to be the smeared annihilator operators acting on a Fock space.

$\mathbf{a}(m)$ is a linear function of m. To preserve this property in the adjoint, we define

$$\mathbf{a}^*(m) := \mathbf{a}(\overline{m})^*, \tag{42}$$

the analogues of smeared creation operators. Here $\overline{m}$ is the function defined by

$$\overline{m}(z) := \overline{m(z)},$$

which is homogeneous since $|z'\rangle = \lambda|z\rangle$ implies $m(z') = m(z)$ by homogeneity of m, hence $\overline{m}(z') = \overline{m(z')} = \overline{m(z)} = \overline{m}(z)$.

5.3 Normal kernels

Let Z be a slender coherent space and $\mathbb{Q}(Z)$ be a quantum space of Z. A kernel $X : Z \times Z \to \mathbb{C}$ is called **homogeneous** if, for all $z \in Z$, the functions $X(\cdot, z), X(z, \cdot)$ are homogeneous in the sense defined in Subsection 4.1.

5.11 Theorem. *For every homogeneous kernel X there is a unique linear operator $N(X)$ from $\mathbb{Q}(Z)$ to its algebraic antidual, called the* ***normal ordering*** *of X, such that*

$$\langle z|N(X)|z'\rangle = X(z, z')K(z, z') \quad \textit{for } z, z' \in Z. \tag{43}$$

(Equivalently, $N(X)$ defines a Hermitian form on $\mathbb{Q}(Z)$.)

Proof. This follows from Theorem 2.6(ii) and slenderness. Indeed, define for vectors $\phi = \sum_k c'_k|z'_k\rangle$ and $\psi = \sum_\ell c_\ell|z_\ell\rangle \in \mathbb{Q}(Z)$ the complex number

$$(\psi, \phi)_X := \sum_\ell \sum_k \overline{c_\ell} c'_k X(z_\ell, z'_k) K(z_\ell, z'_k).$$

We first claim that $(\psi, \phi) \to (\psi, \phi)_X$ is well-defined. Because $(.,.)_X$ is a Hermitian form in the c_ℓ and the c'_k, it is enough to show that $\phi = 0$ implies $(\psi, \phi)_X = 0$. By Proposition 5.5, if $\phi = \sum_k c'_k |z'_k\rangle = 0$, there is a partition of $I := \{k : c'_k \neq 0\}$ into nonempty subsets I_t $(t \in T)$ such that $k \in I_t$ implies $|z'_k\rangle = \alpha_k |z'_t\rangle$ with $\sum_{k \in I_t} c'_k \alpha_k = 0$ for all $t \in T$. Using the homogeneity assumption of $X(z_\ell, \cdot)$ we find for each $t \in T$ and each $k \in I_t$,

$$X(z_\ell, z'_k) K(z_\ell, z'_k) = \alpha_k X(z_\ell, z'_t) K(z_\ell, z'_t).$$

Therefore

$$\begin{aligned}
\sum_\ell \sum_k \overline{c_\ell} c'_k X(z_\ell, z'_k) K(z_\ell, z'_k) &= \sum_\ell \sum_{t \in T} \sum_{k \in I_t} \overline{c_\ell} c'_k X(z_\ell, z'_k) K(z_\ell, z'_k) \\
&= \sum_\ell \sum_{t \in T} \sum_{k \in I_t} \overline{c_\ell} c'_k \alpha_k X(z_\ell, z'_t) K(z_\ell, z'_t) \\
&= \sum_\ell \sum_{t \in T} \overline{c_\ell} \Big(\sum_{k \in I_t} c'_k \alpha_k \Big) X(z_\ell, z'_t) K(z_\ell, z'_t) = 0.
\end{aligned}$$

Using the homogeneity assumption of $X(\cdot, z'_k)$, a similar argument shows that if $\psi = \sum c_\ell |z_\ell\rangle = 0$ then $(\psi, \phi)_X = 0$. Hence, $(\psi, \phi) \to (\psi, \phi)_X$ defines a well-defined Hermitian form on $\mathbb{Q}(Z)$. $\square$

The interesting case is when $N(X)$ maps $\mathbb{Q}(Z)$ to $\mathbb{Q}^\times(Z)$. When this holds, we call the kernel X **normal**.

5.12 Proposition. *Let Z be a slender coherent space whose coherent product vanishes nowhere. Then any linear operator $\mathbf{X} : \mathbb{Q}(Z) \to \mathbb{Q}(Z)^\times$ is the normal ordering of a unique homogeneous kernel X.*

Proof. The kernel X defined by

$$X(z, z') := \frac{\langle z | \mathbf{X} | z' \rangle}{K(z, z')}$$

has the required properties. $\square$

5.13 Proposition. *If $A : Z \to Z$ is coherent, homogeneous and invertible then $\Gamma(A)$ is invertible, and for every normal kernel X, the kernel AX defined by*

$$AX(z, z') := X(A^* z, A^{-1} z') \quad \textit{for } z, z' \in Z \tag{44}$$

is normal, and

$$N(AX) = \Gamma(A) N(X) \Gamma(A)^{-1}.$$

Moreover, if $B : Z \to Z$ is also coherent, homogeneous and invertible then

$$(AB)X = A(BX).$$

Proof. This follows from (44) since

$$\begin{aligned}\langle z|N(AX)\Gamma(A)|z'\rangle &= \langle z|N(AX)|Az'\rangle = AX(z,Az')K(z,Az') \\ &= X(A^*z,z')K(A^*z,z') = \langle A^*z|N(X)|z'\rangle = \langle z|\Gamma(A)N(X)|z'\rangle\end{aligned}$$

and

$$\begin{aligned}(AB)X(z,z') &= X((AB)^*z,(AB)^{-1}z') = X(B^*A^*z,B^{-1}A^{-1}z') \\ &= BX(A^*z,A^{-1}z') = A(BX)(z,z').\end{aligned}$$

□

Define for $f,g:Z\to\mathbb{C}$,

$$(fX)(z,z') := f(z)X(z,z'), \quad (Xf)(z,z') := X(z,z')f(z'),$$

Then

$$(fX)^* = X^*f^*, \quad (Xf)^* = f^*X^*.$$

We write λ for a **constant kernel** with constant value $\lambda\in\mathbb{C}$. Note that $f1$ and $1f$ are different normal kernels!

5.14 Proposition.
(i) Normal kernels form a vector space $\mathbb{X}(Z)$, and the normal ordering operator $N:\mathbb{X}(Z)\to$ $\mathrm{Lin}^\times\,\mathbb{Q}(Z)$ is linear.

(ii) If X is normal then X^* is normal and

$$N(X^*) = N(X)^*.$$

(iii) Any constant kernel λ is normal, and $N(\lambda)=\lambda$.

(iv) If N is normal then mNm' is normal for all homogeneous m,m', and

$$N(mXm') = \mathbf{a}^*(m)N(X)\mathbf{a}(m'). \tag{45}$$

(v) If $X_\ell\to X$ pointwise and all X_ℓ are normal then X is normal, and

$$N(X_\ell)\to N(X).$$

Proof. Statements (i)–(iii) are straightforward.

(iv) Using (41) and (42), we find

$$\begin{aligned}\langle z|\mathbf{a}^*(m)N(X)\mathbf{a}(m')|z'\rangle &= \langle z|\mathbf{a}(m^*)^*N(X)\mathbf{a}(m')|z'\rangle = \langle z|m(z)N(X)m'(z')|z'\rangle \\ &= m(z)\langle z|N(X)|z'\rangle m'(z') = m(z)X(z,z')K(z,z')m'(z') \\ &= (mXm')(z,z')K(z,z').\end{aligned}$$

(v) Let $X_\ell\to X$ pointwise with all X_ℓ normal. Then X is homogeneous. Indeed, for $z,z'\in Z$ and $c,c'\in\mathbb{C}$, we have

$$X(cz,c'z') = \lim_\ell X_\ell(cz,c'z') = \lim_\ell X_\ell(z,z') = X(z,z').$$

We then have

$$\lim_{\ell}\langle z|N(X_\ell)|z'\rangle = \Big(\lim_{\ell} X_\ell(z,z')\Big)K(z,z') = X(z,z')K(z,z') = \langle z|N(X)|z'\rangle,$$

for all $z, z \in Z$. □

5.15 Theorem. *Let Z be a slender coherent space. Let S be a set, $d\mu$ a measure on S. Suppose that the $f_\ell, g_\ell : S\times Z \to \mathbb{C}$ are measurable in the first argument and homogeneous in the second argument, and*

$$X(z,z') := \lim_{\ell}\int d\mu(s) g_\ell(s,z) f_\ell(s,z')$$

exists for all $z, z' \in Z$. Then, with notation as in (41),

$$N(X) = \lim_{\ell}\int d\mu(s)\mathbf{a}(g_\ell(s,\cdot))^*\mathbf{a}(f_\ell(s,\cdot))$$

is a linear operator from $\mathbb{Q}(Z)$ to its algebraic antidual.

Proof. First, we claim that X is homogeneous. To this end, let $z, z' \in Z$ and $\alpha \in \mathbb{C}$ such that $|z'\rangle = \alpha|z\rangle$. Using the homogeneity assumption on each f_ℓ, we have for each $w \in Z$

$$\begin{aligned} X(w,z')|z'\rangle &= \lim_{\ell}\int d\mu(s) g_\ell(s,w) f_\ell(s,z')|z'\rangle \\ &= \alpha\lim_{\ell}\int d\mu(s) g_\ell(s,w) f_\ell(s,z)|z\rangle = \alpha X(w,z)|z\rangle, \end{aligned}$$

which implies that $X(w,\cdot)$ is a homogeneous function. A similar argument, using homogeneity assumption of each g_ℓ, guarantees that $X(\cdot,w)$ is a homogeneous function as well. Now, using Theorem 5.11, there exists a unique linear operator $N(X)$ from $\mathbb{Q}(Z)$ into its algebraic antidual of $\mathbb{Q}(Z)$, such that

$$\langle z|N(X)|z'\rangle = X(z,z')K(z,z'),$$

for all $z, z' \in Z$. Hence, for $z, z' \in Z$, we get

$$\begin{aligned} \langle z|N(X)|z'\rangle &= X(z,z')K(z,z') = \lim_{\ell}\int d\mu(s) g_\ell(s,z) f_\ell(s,z')K(z,z') \\ &= \int d\mu(s)\Big(\mathbf{a}(g(s,\cdot))^*\langle z|\Big)\mathbf{a}(f(s,\cdot))|z\rangle. \end{aligned}$$

□

6 Klauder spaces and bosonic Fock spaces

In this section we discuss in some detail Klauder spaces, a class of coherent spaces with a large semigroup of coherent maps. The quantum spaces of Klauder coherent spaces are the bosonic Fock spaces, relevant for quantum field theory (Baez et al. [2], Glimm & Jaffe [9]) and the theory of Hida distributions in the white noise calculus for classical stochastic processes (Hida & Si [10], Hida & Streit [11], Obata [22]).

6.1 Klauder spaces

We recall from Neumaier [18, Example 3.2] that the **Klauder space** $KL[V]$ over the Euclidean space V is defined by the set $Z = \mathbb{C} \times V$ of pairs

$$z := [z_0, \mathbf{z}] \in \mathbb{C} \times V$$

with the coherent product

$$K(z, z') := e^{\overline{z}_0 + z'_0 + \mathbf{z}^* \mathbf{z}'}. \tag{46}$$

Klauder spaces are degenerate since

$$|[z_0 + 2\pi ik, \mathbf{z}]\rangle = |[z_0, \mathbf{z}]\rangle \quad \text{for } k \in \mathbb{Z}.$$

6.1 Proposition. *With the scalar multiplication*

$$\alpha[z_0, \mathbf{z}] := [z_0 + \log \alpha, \mathbf{z}],$$

using an arbitrary but fixed branch of log, *Klauder spaces are projective of degree 1. The separable maps are precisely the multiplication maps $z \to \alpha z$, with $\chi(\alpha) = \alpha$.*

Proof. Inserting the definition of the scalar multiplication, one finds $K(z, \lambda z') = \lambda K(z, z')$. The second statement can be verified directly; Proposition 4.10 is not applicable. □

6.2 Theorem. *Klauder spaces are slender.*

Proof. Suppose that the Klauder space $Z = Kl(V)$ is not slender. Then there is a nontrivial finite linear dependence $\sum c_k |z_k\rangle = 0$ such that no two $|z_k\rangle$ are parallel. In view of (56) we may assume w.l.o.g. that $z_k = [0, \mathbf{z}_k]$ and conclude that the $\mathbf{z}_k$ are distinct. Now let $v \in V$ and $z = [0, nv]$ for some nonnegative integer n. Then, with $\xi_k := e^{v^* \mathbf{z}_k}$,

$$0 = \langle z| \sum c_k |z_k\rangle = \sum c_k \langle z|z_k\rangle = \sum c_k e^{nv^* \mathbf{z}_k} = \sum c_k \xi_k^n \quad \text{for } n = 0, 1, 2, \ldots.$$

Since the sum has finitely many terms only, we find a homogeneous linear system with a Vandermonde coefficient matrix having a nontrivial solution. So the matrix is singular, and we conclude that two of the ξ_k must be identical. Thus for every $v \in V$ there are indices $j < k$ such that $e^{v^*\mathbf{z}_j} = e^{v^*\mathbf{z}_k}$, hence, with $z_{jk} := z_j - z_k \neq 0$,

$$v^* z_{jk} \equiv 0 \bmod 2\pi i.$$

Now let $u \in V$. If $u^* z_{jk} \neq 0$ for all $j < k$ then picking $v = \lambda u$ with sufficiently many different $\lambda \in \mathbb{R}$ gives a contradiction. Thus for every $u \in V$ there are indices $j < k$ such that

$$v^* z_{jk} = 0. \tag{47}$$

Since $u \in V$ was arbitrary and the z_{jk} are nonzero, this implies that V is the union of finitely many hyperplanes (47), which is impossible. □

6.2 Oscillator groups

Klauder spaces have a large semigroup of coherent maps, which contains a large unitary subgroup. The **oscillator semigroup** over V is the semigroup $Os[V]$ of matrices

$$A = [\rho, p, q, \mathbf{A}] := \begin{pmatrix} 1 & p^* & \rho \\ 0 & \mathbf{A} & q \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Lin}\,(\mathbb{C} \times V \times \mathbb{C})$$

with $\rho \in \mathbb{C}$, $p \in V^\times$, $q \in V$, and $\mathbf{A} \in \mathrm{Lin}^\times V$; one easily verifies the formulas for the product

$$[\rho, p, q, \mathbf{A}][\rho', p', q', \mathbf{A}'] = [\rho' + \rho + p^* q', \mathbf{A}'^* p + p', q + \mathbf{A}q', \mathbf{A}\mathbf{A}']$$

and the identity $1 = [0, 0, 0, 1]$. Writing

$$[\mathbf{A}] := [0, 0, 0, \mathbf{A}]$$

we find

$$[\mathbf{B}][\alpha, p, q, \mathbf{A}][\mathbf{B}'] = [\alpha, \mathbf{B}'^* p, \mathbf{B}q, \mathbf{B}\mathbf{A}\mathbf{B}'].$$

$Os[V]$ turns elements $z \in Z$ written in the projective form

$$z = [z_0, \mathbf{z}] = \begin{pmatrix} z_0 \\ \mathbf{z} \\ 1 \end{pmatrix} \in \mathbb{C}^\times \times V \times \mathbb{C}$$

into elements the same form, corresponding to the action of $Os[V]$ on $[z_0, \mathbf{z}] \in Kl[V]$ as

$$[\rho, p, q, \mathbf{A}][z_0, \mathbf{z}] := [\rho + z_0 + p^* \mathbf{z}, q + \mathbf{A}\mathbf{z}]. \tag{48}$$

6.3 Proposition. *$Os[V]$ is a *-semigroup of coherent maps of $Kl[V]$, with adjoints defined by*

$$[\rho, p, q, \mathbf{A}]^* = [\overline{\rho}, q, p, \mathbf{A}^*]. \tag{49}$$

Proof. We have

$$\begin{aligned} K(Az,z') &= K([\rho,p,q,\mathbf{A}]z,z') = e^{\overline{\rho+z_0+p^*\mathbf{z}}+z_0'+(q+\mathbf{A}\mathbf{z})^*\mathbf{z}'} \\ &= e^{\overline{z}_0+\overline{\rho}+z_0'+q^*\mathbf{z}'+\mathbf{z}^*(p+\mathbf{A}^*\mathbf{z}')} = K(z,[\overline{\rho},q,p,\mathbf{A}^*]z'). \end{aligned}$$

Hence the elements of $Kl(Z)$ are coherent maps, with the stated adjoints. □

The **linear oscillator group** $LOs(V)$ over V consists of the elements $[\rho,p,q,\mathbf{A}]$ with invertible $\mathbf{A}$. One easily checks that the inverse is given by

$$[\rho,p,q,\mathbf{A}]^{-1} = [p^*\mathbf{A}^{-1}q-\rho,-\mathbf{A}^{-*}p,-\mathbf{A}^{-1}q,\mathbf{A}^{-1}], \tag{50}$$

where

$$\mathbf{A}^{-*} = (A^{-1})^* = (A^*)^{-1}.$$

The **unitary oscillator group** $UOs(V)$ over V consists of the unitary elements of $LOs(V)$.

6.4 Proposition.
(i) $UOs(V)$ consists of the coherent maps of the form

$$[\alpha,q,\mathbf{A}] := [\tfrac{1}{2}(\alpha-q^*q),-\mathbf{A}^*q,q,\mathbf{A}] \tag{51}$$

with unitary $\mathbf{A}\in \mathrm{Lin}\,V$, $q\in V$, and $\alpha\in i\mathbb{R}$.

(ii) Product, inverse, and adjoint of unitary elements are given by

$$[\alpha,q,\mathbf{A}][\alpha',q',\mathbf{A}'] = [\alpha+\alpha'-q^*\mathbf{A}q'+q'^*\mathbf{A}^*q,q'+\mathbf{A}q,\mathbf{A}\mathbf{A}'], \tag{52}$$

$$[\alpha,q,\mathbf{A}]^{-1} = [\alpha,q,\mathbf{A}]^* = [-\alpha,-\mathbf{A}^{-1}q,\mathbf{A}]. \tag{53}$$

Moreover,

$$[\mathbf{B}][\alpha,q,\mathbf{A}][\mathbf{B}'] = [\alpha,\mathbf{B}q,\mathbf{B}\mathbf{A}\mathbf{B}'].$$

Proof. (i) Equating (49) and (50) gives the unitarity conditions

$$\overline{\rho} = p^*A^{-1}q-\rho, \quad q=-A^{-*}p, \quad p=-A^{-1}q, \quad A^*=A^{-1}.$$

Thus A must be unitary and $p=-A^{-1}q=-A^*q$. In this case, $q=-A^{-*}p$ and

$$p^*A^{-1}q = -q^*A^{-*}A^{-1}q = -q^*q,$$

hence the unitarity conditions reduce to $\overline{\rho} = -q^*q-\rho$, i.e., $2\,\mathrm{Re}\,\rho = -q^*q$. Writing $\alpha = 2i\,\mathrm{Im}\,\rho$, (i) follows.

(ii) (53) follows from the preceding using (49). To obtain the multiplication law we note that

$$\begin{aligned} [\alpha,q,\mathbf{A}][\alpha',q',\mathbf{A}'] &= [\tfrac{1}{2}(\alpha-q^*q),-\mathbf{A}^*q,q,\mathbf{A}][\tfrac{1}{2}(\alpha'-q'^*q'),-\mathbf{A}'^*q',q',\mathbf{A}'] \\ &= [\tfrac{1}{2}(\alpha'-q'^*q')+\tfrac{1}{2}(\alpha-q^*q)-q^*\mathbf{A}q',-\mathbf{A}'^*\mathbf{A}^*q-\mathbf{A}'^*q',q+\mathbf{A}q',\mathbf{A}\mathbf{A}'] \\ &= [\alpha+\alpha'-q^*\mathbf{A}q'+q'^*\mathbf{A}^*q,q'+\mathbf{A}q,\mathbf{A}\mathbf{A}']. \end{aligned}$$

Indeed, since $\mathbf{A}^*\mathbf{A} = 1$, we have $-\mathbf{A}'^*\mathbf{A}^*q - \mathbf{A}'^*q' = -(\mathbf{A}\mathbf{A}')^*(q + \mathbf{A}q')$ and

$$\tfrac{1}{2}(\alpha' - q'^*q') + \tfrac{1}{2}(\alpha - q^*q) - q^*\mathbf{A}q' = \tfrac{1}{2}\Big(\beta - (q + \mathbf{A}q')^*(q + \mathbf{A}q')\Big),$$

where

$$\beta := \alpha' - q'^*q' + \alpha - q^*q - 2q^*\mathbf{A}q' + (q + \mathbf{A}q')^*(q + \mathbf{A}q') = \alpha + \alpha' - q^*\mathbf{A}q' + q'^*\mathbf{A}^*q.$$

□

The subset of coherent maps of the form

$$W_\lambda(q) := [i\lambda, q, 1] := [\tfrac{1}{2}(i\lambda - q^*q), -q^*, q, 1] \quad (q \in V, \lambda \in \mathbb{R})$$

is the **Heisenberg group** $H(V)$ over V. The n-dimensional **Weyl group** is the subgroup of $H(\mathbb{C}^n)$ consisting of the $W_\lambda(q)$ with real q and λ.

6.5 Proposition. *With the symplectic form*

$$\sigma(q, q') := 2\,\mathrm{Im}\, q^*q', \tag{54}$$

we have

$$W_\lambda(q)W_{\lambda'}(q') = W_{\lambda+\lambda'+\sigma(q,q')}(q + q'),$$
$$W_\lambda(q)^{-1} = W_\lambda(q)^* = W_{-\lambda}(-q),$$
$$W_\lambda(q)[z_0, \mathbf{z}] = [\tfrac{1}{2}(q^*q + \lambda) + z_0 - q^*z, q + \mathbf{z}],$$
$$[\mathbf{B}]W_\lambda(q)[\mathbf{B}]^{-1} = W_\lambda(\mathbf{B}q) \quad \textit{if } \mathbf{B} \textit{ is invertible.}$$

Proof. Specialize Proposition 6.4. □

6.3 Bosonic Fock spaces

A very important class of Hilbert spaces, indispensable in applications to stochastic processes and quantum field theory, is the family of bosonic Fock spaces. In this section we show that bosonic Fock spaces appear naturally as the quantum spaces of Klauder spaces. We identify operators on these quantum spaces corresponding to creation and annihilation operators in Fock space, and prove their basic properties. In particular, we prove the Weyl relations, the canonical commutation relations, without the need to know a particular realization of the quantum space. We also show that the abstract normal ordering introduced earlier reduced for Klauder spaces to that familiar from traditional second quantization. (Analogous statements for fermionic Fock spaces will be proved in another paper of the present series.)

A **bosonic Fock space** is a quantum space of a Klauder space $Kl[V]$. The quantization map on a Klauder space defines on the corresponding Fock spaces both a representation of the linear oscillator semigroup and a unitary representation of the unitary oscillator group $UOs(V)$. The quantization of the coherent maps in the linear oscillator semigroup leads to linear operators $\Gamma([\rho, p^T, q, A]) \in \mathrm{Lin}\,\mathbb{Q}(Z)$. Since Klauder spaces are slender, additional linear operators $\in \mathrm{Lin}^\times\,\mathbb{Q}(Z)$ come from the quantization of normal kernels.

In the following, we work with an arbitrary quantum space $\mathbb{Q}(Z)$, to demonstrate that everything of interest follows on this level, without any need to use any explicit integration.

However, to connect to tradition, we note that for $V = \mathbb{C}^n$, an explicit completed quantum space is the space $L^2(\mathbb{R}^n, \mu)$ of square integrable functions of $\mathbb{R}$ with respect to the measure μ given by $d\mu(x) = (2\pi)^{-n/2} e^{\frac{1}{2}x^T x} dx$ and the inner product

$$f^*g := \int d\mu(x)\overline{f(x)}g(x).$$

To check this we show that the functions

$$f_z(x) := e^{z_0 - \frac{1}{2}(x-\mathbf{z})^2}$$

constitute the coherent states of finite-dimensional Klauder spaces. Indeed, using definition (46), we have

$$f_z^* f_{z'} = \int d\mu(x) e^{\overline{z}_0 - \frac{1}{2}(x-\overline{\mathbf{z}})^2 + z_0' - \frac{1}{2}(x-\mathbf{z}')^2} = e^{\overline{z}_0 + z_0' + \mathbf{z}^*\mathbf{z}'} \int d\mu(x) e^{-\frac{1}{2}(x-\overline{\mathbf{z}})^2 - \frac{1}{2}(x-\mathbf{z}')^2 - \mathbf{z}^*\mathbf{z}'}.$$

Expanding into powers of x and using the Gaussian integration formula

$$\int \frac{dx}{(2\pi)^{n/2}} e^{-\frac{1}{2}(x-u)^*(x-u)} = 1 \quad \text{for } u \in \mathbb{C}^n$$

with $u = \overline{\mathbf{z}} + \mathbf{z}'$, the last integral can be evaluated to 1, hence $f_z^* f_{z'} = K(z, z')$. This proves that K is a coherent product and the f_z are a corresponding family of coherent states.

In the special case $n = 1$, we find for $z = [i\omega\tau - \frac{1}{2}\omega^2, \tau + i\omega]$ that

$$f_z(t) = e^{i\omega\tau - \frac{1}{2}\omega^2 - \frac{1}{2}(t-\tau-i\omega)^2} = e^{i\omega t} e^{-\frac{1}{2}(t-\tau)^2}$$

is the time-frequency shift by $(\tau, \omega) \in \mathbb{R}^2$ of the standard Gaussian $e^{-\frac{1}{2}t^2}$. Thus the general coherent state is a scaled time-frequency shifted standard Gaussian.

In any quantum space $\mathbb{Q}(Z)$ of a Klauder space, we write

$$|\mathbf{z}\rangle := |[0, \mathbf{z}]\rangle$$

and find from (46) that

$$\langle \mathbf{z}|\mathbf{z}'\rangle = e^{\mathbf{z}^*\mathbf{z}'}, \tag{55}$$

$$|z\rangle = e^{z_0}|\mathbf{z}\rangle. \tag{56}$$

Because of (56), the coherent subspace Z_0 consisting of the $[0, \mathbf{z}]$ with $\mathbf{z} \in V$ has the same quantum space as $Kl[V]$. We call the coherent spaces Z_0 **Glauber spaces** since the associated coherent states (originally due to SCHRÖDINGER [27]) were made prominent in quantum optics by GLAUBER [8]. Glauber spaces give a more parsimonious coherent description of the corresponding Fock space, but oscillator spaces are much more versatile since they have a much bigger symmetry group, with corresponding advantages in the applications.

6.4 Lowering and raising operators

In the quantum space of a Klauder space, we introduce an abstract **lowering symbol** a and its formal adjoint, the abstract **raising symbol** a^*. For $f : V \to \mathbb{C}$, we define the homogeneous map $\widetilde{f} : Z \to \mathbb{C}$ with $\widetilde{f}(z) := f(\mathbf{z})$. Since homogeneous maps and kernels of Klauder spaces are independent of z_0 and z_0', we may put

$$f(a) := \mathbf{a}(\widetilde{f}), \quad f(a^*) := \mathbf{a}^*(\widetilde{f}),$$

where $\mathbf{a}$ and $\mathbf{a}^*$ are given by (41) and (42). From the above we find that $\overline{\widetilde{f}} = \widetilde{\overline{f}}$, hence (42) gives $f(a)^* = \mathbf{a}(\widetilde{f})^* = \mathbf{a}(\overline{\widetilde{f}}) = \mathbf{a}(\widetilde{\overline{f}}) = \overline{f}(a^*)$, so that

$$f(a)^* == \overline{f}(a^*).$$

For any map $F : V \times V \to \mathbb{C}$, we define the homogeneous kernel $\widetilde{F} : Z \times Z \to \mathbb{C}$ with $\widetilde{F}(z, z') := F(\mathbf{z}, \mathbf{z}')$, and put

$$:F(a^*, a): \; := N(\widetilde{F})$$

as an operator in $\mathrm{Lin}^\times \mathbb{Q}(Z)$ if $\widetilde{F}$ is normal; otherwise as a Hermitian form on $\mathbb{Q}(Z)$. Here the pair of colons is the conventional notation for normal ordering.

6.6 Theorem. *Let $Z = Kl[V]$. Then:*

(i) Every linear operator $A \in \mathrm{Lin}^\times \mathbb{Q}(Z)$ can be written uniquely in normally ordered form $A = :F(a^*, a):$.

(ii) The map $F \to :F:$ is linear, with $:1: = 1$ and

$$:f(a)^* F(a^*, a) g(a): \; = \; f(a)^* \; :F(a^*, a): \; g(a);$$

in particular,

$$:f(a)^* g(a): \; = \; f(a)^* g(a).$$

(iii) The quantized coherent maps satisfy

$$\Gamma(A) \; = \; :e^{\rho + p^* a + a^* q + a^*(\mathbf{A}-1)a}: \quad \text{for } A = [\rho, p, q, \mathbf{A}] \in Os[V].$$

*(iv) We have the **Weyl relations***

$$e^{p^* a} e^{a^* q} = e^{p^* q} e^{a^* q} e^{p^* a}$$

*and the **canonical commutation relations***

$$(p^* a)(q^* a) = q^*(a)(p^* a), \quad (a^* p)(a^* q) = (a^* q)(a^* p), \tag{57}$$

$$(p^* a)(a^* q) - (a^* q)(p^* a) = \sigma(p, q) \tag{58}$$

hold, with the symplectic form (54).

Proof. (i) follows from Proposition 5.12 since the coherent product vanishes nowhere and homogeneous kernels are independent of z_0 and z'_0.

(ii) Let $F, G : V \times V \to \mathbb{C}$. We then have

$$:F+G: = N(\widetilde{F+G}) = N(\widetilde{F}+\widetilde{G}) = N(\widetilde{F}) + N(\widetilde{G}) = N(\widetilde{F}) + N(\widetilde{G}) = :F: + :G:,$$

$$:cF: = N(\widetilde{cF}) = N(c\widetilde{F}) = cN(\widetilde{F}) = c:F:,$$

which implies that the map $F \to :F:$ is linear.

(iii) holds since (46) implies

$$\langle z|A|z'\rangle = \langle z|[\rho, p^*, q, \mathbf{A}]|z'\rangle = e^{\overline{z}_0+\rho+z'_0+p^*\mathbf{z}'+\mathbf{z}^*(q+\mathbf{A}\mathbf{z}')} = X(z,z')K(z,z')$$

with $X(z,z') := e^{\rho+p^*\mathbf{z}'+\mathbf{z}^*q+\mathbf{z}^*(\mathbf{A}-1)\mathbf{z}'}$.

(iv) The Weyl relations follow from

$$\begin{aligned} e^{p^*a}e^{a^*q} &= :e^{p^*a}::e^{a^*q}: = \Gamma([0,p,0,1])\Gamma([0,0,q,1]) = \Gamma([p^*q,p,q,1]) \\ &= :e^{p^*q+p^*a+a^*q}: = :e^{p^*q}e^{a^*q}e^{p^*a}: = e^{p^*q}e^{a^*q}e^{p^*a}. \end{aligned}$$

The canonical commutation relations (58) are obtained by replacing p and q by εp and εq with $\varepsilon > 0$, expanding their exponentials to second order in ε, and comparing the coefficients of ε^2. (57) follows directly from the definition of the $f(a)$ and $f(a^*)$. □

Fock space is also the quantum space of a bigger coherent space containing the labels for all squeezed states (cf. ZHANG et al. [29]), in which the metaplectic group is realized by coherent maps. In this space, all normally ordered exponentials of quadratics in a^* and a are realized as coherent maps. Details will be discussed elsewhere.

If $V = \mathbb{C}^n$ we define

$$a_k := e_k(a), \quad a_k^* := e_k(a^*),$$

where e_k maps $\mathbf{z}$ to $\mathbf{z}_k$. Thus formally, a is a symbolic column vector with n symbolic **lowering operators** a_k, also called **annihilation operators**. Similarly, a^* is a symbolic row vector with n symbolic **raising operators** a_k, also called **creation operators**. They satisfy the standard canonical commutation relations

$$a_j a_k = a_k a_j, \quad a_j^* a_k^* = a_k^* a_j^*,$$

$$a_j a_k^* - a_k^* a_j = \delta_{jk}$$

following from (57) and (58).

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