Quantum field theory as eigenvalue problem

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March 10, 2003

Abstract. A mathematically well-defined, manifestly covariant theory of classical and quantum field is given, based on Euclidean Poisson algebras and a generalization of the Ehrenfest equation, which implies the stationary action principle. The theory opens a constructive spectral approach to finding physical states both in relativistic quantum field theories and for flexible phenomenological few-particle approximations.

In particular, we obtain a Lorentz-covariant phenomenological multiparticle quantum dynamics for electromagnetic and gravitational interaction which provides a representation of the Poincaré group without negative energy states. The dynamics reduces in the nonrelativistic limit to the traditional Hamiltonian multiparticle description with standard Newton and Coulomb forces.

The key that allows us to overcome the traditional problems in canonical quantization is the fact that we use the algebra of linear operators on a space of wave functions slightly bigger than traditional Fock spaces.

Keywords: action, axiomatic physics, classical field, classical limit, conservative quantum state, constrained Schrödinger equation, covariant interaction, deformation quantization, density, multiparticle Dirac equation, Ehrenfest equation, eigenvalue problem, electrodynamics, Euclidean Poisson algebra, expectation, form factors, gravitation, Hamiltonian dynamics, integral, Liouville equation, manifestly covariant theory, momentum state, noncommutative algebra, nonrelativistic limit, phase space quantization, phenomenological relativistic dynamics, physical system, quantum field theory, quantum gravity, relativistic Coulomb potential, relativistic Newton potential, running coupling constants, self-energy, stationary action principle, Wightman axioms, Wigner transform

E-print Archive No.: gr-qc/0303037
2003 PACS Classification: 03.70, 04.60.-m, 11.10.Cd, 11.10.Ef
2000 MSC Classification: primary 81T05, secondary 70S05, 81S30, 81V10, 81V17

1 Introduction

. . . the ancients (as we are told by Pappus) esteemed the science of mechanics of greatest importance in the investigation of natural things, and the moderns, rejecting substantial forms and occult qualities, have endeavored to subject the phenomena of nature to the laws of mathematics . . .
Isaac Newton, 1686 [30]

Renormalized quantum electrodynamics is by far the most successful theory we have today. This very impressive fact, however, does not make the whole situation less strange. We start out from equations which do not make sense. We apply certain prescriptions to their solutions and end up with a power series of which we do not know that it makes sense. The first few terms of this series, however, give the best predictions we know.
Res Jost, 1965 [16]

In the more than 300 years that passed since Newton wrote this in his *Principia Mathematica*, the moderns have been very successful at the endeavor to subject the phenomena of nature to the laws of mathematics – with exception of quantum field theory. As the second quote (which could have as well been written in 2002) shows, quantum field theory so far resisted a quantitative, mathematically rigorous foundation.

In the present paper, an axiomatic approach is outlined that, I believe, provides foundations on which quantum field theory can be given a rigorous mathematical treatment. The present paper gives the elementary part and exhibits the connections to the traditional settings. A deeper study of the consequences and use of the concepts presented here will be given elsewhere.

In the new approach, each (classical or quantum) conservative physical system is characterized by two Hermitian quantities: a density and an action. A generalized Liouville equation defines the dynamics and implies Ehrenfest equations for expectations.

For a classical (but not a quantum) field theory, the Ehrenfest equations in a symplectic Poisson algebra imply the traditional field equations by the stationary action principle. In particular, all traditional systems derivable from the stationary action principle can be modelled in our setting. In a similar way, one can get from suitable Lie-Poisson algebras relativistic and nonrelativistic Euler equations, Vlasov-Maxwell, Vlasov-Einstein, and Euler-Poincaré equations.

A new phase space quantization principle (generalizing the Wigner transform) allows the simple quantization of arbitrary Poisson algebras, with a good classical limit.

A large class of Poincaré invariant actions on spaces with a reducible representation of the Poincaré group is exhibited. Since it is manifestly covariant but possesses a Hamiltonian nonrelativistic limit, it appears to be well-suited for phenomenological modeling of relativistic few-particle dynamics.

In particular, we obtain a Lorentz-covariant phenomenological multiparticle quantum dynamics for electromagnetic and gravitational interaction which reduces in the non-relativistic limit to the traditional Hamiltonian multiparticle description with standard Newton and Coulomb forces. The key that allows us to overcome the traditional problems in canonical quantization is the fact that we use the algebra of linear operators on a space of wave functions slightly bigger than traditional Fock spaces.

For a quantum system, if the action is translation invariant, one can find pure states of given mass describing isolated systems in a rest frame by solving a constrained Schrödinger equation. This opens a constructive spectral approach to finding physical states both in relativistic quantum field theories and in phenomenological few-particle approximation.

While I have already checked much of what is needed to get the many known results as consequences of the present setting, I am not yet completely sure about the adequacy of the new theory for all aspects of traditional field theory. Thus I'd like to apologize (as Newton did in the preface of [30]) and *"heartily beg that what I have here done may be read with forbearance; and that my labors in a subject so difficult may be examined, not so much with the view to censure, as to remedy their defects."*

2 Prelude: Covariant transmutation

> *Do not imagine, any more than I can bring myself to imagine, that I should be right in undertaking so great and difficult a task. Remembering what I said at first about probability, I will do my best to give as probable an explanation as any other – or rather, more probable; and I will first go back to the beginning and try to speak of each thing and of all.*
> Plato, ca. 367 B.C. [32]

We begin with traditional nonrelativistic quantum mechanics of a multiparticle system and rewrite it in a formally covariant way that foreshadows the axiomatic setup developed afterwards. (This section serves as a heuristic motivation only, without any claims to rigor.)

Let H be a translation invariant and time-independent Hamiltonian, $\mathbf{p}$ the 3-momentum operator, the generator of the spatial translations, ψ the energy eigenstate in the rest frame of a system with energy $E > 0$ and total mass m. The Schrödinger equation gives $H\psi = E\psi$, and the condition that the system is in a rest frame says $\mathbf{p}\psi = 0$.

To make these statements covariant, we extend wave functions by an additional argument E. Then we may consider the energy E as an operator acting on functions $\psi(E,x)$ by multiplication with E, with time $t = i\hbar\partial/\partial E$ as conjugate operator. By introducing the operator

$$L := E - H, \tag{1}$$

we may write the Schrödinger equation together with the rest frame condition in the form

$$L\psi = 0, \quad \mathbf{p}\psi = 0.$$

We now introduce a 4-momentum vector $p = \binom{p_0}{\mathbf{p}}$, where p_0 is related to the energy E by the relation

$$E = p_0 c - mc^2. \tag{2}$$

On writing $p^2 = p_0^2 - \mathbf{p}^2$ (where $\mathbf{p}^2 = \mathbf{p}\cdot\mathbf{p}$) for the Lorentz square of a 4-vector, and applying a Lorentz transform, we see that this is equivalent with the condition

$$L\psi = 0, \quad p\psi = k\psi$$

for a pure momentum state ψ with definite 4-momentum $k > 0$ (in the forward cone, i.e., $k_0 > |\mathbf{k}|$) and energy

$$E = c\sqrt{k^2} - mc^2.$$

A general pure state is a superposition $\psi = \int dk\psi_k$ of (unnormalized) momentum states with 4-momentum k, hence any nonzero ψ with

$$L\psi = 0, \quad \text{i.e.,} \quad \psi \in \mathbb{H} := \ker L.$$

A general mixed state is a mixture $\rho = \displaystyle\int d\pi(\alpha)\psi_\alpha\psi_\alpha^*$ of pure states ψ_α weighted by a nonnegative measure $d\pi(\alpha)$. Thus $L\rho = \rho L = 0$. In particular, with the standard expectation

$$\langle f\rangle_\rho := \operatorname{tr}\rho f,$$

of linear operators f on functions $\psi(E,x)$ we have

$$[L,\rho] = 0, \quad \langle L\rangle_\rho = 0. \tag{3}$$

The equations (3) will be the starting point for our axiomatic setting. Slightly generalized, they will allow us to formulate not only nonrelativistic quantum mechanics, but also classical mechanics, classical field theory, and quantum field theory.

3 Axiomatic physics

Das Streben nach Strenge zwingt uns eben zur Auffindung einfacherer Schlußweisen; auch bahnt es uns häufig den Weg zu Methoden, die entwickelungsfähiger sind als die alten Methoden von geringerer Strenge. [...]
Durch die Untersuchungen über die Grundlagen der Geometrie wird uns die Aufgabe nahe gelegt, nach diesem Vorbilde diejenigen Disziplinen axiomatisch zu behandeln, in denen schon heute die Mathematik eine hervorragende Rolle spielt; dies sind in erster Linie die Wahrscheinlichkeitsrechnung und die Mechanik.
David Hilbert, 1900 [14]

We now begin the axiomatic treatment; from now on, all concepts have a precise, unambiguous meaning. Here we concentrate on the conservative, (classical and quantum) mechanics part of Hilbert's 6th problem, quoted above; for the probability part, viewed in the present context, see NEUMAIER [29]. In this paper, we only give the outlines and general flavor of the theory. A much more extensive version with full details, and a treatment of the dissipative case are in preparation.

The **quantities** of interest are elements of a **Euclidean Poisson algebra** $\mathbb{E}$ containing the complex numbers as **constants**. Apart from an associative product (commutative only in the classical case) one has an **involution** $*$ reducing on the constants to complex conjugation, a complex-valued **integral** $\int$ defined on a subalgebra $I\mathbb{E}$ of **integrable** quantities, and a **Lie product** (or **bracket**) $\neg$. The subalgebra $B\mathbb{E}$ of **bounded** quantities consists of all $f \in \mathbb{E}$ with $f^*f \leq \alpha^2$ for some $\alpha \in \mathbb{R}$. Quantities f with $f^* = f$ are called **Hermitian**. (For reasons given in NEUMAIER [29], we avoid using the customary word 'observables', and follow instead the International System of Units (SI) [43] in our terminology.)

Apart from the standard rules for $*$-algebras and the linearity of the integral and the Lie product, one assumes the following axioms. (The product has priority over the Lie product, and both have priority over the integral. The partial order is defined by $f \geq 0$ iff $f^* = f$ and $\int g^*fg \geq 0$ for all $g \in I\mathbb{E}$, and the monotonic limit is defined by $f_l \downarrow 0$ iff, for every $g \in I\mathbb{E}$, the sequence (or net) $\int g^*f_lg$ consists of real numbers converging monotonically to zero.)

Axioms for a Euclidean $*$-algebra:
(E1) $f \in B\mathbb{E},\ g \in I\mathbb{E} \quad\Rightarrow\quad g^*, fg, gf \in I\mathbb{E}$
(E2) $(\int g)^* = \int g^*, \quad \int fg = \int gf$
(E3) $\int g^*g > 0$ if $g \neq 0$
(E4) $\int g^*fg = 0$ for all $g \in I\mathbb{E} \quad\Rightarrow\quad f = 0$ **(nondegeneracy)**
(E5) $\int g_l^*g_l \to 0 \quad\Rightarrow\quad \int fg_l \to 0,\ \int g_l^*fg_l \to 0$
(E6) $g_l \downarrow 0 \quad\Rightarrow\quad \inf \int g_l = 0$ **(Dini property)**

Additional axioms for a Euclidean Poisson algebra:
(P1) $(f \neg g)^* = f^* \neg g^*$
(P2) $f \neg g = -g \neg f$ **(anticommutativity)**
(P3) $f \neg (g \neg h) = (f \neg g) \neg h + g \neg (f \neg h)$ **(Jacobi identity)**
(P4) $f \neg gh = (f \neg g)h + g(f \neg h)$ **(Leibniz identity)**
(P5) $f^*f = 0 \Rightarrow f = 0$ **(nondegeneracy)**
(P6) $f \in I\mathbb{E},\ g \in \mathbb{E} \Rightarrow f \neg g \in I\mathbb{E}$,
(P7) $\int f \neg g = 0$ if $f \in I\mathbb{E}$ **(partial integration)**
As a consequence,
(P8) $\int f(g \neg h) = \int (f \neg g)h$.

Note that (E3) implies the Cauchy-Schwarz inequality

$$\int (fg)^*(fg) \le \int f^*f \ \int g^*g,$$

which implies thast $I\mathbb{E}$ is contained in $B\mathbb{E}$.

The present definition of a Euclidean Poisson algebra is a modification of the concept of a Poisson algebra as discussed in VAISMAN [45] and DA SILVA & WEINSTEIN [5] in that commutativity is dropped but integration requirements imposing a Euclidean structure are added. This modification enables us to treat classical and quantum physics on the same footing. (For a related attempt in this direction, see LANDSMAN [21].) Moreover, we introduced the symbol $\neg$ (an inverted stylized L, read 'Lie') to replace the Poisson bracket notation, which would be much more cumbersome if used extensively (as in as yet unpublished work).

In the present, elementary paper, we make use only of some of the above axioms (mainly those not involving limits). However, as will be shown elsewhere, all are needed for the deeper analysis of our conceptual basis.

To show that the axioms are rich in contents, we describe two basic realizations of them.

The quantum Poisson algebra. Let $\mathbb{H}$ be a Euclidean (= pre-Hilbert) space. We define the commutator $[f, g] := fg - gf$, and let $\iota := i/\hbar$ with a positive real number $\hbar$ called **Planck's constant**. Then the algebra $\mathbb{E} = \mathrm{Lin}\ \mathbb{H}$ of continuous linear operators on $\mathbb{H}$ is a Poisson algebra with **quantum bracket**

$$f \neg g = \iota[f, g],$$

and Euclidean with **quantum integral**

$$\int f = \mathrm{tr}\, f,$$

Integrable quantities are the operators $f \in \mathbb{E}$ for which all gfh with $g, h \in \mathbb{E}$ are trace class. (This includes all operators of finite rank.) The axioms are easily verified.

Nonrelativistic quantum mechanics. Nonrelativistic quantum physics is usually described by a rigged Hilbert space (see, e.g., BOHM [2]), if one wants to have direct access to the unbounded operators. Hence let $\mathbb{H}_0$ be a Euclidean space (the nuclear part of the rigged Hilbert space) with nuclear topology; we put $\mathbb{H} = C^\infty(\mathbb{R}, \mathbb{H}_0)$. For the standard position representation and $p_0 = i\hbar\partial_t/c$, $q_0 = ct$, we have

$$p_\mu \angle q_\nu = \eta_{\mu\nu} \quad \text{for } \mu, \nu = 0, 1, \ldots, \tag{4}$$

with the metric

$$\eta = \mathrm{Diag}(1, -1, \ldots, -1).$$

Restricted to $\mathbb{E}_0 = \mathrm{Lin}\ \mathbb{H}_0$, this gives the setting of traditional quantum mechanics.

Nonrelativistic classical mechanics. As discussed, e.g., in MARSDEN & RATIU [24], classical physics can be most conveniently described in terms of a Poisson manifold Ω. Let $\{\bullet, \bullet\}$ be the associated Poisson bracket on the algebra $\mathbb{E}_0 := C^\infty(\Omega)$ of infinitely differentiable complex-valued functions on Ω. Then $\mathbb{E} = C^\infty(\mathbb{R}^2, \mathbb{E}_0)$ (defined as in KRIEGL & MICHOR [19]) is a Poisson algebra with **classical bracket**

$$f \angle g := \frac{\partial f}{\partial t}\frac{\partial g}{\partial E} - \frac{\partial f}{\partial E}\frac{\partial g}{\partial t} + \{f, g\}$$

for $f = f(t, E)$, $g = g(t, E)$, and Euclidean with **classical integral**

$$\textstyle\int f = \displaystyle\int dt\ dE \int_\Omega f(t, E)$$

(where $\int_\Omega$ is the Liouville measure). Integrable quantities are the Schwartz functions on $\mathbb{E}$. (Thus integrability in the present sense is much stronger than Lebesgue integrability. This is due to our requirement (E1) which implies that $I\mathbb{E}$ must be an ideal in $\mathbb{E}$.) Again, the axioms are easily verified. With $p_0 = E/c$, $q_0 = ct$ and the standard symplectic Poisson bracket, we get again (4).

4 Physical systems

. . . daß die übliche Quantisierungsvorschrift sich durch eine andere Forderung ersetzen läßt [...]
Die neue Auffassung ist verallgemeinerungsfähig und rührt, wie ich glaube, sehr tief an das wahre Wesen der Quantenvorschriften.
Erwin Schrödinger, 1926 [38]

Motivated by the prelude, and consistent with the introductory remark in the seminal paper "Quantisierung als Eigenwertproblem" ("Quantization as eigenvalue problem") by SCHRÖDINGER [38], we generalize the Schrödinger picture of traditional quantum mechanics as follows. A **physical system** is characterized by a Hermitian **density** $\rho \in I\mathbb{E}$ with $\rho \geq 0$. The density, or any set of parameters from which the density can be uniquely reconstructed by a well-defined recipe, is referred to as the **state** of the system. A physical system with density ρ defines **expectations**

$$\langle f \rangle := \int \rho f = \int f \rho. \tag{5}$$

The **centralizer** $\mathbb{E}(S)$ of a quantity S (or a vector of Lie commuting quantities) is the set of all quantities Lie commuting with (all components of) S,

$$\mathbb{E}(S) = \{f \in \mathbb{E} \mid S \angle f = 0\}.$$

Clearly, $\mathbb{E}(S)$ is again a Poisson algebra. For a quantity $f \in \mathbb{E}(S)$, the **conditional expectation** at a fixed value s of S is defined by

$$\langle f \rangle_{S=s} = \langle f\delta(S-s)\rangle / \langle \delta(S-s)\rangle,$$

defined via a limit of integrable functions approaching the delta function. For example, if S is Hermitian with real spectrum then

$$\langle f \rangle_{S=0} = \lim_{\varepsilon \downarrow 0} \langle (S-i\varepsilon)^{-1} f (S+i\varepsilon)^{-1}\rangle / \langle (S^2+\varepsilon^2)^{-1}\rangle.$$

By construction, conditional expectations always satisfy $\langle 1 \rangle_{S=s} = 1$; they satisfy the axioms for an ensemble given in NEUMAIER [29].

Dynamical predictions are possible only in a system with a well-controlled environment. For a system in interaction with an arbitrary environment, the expectation satisfies a dynamics determined by the **Ehrenfest equations**

$$\langle \mathcal{D}\{f\} \rangle = 0 \quad \text{for all } f \in B\mathbb{E} \tag{6}$$

with a **forward derivation** $\mathcal{D}$, i.e, a continuous linear mapping $\mathcal{D}\{\cdot\} : \mathbb{E} \to \mathbb{E}$ mapping bounded quantities to bounded quantities and satisfying

$$\mathcal{D}\{f\}^* = \mathcal{D}\{f^*\}, \quad \mathcal{D}\{f^*f\} \geq \mathcal{D}\{f^*\}f + f^*\mathcal{D}\{f\}$$

for all $f \in \mathbb{E}$. Physical systems with the same forward derivation (but in general different densities) are said to follow the same **physical law**. Written in terms of the density, (6) becomes the **generalized Liouville equation**

$$\mathcal{D}^*\{\rho\} = 0 \tag{7}$$

with the **Liouville operator** $\mathcal{D}^*$ defined (uniquely by (E4)) by

$$\textstyle\int \mathcal{D}^*\{\rho\}f = \int \rho\mathcal{D}\{f\}.$$

We shall discuss general physical systems and their (dissipative) properties elsewhere.

Here we consider isolated systems only, where the physical law is characterized by a Hermitian **action** $L \in \mathbb{E}$ which determines the forward derivation. A physical system with density ρ is called **isolated** (and ρ is called a **conservative density**) if

$$\langle L \rangle = \textstyle\int L\rho = 0$$

and the generalized Liouville equation

$$L \neg \rho = 0 \tag{8}$$

holds. Since

$$\langle L \neg f \rangle = \textstyle\int \rho(L \neg f) = \int (\rho \neg L)f = -\int (L \neg \rho)f = 0,$$

expectations in isolated systems satisfy the Ehrenfest equations

$$\langle L \neg f \rangle = 0 \quad \text{for all } f \in I\mathbb{E}.$$

The axioms for a Euclidean expectation algebra imply that $\mathcal{D}_\pm\{f\} = \pm L \neg f$ is a forward derivation for both signs; as discussed elsewhere, this reflects the reversible, conservative nature of isolated systems.

If ρ is a conservative density and $f \in \mathbb{E}(L)$ then

$$\rho^f := f\rho f^* \tag{9}$$

is also a conservative density. Thus a large class of conservative densities can be constructed from a single one if some quantities f_l in the centralizer $\mathbb{E}(L)$ are known, since we may apply (9) with any polynomial constructed from the f_l. This generalizes the traditional construction of states from the vacuum by means of creation operators. It is applicable even where – such as for interacting quantum fields in 4 dimensions – no precise mathematical meaning can be given to the latter construction.

5 Hamiltonian systems

Our axioms cover the traditional physics of Hamiltonian systems. The action corresponding to an arbitrary time-dependent Hamiltonian $H(t)$ is defined as

$$L = p_0 - H(t),$$

where $p_0 = E$ in the classical case and $p_0 = i\hbar\partial_t$ in the quantum case. In both cases,

$$p_0 \neg f = -\dot{f}.$$

(Strictly speaking, the name 'action' fits tradition only for field theories. For multi-particle systems, the above expression for L is unrelated to traditional action principles. But applying the same machinery which gives the field equations of field theory to this unorthodox action happens to produce the correct multi-particle dynamics.)

For conservative quantum systems, $L \neg \rho = 0$ implies for $L = p_0 - H$:

$$\dot{\rho} = -p_0 \neg \rho = -(L + H) \neg \rho = -H \neg \rho,$$

and we get the standard **quantum Liouville equation**

$$i\hbar\dot{\rho} = [H, \rho] \tag{10}$$

for a conservative nonrelativistic quantum system with Hamiltonian H. The Ehrenfest equations reduce to their traditional form

$$i\hbar\frac{d}{dt}\langle f\rangle = \langle [f, H]\rangle,$$

showing that expectations follow a deterministic law. For conservative classical systems, exactly the same derivation applies, and we get the **classical Liouville equation**

$$\dot{\rho} = \{H, \rho\}. \tag{11}$$

6 Pure states

Pure states are the limiting situation (in a suitable completion of the space of integrable quantities) of densities extremal with respect to the natural order relation. They are of mathematical interest since any density can be written as a convex combination of pure states, and of physical interest for few-particle systems, where states can often be considered as approximately pure. (However, states at positive temperature are never pure, and the decomposition into pure states is, in the quantum case, not unique. Thus pure states describe idealized situations only.)

Pure classical states. Here extreme states are distributional limits of densities; the expectations are algebra homomorphisms into $\mathbb{C}$ (i.e., characters of the algebra) satisfying $\langle L\rangle = 0$. For nonrelativistic classical physics with phase space variable z,

$$\langle f\rangle = f(t, E, z),$$

and the condition $\langle L\rangle = 0$ fixes the value of E to $E = H(t, z)$. Hence we may assume f to be independent of E.

Thus pure states of a classical nonrelativistic system are characterized by a pair (t, z) consisting of a time t and the phase space location z of the system at this time. The Ehrenfest equations reduce in the limit of pure states to the **Hamiltonian dynamics**

$$\dot{f} = \{f, H\}.$$

Pure quantum states. Extreme states are limiting rank 1 densities

$$\rho = \psi\psi^*, \quad \psi \in \mathbb{H}^*.$$

The equation $L \neg \rho = 0$ implies that ψ is a generalized eigenvector of L. (See, e.g., Maurin [25] for a mathematical treatment in terms of nuclear spaces.) The condition $\langle L\rangle = 0$ then implies that the eigenvalue vanishes. Note that, since generalized eigenvectors need not be in $\mathbb{H}$, not all expectations need to exist in a pure state; the latter are to be regarded only as idealized limits of physical states.

Thus pure states of an isolated quantum system are characterized by a generalized Schrödinger equation

$$L\psi = 0, \quad \psi \in \mathbb{H}^*. \tag{12}$$

We call solutions of (12) **pure conservative quantum states**.

As discussed in the prelude, if the action L is translation invariant and p is the generator of the translations, one can find pure conservative quantum states of definite 4-momentum k by solving the equations.

$$p_0\psi = mc\psi, \quad \mathbf{p}\psi = k\psi. \tag{13}$$

In particular, pure states of mass m in a rest frame can be found by solving the eigenvalue problem

$$p_0\psi = mc\psi, \quad \mathbf{p}\psi = 0, \quad L\psi = 0.$$

This is a **constrained Schrödinger equation**, cf. Section 13 below.

The pure conservative quantum states form a vector space $\mathbb{H}^{\rm cons}$ on which the centralizer $\mathbb{E}(L)$ acts since $f \in \mathbb{E}(L)$ and $\psi \in \mathbb{H}^{\rm cons}$ imply $Lf\psi = fL\psi = f0 = 0$. In the quantum case, $f \in \mathbb{E}(L)$ iff f commutes with L; thus quantities in $\mathbb{E}(L)$ can be found, e.g., by solving the eigenvalue problem for L. Thus we can create from any particular conservative quantum state a large class of other conservative quantum states provided we know enough quantities commuting with L.

7 Classical fields

We discuss here only boson fields. By using super Poisson algebras and super versions of all concepts, fermion fields can be handled in an analogous fashion.

Let $\mathbb{H} := S(\mathbb{R}^{1,3})$ be the algebra of infinitely differentiable, fast decaying Schwartz functions on Minkowski space $\mathbb{R}^{1,3}$, and let V be a finite-dimensional symplectic space with symplectic form Δ. Then the field algebra $\mathbb{E} := C^\infty_{\rm pol}(\mathbb{H}\otimes V^*)$ of infinitely differentiable functions f of the field argument $\Phi \in \mathbb{H}$ with $\partial^n f \in C^\infty(\mathbb{H}\otimes V^*, (\mathbb{H}\otimes V)^{\otimes n})$ and at most polynomial growth is a Poisson algebra with

$$f \angle g = \int dx\, \Delta\left(\frac{\partial f}{\partial \Phi(x)}, \frac{\partial g}{\partial \Phi(x)}\right).$$

With expolynomial functions (linear combinations of products of polynomials with the exponential of a negative definite, quadratic polynomial) as integrable functions, $\mathbb{E}$ is Euclidean with an integral definable via infinite-dimensional Gaussian measures.

Pure classical field states. A pure state over the field algebra $\mathbb{E} = C^\infty_{\rm pol}(\mathbb{H}\otimes V^*)$ assigns to each $f \in \mathbb{E}$ the value $f(\Phi)$ at a particular field $\Phi \in \mathbb{H}\otimes V^*$. The Ehrenfest equations reduce in the limit of pure states over the field algebra to the equations

$$L \angle f = 0 \quad \text{for all } f \in \mathbb{E}.$$

Inserting the linear function $f = a(\Phi)$, where

$$a(\Phi) := \int dx\, a(x)^T \Phi(x),$$

into $L \angle f = 0$ we get

$$\Delta\left(\frac{\partial L}{\partial \Phi(x)}, a(x)\right) = 0$$

for $a \in \mathbb{H} \otimes V$ with compact support. Since Δ is nondegenerate, we conclude

$$\frac{\partial L}{\partial \Phi(x)} = 0 \quad \text{for all } x \in \mathbb{R}^{1,3}.$$

This is the traditional **stationary action principle**. In the current setting, it is not a postulate but a consequence of the Ehrenfest equations. (The equations for other choices of f are consequences of this.)

To get the traditional field theories, we simply need to find the right symplectic structure for each type of field. The field components must appear in conjugate pairs, which we arrange to two conjugate vectors Φ and Φ^c (in place of the single Φ used before). Then adequate commutation relations are

$$a(\Phi) \neg b(\Phi) = a(\Phi^c) \neg b(\Phi^c) = 0,$$
$$a(\Phi^c) \neg b(\Phi) = (a|b) := \int dx\, a(x)^T b(x),$$

where $\Phi^c = \Phi^*$ for complex fields (which come in complex conjugate pairs), while for real fields Φ and Φ^c are independent. For real fields which have no conjugate partner in the Lagrangian, one adds additional conjugate partners to the algebra of quantities. These additional fields are – like gauge degrees of freedom – unobservable and do not affect the field equations for the original fields.

Hence the present framework allows a consistent implementation of all classical field equations derivable from the stationary action principle. (*Note:* If we apply this to the electromagnetic 4-vector potential, we get, in contrast to the approach in canonical quantization, a conjugate 4-vector potential, with standard symplectic Lie bracket for each component!)

By extending the above framework to Euclidean super Poisson algebras, one can also incorporate classical fermion fields. In particular, we can implement a **classical** version of the **standard model**, **including gravitation** within the present setting.

If we use in place of symplectic Poisson algebras suitable Lie-Poison algebras, the Ehrenfest equations produce in the limit of pure classical states for appropriate actions both the relativistic [26] and nonrelativistic [27] **Euler equations** for perfect fluids and the **Euler-Poincaré equations** [24]. Using suitable Lie-Poison algebras of functions of phase space fields, it is possible to define natural actions for which the Ehrenfest equations produce in this way the Vlasov equations. In suitable tensor products one can then form actions that define Vlasov equations interacting with electromagnetic and/or gravitational fields, giving **Vlasov-Maxwell equations** (cf., e.g., [34]) and **Vlasov-Einstein equations** (cf., e.g., [1]).

Details will be given elsewhere.

8 Phase space quantization

There are many ways to quantize a classical system. From the point of view of being able to do analysis (i.e., error estimates), the mathematically most developed form is deformation quantization (see, e.g., RIEFFEL [37]), which deforms a commutative product into a Moyal product. In the following, we propose an alternative deformation approach which, instead, **deforms the operators** $f \in \mathbb{E}$ by embedding $\mathbb{E}$ into Lin $\mathbb{E}$, identifying $f \in \mathbb{E}$ with the multiplication mapping $g \to fg$. This can be done with surprising ease.

The superoperators M_f and D_f defined by

$$M_f\{g\} := fg, \quad D_f\{g\} := f \neg g$$

belongs to Lin $\mathbb{E}$. For $f \in \mathbb{E}$, we define the **quantization** $\widehat{f}$ of f by

$$\widehat{f} := M_f - \frac{i\hbar}{2} D_f \in \text{Lin } \mathbb{E}.$$

The expectations

$$\langle \widehat{f} \rangle = \langle f \rangle - \frac{i\hbar}{2} \langle D_f \rangle$$

differ from those of f by a term of order $O(\hbar)$, justifying an interpretation in terms of "deformation". In particular, we automatically have a good classical limit.

To actually quantize a classical theory, one may choose a Lie algebra of relevant quantities generating the Poisson algebra, quantizes its elements by the above rule, expresses the classical action as a suitably ordered polynomial expression in the generators, and uses as quantum action this expression with all generators replaced by their quantizations.

In general, the above recipe for phase space quantization gives an approximate Poisson isomorphism, up to $O(\hbar)$ terms. But Lie subalgebras are mapped into (perhaps slightly bigger) Lie algebras, and one gets a true isomorphism for all embedded Heisenberg Lie algebras, i.e., Lie algebras where all Lie products are multiples of a central element 1.

Quantization Theorem. If $\mathbb{E}$ is commutative then the quantum bracket

$$A \neg B = \iota [A, B] \quad \text{for } A, B \in \text{Lin } \mathbb{E}$$

satisfies, for $f, g \in \mathbb{E}$,

$$\widehat{f} \neg \widehat{g} = M_{f \neg g} - \frac{i\hbar}{4} D_{f \neg g} = \frac{1}{2}(M_{f \neg g} + \widehat{f \neg g}),$$

Any Lie subalgebra $\mathbb{L}$ of $\mathbb{E}$ defines a Lie algebra

$$\widehat{\mathbb{L}} = \{M_{f \neg g} + \widehat{h} \mid f, g, h \in \mathbb{L}\}$$

under the quantum bracket. If $\mathbb{L}$ is a Heisenberg Lie algebra then $\widehat{\ }: \mathbb{L} \to \widehat{\mathbb{L}}$ is a Lie isomorphism.

The proof is not difficult but will be given elsewhere.

In particular, for the standard symplectic Poisson algebra $\mathbb{E} = C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, phase space quantization amounts to using the reducible representation

$$\widehat{p} = p - \frac{i\hbar}{2}\partial_q, \quad \widehat{q} = q + \frac{i\hbar}{2}\partial_p$$

of the canonical commutation rules on phase space functions instead of the traditional irreducible representation

$$\tilde{p} = -i\hbar\partial_x, \quad \tilde{q} = x$$

on configuration space functions. It will be shown elsewhere that these representations are related by a **Wigner transform** (cf. WIGNER [53]).

By quantizing in phase space, one gives up irreducibility (and hence the description of a state by a *unique* density) but gains in simplicity. Perhaps this is comparable to the situation in gauge theory, where the description by gauge potentials introduces some arbitrariness with which one pays for the more elegant formulation of the field equations but which does not affect the observable consequences.

9 Quantum field theory

A good many physicists are now working on the problem of trying to set up a quantum field theory independently of any Hamiltonian. [...]
I still think that in any future quantum theory there will have to be something corresponding to Hamiltonian theory, even if it is not in the same form as at present.
Paul Dirac, 1964 [7]

Actions for classical or quantum field theories are based on representations of a symmetry group and corresponding invariant actions. In any fundamental theory, the symmetry group must contain either the Galilei group (for nonrelativistic fields) or the Poincaré group (for relativistic fields); if gravitation is involved, the symmetry group must also contain the group of all diffeomorphisms of some spacetime manifold.

Having a symmetry group is equivalent with having nonuniqueness in the description of a physical system. Different states providing equivalent descriptions (satisfying the same laws but in different coordinates) are commonly said to correspond to different choices of an inertial system. Changing the inertial system used to coordinatize a system changes the state and hence the expectations; for example, moving an inertial system O (illustrated by an observing intelligent robot) in time produces a change in observed expectations which O conceives of as the intrinsic dynamics of the environment, while moving (rotating or translating) the inertial system O in space produces a change in observed expectations which O conceives of as the illusion of the space moving around it caused by the motion of its moving head. We now formalize these considerations.

Let $\mathbb{L}$ be the Lie algebra of the Galilei group, the Poincaré group or any assumed symmetry group containing one of these groups, with Lie product $\neg$. Let $p \in \mathbb{L}^{1,3}$ be the generator of the translation subgroup in the canonical basis. Let J be a Poisson representation of $\mathbb{L}$ in a Euclidean Poisson algebra $\mathbb{E}$, defined by

$$J(\delta) \neg J(\delta') = J(\delta \neg \delta') \quad \text{for all } \delta, \delta' \in \mathbb{L}. \tag{14}$$

$P := J(p)$ (taken componentwise) defines the (total) **physical 4-momentum**. A smooth change of the inertial system (modeling a virtual motion of the robots head) is described by an arbitrary continuously differentiable mapping $\delta : [0,1] \to \mathbb{L}$ specifying the infinitesimal motions $\delta(\tau) \in \mathbb{L}$ of the inertial systems at instant $\tau \in [0,1]$. A corresponding assignment of densities $\rho(\tau) \in I\mathbb{E}$ at instant τ is called **consistent** if it satisfies the differential equation

$$\frac{d}{d\tau}\rho(\tau) = \rho(\tau) \neg J(\delta(\tau)). \tag{15}$$

In classical physics, this describes a canonical, in quantum physics a unitary transformation representing a general element of the (connected part of the) symmetry group. In particular, an observer moving in space-time with uniform velocity $u \in \mathbb{R}^{1,3}$ finds the density changing according to the **covariant Liouville equation**

$$\frac{d}{d\tau}\rho(\tau) = \rho(\tau) \neg J(u \cdot p). \tag{16}$$

Thus we have a covariant generalization of the nonrelativistic situation considered in Section 5.

Since such a change of inertial systems should not affect the physics, we require that quantities (and in particular the action L, i.e., the physical law) are unaffected by these changes, and that an isolated system remains isolated. The former condition is simply the requirement that we base our setting on the **Schrödinger picture**, and the latter condition amounts to

$$L \neg \rho(\tau) = 0 \quad \text{for all } \tau \tag{17}$$

whenever $L \neg \rho(0) = 0$. To analyze this condition, let $\rho(0)$ be the density of an isolated system, and put

$$e(\tau) := L \neg \rho(\tau).$$

Then $e(0) = 0$ and

$$\begin{aligned}\frac{d}{d\tau}e(\tau) &= \frac{d}{d\tau}(L \neg \rho(\tau)) = L \neg \frac{d}{d\tau}\rho(\tau) = L \neg (\rho(\tau) \neg J(\delta(\tau)))\\ &= (L \neg \rho(\tau)) \neg J(\delta(\tau)) + \rho(\tau) \neg (L \neg J(\delta(\tau)))\end{aligned}$$

so that

$$\frac{d}{d\tau}e(\tau) = e(\tau) \neg J(\delta(\tau)) + \rho(\tau) \neg (L \neg J(\delta(\tau))). \tag{18}$$

If (17) holds then $e(\tau)$ vanishes identically, and this reduces to $\rho(\tau) \neg (L \neg J(\delta(\tau))) = 0$. The requirement that this holds for arbitrary densities and arbitrary smooth changes of the inertial system therefore demands that

$$L \neg J(\delta) \in C\mathbb{E} \quad \text{for all } \delta \in \mathbb{L}, \tag{19}$$

where $C\mathbb{E}$ denotes the **Lie center** of $\mathbb{E}$, the algebra of quantities which Lie commute with all quantities. An action L satisfying (19) is called **$\mathbb{L}$-invariant**. Conversely, if the action L is $\mathbb{L}$-invariant, then (18) reduces to

$$\frac{d}{d\tau}e(\tau) = e(\tau) \neg J(\delta(\tau)).$$

Under conditions which guarantee the unique solvability of the initial value problem (18), we conclude that $e(\tau) = 0$ for all τ, proving (17). Thus the $\mathbb{L}$-invariance of the action is essentially equivalent to the requirement that being isolated is a covariant concept.

In particular, using in our setting a Poincaré invariant action L defines a relativistic physical theory. As shown in Sections 4 and 6, we can use an arbitrary conservative density (resp. pure quantum state) and a set of quantities in the centralizer $\mathbb{E}(L)$ to construct a large class of conservative densities (resp. pure quantum states) as possible states of an isolated physical system with given action.

Having phase space quantization as a universal generalization of the Wigner transform, we can use it to quantize the basic fields of any (Galilei or Poincaré invariant) classical field theory. This gives well-defined mathematical definitions of the various (nonrelativistic or relativistic) quantum field theories in current physical usage.

Using a Galilei invariant action one gets nonrelativistic field theory. As explained nonrigorously in many textbooks (e.g., UMEZAWA et al. [44]), nonrelativistic quantum field theory is in principle equivalent to nonrelativistic quantum mechanics. Therefore, one uses for nonrelativistic problems field theory only to describe bulk matter, while scattering and bound state problems are handled with the Schrödinger equation. This is much simpler than solving the full operator dynamics of field theory.

In relativistic quantum field theory, there has been in the past no analogue of the Schrödinger equation that could have been used for this purpose. Thus even simple scattering problems were formulated in a field theoretic language accessible to a perturbative treatment, and bound state problems (see, e.g., WEINBERG [47]) could be described only very indirectly through poles in the S-matrix. For the latter, there is no sound mathematical basis since in traditional quantum field theory, the S-matrix is only defined perturbatively in terms of a presumably divergent (DYSON [8]) asymptotic expansion, so that, mathematically, talking about its poles is nonsense.

The results of the present paper show, however, that to each quantum (field or particle) theory there is a corresponding constrained Schrödinger equation from which one can construct pure conservative quantum states with definite momentum in complete analogy to the nonrelativistic case, and without restriction to a particular symmetry group. (Mathematically, it is suspect if certain techniques work for a particular, highly nontrivial group but not for all groups. Already from this perspective one could see that something was missing from current quantum field theory!)

10 Wightman axioms

> *The quantum theory of fields never reached a stage where one could say with confidence that it was free from internal contradictions – nor the converse. In fact, the Main Problem [...] turned out to be [...] to show that the idealizations involved in the fundamental notions of the theory are incompatible in some physical sense, or to recast the theory in such a form that it provides a practical language for the description of elementary particle dynamics.*
> R.F. Streater and A.S. Wightman, 1963 [42]

Traditionally, mathematical physicists approach relativistic quantum field theory via an axiomatic approach discussed in detail by STREATER & WIGHTMAN [42]. The Wightman axioms (WIGHTMAN [52]) are an interpretation of field theory not in terms of field equations but in terms of correlation functions. Relations to the Lagrangian approach have been lacking so far. But one would have such relations if one could combine tradition with the present formulation of quantum field theory. Thus one would like to realize the Wightman axioms by identifying the vacuum with a pure conservative quantum state ψ_0 with zero momentum, i.e.,

$$L\psi_0 = 0, \quad P\psi_0 = 0, \tag{20}$$

and (in view of the remarks at the end of Section 6) Wightman field operators by suitable Hermitian quantities in the centralizer $\mathbb{E}(L)$.

It is not clear whether the Wightman axioms describe correctly the structure of relativistic quantum states. Apart from generalized free fields, no realization of the Wightman axioms in 4-dimensional space-time is known (see, e.g., Rehren [33]), and there are no-go theorems – stating, for example, that there is no natural interaction picture [42, Theorem 4-16] – pointing to the possibility that these axioms are indeed too strong to describe realistic theories.

To prove that the assumptions defining a Wightman field can (or cannot) be satisfied in the present context is therefore a highly nontrivial task. But at least it is embedded into a well-defined functional analytic context, where the Poincaré representation is already fixed. This might make it tractable for systems like QED, which are close to nonrelativistic quantum mechanics. Therefore, one might be able to adapt the insights from nonrelativistic scattering theory (which provides a diagonalization of the action and hence full control over its centralizer) to the new situation.

On the other hand, even without knowing the existence of Wightman fields (and even if one could prove that they do not exist), the setting presented here makes sense and defines for arbitrary actions a good quantum field theory, closely related to physical practice. In particular, one can try to generalize to the new constrained Schrödinger equations the supply of techniques available for ordinary Schrödinger equations, and in this way complement the current perturbative techniques of quantum field theory by techniques known from nonrelativistic quantum mechanics. A first step in this direction – the generalization of the projection formalism – has been done already; see Neumaier [28]. Work on scattering theory is under way.

11 Phenomenological relativistic dynamics

In spite of the acceptance of field theories as a matter of principle, most realistic dynamical calculations in nuclear physics, and many in particle physics, utilize the nonrelativistic Schrödinger equation. [...] Relativistic direct interaction theories of particles lie between local field theoretical models and nonrelativistic quantum mechanical models.
B.D. Keister and W.N. Polyzou, 1991 [18]

While fields are usually used to describe nature on a fundamental level, practical work (especially for bound states and resonances) requires phenomenological few-particle equations, which are frequently related only loosely to underlying fields; see the references in the next section. It is therefore interesting to see that a variety of covariant phenomenological few-particle equations can be easily built in the present framework. We do this by using Poincaré invariant actions on Hilbert spaces carrying a suitable Poincaré representation without states of negative energy.

The possible irreducible Poincaré representations (modeling elementary particles) were classified by WIGNER [54]. The representations of positive (relativistic) energy take their simplest form in momentum space; the momenta p are restricted to a mass shell

$$\Omega(\tilde{p}) = \{p \in \mathbb{R}^{1,3} \mid p^2 = \tilde{p}^2,\ p_0 > 0\}, \tag{21}$$

the orbit of a 4-vector $\tilde{p}$ under the Poincaré group. It is possible to combine these irreducible Poincaré representations in many ways to obtain reducible momentum space representations for few-particle systems. Traditionally (see, e.g., WEINBERG [48] for the canonical field quantization approach and the review in KEISTER & POLYZOU [18] for the direct relativistic Hamiltonian few-body approach), this is done by breaking the manifest invariance to a maximal subgroup of the Poincaré group, with all the awkwardness this entails.

The key that allows us to preserve a manifestly covariant formalism, thus overcoming the traditional problems in canonical quantization, is the fact that we use as algebra of quantities the linear operators on a space of wave functions slightly bigger than traditional Fock spaces. This is done in the following by adding a velocity vector u as a dynamical parameter, which allows us to deform the bare mass shell $p^2 = (mc)^2$ (where c is the speed of light) to $p^2 = (mu)^2$, which in turn permits the conservation of total 4-momentum in interactions.

A phenomenological realization of a system of N massive scalar particles with **rest masses** $m^1, \ldots, m^N > 0$ and **charges** $Q^1, \ldots, Q^N$ is now realized by wave functions

$$\psi = \psi(u, p^{1:N}) = \psi(u, p^1, \ldots, p^N)$$

whose coordinates are a global **4-velocity** vector u with $0 < u \in \mathbb{R}^{1,3}$ and the particle **4-momentum** vectors p^a in the **dynamic mass shells** $\Omega(m^a u)$ whose scale depends on u. The **total 4-momentum** $\sum p^a$ is required to be parallel to the 4-velocity u. Thus the space of wave functions is

$$\mathbb{H} = C^\infty(\Omega^N), \tag{22}$$

where Ω^N is the set of all tuples

$$(u, p^{1:N}) = (u, p^1, \ldots, p^N)$$

with

$$p^a \in \Omega(m^a u) \ \text{ for } a = 1, \dots, N, \qquad 0 < u \ \parallel \sum p^a.$$

The (not everywhere defined) Hermitian inner product – from which a Hilbert space can be constructed by completing the space of vectors of finite norm – is given by

$$\phi^*\psi := \int dm\ du Dp^1 \dots Dp^N \delta\Big(mu - \sum p^a\Big) \overline{\phi(u, p^{1:N})} \psi(u, p^{1:N}),$$

where

$$Dp = dp\ \delta(p^2 - (mu)^2) = \frac{d\mathbf{p}}{2p_0} = \frac{d\mathbf{p}}{2\sqrt{(mu)^2 + \mathbf{p}^2}} \tag{23}$$

is the invariant measure on a dynamic mass shell $\Omega(mu)$. The **one-particle operators** are defined as

$$J(f) := \sum_a f(u, Q^a, m^a, p^a, M^a),$$

where the diagonal operator $f = f(u, Q, m, p, M)$ is a function of 4-velocity u, charge Q, mass m, 4-momentum p and **4-angular momentum**

$$M := p \wedge \frac{\partial}{\partial p} \tag{24}$$

with components

$$M_{\mu\nu} = p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu},$$

and the superscript a indicates application to the coordinates of the ath particle. (Note that the global 4-velocity u carries no superscript; it is shared by all particles.) Since the p_μ are the Poincaré generators of translation in the direction of the μ-axis and the $M_{\mu\nu}$ are the standard generators of the Lorentz transformations, it is easy to see that the total 4-momentum $J(p)$ and **total 4-angular momentum** $J(M)$ define a representation of the Poincaré group *without negative energy states*. In the terminology of Dirac [6], it is a representation in the point form. (It shares this property with the representations of Ruijgrok [36] which are based on Lippmann-Schwinger equations. But his translation generators are much more complicated than the present ones.)

On the space (22), one can now define actions of the form

$$L = L_0 - V, \tag{25}$$

where the **kinetic action** L_0 is a Poincaré invariant one-particle operator, and the **interaction** V is a Poincaré invariant integral operator.

12 Poincaré invariant multiparticle interactions

For scalar particles, the simplest covariant kinetic action is

$$L_0 = J\Big(\frac{p^2-(mc)^2}{2m}\Big) = J\Big(\frac{m}{2}(u^2-c^2)\Big), \tag{26}$$

with a constant $c > 0$, the **speed of light**. However, more complicated covariant formulas with rational or analytic dependence on m and p^2 are admissible, too, if they vanish for $p^2 = (mc)^2$ and nowhere else. In this case, the generalized Schrödinger equation $L\psi = 0$ implies for noninteracting particles, where $V = 0$, the relation $u^2 = c^2$, forcing the dynamic mass shells to equal the bare mass shells.

To construct a versatile class of Poincaré invariant interactions, we first note that the vector

$$p_m := p + u\frac{-p\cdot u + \sqrt{(p\cdot u)^2 - p^2u^2 + (mu^2)^2}}{u^2} \tag{27}$$

is in the dynamic mass shell $\Omega(mu)$. Indeed, it suffices by covariance to check the case where $\mathbf{u} = 0$; then $u_0 > 0$, $u^2 = u_0^2$, $p\cdot u = p_0u_0$,

$$(p_m)_0 = p_0 + u_0\frac{-p_0u_0 + \sqrt{\mathbf{p}^2u_0^2 + (mu_0^2)^2}}{u_0^2} = \sqrt{\mathbf{p}^2 + (mu_0)^2} > 0,$$

and since $\mathbf{p}_m = \mathbf{p}$, we find $p_m^2 = (mu_0)^2 = (mu)^2$. Thus the mapping $p \to p_m$ (the dependence on u is not written explicitly) is a nonlinear projection to the dynamic mass shell $\Omega(mu)$.

The simplest choice for a nontrivial interaction is a sum of pair interactions,

$$V = \sum_{a<b} V^{ab}, \tag{28}$$

where $V^{ab} = V^{ba}$ acts on the coordinates of particles a and b as

$$(V^{ab}\psi)(u, p^a, p^b) = \int dq\ \delta(u\cdot q)U^{ab}(q)\psi(u, (p^a+q)_{m^a}, (p^b-q)_{m^b}), \tag{29}$$

where the projections are to be taken with respect to the common 4-velocity argument u, and $U^{ab}(q)$ is also allowed to depend on mass, momentum and charge of the particles a and b. The delta function removes a redundancy in the projections, which do no change if a multiple of u is added to q. The construction is such that V^{ab} is automatically translation invariant. In particular, if all U^{ab} are Hermitian and Lorentz invariant then

V and hence the action (25) is Hermitian and Poincaré invariant. For example, this is the case in pair potentials of the form

$$U^{ab}(q) = \text{Re}\ \frac{\beta_S(m^a m^b)^2 + (\beta_V m^a m^b + \alpha Q^a Q^b) p^a \cdot p^b + \beta_T (p^a \cdot p^b)^2}{m^a m^b (q^2 + i\varepsilon)}, \qquad (30)$$

where the limit $\varepsilon \downarrow 0$ is to be taken in (29) to regularize the potential near $q = 0$. These potentials describe relativistic electromagnetic and gravitational forces; the coupling constants α and $\beta_S, \beta_V, \beta_T$ determine the strength of the **electromagnetic** and the scalar, vector, and tensor **gravitational interaction**, respectively. (This will be justified in the next section by considering the nonrelativistic limit.) By making these coupling constants q-dependent (**running coupling constants**), one can also account covariantly for phenomenological self-energy contributions; cf. the discussion in PESKIN & SCHROEDER [31, pp. 252–255].

Note that after Fourier transform into spacetime, we get – in contrast to field theories – a nonlocal (but still Poincaré invariant) action.

This basic setting can be extended in various ways. Particles with positive spin or with internal symmetries are easily accommodated, especially when using the representations discussed in WEINBERG [48, 49]. (They are of course equivalent to Wigner's representations but computationally more tractable.) Particles with positive integral spin are handled in exactly the same way, except that the wave functions have additional indices, the angular momentum gets an additional intrinsic spin term operating on these indices, and the inner product has a slightly different form. It is easy to specify $\mathbb{L}$-invariant interaction terms similar to (30) for particles with positive spin and for particles with inner symmetries (and corresponding matrix-valued charges Q^a); but such interactions are now also restricted by Clebsch-Gordan rules (cf. WEINBERG [50]).

Fermion particles with half-integral spin are handled similarly, using spinor components in the wave functions and kinetic actions such as

$$L_0 = J(p \cdot \gamma c - mc^2). \qquad (31)$$

The resulting constrained Schrödinger equations

$$J(p \cdot \gamma c - mc^2)\psi = V\psi, \quad p\psi = k\psi$$

generalize the Dirac equations to the multiparticle case. Details about the handling of spin will be given elsewhere.

Massless particles are handled in the same way, except that the kinetic part of the action is absent, since these particles never go off-shell in our phenomenological setting.

For indistinguishable particles, symmetrization and antisymmetrization can be done in the standard way. Different kinds of particles are handled by adding to the sum of their self-actions another interaction. Few-particle systems in which the particle number is not conserved can be modelled by using a direct sum of Hilbert spaces of the type (22) and covariant interactions changing the particle number. For example, we may model the emission and absorption of a photon of momentum p by a massive scalar particle of charge Q^a with the Hermitian and Poincaré invariant interaction proportional to

$$(V\psi)(u,p,p^a) = F(p)\frac{Q^a p^a}{m^a}\psi(u,(p^a+p)_{m^a}),$$

$$(V\psi)(u,p^a) = \int dp\ \delta(p^2)F(p)\frac{Q^a p^a}{m^a}\cdot\psi(u,p,(p^a-p)_{m^a}),$$

where the **form factor** $F(p)$ is an arbitrary covariant scalar C^∞-function formed from p, p^a and u. (Note that the photon wave function has additional vector components, with respect to which the inner product $\cdot$ is taken.) Previous covariant few-particle models could not handle this situation (Keister & Polyzou [18, p. 392]).

In a multiparticle system, one can model in the same way the interactions corresponding to Feynman diagrams with a single vertex of degree 3, and in a similar way also interactions corresponding to more complex vertices. Note that because momentum is conserved and all particle energies are positive, particles cannot be created from a vacuum state (with 0 particles), nor can particles be annihilated without creating (or preserving) at least one particle. Thus the phenomenological approach does not have the problems which field theories have with the presence of an interacting ('fluctuating') vacuum.

We see that the possibilities for the new action-based relativistic models fully match (and even exceed) the freedom available for nonrelativistic Hamiltonian systems. Since they are manifestly Poincaré invariant, they are much simpler than various relativistic Hamiltonian models that have been constructed in the past (see, e.g., the review in Keister & Polyzou [18] for nuclear physics, Crater et al. [4] for QED, and Ruijgrok [36] for a Lippmann-Schwinger based model), but have the same advantages as the latter: consistency with relativity theory, tractable few-body calculations, easy treatment of bound states, resonances, and particle production, and easy fit to parametric models. In addition, they can be used to give phenomenological models of quantum systems in which the particle number is not preserved, or the spin is > 1.

In time, such action-based relativistic models may therefore replace the many nonrelativistic (e.g., Isgur [15], Karl [17]), semirelativistic (e.g., Lucha et al. [22, 23]) and relativistic (e.g., Keister & Polyzou [18]) Hamiltonian approximations, and approximations based on Bethe-Salpeter equations (e.g., Kummer & Mödrich [20]) or

Dyson-Schwinger equations (e.g., ROBERTS & WILLIAMS [35]) now in vogue for the phenomenological description of quarks, mesons, baryons, and other relativistic matter. Since our phenomenological actions are easily made manifestly symmetric under the full symmetry group of a system, it may also give more workable low energy effective theories for the standard model, such as chiral perturbation theory (e.g., ECKER [10, 11]) or quantum hadrodynamics (e.g., SEROT [39], SEROT & WALECKA [39, 40]).

The relation between the above action-based relativistic multiparticle models and the field-theoretic models discussed earlier is not clear at present. It is expected that the projection techniques from NEUMAIER [28] relate the field theories from Section 9 to corresponding effective N-particle theories modeled as in the present section. On the other hand, it is also conceivable that the field theories should rather be regarded as limits of N-particle theories in the thermodynamic limit $N \to \infty$. There are indications that this might be the case for QED (since radiation phenomena are always dissipative) and for gravitation (since black hole thermodynamics changes pure states to mixed states, cf. WALD [46, pp. 180–185]; the traditional coupling to a hydrodynamic model is also meaningful only in a thermodynamic limit).

13 Constrained Schrödinger equations

States of fixed total 4-momentum $J(p)$ can be obtained by solving (13). With a Lorentz boost, we may transform the system to a rest frame; the resulting constraint $J(\mathbf{p}) = 0$ can be imposed kinematically by restricting the 4-velocity to $\mathbf{u} = 0$. Since c and $J(mc^2)$ are constants, the wave function is an eigenstate of the **rest frame energy** $J(p_0c - mc^2)$ (a shifted relativistic energy p_0c, introduced in analogy to the prelude), and we are left with the (still rotation invariant) **constrained Schrödinger equations**

$$\psi = \delta(\mathbf{u})\psi_0, \quad L\psi = 0, \quad J(p_0c - mc^2)\psi = E\psi, \tag{32}$$

the relativistic analogue of the nonrelativistic multiparticle Schrödinger equation after separation of the motion of the center of mass. Thus our phenomenological approach is a covariant version of the situation in the prelude: The mass shells form 3-dimensional manifolds, and the momenta p^a can be considered as relativistic analogues of 3-momentum vectors. Since $\mathbf{u} = 0$, the 4-velocity contributes only one additional degree of freedom u_0, which replaces the energy degree of freedom of the nonrelativistic situation. Thus, in contrast to the realizations of quantum field theory discussed above, to traditional Bethe-Salpeter equations, and to proper time based relativistic multiparticle dynamics (see, e.g., FANCHI [12]), there are no superfluous degrees of freedom, but the treatment is still manifestly covariant.

The delta function in the interaction (29) forces $q_0 = 0$. Dropping the redundant coordinates $\mathbf{u} = 0$, $p_0 = \sqrt{(mU_0)^2 + \mathbf{p}^2}$ and $q_0 = 0$ from the notation, the interaction can be written as the 3-dimensional integral

$$(V^{ab}\psi)(u_0, p^a, p^b) = c^{-1} \int d\mathbf{q}\ U^{ab}(\mathbf{q})\psi(u_0, \mathbf{p}^a + \mathbf{q}, \mathbf{p}^b - \mathbf{q}); \tag{33}$$

the prefactor comes from the delta function in (29). If we now Fourier transform in space to get the position representation,

$$\widehat{\psi}(u_0, \mathbf{x}^a, \mathbf{x}^b) = \int d\mathbf{p}^a d\mathbf{p}^b e^{\iota \mathbf{p}^a\cdot \mathbf{x}^a} e^{\iota \mathbf{p}^b \cdot \mathbf{x}^b} \psi(u_0, \mathbf{p}^a, \mathbf{p}^b),$$

we find

$$\widehat{V^{ab}\psi}(u_0, \mathbf{x}^a, \mathbf{x}^b) = \widehat{U^{ab}}(\mathbf{x}^b - \mathbf{x}^a)\widehat{\psi}(u_0, \mathbf{x}^a, \mathbf{x}^b)$$

with the spatial potential

$$\widehat{U^{ab}}(\mathbf{r}) = c^{-1} \int d\mathbf{q}\ e^{\iota \mathbf{q}\cdot\mathbf{r}} U^{ab}(\mathbf{q}). \tag{34}$$

This looks like a nonrelativistic formula, but the covariant nature of the model is visible in the form (29) of $U^{ab}(q)$ and also shows in the constraint nature of (32). Compared to the nonrelativistic case, this is now a general linear eigenvalue problem for the eigenvalue E, and its solution is slightly more demanding. But numerical methods are available; see, e.g., Golub & van Loan [13].

The nonrelativistic limit. To deepen the analogy, we give a rough, heuristic derivation of the nonrelativistic limit $c \to \infty$; it would be interesting to have a rigorous version of this from which one can obtain error bounds. The equation $L\psi = 0$ can be written (for bosons) as $J(\frac{1}{2}m(u^2 - c^2))\psi = V\psi$. For small potential energies, $V \ll J(m)c^2$ and small spatial momenta, $p^2 \ll (mc)^2$, this gives $u^2 = c^2 + O(1)$, hence $p^2 = (mu)^2 = (mc)^2 + O(1)$ and $p_0 = \sqrt{(mc)^2 + \mathbf{p}^2} = mc + O(c^{-1})$. Therefore,

$$\frac{p_0^2 - (mc)^2}{2m} = (p_0 - mc)\frac{p_0 + mc}{2m} = (p_0 - mc)(c + O(c^{-1})) = p_0 c - mc^2 + O(c^{-2}),$$

$$\begin{aligned} L &= J\Big(\frac{p^2 - (mc)^2}{2m}\Big) = J\Big(\frac{p_0^2 - (mc)^2}{2m}\Big) - J\Big(\frac{\mathbf{p}^2}{2m}\Big) \\ &\approx J(p_0 c - mc^2) - J(\mathbf{p}^2/2m) = E - J(\mathbf{p}^2/2m) \end{aligned}$$

with the rest frame energy $E = J(p_0 c - mc^2)$. Thus, the constraint Schrödinger equation reduces in the nonrelativistic limit to the standard Schrödinger equation for a multiparticle system with Hamiltonian

$$H = J(\mathbf{p}^2/2m) + \sum_{a<b} \widehat{U^{ab}}(\mathbf{x}^b - \mathbf{x}^a).$$

Arbitrary local pair interactions can be obtained in the nonrelativistic limit by choosing U^{ab} appropriately (and in a non-unique way). For larger kinetic energies, the potential in position space acquires additional, nonlocal terms (that can be approximated using derivatives in the interaction). Thus we have a flexible covariant theory with a good nonrelativistic limit.

In particular, from the covariant potential (28) with pair interactions of the form (30), we recover in the nonrelativistic limit the **standard nonrelativistic multiparticle dynamics** in the presence of electromagnetic and gravitational forces.

14 Golden opportunities

> *Behind it all is surely an idea so simple, so beautiful, that when – in a decade, a century, or a millennium – we grasp it, we will all say to each other, how could it have been otherwise?*
> John Archibald Wheeler, 1987 [51]

> *Eine mathematische Theorie ist nicht eher als vollkommen anzusehen, als bis du sie so klar gemacht hast, daß du sie dem ersten Manne erklären könntest, den du auf der Straße triffst.*
> David Hilbert, 1900 [14]

I do not know whether the perfection requested by Hilbert can be achieved in deep theories. But, having discovered the unexpected beauty of the present approach, I hope that the insights presented will contribute to the perfection of quantum field theory.

In 1972, Freeman DYSON [9] gave a lecture called "*Missed opportunities*", where he talked "*about the contribution that mathematics ought to have made*" to physics "*but did not*". I believe the present contribution widely opens the door for mathematicians to contribute to quantum field theory, and creates golden opportunities for those interested in mathematical physics.

The present setting gives a mathematically consistent point of view from which to study the laws of physics, which complements the point of view taken by past history. On the new basis, it is likely that scientists will resolve in the near future the most basic challenges current theoretical physics poses to mathematicians and mathematical physicists:

- the existence of QED and derivation of its properties,
- bound states and resonances in quantum field theories,

- a unified quantum field theory of all forces of nature,
- the existence and mass gap in quantum Yang-Mills theory – one of seven Clay millenium prize problems [3], a golden opportunity in the most concrete sense.

15 Thanks

It is a great pleasure for me to be able to participate in the revelation of the laws the Creator has built into our universe. I want to thank God for the call, vision, open-mindedness, patience, persistence and joy I got (and needed) for going successfully through the journey in the platonic world of precise ideas (that, for a long time, appeared to me all too foggy in the regions where quantum field theory is located) that lead to the results presented here.

I also want to thank the maintainers of (and the contributors to) the Los Alamos National Laboratory e-Print archive for this wonderful on-line source containing most physics manuscripts of the last few years. It saved me many hours of work by giving me quick access to the many thousands of papers that I glanced at, leaved through, or read more thoroughly while searching for the path to success.

I'd like to thank Dr. Hermann Schichl (Wien) for many discussions on various pieces of the puzzle that helped me to clarify my thoughts. Thanks also to Prof. Peter Michor (Wien) who pointed me to the book by da Silva & Weinstein [5] on Poisson algebras, to Prof. Walter Thirring (Wien) for his treatise on mathematical physics which I used over and over again, to Prof. Gerhard Ecker (Wien) for useful discussions on quantum field theory long ago, and to Prof. Hartmann Römer (Freiburg), who introduced me many years ago to the idea that an elementary particle 'is' [41, p.149] an irreducible unitary representation of the Poincaré group.

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