Continuity notions for multi-valued mappings

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Abstract

If we regard a multi-valued mapping as a generalization of a single-valued mapping, we may ask for generalizations of important properties as well, concerning fixed-points, projection, composition etc. The main interest in this paper are continuity notions for multi-valued mappings which allow non-connected images, but still have a certain fixed-point property. The starting point is a continuity notion introduced by the second author in [23], which was stated to possess a zero-property. However, we provide a counterexample to this theorem. Two other continuity notions are introduced and examined, and applied in a logical field.

Keywords: Continuity, fixed points, multi-valued mappings, constraints.
Mathematics Subject Classification: 54C60, 26E25, 54H25.

Introduction

A multi-valued mapping F from X to Y ($F : X \multimap Y$) maps a point in X to a nonempty subset of Y. Our goal is to generalize the following two well-known theorems for single-valued mappings:

Brouwers Fixed-Point Theorem [5] Let S be an closed, convex and nonempty subset of $\mathbb{R}^n$, let $f : S \to S$ be a continuous single-valued mapping. Then f has a fixed point.

We use the following notation: $\mathbb{IR}$ is the set of all closed intervals in $\mathbb{R}$, $\mathbb{IR}^n$ the set of all n-dimensional **boxes** $\mathbf{x}$, i.e. cartesian products of n closed intervals; $\mathbf{x}_i := [\underline{\mathbf{x}}_i, \overline{\mathbf{x}}_i]$.

Mirandas Intermediate Value Theorem [22] Let $\mathbf{x} \in \mathbb{IR}^n$, let $f : \mathbf{x} \to \mathbf{x}$ be a continuous single-valued mapping such that for $i = 1, \dots, n$,

$$\begin{array}{ll} f_i(x) \leq 0 & \text{for all } x \in \mathbf{x} \ \text{ with } x_i = \underline{\mathbf{x}_i}, \\ f_i(x) \geq 0 & \text{for all } x \in \mathbf{x} \ \text{ with } x_i = \overline{\mathbf{x}_i}. \end{array}$$

Then f has a zero, i.e. there exists a point $x^* \in \mathbf{x}$ such that $f(x^*) = 0$.

We take a short look at the proof: Let Ω be an open and bounded subset of $\mathbb{R}^n$. Let $f \in C^1(\overline{\Omega}, \mathbb{R}^n)$ be a single-valued mapping such that $f(x) \neq 0$ for every $x \in \partial\Omega$ and every zero of f is a regular value of f. Then we define the **index** $\operatorname{ind}(f, \Omega)$ of f on the set $\Omega \subseteq \mathbb{R}^n$ by $\operatorname{ind}(f, \Omega) := \sum_{f(x)=0} \operatorname{sgn}(\det f'(x))$.
The index has the properties:

(i) If $\operatorname{ind}(f,\Omega) \neq 0$ then f has at least one zero on Ω.

(ii) If $f : \Omega \times [0,1] \to \mathbb{R}^n$ is a homotopy such that $f(x,t) \neq 0$ for every $x \in \partial\Omega$ and all $t \in [0,1]$, then $\operatorname{ind}(f(\cdot,t_1),\Omega) = \operatorname{ind}(f(\cdot,t_2),\Omega)$ for all $t_1, t_2 \in [0,1]$.

By approximation, we can expand the index to all continuous single-valued mappings, and the properties (i) and (ii) still hold.
To prove Brouwers Fixed-point Theorem, let S be an open, convex and nonempty subset of $\mathbb{R}^n$, let $f : \overline{S} \to \overline{S}$ be a continuous single-valued mapping. Then for an arbitrary $x_0 \in S$, consider the homotopy $f(x,t) = tf(x) + (1-t)x_0 - x$. Clearly $f(x,0) = x_0 - x$ and $f(x,1) = f(x) - x$, and $f(x,0)$ has zero at $x_0 \in S$, hence $\operatorname{ind}(f(\cdot,0),S) \neq 0$. Since S is convex, the assumption of item (ii) holds, therefore $\operatorname{ind}(f(\cdot,1),S) \neq 0$ and $f(x,1)$ has a zero in S, i.e. $f(x^*,1) = f(x^*) - x^* = 0$ and hence x^* is a fixed point of f. To prove Miranda's Theorem, simply choose the homotopy $f(x,t) = tf(x) + (1-t)x_0$. For details, see Chapter 6 in Ortega & Rheinboldt [24].

If for a multi-valued mapping $F : X \multimap Y$ there exists a continuous single-valued mapping $f : X \to Y$ with $f(x) \in F(x)$ for every $x \in X$, we say that f is a **selection** of F, and we apply the Brouwer Theorem or the Miranda Theorem to the selection f to obtain a fixed-point or a zero property for the multi-valued mapping F, see [20], [21], [17] and [24]. If for a multi-valued mapping $F : X \multimap Y$ for every $\varepsilon > 0$ there exists a continuous single-valued mapping $f : X \to Y$ with $\operatorname{graph}(f) \subseteq \mathcal{O}_\varepsilon(\operatorname{graph} F)$, we say that F is **approxable**. Via a generalization of the index called the **Lefschetz number** we obtain a fixed-point theorem for approxable multi-valued mappings, see [12]. The most common continuity notions for multi-valued mappings are upper semicontinuity (u.s.c.) and lower semiconitinuity (l.s.c.), for details see [1], [3], [6] and [19]. In view of fixed-point properties, we cannot obtain strong results for mere u.s.c. or l.s.c. multi-valued mappings, unless we make assumptions to the topological properties of the sets $F(x)$. If for all $x \in X$ the set $F(x)$ is compact and convex, we have the well-known fixed-point theorem by Kakutani, see [18], [10] and [27]. If for all $x \in X$ the set $F(x)$ is **acyclic**, i.e. it has the same homology groups as the one-point space (hence $F(x)$ is connected) we obtain fixed-pint properties via a generalization of the Lefschetz number, for example the theorem by Eilenberg-Montgomery [9], [2] and [13]. There exist certain fixed-point properties for multi-valued mappings with non-acyclic images, see e.g. [8], [4], [7]. For details about homology theory, we recommend [15] and [26]. A recent overview of the fixed-point theory for multi-valued mappings gives Gôrniewicz in [12], which also contains a current bibliography of publications concerning the fixed point theory of multi-valued mappings.

In this paper, we are concerned with continuity notions which are not defined by a topological property of the image of each point, but with a property of the graph of the multi-valued mapping. Specifically, we are only interested in continuity notions which allow disconnected images. For all continuity notions discussed here, we demand that if f is a continuous single-valued mapping, then the multi-valued mapping defined by $F(x) := \{f(x)\}$ is continuous in the multi-valued meaning. In this sense, all continuity notions considered here are generalizations of the usual continuity of single-valued mappings.

The second author in [23] introduced a continuity notion for multi-valued mappings, called 'c-continuity'. A multi-valued mapping $F : \mathbf{x} \multimap \mathbf{y}$ is called c-continuous if for every continuum C in $\mathbf{x}$ and $\tau, \tau' \in C$, the set $\operatorname{graph}(F|C)$ connects $\{\tau\} \times \mathbf{y}$ with $\{\tau'\} \times \mathbf{y}$. He stated a generalization of Miranda's theorem; unfortunately the proof is not valid (see Section 2).

Overview. The discussion of the continuity notions in Section 1 is guided by properties of multi-valued mappings which were suggested by Alexandre Goldsztejn and the second author. These are:

(PABR)	A multi-valued version of Brouwer's fixed point theorem,
(SETM)	a multi-valued version of Miranda's intermediate value theorem,
(REDO)	the presevation of continuity on a restiction of the domain,
(ENDO)	the preservation of continuity on a larger domain,
(JOINT)	a generalization of (PABR),
(PROD)	the preservation of continuity on the Cartesian products,
(PROJ)	the preservation of continuity under projection, and
(COMP)	the preservation of continuity under composition of multi-valued mappings.

The multivalued version of the Miranda theorem is a concept by the second author in [23], the other properties are ideas of GOLDSZTEJN in [11]. After introducing precise versions of these properties, we show that certain of these properties imply others.

In Section 2, we show that c-continuity has two of the properties above, namely the (REDO) and (PROJ), but fails to satisfy the others. Most important, (SETM) does not hold, although it was stated by NEUMAIER in [23]. We give a counterexample.

In Section 3, we introduce a continuity notion called 'is-continuity' which has five of the properties above, but three of them are still open questions. The property (SETM) is one of the open questions, but we show that a weaker version holds. A multi-valued mapping $F : \mathbf{x} \multimap \mathbf{y}$ is called is-continuous if there exists a continuous single-valued mapping $f : \mathbf{x} \times \mathbf{y} \to \mathbb{R}^{\dim \mathbf{y}}$ such that $f^{-1}(0)$ is a subset of graph$F$, plus additional assumptions to prevent that $f^{-1}(0)$ is empty. Since everything is expressed in terms of continuous single-valued mappings, the proofs are all rather elementary and straightforward.

Section 4: We introduce a continuity notion by GOLDSZTEJN called ma-continuity. Multi-valued mappings are ma-continuous if their graph can be approximated by a sequence of smooth manifolds with boundary. GOLDSZTEJN proved the (JOINT)-property.

Section 5: RATSCHAN in [25] applied the second author's (wrong) multi-valued version of the Miranda Theorem for c-continuous multi-valued mappings to constraints, i.e. formulae in first-order predicate language. A witness function for some constraint is a continuous single-valued mapping F such that for every point in the image of F, the constraint is true. The central theorem in RATSCHAN [25] is that if there exists a witness function for two constraints, then there exists also a witness function for the conjunction of these two constraints. Unfortunately RATSCHAN's theorem is wrong in general, we give a counterexample. Then we introduce and prove a modified version of RATSCHAN's theorem: instead of continuous single-valued mappings, we use is-continuous multi-valued mappings.

Notation. For subsets $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ we say that a single-valued mapping of X into $\mathcal{P}(Y) \setminus \emptyset$ is a **multi-valued mapping** (or **set-valued map** or **multimap**) of X into Y and we write $F : X \multimap Y$. In general, we use lower-case letters $f, g, h, \ldots$ for single-valued mappings and upper-case letter $F, G, H, \ldots$ for multi-valued mappings. Lower-case greek letters $\varphi, \psi, \ldots$ usually denote paths. We use upper-case letters $X, Y, Z, \ldots$ for

Table 1 gives a summary of the results of Sections 2, 3 and 4. We discuss three continuity notions for multi-valued mappings: c-continuity, is-continuity and ma-continuity. Eight properties, which will be described in the following section, are examined for each continuity notion.

Table 1: Summary

	PABR	SETM	REDO	ENDO	JOINT	PROD	PROJ	COMP
c-continuity	no	no	yes	no	no	no	yes	no
is-continuity	yes	**	yes	yes	yes	yes		
ma-continuity	yes			*	yes	*	*	*

* These are conjectured to be true by GOLDSZTEJN [11].

** Although (SETM) remains an open question we prove a slightly weaker version of the set-valued Miranda theorem.

spaces (usually subsets of $\mathbb{R}^n$, if not asserted otherwise). A **fixed point** of a single-valued mapping $f : X \to X$ is a point $x^* \in X$ such that $f(x^*) = x^*$. For a single-valued mapping $f : X \to Y$, let $\mathrm{graph} f := \{(x, y) \in X \times Y \mid x \in X, y = f(x)\}$. For a multi-valued mapping $F : X \multimap Y$ let $\mathrm{graph} F := \{(x, y) \in X \times Y \mid x \in X, y \in F(x)\}$, and for a **continuum**, i.e. a compact and connected set $C \subseteq X$, let $\mathrm{graph}(F|C) := \{(x, y) \in X \times Y \mid x \in C, y \in F(x)\}$. For a metric space X, we write $\mathrm{dist}(x, y)$ for the **distance** of the points x and y. For a set $A \subseteq X$, $\mathrm{dist}(x, A) := \inf\{\mathrm{dist}(x, y) \mid y \in A\}$ is the distance of x and A.
Other often used symbols are: $B_r(x) := \{y \in X \mid \|x - y\| < r\}$, the open ball with center x and radius r, $\mathcal{O}_\varepsilon(A) := \bigcup_{a\in A} B_\varepsilon(a)$, an '$\varepsilon$-close' neighbourhood of the set A, $\mathrm{proj}_X A$ is the projection of the set A to the space X, $\mathrm{Im} f := \bigcup_{x\in X} f(x)$, the image of the mapping $f : X \to Y$ and for the multi-valued mapping $F : X \multimap Y$, the set $F_i(x)$ is the projection of the set $F(x)$ to the ith component of Y.

1 Properties

Since we deal with different continuity notions, we will formulate the properties generally for ω-continuity. First, for any continuity notion ω we demand that if $f : \mathbf{x} \to \mathbf{y}$ is continuous, then $F : \mathbf{x} \multimap \mathbf{y}$ defined by $F(x) := \{f(x)\}$ is ω-continuous. Now we introduce some properties desireable for a good continuity notion ω for multi-valued mappings.

We shall first consider properties of general continuity notions, and then (in chapters 2, 3 and 4) the specific continuity notions with $\omega \in \{\text{c, is, ma}\}$ as stated in Table 1. Some of the following properties are desirable for a continuity notion ω for multi-valued mappingsto be useful:

(PABR) Let $F : \mathbf{x} \times \mathbf{y} \multimap \mathbf{y}$ be ω-continuous. Then the multi-valued mapping $H : \mathbf{x} \multimap \mathbf{y}$ defined by

$$H(x) := \{y \in \mathbf{y} \mid y \in F(x, y)\}$$

is ω-continuous.
This is a parametric version of the Brouwer fixed-point theorem.

(SETM) Let $F : \mathbf{x} \times \mathbf{y} \multimap \mathbf{y}$ be a ω-continuous multi-valued mapping, let $0 \in \mathbf{y}$, and let

$$\begin{aligned} \sup(F_i(x,y)) \leq 0 \quad &\text{if} \quad y_i = \overline{\mathbf{y}}_i, \\ \inf(F_i(x,y)) \geq 0 \quad &\text{if} \quad y_i = \underline{\mathbf{y}}_i. \end{aligned}$$

Then the multi-valued mapping $H : \mathbf{x} \multimap \mathbf{y}$ defined by

$$H(x) := \{y \in \mathbf{y} \mid 0 \in F(x,y)\}$$

is ω-continuous.
This is a set-valued version of the Miranda theorem.

(REDO) If $F : \mathbf{x} \multimap \mathbf{y}$ is ω-continuous on a continuum (i.e. a connected and compact set) $C \subseteq \mathbf{x}$, then F is also ω-continuous on any continuum which is a subset of C.
This is the preservation of ω-continuity on a smaller domain.

(ENDO) Let $F : \mathbf{x} \multimap \mathbf{z}$ be ω-continuous. Then the multi-valued mapping $H : \mathbf{x} \times \mathbf{y} \multimap \mathbf{z}$ defined by

$$H(x,y) := F(x)$$

is ω-continuous.
This is the preservation of ω-continuity on a larger domain.

(JOINT) Let $F : \mathbf{x} \multimap \mathbf{y}$ be ω-continuous, such that $\mathbf{y}_1 \subseteq \mathbf{x}_1$. Then the multi-valued mapping $H : \mathbf{x}_2 \times \ldots \times \mathbf{x}_n \multimap \mathbf{y}$ defined by

$$H(x_2, \ldots, x_{\mathrm{dim}\mathbf{x}}) := \{y \in \mathbf{y} \mid y \in F(y_1, x_2, \ldots, x_{\mathrm{dim}\mathbf{x}})\}$$

is ω-continuous.
This is a generalization of the (PABR)-property.

(PROD) Let $F : \mathbf{x} \multimap \mathbf{u}$ and $G : \mathbf{y} \multimap \mathbf{v}$ be two ω-continuous multi-valued mappings. Then the multi-valued mapping $H : \mathbf{x} \times \mathbf{y} \multimap \mathbf{u} \times \mathbf{v}$ defined by

$$H(x,y) := F(x) \times G(y)$$

is ω-continuous.
This is the preservation of ω-continuity on the Cartesian product of two multi-valued mappings.

(PROJ) Let $F : \mathbf{x} \multimap \mathbf{y}$ be a ω-continuous multi-valued mapping. Then for any sequence $I = \{i_1, \ldots, i_k\}$ of integers $0 < i_1 < \ldots < i_k \leq \mathrm{dim}\mathbf{y}$, the multi-valued mapping $H : \mathbf{x} \multimap \mathbf{y}_{i_1} \times \ldots \times \mathbf{y}_{i_k}$ defined by

$$H(x) := \mathrm{proj}_{\mathbf{y}_I}(F(x))$$

is ω-continuous.
This is the preservation of ω-continuity with respect to arbitrary projections.

(COMP) Let $F : \mathbf{x} \multimap \mathbf{y}$ and $G : \mathbf{y} \multimap \mathbf{z}$ be two ω-continuous multi-valued mappings. Then the multi-valued mapping $H : \mathbf{x} \multimap \mathbf{z}$ defined by

$$H(x) = G \circ F\ (x) := \bigcup_{y \in F(x)} G(y)$$

is ω-continuous.
This is the preservation of ω-continuity with respect to composition of multi-valued mappings.

(COMP') Let $F : \mathbf{x} \multimap \mathbf{y}$ and $G : \mathbf{x} \times \mathbf{y} \multimap \mathbf{z}$ be two ω-continuous multi-valued mappings. Then the multi-valued mapping $H : \mathbf{x} \multimap \mathbf{z}$ defined by

$$H(x) = G \circ F\ (x) := \bigcup_{y \in F(x)} G(x, y)$$

is ω-continuous.
This is the preservation of ω-continuity with respect to an alternative composition of multi-valued mappings.

Some implications

Without further requestions on a continuity notions ω for multi-valued mappings, some properties imply others. Theorem 1.1 and Theorem 1.2 item (i) and (iii) are due to Goldsztejn. In the following list of implications, we do not claim completeness. If not stated otherwise, let $\mathbf{x} \in \mathbb{IR}^n$, $\mathbf{y} \in \mathbb{IR}^m$ for $n, m \geq 1$.

1.1 Theorem.

- *(i) The property (JOINT) implies (PABR).*
- *(ii) The properties (PABR) together with (ENDO) imply (JOINT).*

Proof. For (i) we assume an ω-continuous multi-valued mapping $F : \mathbf{x} \times \mathbf{y} \multimap \mathbf{y}$. Now apply the property (JOINT) to each coordinate of $\mathbf{y}$ (trivially $\mathbf{y}_i \subseteq \mathbf{y}_i$). Then the multi-valued mappings

$$\begin{array}{lll} H_1 : \mathbf{x} \times \mathbf{y}_2 \times \ldots \times \mathbf{y}_m \multimap \mathbf{y} & \text{defined by} & H_1(x, y_2, \ldots, y_m) := \{y \in \mathbf{y} \mid y \in F(x, y)\}, \\ \quad \vdots & \quad \vdots & \\ H_m : \mathbf{x} \multimap \mathbf{y} & \text{defined by} & H_m(x) := \{y \in \mathbf{y} \mid y \in F(x, y)\}, \end{array}$$

are all ω-continuous. Since H_m is the same multi-valued mapping as H in (PABR), the proof of (i) is complete.
For (ii) we assume an ω-continuous multi-valued mapping $F : \mathbf{x} \multimap \mathbf{y}$ with $\mathbf{y}_1 \subseteq \mathbf{x}_1$. Now we define a multi-valued mapping $\widehat{F} : \mathbf{x}_2 \times \ldots \times \mathbf{x}_n \times \mathbf{y} \multimap \mathbf{y}$ by

$$\widehat{F}(x_2, \ldots, x_n, y_1, \ldots, y_m) := F(x_1, \ldots, x_n) \qquad \text{for} \ \ x_1 = y_1.$$

Due to (ENDO), $\widehat{F}$ is ω-continuous. Now we apply (PABR) and obtain an ω-continuous multi-valued mapping $H : \mathbf{x}_2 \times \ldots \times \mathbf{x}_n \multimap \mathbf{y}$ defined by

$$\begin{aligned} H(x_2, \ldots, x_n) :\ &= \{y \in \mathbf{y} \mid y \in \widehat{F}(x_2, \ldots, x_n, y)\} \\ &= \{y \in \mathbf{y} \mid y \in F(x_2, \ldots, x_n)\}. \end{aligned}$$

Since H is ω-continuous by assumption, the proof for (ii) is complete. □

1.2 Theorem.

(i) The property (COMP) implies (ENDO).

(ii) The properties (PROD) together with (PROJ) imply (ENDO).

(iii) The properties (JOINT) together with (PROD) and (PROJ) imply (COMP).

Proof. For (i) we assume an ω-continuous multi-valued mapping $F : \mathbf{x} \multimap \mathbf{z}$, and we define a multi-valued mapping $G : \mathbf{x} \times \mathbf{y} \multimap \mathbf{x}$ by $G(x, y) := \{x\}$. Then G is ω-continuous since it is the set-valued version of a continuous single-valued mapping. Now we define the multi-valued mapping $H : \mathbf{x} \times \mathbf{y} \multimap \mathbf{z}$ by

$$H(x, y) := F \circ G(x, y).$$

Then H is ω-continuous because (COMP) applies. Since

$$\begin{aligned} H(x, y) \ &= \textstyle\bigcup_{z \in G(x,y)} F(z) \\ &= \textstyle\bigcup_{z = x} F(z) \\ &= F(x), \end{aligned}$$

the proof of (i) is complete.

For (ii) we assume an ω-continuous multi-valued mapping $F : \mathbf{x} \multimap \mathbf{z}$. Define the multi-valued mapping $G : \mathbf{y} \multimap \{0\}$ by $G(y) := \{0\}$. Then G is the set-valued version of a continuous single-valued mapping and therefore ω-continuous. Now we apply (PROD) to F and G and obtain an ω-continuous multi-valued mapping $\widehat{H} : \mathbf{x} \times \mathbf{y} \multimap \mathbf{z} \times \{0\}$ defined by

$$\widehat{H}(x, y) := F(x) \times \{0\}.$$

Now we apply (PROJ) and obtain that the multi-valued mapping $H : \mathbf{x} \times \mathbf{y} \multimap \mathbf{z}$ defined by

$$\begin{aligned} H(x, y) \ &= \operatorname{proj}_{\mathbf{z}}(F(x) \times \{0\}) \\ &= F(x), \end{aligned}$$

is ω-continuous, which completes the proof for (ii).

For (iii), let $F : \mathbf{x} \multimap \mathbf{y}$ and $G : \mathbf{y} \multimap \mathbf{z}$ be ω-continuous. Define the multi-valued mapping $\widehat{H} : \mathbf{x} \times \mathbf{y} \multimap \mathbf{y} \times \mathbf{z}$ by

$$\widehat{H}(x, y) := F(x) \times G(y).$$

Then $\widehat{H}$ is ω-continuous due to (PROD). Now we apply (JOINT) to all coordinates of $\mathbf{y}$ (trivially $\mathbf{y}_i \subseteq \mathbf{y}_i$), and we obtain that the multi-valued mappings

$$\begin{array}{ll} \widehat{H}_1 : \mathbf{x} \times \mathbf{y}_2 \times \ldots \times \mathbf{y}_m \multimap \mathbf{y} \times \mathbf{z}, & \widehat{H}_1(x, y_2, \ldots, y_m) = \{(y, z) \in \mathbf{y} \times \mathbf{z} \mid (y, z) \in F(x) \times G(y)\}, \\ \qquad \vdots & \qquad \vdots \\ \widehat{H}_m : \mathbf{x} \multimap \mathbf{y} \times \mathbf{z}, & \widehat{H}_m(x) = \{(y, z) \in \mathbf{y} \times \mathbf{z} \mid (y, z) \in F(x) \times G(y)\} \end{array}$$

are all ω-continuous. We apply (PROJ) to $\widehat{H}_m$ and obtain an ω-continuous multi-valued mapping $H : \mathbf{x} \multimap \mathbf{z}$ defined by

$$\begin{aligned} H(x) : \; &= \operatorname{proj}_{\mathbf{z}}(\widehat{H}_m(x)) \\ &= \{z \in \mathbf{z} \mid y \in F(x), z \in G(y)\} \\ &= \{z \in G(y) \mid y \in F(x)\} \\ &= \textstyle\bigcup_{y \in F(x)} G(y) \\ &= G \circ F\ (x), \end{aligned}$$

which completes the proof. $\square$

1.3 Theorem. *If for any ω-continuous multi-valued mapping $F : \mathbf{x} \times \mathbf{y} \multimap \mathbf{y}$ the multi-valued mapping defined by*

$$\widehat{F}(x, y) := F(x, \dot{y}) + \dot{y}$$

for a continuous, piecewise linear single-valued mapping $\dot{y} : \mathbf{y} \to \mathbf{y}$ is also ω-continuous, then the properties (PABR) and (SETM) are equivalent.

Proof. Let $F : \mathbf{x} \times \mathbf{y} \multimap \mathbf{y}$ be ω-continuous with $0 \in \mathbf{y}$ and

$$\begin{aligned} \sup(F_i(x, y)) \leq 0 \quad &\text{if} \quad y_i = \overline{\mathbf{y}}_i, \\ \inf(F_i(x, y)) \geq 0 \quad &\text{if} \quad y_i = \underline{\mathbf{y}}_i. \end{aligned}$$

For each $\mathbf{y}_i$ define $\widehat{\mathbf{y}}_i := [2\underline{\mathbf{y}}_i, 2\overline{\mathbf{y}}_i]$, and $\widehat{\mathbf{y}} := \prod_i \widehat{\mathbf{y}}_i$. Define the multi-valued mapping $\widehat{F} : \mathbf{x} \times \widehat{\mathbf{y}} \multimap \widehat{\mathbf{y}}$ by

$$\widehat{F}(x, y) := F(x, \dot{y}) + \dot{y} \qquad \text{with} \qquad \dot{y}_i := \begin{cases} y_i & \text{if} \quad y_i \in \mathbf{y}_i, \\ \underline{\mathbf{y}}_i & \text{if} \quad y_i \leq \underline{\mathbf{y}}_i, \\ \overline{\mathbf{y}}_i & \text{if} \quad y_i \geq \overline{\mathbf{y}}_i. \end{cases}$$

Then by assumption, $\widehat{F}$ is ω-continuous. We now apply (PABR) and obtain that the multi-valued mapping $\widehat{H} : \mathbf{x} \multimap \widehat{\mathbf{y}}$ defined by

$$\widehat{H}(x) := \{y \in \widehat{\mathbf{y}} \mid y \in \widehat{F}(x, y)\}$$

is ω-continuous. Since $\widehat{F}$ is constant in y for $y \notin \mathbf{y}$, the set $\widehat{H}(x)$ is a subset of $\mathbf{y}$ for all $x \in \mathbf{x}$. Hence

$$\begin{aligned} \widehat{H}(x) \; &= \{y \in \mathbf{y} \mid y \in F(x, y) + y\} \\ &= \{y \in \mathbf{y} \mid 0 \in F(x, y)\} \\ &= H(x). \end{aligned}$$

Hence $H : \mathbf{x} \multimap \mathbf{y}$ is ω-continuous.
For the reverse direction, let $F : \mathbf{x} \times \mathbf{y} \multimap \mathbf{y}$ be ω-continuous, and let $F_i(x, y) > \underline{\mathbf{y}}_i$ for $y_i = \underline{\mathbf{y}}_i$. For each $\mathbf{y}_i$ define $\widehat{\mathbf{y}}_i := [\underline{\mathbf{y}}_i - \overline{\mathbf{y}}_i, \overline{\mathbf{y}}_i - \underline{\mathbf{y}}_i]$, and $\widehat{\mathbf{y}} := \prod_i \widehat{\mathbf{y}}_i$. Hence $0 \in \widehat{\mathbf{y}}$. Define the multi-valued mapping $\widehat{F} : \mathbf{x} \times \widehat{\mathbf{y}} \multimap \widehat{\mathbf{y}}$ by

$$\widehat{F}(x, y) := F(x, \dot{y}) - \dot{y} \qquad \text{with} \qquad \dot{y}_i := \begin{cases} y_i + \underline{\mathbf{y}}_i & \text{if} \quad y_i \geq 0, \\ 0 & \text{if} \quad y_i < 0. \end{cases}$$

Then by assumption, $\widehat{F}$ is ω-continuous.
Note that

$$\begin{aligned} \sup(\widehat{F}_i(x,y)) \le 0 \quad &\text{if} \;\; y_i = \overline{\widehat{\mathbf{y}}}_i, \\ \inf(\widehat{F}_i(x,y)) \ge 0 \quad &\text{if} \;\; y_i = \underline{\widehat{\mathbf{y}}_i}. \end{aligned}$$

We now apply (SETM) and obtain that the multi-valued mapping $\widehat{H} : \mathbf{x} \multimap \widehat{\mathbf{y}}$ defined by

$$\widehat{H}(x) := \{y \in \widehat{\mathbf{y}} \mid 0 \in \widehat{F}(x,y)\}$$

is ω-continuous. Since $\widehat{F}(x,y) > 0$ for $y_i < 0$, the set $\widehat{H}(x)$ is a subset of $\prod_i [0, \overline{\mathbf{y}}_i - \underline{\mathbf{y}}_i]$. for all $x \in \mathbf{x}$. Hence

$$\begin{aligned} \widehat{H}(x) \;\; &= \{y \in \textstyle\prod_i [0, \overline{\mathbf{y}}_i - \underline{\mathbf{y}}_i] \mid 0 \in \widehat{F}(x,y)\} \\ &= \{y \in \textstyle\prod_i [0, \overline{\mathbf{y}}_i - \underline{\mathbf{y}}_i] \mid 0 \in F(x,\dot{y}) - \dot{y}\} \\ &= \{y \in \textstyle\prod_i [0, \overline{\mathbf{y}}_i - \underline{\mathbf{y}}_i] \mid y_i + \underline{\mathbf{y}}_i \in F_i(x, y + \underline{\mathbf{y}}) \quad \text{for every } i\} \\ &= \{y \in \textstyle\prod_i [\underline{\mathbf{y}}_i, \overline{\mathbf{y}}_i] \mid y \in F(x,y)\} \\ &= \{y \in \mathbf{y} \mid y \in F(x,y)\} \\ &= H(x). \end{aligned}$$

Hence $H : \mathbf{x} \multimap \mathbf{y}$ is ω-continuous, and the proof is complete. ☐

2 c-Continuous multi-valued mappings

The concept of c-continuity was introduced by Neumaier in [23]. It was used there in a context of interval arithmetic. We show that c-continuous multi-valued mappings have the properties (REDO) and (PROJ) but do not have the properties (PABR), (SETM), (ENDO), (JOINT), (PROD) and (COMP). As pointed out before, the negative answer to the property (SETM) is the most important one, because in [23], c-continuous multi-valued mappings were claimed to possess the property (SETM). We shall point out why the 'proof ' in [23] is faulty. Note that for single-valued mappings, there exists a notion related to the idea of c-continuity, called 'connectivity maps', see Hamilton [14] and Jordan & Nadler [17].

2.1 Definition. Let $A, B \subseteq \mathbf{x}$. A continuum (i.e. a compact and connected set) $\mathcal{C} \subseteq \mathbf{x}$ **connects** the sets A with B if the sets $A \cap \mathcal{C}$ and $B \cap \mathcal{C}$ are both nonempty.

2.2 Definition. We say that the multi-valued mapping $F : \mathbf{x} \multimap \mathbf{y}$ is **c-continuous** (*connection continuous*) on a set $E \subseteq \mathbf{x}$ if for every continuum and any two points $\tau, \tau' \in C$, the set graph$(F|C)$ contains a continuum $\mathcal{C}$ such that $\mathcal{C}$ connects the sets $\{\tau\} \times \mathbf{y}$ with $\{\tau'\} \times \mathbf{y}$.

2.3 Proposition. *(*Neumaier *[23, Proposition 5.3.3]) If $f : \mathbf{x} \to \mathbf{y}$ is a continuous single-valued mapping, then the multi-valued mapping $F : \mathbf{x} \multimap \mathbf{y}$ defined by*

$$F(x) := \{f(x)\}$$

is c-continuous.

The next proposition gives a sufficient condition for a multi-valued mapping to be c-continuous.

2.4 Proposition. *Let* $\mathbf{x} \in \mathbb{R}^n$ *and let* $\mathbf{y} \in \mathbb{R}^m$. *Let* $F : \mathbf{x} \multimap \mathbf{y}$ *be a multi-valued mapping such that* $\operatorname{graph}(F|C)$ *is closed for every continuum* $C \subseteq \mathbf{x}$, *and* $F(x)$ *is connected for every* $x \in \mathbf{x}$. *Then* F *is c-continuous.*

Proof. Choose an arbitrary continuum $C \subseteq \mathbf{x}$ and $\tau, \tau' \in C$. Then $\operatorname{graph}(F|C)$ is closed by assumption and bounded, hence compact. Suppose that $\operatorname{graph}(F|C)$ does not connect $\tau \times \mathbb{R}^m$ with $\tau' \times \mathbb{R}^m$. Since $\operatorname{graph}(F|C)$ is closed, there exist closed, disjoint sets $A_1, A_2 \subseteq \mathbf{x} \times \mathbf{y}$ such that $A_1 \cup A_2 = \operatorname{graph}(F|C)$. Since for every $x \in C$ the set $F(x)$ is connected, the set $\{x\} \times F(x)$ is either a subset of A_1 or a subset of A_2. Let

$$B_1 := \{x \in C \mid x \times F(x) \subseteq A_1\} \quad \text{and} \quad B_2 := \{x \in C \mid x \times F(x) \subseteq A_2\}.$$

Then B_1 and B_2 are disjoint sets that cover C. Since B_i is the projection of A_i to $\mathbf{x}$ for $i = 1, 2$, the sets B_1 and B_2 are also closed, hence C is not connected, a contradiction. □

2.5 Corollary. *Let* $\mathbf{x} = [-1, 1]$, *let* $\mathbf{y} := [-1, 1] \times [-1, 1]$. *For* $r > 0$, $\alpha < 1$ *and* $\delta < \frac{r}{2}$ *we define the multi-valued mapping* $G : \mathbf{x} \times \mathbf{y} \multimap \mathbf{y}$ *by:*

$$G(x, y) := \begin{cases} \overline{B_r}(0) \setminus B_\delta(y) & \text{for} \quad x \in (-\alpha, \alpha), \\ \overline{B_r}(0) & \text{otherwise.} \end{cases}$$

Then G *is c-continuous.*

Proof. Note that $G(x, y) \subseteq \mathbf{y}$ is connected and nonempty for all $(x, y) \in \mathbf{x} \times \mathbf{y}$ because $\delta < \frac{r}{2}$. The set $\operatorname{graph}(G|\mathbf{x} \times \mathbf{y})$ is closed, because

$$\begin{aligned} \operatorname{graph}(G|\mathbf{x} \times \mathbf{y}) &= \{(x, y) \times G(x, y) \mid x \in \mathbf{x}, y \in \mathbf{y}\} \\ &= \{(x, y) \times \overline{B_r}(0) \mid x \in \mathbf{x}, y \in \mathbf{y}\} \setminus \{(x, y) \times B_\delta(y) \mid x \in (-\alpha, \alpha), y \in \mathbf{y}\} \\ &= R \setminus S, \end{aligned}$$

where $R := \{(x, y) \times \overline{B_r}(0) \mid x \in \mathbf{x}, y \in \mathbf{y}\}$ and $S := \{(x, y) \times B_\delta(y) \mid x \in (-\alpha, \alpha), y \in \mathbf{y}\}$. We define $S' \subseteq S$ by $S' := \{(x, y) \times B_\delta(y) \mid x \in (-\alpha, \alpha), y \in \mathcal{O}_\varepsilon(\mathbf{y})\}$ for some $\varepsilon > 0$. Since $R \cap S = R \cap S'$ we infer $R \setminus S' = R \setminus S = \operatorname{graph}(G|\mathbf{x} \times \mathbf{y})$. Since $R = \mathbf{x} \times \mathbf{y} \times \overline{B_r}(0)$, the set R is closed. Via the homeomorphism $h : (x, y, z) \mapsto (x, y, z - y)$, S' is homeomorphic to $(-\alpha, \alpha) \times \mathcal{O}_\varepsilon(\mathbf{y}) \times B_\delta(0)$, and hence S' is open. Therefore $\operatorname{graph}(G|\mathbf{x} \times \mathbf{y}) = R \setminus S'$ is closed. Now Proposition 2.4 applies and G is c-continuous. □

(PABR) Not every c-continuous multi-valued mapping satisfies the property (PABR), as the following counterexample shows.

2.6 Example. Let G be defined as in Corollary 2.5. By definition of H,

$$H(x) = \{y \in \mathbf{y} \mid y \in G(x, y)\} = \begin{cases} \emptyset & \text{for} \quad x \in (-\alpha, \alpha), \\ \overline{B_r}(0) & \text{otherwise.} \end{cases}$$

For $C = \mathbf{x}$ and $\tau = -1, \tau' = 1$, the set $\operatorname{graph}(H|C) = ([-1, -\alpha] \cup [\alpha, 1]) \times \overline{B_r}(0)$ does not connect $\mathbf{x} \times \tau$ with $\mathbf{x} \times \tau'$. Hence H is not c-continuous.

(SETM) In Neumaier [23], c-continuous multi-valued mappings were claimed to possess the property (SETM) (Theorem 5.3.7) We give a counterexample.

2.7 Corollary. *Let* $\mathbf{x} = [-1,1] \times [-1,1]$*. For* $r > 0$ *and* $\delta < \frac{r}{2}$ *we define the multi-valued mapping* $G_0 : \mathbf{x} \multimap \mathbf{x}$ *by*

$$G_0(x) := \overline{B_r}(x) \setminus B_\delta(0).$$

Then G_0 *is c-continuous.*

Proof. To show that G_0 is c-continuous, we apply Proposition 2.4. Since $\delta < \frac{r}{2}$, the set $G_0(x)$ is nonempty and connected for every $x \in \mathbf{x}$.

$$\begin{aligned}\operatorname{graph}(G_0|\mathbf{x}) &= \textstyle\bigcup_{x\in\mathbf{x}}\{x\} \times (\overline{B_r}(x) \setminus B_\delta(0))\\ &= \textstyle\bigcup_{x\in\mathbf{x}}\{x\} \times \overline{B_r}(x) \setminus \bigcup_{x\in\mathbf{x}}\{x\} \times B_\delta(0)\\ &= R \setminus S,\end{aligned}$$

where $R := \bigcup_{x\in\mathbf{x}}\{x\} \times \overline{B_r}(x)$ and $S := \bigcup_{x\in\mathbf{x}}\{x\} \times B_\delta(0)$.
We define a set S' with $S' \supseteq S$ by

$$S' := \bigcup_{x\in\mathcal{O}_\varepsilon(\mathbf{x})} \{x\} \times B_\delta(0).$$

Since $R \cap S = R \cap S'$ we infer $R \setminus S = R \setminus S' = \operatorname{graph}(G_0|\mathbf{x})$. Since R is homeomorphic to $\mathbf{x} \times \overline{B_r}(0)$, the set R is closed, and $S' = \mathcal{O}_\varepsilon(\mathbf{x}) \times B_\delta(0)$ is open. Hence the set $\operatorname{graph}(G_0|\mathbf{x}) = R \setminus S'$ is closed, and the set $\operatorname{graph}(G_0|C) = \operatorname{graph}(G_0|\mathbf{x}) \cap C \times \mathbf{x}$ is also closed for any continuum $C \subseteq \mathbf{x}$. Now Proposition 2.4 applies and G_0 is c-continuous. ☐

2.8 Corollary. *Let* $\mathbf{y} = [-1,1] \times [-1,1]$*, let* $\mathbf{x} := \{0\}$*. For* $r < 1$ *and* $\delta < \frac{r}{2}$ *we define the multi-valued mapping* $G : \mathbf{x} \times \mathbf{y} \multimap \mathbf{x}$ *by*

$$G(x,y) := G_0(y) = \overline{B_r}(y) \setminus B_\delta(0),$$

with G_0 *as defined in 2.7. Then* G *is c-continuous.*

Proof. Since $\operatorname{graph}(G|\mathbf{x} \times \mathbf{y}) = \{0\} \times \operatorname{graph}(G_0|\mathbf{y})$ we can again apply Proposition 2.4 and obtain the c-continuity of G. ☐

2.9 Example. Let G be defined as in Corollary 2.8. Since $r < 1$, for $i = 1, 2$,

$$\begin{aligned}\sup(G_i(x,y)) \le 0 \quad &\text{if} \quad y \in \mathbf{y}^{(i)},\\ \inf(G_i(x,y)) \ge 0 \quad &\text{if} \quad y \in \mathbf{y}_{(i)}.\end{aligned}$$

Hence G satisfies the hypothesis of (SETM), but by definition of G there exists no point $(x_0, y_0) \in \mathbf{x} \times \mathbf{y}$ with $0 \in G(x_0, y_0)$. Therefore H maps every point to the empty set, hence is not c-continuous.

The counterexample shows that the proof given by the second author in [23] cannot be correct. In fact, the stated equivalence of the term in line 15 with the term in line 16 on page 197 in [23] is incorrect. However, the error has only consequences for the following three theorems in [23], since the rest of the book only uses the well-known theorem of Leray & Schauder. Other parts than Section 5.3 of [23] are not concerned by the error.

(REDO)

2.10 Proposition. *Every c-continuous multi-valued mapping has the property (REDO).*

Proof. This property follows immediately from the definition of c-continuity. $\square$

(ENDO) Not every c-continuous multi-valued mapping satisfies the property (ENDO). We give a counterexample, but first need two Lemmas by NEUMAIER [23].

2.11 Lemma. *(*NEUMAIER *[23, Lemma 5.3.1.(iii)])*
Let A, B be closed subsets of $E \subseteq \mathbb{R}^n$. If E is compact and $C_l (l = 0, 1, 2, \ldots)$ is an infinite sequence of subsets of E such that each C_l connects A with B then the set C of all accumulation points of all sequences $t^l (l = 0, 1, 2, \ldots)$ with $t^l \in C_l$ for $l \geq 0$ connects A with B.

2.12 Lemma. *(*NEUMAIER *[23, Lemma 5.3.2.])*
Let a, b be closed intervals and let Σ be a closed subset of $a \times b$. If for every continuous mapping $\varphi : [0, 1] \to a \times b$ with $\mathrm{proj}_b(\varphi(0)) \in \underline{b}, \mathrm{proj}_b(\varphi(1)) \in \overline{b}$ there is a number $s \in [0, 1]$ such that $\varphi(s) \in \Sigma$ then Σ connects $\{\underline{a}\} \times \mathbb{R}$ with $\{\overline{a}\} \times \mathbb{R}$.

2.13 Proposition. *Let $\mathbf{x} = [-2, 2] \times [-2, 2], \mathbf{z} = [-2, 2]$. Let $\varphi : [0, 1] \to \mathbf{x} \times \mathbf{z}$ be defined by*

$$\varphi(t) := (\cos(3\pi t), \cos(2\pi t), 2t - 1).$$

For $0 < \varepsilon < \frac{1}{2}$ define the set $\Gamma \subseteq \mathbf{x} \times \mathbf{z}$ by

$$\Gamma := \mathbf{x} \times [-1, 1] \setminus \mathcal{O}_\varepsilon(\mathrm{Im}\varphi).$$

Now let the multi-valued mapping $F : \mathbf{x} \multimap \mathbf{z}$ be defined by

$$F(x) := \mathrm{proj}_{\mathbf{z}}(\{x\} \times \mathbf{z} \cap \Gamma).$$

Then F is c-continuous.

Proof. (i) First notice that $\mathrm{graph}F = \Gamma$, and that $\mathrm{proj}_{\mathbf{x}}(\mathrm{Im}\varphi)$ has a shape like '$\propto$'. Suppose that there exists a path $\alpha : [0, 1] \to \mathbf{x} \times \mathbf{z}$ connecting $\mathbf{x} \times \underline{\mathbf{z}}$ with $\mathbf{x} \times \overline{\mathbf{z}}$, and $\mathrm{Im}\alpha \cap \mathrm{graph}F = \emptyset$. Then by construction of $\mathrm{graph}F$, the projection of $\mathrm{Im}\alpha$ to $\mathbf{x}$ has at least one double-point, hence $\mathrm{proj}_{\mathbf{x}}(\mathrm{Im}\alpha)$ is not homeomorphic to $[0, 1]$.
(ii) Suppose that there exists a path $\beta : [0, 1] \to \mathbf{x}$ and two points $\tau, \tau' \in \mathrm{Im}\beta$ such that $\mathrm{graph}(F|\mathrm{Im}\beta)$ does not connect $\{\tau\} \times \mathbf{z}$ with $\{\tau'\} \times \mathbf{z}$. W.l.o.g., the path β is doublepoint-free, and hence the mapping *hom* $: [0, 1] \times \mathbf{z} \to \mathrm{Im}\beta \times \mathbf{z}$ defined by $(t, z) \mapsto (\beta(t), z)$ is a homeomorphism. Since the homeomorphic image of a continuum is a continuum, the set $hom(\mathrm{graph}(F|\mathrm{Im}\beta)) \subseteq [0, 1] \times \mathbf{z}$ does not connect $\{0\} \times \mathbf{z}$ with $\{1\} \times \mathbf{z}$ by assumption. Therefore, Lemma 2.12 applies and there exists a path $\gamma : [0, 1] \to [0, 1] \times \mathbf{z}$ connecting $[0, 1] \times \underline{\mathbf{z}}$ with $[0, 1] \times \overline{\mathbf{z}}$. But then the path $\delta := hom^{-1}(\gamma) : [0, 1] \to \mathrm{Im}\beta \times \mathbf{z}$ is a path connecting $\mathbf{x} \times \underline{\mathbf{z}}$ with $\mathbf{x} \times \overline{\mathbf{z}}$, and $\mathrm{Im}\delta \cap \mathrm{graph}F = \emptyset$. Consequently, by (i), the set $\mathrm{proj}_{\mathbf{x}}(\mathrm{Im}\delta)$ is not homeomorphic to $[0, 1]$ and neither is $\mathrm{Im}\beta \supseteq \mathrm{proj}_{\mathbf{x}}(\mathrm{Im}\delta)$, a contradiction. Hence for every path $\beta : [0, 1] \to \mathbf{x}$ connecting $\tau, \tau' \in \mathrm{Im}\beta$, the set $\mathrm{graph}(F|\mathrm{Im}\beta)$ connects $\{\tau\} \times \mathbf{z}$

with $\{\tau'\} \times \mathbf{z}$.
(iii) Suppose that there exists a continuum $C \subseteq \mathbf{x}$ and $\tau, \tau' \in C$ such that graph$(F|C)$ does not connect $\{\tau\} \times \mathbf{z}$ with $\{\tau'\} \times \mathbf{z}$. Let $C_\varepsilon := \mathcal{O}_\varepsilon(C)$ hence $(C_{1/n})_{n\in\mathbb{N}}$ is a sequence of open, connected sets such that $\tau, \tau' \in C_{1/n}$ for all $n \in \mathbb{N}$. Since every $C_{1/n}$ is also path-connected, graph$(F|C_{1/n})$ connects $\{\tau\} \times \mathbf{z}$ with $\{\tau'\} \times \mathbf{z}$ by (ii). Since the sets $\{\tau\} \times \mathbf{z}$ and $\{\tau'\} \times \mathbf{z}$ are closed, we can apply Lemma 2.11 for the sequence $(\mathrm{graph}(F|C_{1/n}))_{n\in\mathbb{N}}$. We obtain that the set of all accumulation points connects $\{\tau\} \times \mathbf{z}$ with $\{\tau'\} \times \mathbf{z}$. Furthermore, this set is a subset of graph$(F|C)$, a contradiction, and hence F is c-continuous. □

The following is a counterexample to (ENDO).

2.14 Example. Let F be the multi-valued mapping defined in 2.13, let $\mathbf{y} = [-2, 2]$. Then the multi-valued mapping $H : \mathbf{x} \times \mathbf{y} \multimap \mathbf{z}$ defined by

$$H(x, y) := F(x)$$

is not c-continuous.

Proof. Let $\psi : [0, 1] \to \mathbf{x} \times \mathbf{y}$ be defined by

$$\psi(s) := (\cos(3\pi s), \cos(2\pi s), 2s - 1).$$

Let $C := \mathrm{Im}\psi \subseteq \mathbf{x} \times \mathbf{y}$ and $\tau = \psi(0), \tau' = \psi(1)$ points in C.

$$\begin{aligned}
\mathrm{graph}(H|C) &= \{(x, y, z) \in \mathbf{x} \times \mathbf{y} \times \mathbf{z} \mid (x, y) \in C, z \in H(x, y)\} \\
&= \{(x, y, z) \mid (x, y) \in \mathrm{Im}\psi, z \in H(x, y)\} \\
&= \{(\psi(s), z) \mid s \in [0, 1], z \in H(\psi(s))\} \\
&= \{(\psi(s), z) \mid s \in [0, 1], z \in F(\mathrm{proj}_{\mathbf{x}\times\mathbf{y}}(\psi(s)))\}
\end{aligned}$$

We show that the set $D := \{(\psi(s), z) \mid s \in [0, 1], z = 2s - 1\}$ does not intersect graph$(H|C)$: For an arbitraty $s \in [0, 1]$,

$$\begin{aligned}
F(\mathrm{proj}_{\mathbf{x}\times\mathbf{y}}(\psi(s))) &= F(\cos(3\pi s), \cos(2\pi s)) \\
&= \mathrm{proj}_{\mathbf{z}}((\cos(3\pi s), \cos(2\pi s) \times \mathbf{z} \cap \Gamma) \\
&= [-1, 1] \setminus \mathrm{proj}_{\mathbf{z}}(\mathcal{O}_\delta(\varphi(s))) \qquad \text{for some } \ \delta > 0 \\
&= [-1, 1] \setminus (2s - 1 - \delta, 2s - 1 + \delta).
\end{aligned}$$

Hence $(2s-1) \notin F(\mathrm{proj}_{\mathbf{x}\times\mathbf{y}}(\psi(s)))$ for every $s \in [0, 1]$ and consequently $D \cap \mathrm{graph}(H|C) = \emptyset$. Since $C \times \mathbf{z} \setminus D$ consists of two components and none of them connects $\tau \times \mathbf{z}$ with $\tau' \times \mathbf{z}$, neither does graph$(H|C) \subseteq C \times \mathbf{z} \setminus D$. So H is not c-continuous. □

(JOINT)

2.15 Corollary. *Not every c-continuous multi-valued mapping satisfies the property (JOINT), since (JOINT) implies (PABR).*

(PROD)

2.16 Corollary. *Not every c-continuous multi-valued mapping has the property (PROD), since (PROD) together with (PROJ) implies (ENDO).*

(PROJ)

2.17 Proposition. *(*Neumaier *[23, Corollary 5.3.6])*
All c-continuous multi-valued mappings have the property (PROJ).

(COMP)

2.18 Corollary. *Not every c-continuous multi-valued mapping has the property (COMP), since (COMP) implies (ENDO).*

(COMP')

2.19 Proposition. *(*Neumaier *[23, Corollary 5.3.5])*
All c-continuous multi-valued mappings have the property (COMP').

3 is-Continuous multi-valued mappings

We introduce a continuity notion called is-continuity. A multi-valued mapping is is-continuous if there exists a continuous single-valued mapping f such that the set of the zeros of f is a subset of graph$F$, plus assumptions to make sure that the set of the zeros of f is not empty. We show that is-continuous multi-valued mappings have the properties (PABR), (REDO), (ENDO), (JOINT) and (PROD), while (PROJ) and (COMP) are open questions. The property (SETM) is also an open question, but we prove a weaker version of (SETM).

3.1 Definition. Let $X \subseteq \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{IR}^m$. A multi-valued mapping $F : X \multimap \mathbf{y}$ is called **is-continuous** (*implicitly selectionable*) on a continuum $C \subseteq X$ if there exists a continuous single-valued mapping $f : C \times \mathbf{y} \to \mathbb{R}^m$ (an **implicit selection**) such that

$$\begin{aligned} f_i(x,y) &\geq 0 \quad \text{for} \quad y_i \in \overline{\mathbf{y}}_i, x \in C, \\ f_i(x,y) &\leq 0 \quad \text{for} \quad y_i \in \underline{\mathbf{y}}_i, x \in C \end{aligned}$$

and

$$f^{-1}(0) \subseteq \text{graph}(F|C).$$

If $X = \mathbf{x}$ is a box and if $F : \mathbf{x} \multimap \mathbf{y}$ is is-continuous on the continuum $C = \mathbf{x}$, we simply say that $F : \mathbf{x} \multimap \mathbf{y}$ is is-continuous.

3.2 Remark. For an is-continuous multi-valued mapping $F : X \multimap \mathbf{y}$ and for a fixed $x \in X$ we can apply the single-valued Miranda theorem [22] to the continuous single-valued mapping f, and obtain that the set $\{y \in \mathbf{y} \mid f(x,y) = 0\}$ is nonempty. This implies that $F(x)$ is nonempty for every $x \in X$.

3.3 Proposition. *Let X be a continuum and let $f : X \to \mathbf{y}$ be a continuous single-valued mapping. Then the multi-valued mapping $F : X \multimap \mathbf{y}$ defined by*

$$F(x) = \{f(x)\}$$

is is-continuous on X.

Proof. Let $m := \dim\mathbf{y}$. Define the single-valued mapping $g : X \times \mathbf{y} \to \mathbb{R}^m$ by

$$g(x, y) := y - f(x).$$

For $y_i \in \overline{\mathbf{y}}_i$ we have $f_i(x) \le y_i$ and hence $g(x, y) = y_i - f_i(x) \ge 0$, and
for $y_i \in \underline{\mathbf{y}}_i$ we have $f_i(x) \ge y_i$ and hence $g(x, y) = y_i - f_i(x) \le 0$.
Obviously, $g(x, y) = 0$ if and only if $y = f(x)$ if and only if $y \in F(x)$, and hence $g^{-1}(0) = \text{graph}F$. Therefore, F is is-continuous. □

(PABR)

3.4 Theorem. *Every is-continuous multi-valued mapping has the property (PABR).*

Proof. Let $f : \mathbf{x} \times \mathbf{y} \times \mathbf{y} \to \mathbb{R}^m$ be the implicit selection for the given multi-valued mapping $F : \mathbf{x} \times \mathbf{y} \multimap \mathbf{y}$. We define the single-valued mapping $h : \mathbf{x} \times \mathbf{y} \to \mathbb{R}^m$ by

$$h(x, y) := f(x, y, y).$$

The mapping h is continuous because f is continuous, and

$$\begin{aligned} h_i(x, y) = f_i(x, y, y) &\ge 0 \quad \text{for} \quad y_i \in \overline{\mathbf{y}}_i, \\ h_i(x, y) = f_i(x, y, y) &\le 0 \quad \text{for} \quad y_i \in \underline{\mathbf{y}}_i. \end{aligned}$$

Hence it only remains to show that $h^{-1}(0) \subseteq \text{graph}H$. We define the set $\mathfrak{D} \subseteq \mathbf{x} \times \mathbf{y} \times \mathbf{y}$ by

$$\mathfrak{D} := \{(x, y, y) \in \mathbf{x} \times \mathbf{y} \times \mathbf{y}\}$$

$$\begin{aligned} h^{-1}(0) \; &= \{(x, y) \in \mathbf{x} \times \mathbf{y} \mid h(x, y) = 0\} \\ &= \{(x, y) \mid f(x, y, y) = 0\} \\ &= \text{proj}_{\mathbf{x}\times\mathbf{y}}(\{(x, y, y) \mid f(x, y, y) = 0\}) \\ &= \text{proj}_{\mathbf{x}\times\mathbf{y}}(\{(x, y, z) \mid f(x, y, z) = 0\} \cap \mathfrak{D}) \\ &= \text{proj}_{\mathbf{x}\times\mathbf{y}}(f^{-1}(0) \cap \mathfrak{D}) \\ &\subseteq \text{proj}_{\mathbf{x}\times\mathbf{y}}(\text{graph}F \cap \mathfrak{D}) \\ &= \text{proj}_{\mathbf{x}\times\mathbf{y}}(\{(x, y, z) \mid (x, y) \in \mathbf{x} \times \mathbf{y}, z \in F(x, y)\} \cap \mathfrak{D}) \\ &= \text{proj}_{\mathbf{x}\times\mathbf{y}}(\{(x, y, y) \mid (x, y) \in \mathbf{x} \times \mathbf{y}, y \in F(x, y)\}) \\ &= \{(x, y) \mid x \in \mathbf{x}, y \in F(x, y)\} \\ &= \{(x, y) \mid x \in \mathbf{x}, y \in H(x)\} \\ &= \text{graph}H \end{aligned}$$

□

(SETM')

While (SETM) remains an open question, we show that all is-continuous multi-valued mappings have the following property (SETM'), which is a weaker version of (SETM). Note that the difference to (SETM) are the relations $>$ and $<$ in place of $\geq$ and $\leq$. Of course, (SETM) implies (SETM').

3.5 Theorem. (SETM') *Let* $F : \mathbf{x} \times \mathbf{y} \multimap \mathbf{y}$ *be a is-continuous multi-valued mapping, let* $0 \in \mathbf{y}$, *and let*

$$\begin{aligned} \sup(F_i(x,y)) < 0 \quad &\text{if } \; y_i = \overline{\mathbf{y}}_i, \\ \inf(F_i(x,y)) > 0 \quad &\text{if } \; y_i = \underline{\mathbf{y}}_i. \end{aligned}$$

Then the multi-valued mapping $H : \mathbf{x} \multimap \mathbf{y}$ *defined by*

$$H(x) := \{y \in \mathbf{y} \mid 0 \in F(x,y)\}$$

is is-continuous.

Proof. Let f be an implicit selection for F. Since $f^{-1}(0) \subseteq \text{graph}F$, and $\text{graph}F \cap \mathbf{x} \times \overline{\mathbf{y}_i} \times \{y \in \mathbf{y}_i \mid y \geq 0\}$ is empty by assumption, and since $f^{-1}(0)$ is closed, there exists an $\varepsilon_1 > 0$ such that $\mathbf{x} \times [\overline{\mathbf{y}_i} - \varepsilon_1, \overline{\mathbf{y}_i}] \times \{y \in \mathbf{y}_i \mid y \geq 0\} \cap f^{-1}(0)$ is empty. By a similar argument, the set $\mathbf{x} \times [\underline{\mathbf{y}_i}, \underline{\mathbf{y}_i} + \varepsilon_2] \times \{y \in \mathbf{y}_i \mid y \leq 0\} \cap f^{-1}(0)$ is empty.
Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. For $i = 1, \dots, m$ define the continuous single-valued mapping $\alpha_i : \mathbf{y}_i \to \mathbf{y}_i$ by

$$\alpha(y_i) := \begin{cases} y_i(\underline{\mathbf{y}_i}/\varepsilon) + \underline{\mathbf{y}_i}(1 + (\underline{\mathbf{y}_i}/\varepsilon)) & \text{for } \; y_i \in [\overline{\mathbf{y}_i} - \varepsilon, \overline{\mathbf{y}_i}], \\ y_i(\overline{\mathbf{y}_i}/\varepsilon) + \overline{\mathbf{y}_i}(1 - (\overline{\mathbf{y}_i}/\varepsilon)) & \text{for } \; y_i \in [\underline{\mathbf{y}_i}, \underline{\mathbf{y}_i} + \varepsilon], \\ 0 & \text{otherwise.} \end{cases}$$

Now define the continuous single-valued mapping $\beta_i : \mathbf{y} \to \mathbf{y}$ by

$$\beta_i(y) := (y_1, y_2, \dots, \alpha_i(y_i), \dots, y_m)$$

and finally the continuous single-valued mapping $g : \mathbf{x} \times \mathbf{y} \to \mathbb{R}^m$ by

$$g_i(x,y) := f_i(x, y, \beta_i(y)).$$

Since

$$\begin{aligned} g_i(x,y) = f_i(x,y,y) \geq 0 \quad &\text{for } \; y_i \in \overline{\mathbf{y}}_i, \\ g_i(x,y) = f_i(x,y,y) \leq 0 \quad &\text{for } \; y_i \in \underline{\mathbf{y}}_i, \end{aligned}$$

it only remains to show that

$$\begin{aligned} g^{-1}(0) \quad &= \{(x,y) \in \mathbf{x} \times \mathbf{y} \mid g(x,y) = 0\} \\ &= \{(x,y) \mid f_i(x,y,\beta_i(y)) = 0 \;\text{ for all } i\} \\ &= \{(x,y) \mid f(x,y,0) = 0\} \\ &= \text{proj}_{\mathbf{x}\times\mathbf{y}}(f^{-1}(0) \cap \mathbf{x} \times \mathbf{y} \times \{0\}) \\ &\subseteq \text{proj}_{\mathbf{x}\times\mathbf{y}}(\text{graph}F \cap \mathbf{x} \times \mathbf{y} \times \{0\}) \\ &= \{(x,y) \mid 0 \in F(x,y)\} \\ &= \{(x,y) \mid y \in G(x)\} \\ &= \text{graph}G. \end{aligned}$$

□

(REDO)

3.6 Proposition. *Every is-continuous multi-valued mapping has the property (REDO)*

Proof. Let $f : C \times \mathbf{y} \to \mathbb{R}^m$ be an implicit selection of F. Let C' be a continuum contained in C. Then the single-valued mapping $h : C' \times \mathbf{y} \to \mathbb{R}^m$ defined by $h := f_{|C' \times \mathbf{y}}$ i.e., the restriction of f to the set $C' \times \mathbf{y}$, satisfies:

$$\begin{aligned} h_i(x,y) = f_i(x,y) \geq 0 \quad \text{for} \quad y_i \in \overline{\mathbf{y}}_i, x \in C' \\ h_i(x,y) = f_i(x,y) \leq 0 \quad \text{for} \quad y_i \in \underline{\mathbf{y}}_i, x \in C' \end{aligned}$$

and

$$\begin{aligned} h^{-1}(0) \quad &= \{(x,y) \in C' \times \mathbf{y} \mid h(x,y) = 0\} \\ &= \{(x,y) \in C' \times \mathbf{y} \mid f(x,y) = 0\} \\ &= \{(x,y) \in C' \times \mathbf{y} \mid f^{-1}(0)\} \\ &\subseteq \{(x,y) \in C' \times \mathbf{y} \mid (x,y) \in \operatorname{graph}(F|C)\} \\ &= \{(x,y) \in C' \times \mathbf{y} \mid x \in C, y \in F(x)\} \\ &= \{(x,y) \in C' \times \mathbf{y} \mid x \in C', y \in F(x)\} \\ &= \operatorname{graph}(F|C'). \end{aligned}$$

Obviously h is continuous, hence F is is-continuous on the continuum $C' \subseteq C$. □

(ENDO)

3.7 Proposition. *Every is-continuous multi-valued mapping has the property (ENDO)*

Proof. Let $\mathbf{x} \in \mathbb{IR}^n$, $\mathbf{y} \in \mathbb{IR}^m$, $\mathbf{z} \in \mathbb{IR}^k$ and let $F : \mathbf{x} \multimap \mathbf{z}$ be is-continuous. Let $f : \mathbf{x} \times \mathbf{z} \multimap \mathbb{R}^k$ be an implicit selection of F. We define the single-valued mapping $h : \mathbf{x} \times \mathbf{y} \times \mathbf{z} \to \mathbb{R}^k$ by

$$h(x,y,z) := f(x,z).$$

Then h is continuous since it is independent of y. Moreover,

$$\begin{aligned} h_i(x,y,z) = f_i(x,z) \geq 0 \quad \text{for} \quad z_i \in \overline{\mathbf{z}}_i, \\ h_i(x,y,z) = f_i(x,z) \leq 0 \quad \text{for} \quad z_i \in \underline{\mathbf{z}}_i. \end{aligned}$$

Then

$$\begin{aligned} h^{-1}(0) \quad &= \{(x,y,z) \in \mathbf{x} \times \mathbf{y} \times \mathbf{z} \mid h(x,y,z) = 0\} \\ &= \{(x,y,z) \mid y \in \mathbf{y}, f(x,z) = 0\} \\ &= \{(x,y,z) \mid y \in \mathbf{y}, (x,z) \in f^{-1}(0)\} \\ &\subseteq \{(x,y,z) \mid y \in \mathbf{y}, (x,z) \in \operatorname{graph} F\} \\ &= \{(x,y,z) \mid y \in \mathbf{y}, x \in \mathbf{x}, z \in F(x)\} \\ &= \{(x,y,z) \mid (x,y) \in \mathbf{x} \times \mathbf{y}, z \in H(x,y)\} \\ &= \operatorname{graph} H, \end{aligned}$$

implies that H is is-continuous. □

(JOINT)

3.8 Corollary. *Every is-continuous multi-valued mapping has the property (JOINT)*

Proof. (JOINT) is a consequence of (PABR) and (ENDO). □

(PROD)

3.9 Proposition. *Every is-continuous multi-valued mapping has the property (PROD)*

Proof. By assumption there exist single-valued mappings $f : \mathbf{x} \times \mathbf{u} \to \mathbb{R}^k$ and $g : \mathbf{y} \times \mathbf{v} \to \mathbb{R}^l$ with $f^{-1}(0) \subseteq \text{graph}F$ and $g^{-1}(0) \subseteq \text{graph}G$. We define a single-valued mapping $h : \mathbf{x} \times \mathbf{y} \times \mathbf{u} \times \mathbf{v} \to \mathbb{R}^k \times \mathbb{R}^l$ by

$$h(x, y, u, v) := f(x, u) \times g(y, v).$$

Then h is continuous since f and g are both continuous. By definition, for $i = 1, \ldots, k$

$$h_i(x, y, u, v) = f_i(x, u), \qquad (u, v)_i = u_i, \qquad (\overline{\mathbf{u} \times \mathbf{v}})_i = \overline{\mathbf{u}}_i,$$

and for $i = k + 1, \ldots, k + l$

$$h_i(x, y, u, v) = g_{i-k}(y, v), \qquad (u, v)_i = v_{i-k}, \qquad (\overline{\mathbf{u} \times \mathbf{v}})_i = \overline{\mathbf{v}}_{i-k}.$$

Hence $h_i(x, y, u, v) \geq 0$ for $(u, v)_i \in (\overline{\mathbf{u} \times \mathbf{v}})_i$ because

$$h_i(x, y, u, v) \quad = \begin{cases} f_i(x, u) \geq 0 & \text{for} \quad u_i \in \overline{\mathbf{u}}_i, \\ g_{i-k}(y, v) \geq 0 & \text{for} \quad v_{i-k} \in \overline{\mathbf{v}}_{i-k}. \end{cases}$$

Similary, we obtain that $h_i(x, y, u, v) \leq 0$ for $(u, v)_i \in (\underline{\mathbf{u} \times \mathbf{v}})_i$.

$$\begin{aligned} h^{-1}(0) \quad &= \{(x, y, u, v) \mid h(x, y, u, v) = 0\} \\ &= \{(x, y, u, v) \mid f(x, u) = 0, g(y, v) = 0\} \\ &= \{(x, y, u, v) \mid (x, u) \in f^{-1}(0), (y, v) \in g^{-1}(0)\} \\ &\subseteq \{(x, y, u, v) \mid (x, u) \in \text{graph}F, (y, v) \in \text{graph}G\} \\ &= \{(x, y, u, v) \mid x \in \mathbf{x}, y \in \mathbf{y}, u \in F(x), v \in G(y)\} \\ &= \{(x, y, u, v) \mid x \in \mathbf{x}, y \in \mathbf{y}, (u, v) \in F(x) \times G(y)\} \\ &= \{(x, y, u, v) \mid (x, y) \in \times\mathbf{y}, (u, v) \in H(x, y)\} \\ &= \text{graph}H \end{aligned}$$

□

(PROJ) and (COMP) These properties are open question, but note that by Proposition 1.2 (iii) and Corollary 3, the property (COMP) holds in the case that (PROJ) holds.

We end this chapter with another useful property of is-continuous multi-valued mappings.

3.10 Proposition. *Let $F : X \multimap \mathbf{y}$ be is-continuous on $C \subseteq X$, and let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a nonsingular linear transformation, hence an $n \times n$-matrix. Let $\widehat{X} := \{Ax | x \in X\}$, let $\widehat{C} := \{Ax | x \in C\}$. Then the multi-valued mapping $G : \widehat{X} \multimap \mathbf{y}$ defined by*

$$G(x) := F(Ax)$$

is is-continuous on $\widehat{C}$.

Proof. Let f be the implicit selection for F. Clearly, the single-valued mapping $g : \widehat{C} \times \mathbf{y} \to \mathbb{R}^n$ by

$$g(x, y) := f(Ax, y) \qquad \text{for } x \in C$$

is continuous. For $y_i \in \overline{\mathbf{y}}_i$, we have $g_i(x, y) = f_i(x, y) \geq 0$, for $y_i \in \underline{\mathbf{y}}_i$, we have $g_i(x, y) = f_i(x, y) \leq 0$. Finally,

$$\begin{aligned} g^{-1}(0) &= \{(x, y) \in \widehat{C} \times \mathbf{y} | g(x, y) = 0\} \\ &= \{(x, y) \in \widehat{C} \times \mathbf{y} | f(Ax, y) = 0\} \\ &\subseteq \{(x, y) \in \widehat{C} \times \mathbf{y} | y \in F(Ax)\} \\ &= \{(x, y) \in \widehat{C} \times \mathbf{y} | y \in G(x)\} \\ &= \operatorname{graph}(G|\widehat{C}). \end{aligned}$$

□

4 ma-Continuous multi-valued mappings

The notion of ma-continuity was introduced by GOLDSZTEJN in [11]. A multi-valued mapping is ma-continuous, if its graph can be approximated by a sequence of C^∞-manifolds with boundary. (For details about manifolds, see HIRSCH [16].) In [11], Goldsztejn shows that a continuous single-valued mapping, understood as a multi-valued mapping, is ma-continuous. Only the property (JOINT) is proved directly in [11], but Goldsztejn makes suggestions about other properties.

4.1 Definition. Let $\mathbf{x} \in \mathbb{IR}^n$. A multi-valued mapping $F : \mathbf{x} \multimap \mathbb{R}^m$ is called **ma-continuous** (*manifold approximation*)if there exists

a closed ball $\mathbf{d} \subseteq \mathbb{R}^n$ such that $\mathbf{x} \subseteq \operatorname{int}\mathbf{d}$,

a sequence $(M_k)_{k\in\mathbb{N}}$ of C^∞ compact n-manifolds with boundary such that ∂M_k is homeomorphic to S^{n-1}, where S^{n-1} denotes the $n-1$-dimensional sphere, and

a sequence $(g_k : M_k \to \mathbb{R}^{n+m})_{k\in\mathbb{N}}$ of C^∞ maps such that g_k restricted to ∂M_k is a C^∞ diffeomorphism between ∂M_k and $\partial\mathbf{d} \times \{0\}$, such that for any sequence $(x_k)_{k\in\mathbb{N}}$ of points in M_k,

> the sequence $(g_k(x_k))_{k\in\mathbb{N}}$ is bounded, and
>
> if $g_k(x_k) \in \mathbf{x} \times \mathbb{R}^n$ for every $k \in \mathbb{N}$, then every accumulation point (x^*, y^*) ot the sequence $(g_k(x_k))_{k\in\mathbb{N}}$ satisfies $y^* \in F(x^*)$.

4.2 Proposition. *(*GOLDSZTEJN *[11, Proposition 4.1])*
Let $f : \mathbf{x} \to \mathbf{y}$ be a continuous single-valued mapping. Then the multi-valued mapping $F : \mathbf{x} \multimap \mathbf{y}$ defined by

$$F(x) = \{f(x)\}$$

is ma-continuous.

(JOINT)

4.3 Proposition. *(*GOLDSZTEJN *[11, Theorem 3.1])*
Every ma-continuous multi-valued mapping has the property (JOINT).

(PABR)

4.4 Corollary. *Every ma-continuous multi-valued mapping has the property (PABR).*

Proof. This follows from Proposition 4.3 and Proposition 1.1 (i). □

(ENDO), (PROD), (PROJ) and (COMP) GOLDSZTEJN conjectures in [11] that the properties (PROD) and (PROJ) hold. In this case, the properties (ENDO) and (COMP) would follow via Proposition 1.2 (ii) and (iii).

5 An application in logic

This section is concerned with the article RATSCHAN [25]. The article deals with constraints, i.e. formulae in first-order predicate language. For example, every equation for real numbers is a constraint. A witness function for a constraint is a mapping f such that the constraint is true for every point of $\mathrm{Im} f$. The central statement in [25] is that if there exists a witness function for the constraint ϕ_1 and one for the constraint ϕ_2, then there exists a witness function for the constraint $\phi_1 \wedge \phi_2$. For the proof, Ratschan used Theorem 5.3.7 in [23] (that is (SETM) for c-continuous multi-valued mappings) which is, as we saw in Chapter 2, not valid in general. We show that Ratschan's statement does not hold by giving a counterexample. The analysis leads to a different concept of 'witness functions' which uses multi-valued mappings and hence will be called 'witness multimappings'. We formulate and prove a statement analogous to the one made by Ratschan in [25], but which uses witness multimappings instead of witness functions.
Note that in this chapter we use Ratschan's notation; hence the upper-case letters $F, G, \ldots$ are not reserved for multi-valued mappings anymore.

5.1 Definition. A **first-order constraint** (or simply: **constraint**) is a formula in the first-order predicate language with predicate symbols $(=, \neq, <, \leq)$, function symbols $(+, \cdot, \sin, \exp, \ldots)$ and their usual interpretation over the real numbers.
Let $I = \{i_1, \ldots, i_N\}$ be a totally ordered index set, let $J = \{i_{l_1}, \ldots, i_{l_k}\}$ be a subset of I. Let $V := \{x_i \mid i \in I\}$ be a set of variables, let $U := \{x_j \mid j \in J\}$ a subset of V. For each variable x_i, let the possible values of x_i be restricted to the closed interval $\mathbf{x}_i \subseteq \mathbb{R}$. We define the box $\mathbf{x} := \mathbf{x}_{i_1} \times \ldots \times \mathbf{x}_{i_N} = \prod_{i \in I} \mathbf{x}_i$ and $\mathbf{x}_U := \mathbf{x}_{l_1} \times \ldots \times \mathbf{x}_{l_k} = \prod_{j \in J} \mathbf{x}_j$.

For a constraint with variables in U, we say that the continuous single-valued mapping $F : [0,1]^k \to \mathbf{x}$ is a **witness function** of the constraint ϕ in the box $\mathbf{x}$, if

(i) for all $t \in [0,1]^k$ and $j \in J$, $t_j = 0$ implies $F_j(t) = \underline{\mathbf{x}}_j$,

(ii) for all $t \in [0,1]^k$ and $j \in J$, $t_j = 1$ implies $F_j(t) = \overline{\mathbf{x}}_j$,

(iii) for all $t \in [0,1]^k$, $F(t) \in \mathbf{x}$, and

(iv) for all $t \in [0,1]^k$, $\phi(F(t))$ is true.

Theorem 1 in Ratschan [25] says that if there are witness functions for the constraints ϕ_1 and for ϕ_2, there also exists a witness function for $\phi_1 \wedge \phi_2$. We give a counterexample.

5.2 Example. Let ϕ_1 be the constraint

$$(x_2 \leq 1 \to x_1 = -1 + x_2) \wedge (1 < x_2 < 3 \to x_1 = 0) \wedge (3 \leq x_2 \to x_1 = -3 + x_2),$$

and let ϕ_2 be

$$x_1 = |x_3| \sin(\frac{\pi}{2x_3}).$$

First we show that there exist witness functions for ϕ_1 and for ϕ_2.
Let $V := \{x_1, x_2, x_3\}$, let $U_1 := \{x_1, x_2\}$ and $U_2 := \{x_1, x_3\}$. Hence $|U_1 \cap U_2| = |\{x_1\}| = 1$, and the elements of $U_1 \cap U_2$ appear first in the order of V. Then there exist a box $\mathbf{x} := [-1,1] \times [0,4] \times [-1,1]$ and continuous functions $F^1 : [0,1]^2 \to \mathbf{x}$ and $F^2 : [0,1]^2 \to \mathbf{x}$ such that F^1 and F^2 are witness functions for ϕ_1 and ϕ_2 respectively. We define

$$\begin{aligned} F^1(t_1, t_2) := &\ (\varphi(t_1), 4t_2, 2t_1 - 1), \\ F^2(t_1, t_2) := &\ (\psi(t_1), 4t_2, 2t_1 - 1), \end{aligned}$$

with $\varphi : [0,1] \to \mathbf{x}_1$ and $\psi : [0,1] \to \mathbf{x}_1$ defined by

$$\begin{aligned} \varphi(t_1) := &\ \begin{cases} 4t_1 - 1 & \text{if } \ t_1 \leq 1/4, \\ 0 & \text{if } \ 1/4 < t_1 < 3/4, \\ 4t_1 - 3 & \text{if } \ 3/4 \leq t_1. \end{cases} \\ \psi(t_1) := &\ |2t_1 - 1| \sin(\tfrac{\pi}{4}(t_1 - \tfrac{1}{2})^{-1}) \end{aligned}$$

Note that the properties (i) and (ii) are satisfied, since

$$\begin{aligned} F_1^1(0, t_2) &= -1 = \underline{\mathbf{x}}_1, & F_1^2(0, t_2) &= -1 = \underline{\mathbf{x}}_1, \\ F_2^1(t_1, 0) &= 0 = \underline{\mathbf{x}}_2, & F_3^2(t_1, 0) &= -1 = \underline{\mathbf{x}}_3, \\ F_1^1(1, t_2) &= 1 = \overline{\mathbf{x}}_1, & F_1^2(1, t_2) &= 1 = \overline{\mathbf{x}}_1, \\ F_2^1(t_1, 1) &= 4 = \overline{\mathbf{x}}_2, & F_3^2(t_1, 1) &= 1 = \overline{\mathbf{x}}_3. \end{aligned}$$

The functions F^1 and F^2 are continuous, and since $\phi_1(F^1(t_1,t_2))$ is true and $\phi_2(F^2(t_1,t_2))$ is also true for all $(t_1,t_2) \in [0,1]^2$, they are witness functions for ϕ_1 and ϕ_2 respectively. Obviously, $\phi_1 \wedge \phi_2$ is the constraint

$$\begin{aligned}&(x_2 \leq 1 \rightarrow x_1 = -1 + x_2) \wedge (1 < x_2 < 3 \rightarrow x_1 = 0)\\ \wedge&(3 \leq x_2 \rightarrow x_1 = -3 + x_2) \wedge (x_1 = |x_3| \sin(\tfrac{\pi}{2x_3})).\end{aligned}$$

We show that there exists no witness function for $\phi_1 \wedge \phi_2$ in the box $\mathbf{x}$ as defined above. Let Γ be the subset of $\mathbf{x}$ for which $\phi_1 \wedge \phi_2$ is true.

$$\begin{aligned}\Gamma = \quad & \{(x_1,x_2,x_3) \in \mathbf{x} \mid x_2 \leq 1, x_1 = -1 + x_2, x_1 = |x_3| \sin(\tfrac{\pi}{2x_3})\}\\ \cup \quad & \{(x_1,x_2,x_3) \in \mathbf{x} \mid 1 < x_2 < 3, x_1 = 0, x_1 = |x_3| \sin(\tfrac{\pi}{2x_3})\}\\ \cup \quad & \{(x_1,x_2,x_3) \in \mathbf{x} \mid 3 \leq x_2, x_1 = -3 + x_2, x_1 = |x_3| \sin(\tfrac{\pi}{2x_3})\}\\ = \quad & \{(x_1,x_2,x_3) \in \mathbf{x}_1 \times [0,1] \times \mathbf{x}_3 \mid x_1 = -1 + x_2, x_1 = |x_3| \sin(\tfrac{\pi}{2x_3})\}\\ \cup \quad & \{(0,x_2,x_3) \in \{0\} \times]1,3[\times \mathbf{x}_3 \mid 0 = |x_3| \sin(\tfrac{\pi}{2x_3})\}\\ \cup \quad & \{(x_1,x_2,x_3) \in \mathbf{x}_1 \times [3,4] \times \mathbf{x}_3 \mid x_1 = -3 + x_2, x_1 = |x_3| \sin(\tfrac{\pi}{2x_3})\}\end{aligned}$$

A witness function for $\phi_1 \wedge \phi_2$ would be a continuous function $F : [0,1] \rightarrow \mathbf{x}$ (hence a path) with

(i) $t = 0$ implies $F_1(t) = \underline{\mathbf{x}}_1$,

(ii) $t = 1$ implies $F_1(t) = \overline{\mathbf{x}}_1$,

(iii) for all $t \in [0,1]$, $F(t) \in \mathbf{x}$, and

(iv) for all $t \in [0,1]$, $\phi(F(t))$ is true, hence $F(t) \in \Gamma$.

Note that Γ is connected, but since every path $p : [0,1] \rightarrow \Gamma$ connecting $\underline{\mathbf{x}}_1$ with $\overline{\mathbf{x}}_1$ must contain the set $\{(0,x_2,x_3) \in \{0\} \times]1,3[\times \mathbf{x}_3 \mid 0 = |x_3| \sin(\tfrac{\pi}{2x_3})\}$, p is not continuous, a contradiction. Hence there exists no witness function for the constraint $\phi_1 \wedge \phi_2$.

Witness multimappings

Now we give a notion for a witness function which is more general and satisfies a corresponding statement like Theorem 1 in Ratschan [25]. Instead of a 'continuous deformation' of $[0,1]^k$, we choose a multi-valued mapping.

5.3 Definition. As before, let $I = \{i_1, \ldots, i_N\}$ be a totally ordered index set, let $J = \{i_{l_1}, \ldots, i_{l_k}\}$ be a subset of I. Let $V := \{x_i \mid i \in I\}$ be a set of variables, let $U := \{x_j \mid j \in J\}$ a subset of V. For each variable x_i, let the possible values of x_i be restricted to the interval $\mathbf{x}_i \in \mathbb{IR}$. We define $\mathbf{x}_U := \mathbf{x}_{l_1} \times \ldots \times \mathbf{x}_{l_k} = \prod_{j \in J} \mathbf{x}_j$.
Given a constraint ϕ with variables in U and a box $\mathbf{x}$ we say that $F : \mathbf{x}_U \multimap \mathbf{x}_{V \setminus U}$ is an **is-continuous witness multimapping** for ϕ if there exists a continuous single-valued mapping $f : \mathbf{x} \rightarrow \mathbb{R}^{|V \setminus U|}$ such that

(a) $f_i(x) \geq 0$ for $x_i \in \overline{\mathbf{x}}_i$, $x_i \in V \setminus U$,

(b) $f_i(x) \leq 0$ for $x_i \in \underline{\mathbf{x}}_i$, $x_i \in V \setminus U$,

(c) $f^{-1}(0) \subseteq \mathrm{graph}F$,

(d) $\phi(x)$ is true for all $x \in \mathrm{graph}F$.

Note that the Miranda Theorem guarantees that $F(x) \neq \emptyset$ for every $x \in \mathbf{x}_U$ and since $F(x) \subseteq \mathbf{x}_{V\setminus U}$, there exists a 'witness' on the lower and the upper bound of $\mathbf{x}_i$. These are the requirements analogous to (i)-(iii) in Definition 5.1, and item (d) is the analogue of (iv).

5.4 Theorem. *If for the constraints ϕ_1 with variable set $U_1 \subseteq V$ and for ϕ_2 with variable set $U_2 \subseteq V$ there exist is-continuous witness multimappings, then there also exists an is-continuous multimapping for the constraint $\phi_1 \wedge \phi_2$ with variable set $V := U_1 \cup U_2$.*

Proof. Let $F^1 : \mathbf{x}_{U_1} \multimap \mathbf{x}_{V\setminus U_1}$ and $F^2 : \mathbf{x}_{U_2} \multimap \mathbf{x}_{V\setminus U_2}$ be the witness multimappings. By assumption there exist continuous single-valued mappings $f^i : \mathbf{x} \to \mathbb{R}^{|V\setminus U_i|}$ and $(f^i)^{-1}(0) \subseteq \mathrm{graph}(F^i)$ for $i = 1, 2$.
Define the single-valued mapping $g : \mathbf{x} \to \mathbb{R}^{|V\setminus U_1|} \times \mathbb{R}^{|V\setminus U_2|}$ by

$$g(x) = (f^1(x), f^2(x)).$$

Obviously g is continuous, and since

$$\begin{aligned}
|V \setminus U_1| + |V \setminus U_2| &= |V| - |U_1| + |V| - |U_2| \\
&= 2|V| - (|U_1| + |U_2|) \\
&= 2|V| - (|U_1 \cup U_2| + |U_1 \cap U_2|) \\
&= 2|V| - (|V| + |U_1 \cap U_2|) \\
&= |V| - |U_1 \cap U_2|
\end{aligned}$$

the range of g is $\mathbb{R}^{|V|-|U_1\cap U_2|}$ and we can write $g : \mathbf{x} \to \mathbb{R}^{|V|-|U_1\cap U_2|}$.
Now we define a multi-valued mapping $G : \mathbf{x}_{U_1\cap U_2} \multimap \mathbf{x}_{V\setminus U_1\cap U_2}$ by

$$G(x) := \{y \mid g(x, y) = 0\}.$$

It remains to show that the multi-valued mapping G is an is-continuous witness multimapping for the constraint $\phi_1 \wedge \phi_2$.

$$g_i(x) = \begin{cases} f_i^1(x) \leq 0 & \text{for} \quad x_i \in \underline{\mathbf{x}}_i \quad \text{if} \quad x_i \in V \setminus U_1, \\ f_i^2(x) \leq 0 & \text{for} \quad x_i \in \underline{\mathbf{x}}_i \quad \text{otherwise,} \end{cases}$$

and

$$g_i(x) = \begin{cases} f_i^1(x) \geq 0 & \text{for} \quad x_i \in \overline{\mathbf{x}}_i \quad \text{if} \quad x_i \in V \setminus U_1, \\ f_i^2(x) \geq 0 & \text{for} \quad x_i \in \overline{\mathbf{x}}_i \quad \text{otherwise.} \end{cases}$$

By definition of G,

$$\begin{aligned}
\mathrm{graph}G &= \{(x, y) \mid y \in G(x)\} \\
&= \{(x, y) \mid g(x, y) = 0\} \\
&= g^{-1}(0).
\end{aligned}$$

In particular, $g^{-1}(0) \subseteq \mathrm{graph}G$.
Let x be an arbitrary point in graph$G$. Since

$$\begin{aligned}\mathrm{graph}G &= g^{-1}(0)\\ &= \{y \mid g(y) = 0\}\\ &= \{y \mid f^1(y) = 0, f^2(y) = 0\}\\ &= (f^1)^{-1}(0) \cap (f^2)^{-1}(0)\\ &\subset \mathrm{graph}(F^1) \cap \mathrm{graph}(F^2),\end{aligned}$$

$\phi_1(x)$ is true and $\phi_2(x)$ is also true. Hence $\phi_1 \wedge \phi_2(x)$ is true, and consequently G is an is-continuous witness multimapping for the constraint $\phi_1 \wedge \phi_2$.

□

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