A distributive interval arithmetic

Arnold Neumaier

Fakultät für Mathematik, Universität Wien
Nordbergstr. 15, A-1090 Wien, Austria
email: Arnold.Neumaier@univie.ac.at
WWW: `http://www.mat.univie.ac.at/~neum/`

Abstract. A way is shown how to modify the notation for an interval and to define addition and mulitplication of intervals such that the commutative, associative, and distributive laws hold.

This is a slightly edited and corrected version of the manuscript with the same title that appeared in the preprint series "Freiburger Intervallberichte". Please quote the paper as:

A. Neumaier,
A distributive interval arithmetic,
Freiburger Intervall-Berichte 82/10 (1982), 31–38.
http://www.mat.univie.ac.at/~neum/publist.html#distr

A scanned version of the unedited original can be found at the above web address.

1 Introduction

Traditional interval arithmetic [1] does not satisfy the distributive law. This note defines a variant of interval arithmetic in which the distributive law holds.

Let $\mathbb{K} = \mathbb{R}$ be the field of real numbers, and write $\mathbb{R}^+$ for the set of nonnegative reals. In analogy with Henrici's complex circular arithmetic Henrici [2], where $\mathbb{K} = \mathbb{C}$, we write an **interval** in the form

$$\langle a, \rho \rangle := \{x \in \mathbb{K} \mid |x - a| \leq \rho\}, \tag{1}$$

and write

$$\mathbb{IK} = \{\langle a, \rho \rangle \mid a \in \mathbb{K}, \rho \in \mathbb{R}^+\}$$

for the set of all intervals over $\mathbb{K}$. We call a the **center** and ρ the **radius** of $\langle a, \rho \rangle$. As in [2], we define four basic operations $+, -, \cdot, /$ such that for $\mathbf{a}, \mathbf{b} \in \mathbb{IK}$ and $\circ \in \{+, -, \cdot, /\}$,

$$\{x \circ y \mid x \in \mathbf{a}, y \in \mathbf{b}\} \subseteq \mathbf{a} \circ \mathbf{b}; \tag{2}$$

the rules are

$$\langle a, \rho \rangle \pm \langle a', \rho' \rangle := \langle a \pm a', \rho + \rho' \rangle, \tag{3}$$

$$\langle a, \rho \rangle \cdot \langle a', \rho' \rangle := \langle aa', |a|\rho' + \rho|a'| + \rho\rho' \rangle, \tag{4}$$

$$\langle a, \rho \rangle / \langle a', \rho' \rangle := \left\langle \frac{a}{a'}, \frac{|a|\rho' + \rho|a'|}{|a'|(|a'| - \rho')} \right\rangle \quad \text{if } |a'| > \rho'. \tag{5}$$

Moreover, inclusion of intervals is characterized by

$$\langle a, \rho \rangle \subseteq \langle a', \rho' \rangle \;\Leftrightarrow\; |a' - a| \leq \rho' - \rho. \tag{6}$$

The operations thus defined are **inclusion isotone**, i.e,

$$\mathbf{a} \subseteq \mathbf{b},\ \mathbf{a}' \subseteq \mathbf{b}' \ \Rightarrow\ \mathbf{a} \circ \mathbf{a}' \subseteq \mathbf{b} \circ \mathbf{b}' \tag{7}$$

for $\circ \in \{+, -, \cdot, /\}$, and satisfy both commutative and associative laws. But in place of the distributive law only the following **subdistributive law**

$$(\mathbf{a} \pm \mathbf{b})\mathbf{c} \ \subseteq\ \mathbf{ac} \pm \mathbf{bc} \quad \text{for } \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{IK}$$

holds.

2 Enforcing distributivity

The key for our approach is the notion of absolute value. Alefeld & Herzberger [1] define the **absolute value** of an interval as

$$|\langle a, \rho\rangle| := |a| + \rho,$$

noting that this is the maximal absolute value of elements of $\langle a, \rho\rangle$. Looking at (4), we find at once that

$$|\langle a, \rho\rangle \cdot \langle a', \rho'\rangle| = |\langle a, \rho\rangle| \cdot |\langle a', \rho'\rangle|.$$

Thus the multiplication rule for the absolute value is multiplicative, whereas that for the radius is more complicated. This suggests to write (a, β) for an interval with center a and absolute value β, i.e.

$$(a, \beta) := \langle a, \beta - |a|\rangle = \{x \in \mathbb{K} \mid |x - a| + |a| \le \beta\},$$

Then muliplication takes the simple form

$$(a, \beta) \cdot (a', \beta') = (aa', \beta\beta'),$$

and since the absolute value of the result in (5) is

$$\frac{|a|\rho' + \rho|a'|}{|a'|(|a'| - \rho')} + \frac{|a|}{|a'|} = \frac{|a||a'| + \rho|a'|}{|a'|(|a'| - \rho')} = \frac{|a| + \rho}{2|a'| - |a'| - \rho'}$$

the division rule becomes

$$(a, \beta)/(a', \beta') = \Big(\frac{a}{a'}, \frac{\beta}{2|a'| - \beta'}\Big).$$

The addition rule, obtained from (3), now reads

$$\begin{aligned} (a, \beta) \pm (a', \beta') &= (a \pm a', \beta + \beta' + |a \pm a'| - |a| - |a'|) \\ &= \begin{cases} (a \pm a', \beta + \beta') & \text{if } \pm aa' \ge 0, \\ (a \pm a', \beta + \beta' - 2\min(|a|, |a'|)) & \text{otherwise.} \end{cases} \end{aligned}$$

This formula suggests a simplification obtained by replacing the absolute value with its upper bound $\beta + \beta'$, and redefining addition and subtraction as

$$(a, \beta) \pm (a', \beta') := (a \pm a', \beta + \beta').$$

The modified formula is different from the original one when $\pm aa' < 0$, in which case it makes the result bigger. Therefore, the modification preserves inclusion isotonicity (7). On the other hand, the commutative, associative, and distributive laws are now obvious. So we exchanged some loss of information in the result for the validity of the distributive law. We summarize:

2.1 Theorem. *In the set*

$$\mathbb{IK} = \{(a,\beta) \mid a \in \mathbb{K}, \beta \in \mathbb{R}^+, |a| \le \beta\}$$

of all intervals written in the form

$$(a,\beta) = \{x \in \mathbb{K} \mid |x-a| + |a| \le \beta\},$$

equipped with the operations $+, -, \cdot, /$ *defined by*

$$(a,\beta) \pm (a',\beta') := (a \pm a', \beta + \beta'),$$

$$(a,\beta) \cdot (a',\beta') := (aa', \beta\beta'),$$

$$(a,\beta)/(a',\beta') := \Big(\frac{a}{a'}, \frac{\beta}{2|a'| - \beta'}\Big) \quad \text{if } \beta' < 2|a'|,$$

the commutative, associative, and distributive laws hold. Moreover, all operations are inclusion isotone.

2.2 Remarks. 1. It is easy to see that the theorem also holds for $\mathbb{K} = \mathbb{C}$; in this case, the "intervals" are actually discs.

2. If we forget about the division rule and the absolute value is replaced by the norm, the theorem also holds for Banach algebras $\mathbb{K}$ over $\mathbb{R}$ or $\mathbb{C}$.

3. Inclusion is now characterized by

$$(a,\beta) \subseteq (a',\beta') \quad \text{iff } |a'-a| + |a'| - |a| \le \beta' - \beta.$$

4. Distributivity forces $1 + (-1) \ne 0$; indeed $(1 + (-1))A = A + (-1)A = A - A \ne 0$ for any interval A with interior points.

5. The elements of $\mathbb{K}$ (= intervals of radius 0) now have the representation $a = (a, |a|)$. The nonzero elements of $\mathbb{K}$ are now characterized as the intervals A for which $AB/A = B$ for all $B \in \mathbb{IK}$. On the other hand, $A + B - A = B$ for all $B \in \mathbb{IK}$ implies now that $A = (0,0)$ in contrast to Henrici's definition. In particular, the addition of elements of $\mathbb{K}$ in $\mathbb{IK}$ is *not* the field addition, e.g.,

$$1 + (-1) = (0,2) \text{ and not } (0,0) = 0.$$

The excess width in $1+(-1) = (0,2)$ is somewhat distressing and makes the above arithmetic useless for practical purpose. In the next section, we therefore consider a modification that improves upon this without losing distributivity.

3 An improved formulation

In view of $1 + (-1) \ne 0$ we can avoid excess width only if we allow a single interval to have several names; then $1 + (-1)$ would be a "nonstandard" zero. This idea can indeed be made to work as follows.

Instead of the set $\mathbb{IK}$ of ordinary intervals over $\mathbb{K}$ we consider a set $\mathbb{I}$ whose elements are **formal intervals**; each formal interval, written $\langle a \rangle$, has a **value** $v\langle a \rangle$ which is an ordinary interval; conversely each ordinary interval $\mathbf{a} = [\underline{a}, \overline{a}]$ is represented by a formal interval $r\mathbf{a} = r[\underline{a}, \overline{a}]$.

3.1 Definition. A set $\mathbb{I}$, whose elements are called **formal intervals**, together with a value map $v : \mathbb{I} \to \mathbb{IK}$ and a representation map $r : \mathbb{IK} \to \mathbb{I}$, and operations $\circ \in \{+, -, \cdot\}$ is called an **interval ring** (over $\mathbb{K}$) if the following axioms hold.

(I1) $(\mathbb{I}, +)$ and $(\mathbb{I}, \cdot)$ are commutative semigroups, i.e., both commutative and associative laws hold.

(I2) $(\langle a\rangle \pm \langle b\rangle)\langle c\rangle = \langle a\rangle\langle c\rangle \pm \langle b\rangle\langle c\rangle \quad$ for all $\langle a\rangle, \langle b\rangle, \langle c\rangle \in \mathbb{I}$.

(I3) $v\langle a\rangle \circ v\langle b\rangle \subseteq v(\langle a\rangle \circ \langle b\rangle), \quad$ for $\langle a\rangle, \langle b\rangle \in \mathbb{I}$, $\circ \in \{+, -, \cdot\}$.

(I4) $v(r\mathbf{a}) = \mathbf{a}$.

(I5) $v(r[a, a] \circ r[b, b]) = [a \circ b, a \circ b] \quad$ for $a, b \in \mathbb{K}, \circ \in \{+, -, \cdot\}$.

Axiom (I3) is the new version of inclusion isotonicity, (I4) states that an ordinary interval $\mathbf{a}$ is represented by a formal interval with value $\mathbf{a}$, and (I5) records the requirement that for representations of degenerate intervals, the operations $+, -, \cdot$ introduce no excess width. Of course (e.g., $1+(-1) \neq 0$), the result $r[a, a]\circ r[b, b]$ may not be the representation $r[a\circ b, a\circ b]$, but at least it has the same value. Note that we do not require that $+$ and $-$ are converse operations; in general, this is false.

A division can be introduced by defining

$$\langle a\rangle/\langle b\rangle := \langle a\rangle \cdot \langle b\rangle^{-1} \quad \text{if } v\langle b\rangle \not\ni 0, \tag{8}$$

where

$$\langle b\rangle^{-1} := r(v\langle b\rangle)^{-1} \quad \text{if } v\langle b\rangle \not\ni 0. \tag{9}$$

This operation is then inclusion isotone, too, i.e., (I3) holds for $\circ = /$. Inclusion isotone unary elementary functions ϕ can be introduced similarly by

$$\phi(\langle b\rangle) := r\phi(v\langle b\rangle),$$

whith the optimal interval extension

$$\phi(\mathbf{a}) := \Box\{\phi(a) \mid a \in \mathbf{a}\}.$$

In the following, two interval rings are exhibited. The straightforward proofs verifying the axioms are omitted. The first construction is inspired by Theorem 2.1.

3.2 Theorem. *The set*

$$\mathbb{I}_1 := \{\langle a, \alpha, \beta\rangle \mid \ |a| \leq \alpha \leq \beta\}$$

becomes an interval ring with the definitions

$$v\langle a, \alpha, \beta\rangle = \ \{x \in \mathbb{K} \mid \ |x - a| + \alpha \leq \beta\},$$

$$r[a - \varepsilon, a + \varepsilon] = \langle a, |a|, |a| + \varepsilon\rangle,$$
$$\langle a, \alpha, \beta\rangle \pm \langle a', \alpha', \beta'\rangle = \langle a \pm a', \alpha + \alpha', \beta + \beta'\rangle,$$
$$\langle a, \alpha, \beta\rangle \cdot \langle a', \alpha', \beta'\rangle = \langle aa', \alpha\alpha', \beta\beta'\rangle.$$

The second construction exploits the facts that nonnegative ordinary intervals are distributive, and that every interval is a difference of nonnegative intervals.

3.3 Theorem. *The set*

$$\mathbb{I}_2 := \{\langle a\rangle = \langle \mathbf{a}^+, \mathbf{a}^-\rangle \mid 0 \le \mathbf{a}^\pm \in \mathbb{IK}\}$$

becomes an interval ring with the definitions

$$v\langle a\rangle = \mathbf{a}^+ - \mathbf{a}^-,$$

$$r\mathbf{a} = \begin{cases} \langle \mathbf{a}, 0\rangle & \textit{if } \mathbf{a} > 0, \\ \langle 0, -\mathbf{a}\rangle & \textit{if } \mathbf{a} < 0, \\ \langle [0,\overline{a}], [0,-\underline{a}]\rangle & \textit{if } \mathbf{a} = [\underline{a},\overline{a}] \ni 0, \end{cases}$$

$$\langle a\rangle + \langle b\rangle = (\mathbf{a}^+ + \mathbf{b}^+, \mathbf{a}^- + \mathbf{b}^-),$$

$$\langle a\rangle - \langle b\rangle = (\mathbf{a}^+ + \mathbf{b}^-, \mathbf{a}^- + \mathbf{b}^+),$$

$$\langle a\rangle \cdot \langle b\rangle = (\mathbf{a}^+\mathbf{b}^+ + \mathbf{a}^-\mathbf{b}^-,\ \mathbf{a}^+\mathbf{b}^- + \mathbf{a}^-\mathbf{b}^+).$$

The realization $\mathbb{I}_2$ has the useful property that if no intermediate result value contains zero then the value of an arithmetic expression computed in $\mathbb{I}_2$ agrees with the arithmetic expression of the values computed in $\mathbb{IK}$.

3.4 Remark. Again it is possible to define interval rings over the complex numbers, and the first realization immediately extends to $\mathbb{K} = \mathbb{C}$. On the other hand, the second realization can be generalized only to ordered fields.

An important consequence of axiom (I3) is that the result of an arithmetic expression of values, computed in $\mathbb{IK}$, is always contained in the value of the result computed in $\mathbb{I}$. This property, together with the distributivity of $\mathbb{I}$, suggests that calculations in an interval ring might facilitate the analysis of algorithms carried out in ordinary interval arithmetic.

References

[1] G. Alefeld and J. Herzberger, Einführung in die Intervallrechnung, Bibliographisches Institut, Mannheim 1974.

[2] P. Henrici, Circular arithmetic and the determination of polynomial zeros, Springer Lecture Notes in Mathematics 228 (1971), 86–92.