

Renormalization without infinities

– an elementary tutorial

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Abstract. Renormalization is an indispensable tool for modern theoretical physics. At the same time, it is one of the least appealing techniques, especially in cases where naive formulations result in divergences that must be cured – a step that is often done in a mathematically dubious way.

In this paper, it is shown how the renormalization procedure works both in singular cases where it removes naive divergences and in regular cases where a naive approach is possible but renormalization improves the quality of perturbation theory. In fact, one can see immediately that the singular situation is simply a limiting case of the regular situation.

After discussing generalities, the paper introduces a large family of toy examples, defined by special perturbations of an arbitrary Hamiltonian with a discrete spectrum. The examples show explicitly many of the renormalization effects arising in realistic quantum field theories such as quantum chromodynamics: logarithmic divergences, running couplings, asymptotic freedom, dimensional transmutation, the renormalization group, and renormalization scheme dependent results at any order of perturbation theory.

Unlike in more realistic theories, everything is derived rigorously and nonperturbatively in terms of simple explicit formulas. Thus one can understand in detail how the infinities arise (if they arise) – namely as an unphysical infinitely sensitive dependence on the bare coupling constants. One also sees that all spurious infinities are cured automatically by the same renormalization process that gives robust physical results in the case where no infinities occur.

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Part I

General observations

1 Introduction

Renormalization (see, e.g., COLLINS [11]) is the fundamental technique that makes relativistic quantum field theories – such as quantum electrodynamics (QED), quantum chromodynamics (QCD), and the standard model – work. Thus it is one of the cornerstones of current theoretical physics. But at the same time, it is one of the least appealing techniques since the conventional textbook presentations result in divergences that must be cured – a step that is often done in a mathematically dubious way. (An important exception, which derives the perturbation theory for QED without meeting a single ultraviolet divergence, is the causal approach to quantum field theory; see GRIGORE [29] for an overview, and for details the book by SCHARF [49].)

Less well known is the fact that the same type of divergences also arise in the naive treatment of much simpler quantum systems, if these have a singular Hamiltonian. Renormalization is even important when there are no singularities, as in the case of the quartic anharmonic oscillator; see, e.g., ZAMASTIL et al. [65]. It is also relevant for purely classical theories, see, e.g., GALLAVOTTI [22].

The literature contains many discussions of singular Hamiltonians and their renormalization at many different levels of rigor; let me recommend DELAMOTTE [12], GLAZEK & WILSON [25], RAJEEV [31, 43], and WEGNER [58, 59]. An older but comprehensive review by FRANK & LAND [19] includes many nonrelativistic examples of physical interest. A recent thesis by GOPALAKRISHNAN [28] gives an overview of the current state of affairs for systems with few degrees of freedom. An exposition of more advanced, action-based non-perturbative (approximate) renormalization techniques in quantum field theory was given by DELAMOTTE [13].

The present paper benefitted from the above expositions, especially from [12], which complements our treatment by emphasizing the perturbative aspects. We address the renormalization problem in a different manner. We aim at a nonperturbative and mathematically rigorous understanding of renormalization, explaining the miraculous cancellations in the traditional approach rather than taking them for a happy accident that saves us from infinities.

By putting the basic approach into the context of finite renormalization (where there are no divergences at all), it is shown how the renormalization procedure works, using a class of toy problems chosen because of their calculational simplicity and tractability. Unlike in more realistic theories, everything can be derived rigorously and nonperturbatively in terms of simple explicit formulas. Thus one can understand in detail how the infinities arise (if they arise) – namely as an unphysical infinitely sensitive dependence on the bare coupling constants. One also sees that all spurious infinities are cured automatically by the same

renormalization process that gives physical results in the case where no infinities occur. By working with discrete spectra only we avoid the additional technical problems involved in particle scattering.

Our model Hamiltonians have a direct physical significance only in a continuum limit (a particular sequence of models converges to the well-known singular delta function potential), and hence have the status of toy models only. But they show many of the renormalization effects arising in realistic quantum field theories such as quantum chromodynamics: running coupling constants, asymptotic freedom, dimensional transmutation, the renormalization group, renormalization scheme dependent results at any order of perturbation theory.

One can see from the present study that, in our class of problems, all fully renormalized formulas derived in the regular case can be used in the singular case without any change. The divergences are therefore only due to having started with a singular parameterization of the family of models. Precisely the same is known or expected to hold for all renormalizable quantum field theories, since only this can explain the often observed fact that very different renormalization schemes lead to the same answers. However, in these not exactly solvable cases, one cannot expect to find an explicit regular reparameterization.

The paper is organized as follows. Familiarity with elementary quantum mechanics including second order perturbation theory is assumed throughout. By restricting attention mainly to three classes of explicitly solvable toy models (type A in Section 3, type B in Section 7, and type C in Section 12), things are kept elementary. In particular, no background in quantum field theory is required, although notions from quantum field theory are alluded to for motivation and to provide some relevant context.

The first part discusses generalities valid for arbitrary Hamiltonians. Section 2 introduces the basic concept of renormalization in a general informal context, and Section 3 applies it to the simplest of all quantum systems, a 2-state system (type A model). Section 4 introduces general aspects of perturbation theory, and Section 5 gives details of second-order perturbation theory in a general framework suitable for both the bare and the renormalized case. Section 6 introduces the basic difficulty – ultraviolet divergences when the Hamiltonian is singular.

The second part discusses explicit formulas for certain solvable models. Type B models are introduced in Section 7, and Section 8 derives their explicit nonperturbative renormalized solution. Up to this point, everything is completely straightforward and free from conceptual problems. Section 9 extends our discussion to a singular version of type B models and shows that we get a rigorous nonperturbative renormalization of the spectrum of the singular model, using cutoff regularization and a controlled limit involving only physically relevant parameters. Remarkably, the renormalized formulas for the regular case hold without change in the singular case, as long as a simple renormalizability condition is satisfied. We also encounter the phenomenon of dimensional transmutation characteristic for renormalizable field theories with dimensionless coupling constants. Section 10 shows that one can explicitly calculate the renormalized solution in terms of a running coupling constant. The freedom in the choice of the running coupling constant translates into the renormalization group equation, a differential equation which takes here an exact and explicitly

solvable form. We then treat in Section 11 Type B models using perturbation theory and find that, for renormalizable singular interactions, renormalized perturbation theory gives finite results, while naive perturbation theory leads to divergent expressions at second order. Section 12 introduces type C models, a variant of Type B that exhibits the phenomenon of mass renormalization. The concluding Section 13 summarizes the advantages of the presentation of renormalization along the present lines.

Explicitly excluded from this paper as too technical are models of quantum field theory (QFT). Instead, we provide a few pointers to the literature related to solvable models and renormalization in QFT. There are a number of simple nonrelativistic QFT models where renormalization issues can be studied in solvable sectors; see, e.g., LEE [38], CHIU et al. [10], VARMA & SUDARSHAN [56]. There are also many exactly solvable local relativistic QFTs in 1+1 dimensions (time and one space dimension), see, e.g., SUTHERLAND [53] or ABDALLA et al. [1]; they are closely related to Yang-Baxter equations and quantum groups; see, e.g., FUCHS [21], MAJID [39]. However, the renormalization of local relativistic QFTs in 1+1 dimensions is quite simple since (GLIMM & JAFFE [27], BAEZ et al. [4]) it just amounts to shifting the Hamiltonian by normal ordering (WILCOX [62]) and deleting the constant term. In higher dimensions, the only known exactly solvable QFTs are linear, and renormalization is much more technical; see COLLINS [11] for the traditional version, and SALMHOFER [47] for a mathematically rigorous approach.

A nice, multifaceted view of the renormalization of quantum field theories in the context of mathematics and physics in general is given in Sections 3.2, 11.5, and 15.4 of ZEIDLER [66]. A history of the concept and philosophy of renormalization in quantum field theory can be found in the book by BROWN [6].

Throughout the paper, we use units such that Planck's constant \hbar and the speed c of light both have the value 1, unless c or \hbar are explicitly mentioned. In these units, the well-known relations $E = mc^2$ and $E = \hbar\omega$ imply that (angular) frequencies ω are the same things as masses m and energies E .

2 The concept of renormalization

In quantum mechanics and quantum field theory, the dynamics of a stationary system is generally given by a **Hamiltonian** H , the infinitesimal generator of time translations of the system. Any particular real quantum system can be described to experimental accuracy by many different Hamiltonians. The typical procedure is to take a model Hamiltonian $H(\theta)$ parameterized by a vector θ of parameters defining particular instances of the model. By solving the Schrödinger equation (or related dynamical equations) analytically or numerically, one obtains predictions $v(\theta)$ for a vector v of experimentally accessible numbers. The quality of the model is then determined by how well one can match v with $v(\theta)$ for a suitable choice of θ , usually determined by some form of data fitting.

Different models differ in the form of the model Hamiltonian, often also in the number of parameters comprising θ . Therefore, fitting the experimental data v to the computational

results $v_j(\theta_j)$ computed from different model Hamiltonians $H_j(\theta_j)$ ($j = 1, 2, \dots$) lead in general to estimated parameter vectors θ_j that may have very little to do with each other. In particular, the parameters have no intrinsic physical relevance and have meaning only in the context of the specific model under investigation.

This explains for example the fact that the physically unobservable quark masses parameterizing quark models come out very different in different quark models – compare, e.g., the masses in RÜSTER et al. [46] and SCADRON et al. [48]. The official tables of quark masses from the Particle Data Group [3] must also be viewed in the context of a specific model; [40] explicitly warns: *”Although one often speaks loosely of quark masses as one would of the mass of the electron or muon, any quantitative statement about the value of a quark mass must make careful reference to the particular theoretical framework that is used to define it. It is important to keep this scheme dependence in mind when using the quark mass values tabulated in the data listings.”*

To take a specific example, we consider the Hamiltonian

$$H(\theta) = \frac{p^2}{2m} + V(q, \theta),$$

describing a single (bosonic) quantum degree of freedom. Here θ is a parameter vector of bare coefficients, and p, q are Hermitian (momentum and position) operators satisfying the **canonical commutation rules (CCR)**

$$qp - pq = i\hbar,$$

where \hbar is Planck’s constant. To study the discrete part of the spectrum, one often transforms the Hamiltonian into one involving annihilation and creation operators a and a^* . This can be done by defining for an arbitrary nonzero real number α and arbitrary real numbers p_0 and q_0 the operator

$$a := \frac{q - q_0}{\alpha} + \frac{i\alpha(p - p_0)}{2\hbar}$$

which satisfies the ”second-quantized” commutation rules

$$aa^* - a^*a = 1. \tag{1}$$

According to standard reasoning, this implies that we can use a basis of ladder states $|k\rangle$ ($k = 0, 1, 2, \dots$) satisfying

$$a|k\rangle = \sqrt{k} |k - 1\rangle, \quad a^*|k\rangle = \sqrt{k + 1} |k + 1\rangle.$$

We may eliminate p and q from the expression for H using

$$p = p_0 + \frac{\hbar}{\alpha i}(a - a^*), \quad q = q_0 + \frac{\alpha}{2}(a + a^*),$$

and obtain a ”second-quantized” form of the Hamiltonian,

$$H(\theta) = H(a^*, a, \theta).$$

We note that many different parameterizations of a polynomial expression in a^* and a describe precisely the same physical situation. Indeed, the above freedom in the definition of a amounts to performing on the Hamiltonian a linear **Bogoliubov transformation** $a \rightarrow \hat{a}$, defined by

$$\hat{a} = \alpha_0 + \beta_0 a + \gamma_0 a^*, \quad \hat{a}^* = \bar{\alpha}_0 + \bar{\beta}_0 a^* + \bar{\gamma}_0 a,$$

where $|\beta_0|^2 - |\gamma_0|^2 = 1$ (so that the commutation rules (1) are preserved), the polynomial degree is preserved but the lower and higher order terms in the new creation and annihilation operators are completely mixed up. As a result, the coefficients can have very different values depending on the choice of the transformation. On the other hand, a Bogoliubov transformation is unitary and hence does not change the physics. In particular, the observable information – encoded in energy differences of discrete spectra and in scattering angles computed from the continuous spectrum – remains unaltered. The possibility of performing arbitrary Bogoliubov transformations also shows that neither a nor \hat{a} can have any physical significance in themselves. More general Bogoliubov transformations are routinely used in statistical physics. They are basic for understanding effects such as superconductivity; see, e.g., NAMBU [41].

Of course, the physics contained in the models must be independent of the parameters that happen to be used in particular models. In many cases of interest, the experimental data can be empirically described in terms of a few physical key parameters, such as basic observable masses and charges. These are generally different from the mass and charge coefficients that appear in particular models. To distinguish these in a general context, we therefore refer to the model-dependent coefficients – such as the quark masses mentioned above – as **bare parameters** and to the model-independent parameters chosen for the physical parameterization – measurable masses, charges, etc., related directly to experiment – as **renormalized** or **dressed** parameters. (As bare persons must usually be dressed to be socially acceptable, so bare parameters must usually be dressed – or, in modern terminology, renormalized – to be physically acceptable.)

The purpose of renormalization is to reparameterize a given family of Hamiltonians in such a way that one can match physical parameters in a numerically robust way. The ideal case is the direct parameterization by a physical parameter; often, however, one reparameterizes in terms of intermediate parameters that are sensitive to physical parameters and allow one to determine the latter by means of simple renormalization conditions. Note that – just as a gauge transformation – a reparameterization does not change the physics, only the form the calculations take.

Writing the renormalized parameter vector as θ_{ren} , we can express to experimental accuracy both θ_{ren} (which is part of $v(\theta)$) as a function of θ ,

$$\theta_{\text{ren}} = T_{\text{ren}}(\theta)$$

and θ as a function of θ_{ren} (via the data fit),

$$\theta = T(\theta_{\text{ren}}).$$

To experimental accuracy, the two transformations T and T_{ren} are inverse to each other. Clearly, the physics is completely contained in the **renormalized Hamiltonian**

$$H_{\text{ren}}(\theta_{\text{ren}}) := H(T(\theta_{\text{ren}})).$$

More precisely, since two Hamiltonians differing only by a unitary transformation describe the same physics (after a corresponding transformation of the states), the physics is determined by the equivalence class of Hamiltonians unitarily equivalent to H_{ren} . In statistical mechanics and quantum field theory, the process of going from H to H_{ren} (or a unitarily equivalent Hamiltonian) is conventionally called **renormalization**.

3 Type A: A 2-state system

Before treating Hamiltonians that resemble those for real systems, we illustrate the basic principle with the simplest of all quantum systems, a 2-state system with a 2-dimensional Hilbert space $\mathbb{H} := \mathbb{C}^2$. Let $\omega > 0$ be the excitation frequency for moving the unperturbed system from the ground state to the excited state. This is described by the unperturbed Hamiltonian

$$H_0 = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}.$$

Suppose that under some force of strength g , the system behaves according to the perturbed Hamiltonian

$$H_\Lambda = H_0 + gV_\Lambda$$

with an external potential of the form

$$V_\Lambda := \begin{pmatrix} -1 & \Lambda \\ \Lambda & 0 \end{pmatrix},$$

where $\Lambda > 0$ is a very large number simulating what in more realistic cases will be an energy cutoff. This defines our **Type A model**. We can easily work out the eigenvalues of

$$H_\Lambda = \begin{pmatrix} -g & g\Lambda \\ g\Lambda & \omega \end{pmatrix}; \quad (2)$$

they are the zeros of the characteristic equation $0 = \det(E - H_\Lambda) = (E + g)(E - \omega) - g^2\Lambda^2$, so that

$$E_{0,1} = \frac{1}{2} \left(\omega - g \pm \sqrt{(\omega + g)^2 + 4g^2\Lambda^2} \right). \quad (3)$$

If Λ is very large, the formulas are extremely sensitive to changes in g since these are magnified by the factor Λ . *Such an extreme sensitivity is characteristic for a poor choice of parameterization.* We therefore consider the parameters ω and g as bare parameters in our physical model, and look for a better parameterization. One possibility is to use as parameters physical data; most natural are in the present case the renormalized energies E_0 and E_1 . Since

$$E_0 + E_1 = \omega - g, \quad E_0 E_1 = -g\omega - g^2\Lambda^2,$$

we find that $\omega = E_0 + E_1 + g$ and g must be a solution of the quadratic equation

$$(\Lambda^2 + 1)g^2 + (E_0 + E_1)g + E_0 E_1 = 0.$$

Thus g and ω depend on Λ , and one sees that, given physically reasonable values of E_0 and E_1 , we have $g = O(\Lambda^{-1})$ when Λ is large.

This form of the dependence suggests a simpler reparameterization in terms of a **renormalized coupling constant** $g_{\text{ren}} := g\Lambda$. Eliminating the **bare coupling constant** g in favor of g_{ren} , we obtain the family of **renormalized Hamiltonians**

$$H_{\Lambda}^{\text{ren}} = \begin{pmatrix} -g_{\text{ren}}\Lambda^{-1} & g_{\text{ren}} \\ g_{\text{ren}} & \omega \end{pmatrix}. \quad (4)$$

Except for the reparameterization, the renormalized family of Hamiltonians is exactly equivalent to the original family of Hamiltonians; Every physical situation that can be described by a member of the original family can also be described as a member of the renormalized family, and conversely.

The renormalized Hamiltonians behave perfectly for arbitrarily large Λ , since the sensitivity under changes of g_{ren} decreases when Λ increases. Indeed, we can even perform the limit $\Lambda \rightarrow \infty$, and obtain a well-defined limiting Hamiltonian

$$H_{\infty}^{\text{ren}} := \lim_{\Lambda \rightarrow \infty} H_{\Lambda} = \begin{pmatrix} 0 & g_{\text{ren}} \\ g_{\text{ren}} & \omega \end{pmatrix}.$$

This limiting renormalized Hamiltonian has the spectrum

$$E_{0,1} = \frac{1}{2} \left(\omega \pm \sqrt{\omega^2 + 4g_{\text{ren}}^2} \right), \quad (5)$$

determined from the characteristic equation $0 = \det(E - H_{\infty}^{\text{ren}}) = E(E - \omega) - g_{\text{ren}}^2$. On the other hand, in the original parameterization, one cannot take the limit $\Lambda \rightarrow \infty$ without getting meaningless infinities.

Thus we derived *the basic idea of renormalization: It consists in reparameterizing a given family of Hamiltonians in a way that reduces the sensitivity with respect to the new, renormalized parameters*. Renormalization is a practical necessity whenever a family of model Hamiltonians depends on a very large parameter Λ (a cutoff energy or – in thermodynamical calculations – a volume) and additional bare parameters, in a way that some results computed from the model that can be compared with experiments are extremely sensitive to changes in the bare parameters. If the model Hamiltonians match a physical situation, such an extreme sensitivity is an artifact of the particular parameterization. By expressing the bare parameters as appropriate functions of Λ and additional renormalized parameters, one can usually reduce the sensitivity to a meaningful level. Moreover, in the renormalized family, one can often take the limit $\Lambda \rightarrow \infty$, thereby simplifying the model and getting rid of a parameter that, in the renormalized version, hardly affects the results. The specific form of the cutoff-dependence can be freely chosen as long as certain renormalization conditions are respected that guarantee that the limit can safely be taken after renormalization.

The renormalization procedure is especially important in cases where the physics is modelled by a system in which arbitrarily large energy scales play a role, although the model is correct only for small energies. In this case, the true physics is determined by an approximation in which large energy contributions beyond an energy scale Λ is cutoff, but the precise value of the cutoff is considered irrelevant as the results should not depend significantly on the value

of the cutoff scale. However, the fit to the bare parameters is typically strongly dependent on Λ , a clear sign for the necessity of a reparameterization.

The ability to take the limit $\Lambda \rightarrow \infty$ in the reparameterized model is proof of the irrelevance of the cutoff scale, and characterizes a successful renormalization. But at fixed renormalized parameters, the results are in this case physically indistinguishable for large enough Λ , so that it is irrelevant whether one takes the limit or keeps Λ finite but very large. This is in sharp contrast to what happens at fixed bare parameters, where magnifying a large Λ by another large factor drastically changes the observable results. Since a precise value of the cutoff is usually impossible to obtain (whether from theory or from a fit to experimental data), we conclude that the renormalized parameters are the ones that carry the relevant experimental information.

Note that in our bare 2-state model, the limit of the potential does not exist as an operator on \mathbb{H} , and the limit of the coupling constant must be zero if we want to maintain finite energies. So the limiting problem is singular, analogous to more realistic singular interactions present in QED or QCD. In these realistic theories, and in contrast to our 2-state model, the unperturbed system and the postulated forces on it (defined by the interaction) are unobservable, hence fictional, chosen simply in order to define the phenomenologically successful full Hamiltonian. Only the full Hamiltonian makes physical sense; in the case of QED, it is amply justified by the fact that the renormalized results computed from the Hamiltonian match the corresponding experimental results to extremely high accuracy. Note that the well-known divergent expressions one gets for QED from naive (bare) second-order perturbation theory have a correspondence in our 2-state problem, where in the bare setting, the two eigenvalues E_0 and E_1 diverge to $\pm\infty$ as $\Lambda \rightarrow \infty$.

The limiting situation becomes in some sense exact for problems where the natural formulation is singular although the physical system does not exhibit a singularity. One may compare the reparameterization that lifts the singularity by renormalization to the coordinate transformation from polar coordinates to Cartesian coordinates. The singularity of polar coordinates in the origin is removed in Cartesian coordinates, and shows up instead in the singular behavior of the coordinate transformation.

4 Perturbation theory

Since realistic models are usually not exactly solvable, one usually turns to some form of perturbation theory to compute approximations to the results of interest. In perturbation theory, one chooses a free (or at least an explicitly solvable) **reference Hamiltonian** $H_0 := H_0(\mu)$, and defines the **interaction**

$$V := V(\theta, \mu) = H(\theta) - H_0(\mu).$$

Hoping that the effect of the interaction is small, one expands the numbers of interest into a perturbation series around the corresponding numbers computed from the reference Hamiltonian.

For example, expansion of the eigenvalues (3) of the Hamiltonian of the 2-state model (with $\theta = \left(\frac{\omega}{g}\right)$ and $\mu = \omega$) gives for small g :

$$E_0 = -g - \frac{g^2 \Lambda^2}{\omega} + O(g^3), \quad E_1 = \omega + \frac{g^2 \Lambda^2}{\omega} + O(g^3),$$

Note that although g is small, the second order term is small only if $g \ll \Lambda^{-1}$; thus the validity of the expansion restricts g to very small values, and in the limit of an infinite cutoff, any nonzero g gives meaningless results. This is precisely the situation that occurs when applying perturbation theory naively to field theories like QED. However, after renormalization, i.e., when expanded in the renormalized coupling constant g_{ren} , the eigenvalues (5) satisfy

$$E_0 = -\frac{g_{\text{ren}}^2}{\omega} + O(g_{\text{ren}}^4), \quad E_1 = \omega + \frac{g_{\text{ren}}^2}{\omega} + O(g_{\text{ren}}^4),$$

showing that the renormalized model behaves well in the sense of perturbation theory. Thus a lot depends on the proper choice of the parameters in which perturbation theory is carried out – the naive choice may be a very poor choice!

Naively – and especially in most introductory discussions – one chooses the reference parameters comprising μ simply as a subset of the list of bare parameters, and the reference Hamiltonian by simply setting the remaining parameters equal to zero. But this is usually a poor choice; the only exceptions are traditional textbook examples where the ignored parameters are tiny and the resulting interaction V is relatively compact with respect to H_0 .

In general, the freedom to choose μ arbitrarily can be exploited by following a very successful approach to quantum computations called **variational perturbation theory** – see, e.g., KLEINERT [36, 35, 34], BUCKLEY et al. [7] –, where, for anharmonic oscillators, the value of μ is considered to be an adjustable parameter in an approximative perturbative analysis, which is optimized to get best approximate results. The approach is more general, however, being based on a very general **principle of minimal sensitivity** of STEVENSON [52]. This principle asserts that, for any computation in which the exact results would be independent of μ , the best choice of μ in a corresponding approximate calculation should be the one which minimizes the sensitivity of the results to changes in μ – an expectation frequently born out in practice. The optimal choice is taken to be either a stationary point or an inflection point; the resulting approximations are often *much* better than those from the naive approach.

Note that the application of the principle of minimal sensitivity depends on the quantity Q (e.g., an energy difference) whose sensitivity is made small. There are cases where the principle breaks down. This happens when Q depends on μ in a way that Q has neither a stationary point nor an inflection point. In these cases, some parameters in μ effectively behave as new degrees of freedom for the approximate theory, and must be determined (as θ) by matching experimental data.

In general, at every order in perturbation theory, the results depend on the choice of both θ and μ . These are basically independent sets of parameters (although they may be chosen to partially overlap): θ selects the specific instance of the model, while μ should be chosen

primarily with the aim of getting a good approximate perturbative solution. Reasonable results can be expected only if H and H_0 have a related spectrum; in particular, the spectra must have the same topological structure in the energy range of interest. This dictates the choice of H_0 .

For example, in a scattering experiment one expects a continuous spectrum, and often chooses a multiple of the squared momentum, $H_0 = -\mu(a - a^*)^2$, where μ is proportional to a bare mass. On the other hand, for an anharmonic oscillator far below the dissociation threshold (if one exists), one expects a discrete spectrum and usually chooses a harmonic oscillator Hamiltonian $H_0 = \mu a^* a$, where μ is a bare frequency. Apart from being consistent with the topology of the spectrum, which requires in both cases that $\mu \geq 0$, the choice of μ is in both cases completely arbitrary. But different values of μ may lead to very different results in a truncated perturbative expansion. This seems counterintuitive at first since after summing the full perturbation theory (assuming it converges), the results must of course be those for H . They are therefore obviously independent of parameters in the more or less arbitrary reference Hamiltonian H_0 . However, since truncating two different series with the same limit generally gives different results, the choice of μ matters in perturbation theory at finite order.

In the case of a single quantum degree of freedom with discrete spectrum $E_0 < E_1 < \dots$, what is measurable (and therefore physical) are energy differences such as the excitation energy $\epsilon := E_1 - E_0$. If the model Hamiltonian contains only a single bare parameter, we may choose the excitation energy as the single renormalized parameter, $\theta_{\text{ren}} = \epsilon$. The renormalized Hamiltonian is then expressible as a function $H_{\text{ren}}(\epsilon)$ of the excitation energy. Note that for any real-valued function f , the operator $H_0 = f(a^* a)$ has the eigenvalues $E_j = f(j)$ ($j = 0, 1, 2, \dots$) and corresponding eigenvectors $|j\rangle$, hence can serve as an explicitly solvable reference Hamiltonian $H_0 = f(a^* a)$. For example, using $H_0 = \omega_0 a^* a + E_0$ as reference Hamiltonian, we have several free parameters (μ, p_0, q_0 from the transformation to second-quantized form, ω_0 , and E_0) for a subsequent variational perturbation theory.

5 Renormalized perturbation theory

In this section, we give an elementary presentation of second-order perturbation theory, in a generalized setting that allows us to treat within the same formalism both naive perturbation theory (as presented in most textbooks) and the renormalized version.

The Hamiltonian $H = H(z)$ may be an arbitrary infinitely differentiable operator-valued function of a perturbation parameter z . In the typical (naive, bare) textbook treatment, z is taken to be a coupling constant g , but in all but the simplest cases, alternative choices are better, and may provide the renormalization needed for getting meaningful approximate results. Moreover, in cases where there are multiple coupling constants, it is not clear what would be the most natural expansion parameter, so not tying z to a coupling constant is appropriate.

The motivation and primary reason for choosing z differently from a coupling constant is

the fact that the perturbative expansion of the same expression may have a very different behavior depending on the choice of the perturbation parameter. For example, the power series expansion

$$\log(1 + g) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} g^k$$

converges only for $|g| < 1$, but if we expand in terms of $z = g/(g+2)$, so that $g = 2z/(1-z)$ we get

$$\log(1 + g) = \log\left(\frac{1+z}{1-z}\right) = \sum_{k=0}^{\infty} \frac{2}{2k+1} z^{2k+1},$$

which converges for $|z| < 1$, hence in the much larger domain $\text{Re } g > -1$. Moreover, convergence is typically much faster. As observed by WEINBERG [60], one can obtain convergence of the perturbative expansion of *arbitrary* two-body problems by expanding in terms of a z related to g by means of a suitable conformal transformation chosen to avoid the complex singularities that limit the convergence of a power series.

We assume that the Hamiltonian $H = H(z)$, some eigenvalue $\epsilon(z)$, and the associated eigenvector $\psi(z)$ can all be expanded into power series in z ,

$$H(z) = \sum_{j=0}^{\infty} z^j H_j, \quad \epsilon(z) = \sum_{j=0}^{\infty} z^j \epsilon_j, \quad \psi(z) = \sum_{j=0}^{\infty} z^j \psi_j.$$

Since ψ is determined only up to an arbitrary scalar multiple that may depend on z , we may demand without loss of generality that

$$\psi_0^* \psi_j = 0 \quad \text{for } j = 1, 2, \dots \quad (6)$$

Substituting the power series into the eigenvalue equation $(\epsilon(z) - H(z))\psi(z) = 0$ gives the equation

$$\left(\sum_{j'=0}^{\infty} z^{j'} (\epsilon_{j'} - H_{j'}) \right) \left(\sum_{j''=0}^{\infty} z^{j''} \psi_{j''} \right) = 0.$$

Multiplying out the sums and comparing the coefficient of z^j then gives the equations

$$\sum_{j'=0}^j (\epsilon_{j'} - H_{j'}) \psi_{j-j'} = 0 \quad \text{for } j = 0, 1, 2, \dots \quad (7)$$

Setting $j = 0$ gives the reference eigenvalue equation

$$(\epsilon_0 - H(0))\psi_0 = 0. \quad (8)$$

Thus, to get started, we need to know the spectrum of the **reference Hamiltonian**

$$H_0 := H(0),$$

or at least some of its eigenvalues and eigenvectors. For simplicity, we assume that H_0 has a discrete, nondegenerate spectrum; the general case is not much different but technically

more complicated (additional labels in the basis kets to distinguish degenerate states; integrals in place of sums to account for continuous spectra). We consider the diagonal representation of H_0 in a basis of normalized eigenstates $|k\rangle$ (for k in some label set K) with

$$H_0|k\rangle = E_k|k\rangle, \quad \langle k|k'\rangle = \delta_{kk'}.$$

We refer to the eigenvalues E_k and eigenvectors $|k\rangle$ of the reference Hamiltonian H_0 as **bare** eigenvalues and eigenvectors, and to those of the interacting Hamiltonian H as **renormalized** eigenvalues and eigenvectors. H_0 acts on the Euclidean space \mathbb{H} consisting of all vectors

$$\psi = \sum_k \psi_k |k\rangle \quad (9)$$

with only finitely many ψ_k nonzero. The orthogonality relations imply the standard inner product

$$\phi^* \psi = \sum_k \bar{\phi}_k \psi_k;$$

the sum is finite since almost all terms vanish. (We use the notation customary from linear algebra rather than Dirac's bra-ket notation, which would write this as $\langle \phi | \psi \rangle$.)

By completion, we get from \mathbb{H} the Hilbert space \mathbb{H}_{phys} of **square summable** vectors ψ with finite $\sum_k |\psi_k|^2$. Since the energies are unbounded, there are square summable vectors $(\psi_0, \psi_1, \psi_2, \dots)$ for which the vector $(E_0\psi_0, E_1\psi_1, E_2\psi_2, \dots)$ is no longer square summable. (For example, this is the case for $\psi_k = 1/(\alpha + E_k)$ when $\alpha > 0$ and $E_k = \omega k$ is the spectrum of the harmonic oscillator.) Therefore H_0 is not definable as a linear operator on \mathbb{H}_{phys} , but only on the dense subspace of \mathbb{H}_{phys} consisting of the vectors ψ for which $\sum |E_k \psi_k|^2$ converges. But as a densely defined operator, both H_0 and H are self-adjoint; hence $e^{itH/\hbar}$ exists for all t and we have a good quantum dynamics.

Having made our assumptions precise, we return to (8), and find (after rescaling ψ by a z -independent constant) that

$$\epsilon_0 = E_\ell, \quad \psi_0 = |\ell\rangle \quad (10)$$

for some $\ell \in K$. The condition (6) can now be written as

$$\psi_j = \sum_{k \in K} \psi_{jk} |k\rangle, \quad \psi_{j\ell} = 0. \quad (11)$$

For $j = 1$, we obtain from (7) the equation

$$(\epsilon_0 - H_0)\psi_1 + (\epsilon_1 - H_1)\psi_0 = 0,$$

which, using (11), has the unique solution

$$\epsilon_1 = \langle \ell | H_1 | \ell \rangle, \quad \psi_{1k} = \begin{cases} 0 & \text{if } k = \ell, \\ \frac{\langle k | H_1 | \ell \rangle}{E_\ell - E_k} & \text{otherwise.} \end{cases} \quad (12)$$

For $j = 2$, we obtain from (7) the equation

$$(\epsilon_0 - H_0)\psi_2 + (\epsilon_1 - H_1)\psi_1 + (\epsilon_2 - H_2)\psi_0 = 0. \quad (13)$$

We may write

$$H_2\psi_0 + H_1\psi_1 = \sum_{k \in K} \alpha_k |k\rangle,$$

where

$$\begin{aligned} \alpha_k &:= \langle k | H_2 \psi_0 + \langle k | H_1 \psi_1 = \langle k | H_2 | \ell \rangle + \sum_j \psi_{1j} \langle k | H_1 | j \rangle \\ &= \langle k | H_2 | \ell \rangle + \sum_{j \neq \ell} \frac{\langle k | H_1 | j \rangle \langle j | H_1 | \ell \rangle}{E_\ell - E_j}. \end{aligned}$$

Using (11), we find that (13) has the unique solution

$$\epsilon_2 = \alpha_\ell, \quad \psi_{2k} = \begin{cases} 0 & \text{if } k = \ell, \\ \frac{\alpha_k - \epsilon_1 \psi_{1k}}{E_\ell - E_k} & \text{otherwise.} \end{cases} \quad (14)$$

This determines the eigenvalue $\epsilon(z)$ and the eigenvector $\psi(z)$ with an error of order $O(z^3)$.

In many cases of interest (and the only case we consider later), the family of Hamiltonians is given in the form

$$H = Bc = \sum_s B^{(s)} c^{(s)} \quad (15)$$

with finitely many bare coefficients $c^{(s)}$ and corresponding operators $B^{(s)} \in \text{Lin}(\mathbb{H}, \mathbb{H}^*)$, and there is no prespecified bare Hamiltonian. In this case, we reparameterize the vector c whose components are the $c^{(s)}$ in terms of a power series

$$c = c(z) = \sum z^j c_j,$$

so that the power series coefficients of $H = Bc(z)$ are

$$H_j = Bc_j \quad (j = 0, 1, 2, \dots).$$

In concrete calculations, c_0 must be chosen such that $H_0 = Bc_0$ has a known spectral resolution. This typically forces some components of c_0 to vanish,

$$c_0^{(s)} = 0 \quad \text{for } s \in S_0.$$

The independent parameters are the $c_0^{(s)}$ with $s \notin S_0$ (typically defining masses) and the $c_1^{(s)}$ with $s \in S_0$ (typically defining charges or other coupling constants).

The remaining coefficients $c_j^{(s)}$ can be freely chosen; we refer to them as **calibration terms**.

Note that truncation at second (or any higher) order produces approximations that depend (often very strongly) on the choice of the calibration parameters since the discarded higher order terms are not independent of them; cf. EPELBAUM & GEGELIA [18].

In **naive perturbation theory**, all calibration terms are taken as zero, and we essentially get the bare parameterization; the expansion parameter z simply scales the coupling constants and can be set to $z = 1$. However, the bare parameterization is frequently far too

sensitive to be of practical use. In **renormalized perturbation theory**, the calibration terms $c_j^{(s)}$ are therefore chosen such that the sensitivity is reduced to an acceptable level.

At first sight, it would seem that renormalized perturbation theory is completely unpredictable since the finite-order results depend on the choice of the arbitrary calibration terms. However, the situation is completely analogous to that in variational perturbation theory mentioned before, which leads to highly accurate prediction provided the free parameters are chosen in a region of parameter space where the results do not vary much. One way of doing this is by specific choices cancelling the most sensitive terms in the perturbative expansion; this is the standard practice in quantum field theory, where the calibration terms are generally referred to as **counterterms**. Alternatively, the calibration terms can often be determined numerically by means of the principle of minimal sensitivity explained in Section 4. (For a specific example in the context of similarity renormalization, see GLAZEK & MLYNIK [23].)

6 Singular Hamiltonians and their treatment

In nonrelativistic quantum mechanics, renormalizing transformations can be computed for sufficiently regular Hamiltonians with traditional perturbation theory, posing no particular conceptual difficulties. However, in case of singular Hamiltonians and in relativistic quantum field theory (where all interactions are singular due to causality requirements), naively mimicking the regular perturbation theory introduces troubling aspects, so-called **ultraviolet (UV) divergences**. (There may also be so-called infrared divergences, but only if there are long range interactions or massless particles.) These divergences only show up when doing (as done in most quantum field theory textbooks) the analysis in a formal way, disregarding topological issues about the convergence of the terms involved. The reason is that the appropriate bare parameters to achieve experimental accuracy are huge in a model with an additional parameter that regularizes the singularities, and in the formal limit where the regularization is turned off, the bare parameters diverge. (As we shall see, a correct treatment must not take the limit before all results of interest have been renormalized.)

HEISENBERG & PAULI [32] noticed in 1929 the first divergences in quantum field theory; in the following years, many more divergences were encountered. In 1934, DIRAC [15] and WEISSKOPF [61] used bare QED to calculate the second-order corrections to the Compton effect. It was noted that the second order correction was infinite, which puzzled the physics community for a long time. This was all the more puzzling as the first order results were in reasonable agreement with measurements. Indeed, our first order perturbation formulas (12) make sense for almost arbitrary interactions, and are analogous to the case of quantum electrodynamics, where the spectrum is continuous and describes scattering.

History oscillated between attempts to get rid of the divergences and attempts to alter the basis of quantum field theory – until it was finally discovered how to tame the infinities by regularization and renormalization. It took almost twenty years until, in 1948, SCHWINGER [50] found the correct approach to get UV finite results, using the newly created renormal-

ized QED that restored confidence in quantum field theory, and for which Feynman, Tomonaga and Schwinger received the Nobel prize. The same techniques, suitably extended, were later shown to apply to a large class of renormalizable theories, including QCD and the standard model. (It should be mentioned that QED also suffers from infrared divergences. These are related to the large distance behavior and occur only in field theories with massless particles or long-range nonlocal interactions. They are not cured by renormalization and are outside of the scope of the present article. Currently, the most adequate treatment of infrared divergences in QED uses coherent state techniques; see KULISH & FADDEEV [37] and STEINMANN [51]).

A perturbative approach to renormalized Hamiltonians (as opposed to the renormalized scattering theory developed by Tomonaga, Feynman, and Schwinger) was developed first for the special case of QED in almost forgotten papers by DYSON [16] (cf. WALHOUT [57]). It was later rediscovered and popularized by GLAZEK & WILSON [25] under the name of **similarity renormalization**. In the framework of similarity renormalization, one goes from a formal but ill-defined Hamiltonian to a better defined Hamiltonian by means of formally unitary similarity transformations. These similarity transformations are also ill-defined, as they intertwine between ill-defined and well-behaving objects. But they can be handled rigorously by regularization, postponing the limit to the stage where the renormalization has been completed. The renormalized Hamiltonian is then defined as a limit of the similarity-transformed regularized Hamiltonians.

From a mathematically rigorous point of view, singular Hamiltonians are no longer densely defined self-adjoint operators in a Hilbert space but only so-called quadratic forms. In quantum field theory, we are given a family of such singular Hamiltonians $H(g)$ depending on a vector of coupling constants. Since only self-adjoint Hamiltonians have a good spectral theory and hence a well-defined dynamics via the solution of the Schrödinger equation, the question to be addressed is how one can associate in a canonical way with a family $H(g)$ of quadratic forms a family of renormalized self-adjoint Hamiltonians.

This problem is unsolved in full generality, and in particular for QED. Available in case of QED is only a rigorous perturbative treatment (see, e.g., SALMHOFER [47]), which leaves convergence questions open. Indeed, it is generally believed since the analysis by DYSON [17]) that the perturbation series obtained is only asymptotic, thus having no clear mathematical content. But the problem is solved satisfactorily in many simpler instances (CASE [9], FRANK & LAND [19], JACKIW [33], and GUPTA [30] are some of many possible references) in terms of so-called **self-adjoint extensions** (see, e.g., REED & SIMON [45, Theorem X.2], BONNEAU et al. [5], THIRRING [54, Section 2.5], ALBEVERIO et al. [2]), though this latter technique is not always equivalent to renormalization (CAMBLONG et al. [8], GOPALAKRISHNAN [28]). Another interesting and now very popular regularization technique is **dimensional regularization**; again, it is not always equivalent to standard renormalization techniques involving an energy cutoff; see PHILLIPS et al. [42].

In relativistic quantum field theories, and in particular for QED and QCD, the picture of renormalization usually drawn is dominated by the fact that one must usually work perturbatively; explicitly solvable models exist only in one space and one time dimension, and are far from being representative for the general case.

However, renormalization problems involving divergences – if done in a naive way – already occur in much simpler situations. In the remainder of this paper, we therefore discuss in detail a class of toy examples, first in the regular case and then in the singular case. Everything is fully transparent since even the singular case is explicitly solvable. This allows us to derive closed nonperturbative formulas, and to avoid the technical framework of self-adjoint extensions. By comparing (in Section 11) the nonperturbative results with the results from formal (in the singular case ill-defined) perturbation theory, we can see *why* things go wrong in the naive approach.

Part II

Solvable models

7 Type B: A separable potential

We now introduce our models of Type B. They generalize a model introduced in a forgotten paper by TRUBATCH [55] to understand the renormalization process, and later used by GLAZEK & WILSON [24] in the context of similarity renormalization. They are also related to the free Hamiltonian perturbed by a separable potential [20, 63, 64], modified in that the continuous spectrum has been replaced by a discrete one. We use the notation from Section 5.

Let H_0 be an arbitrary reference Hamiltonian with a discrete, nondegenerate spectrum of eigenvalues E_k ($k \in K$), where K is a countably infinite index set. We assume that the spectrum is unbounded but bounded below, and that the set of accumulation points of the spectrum is bounded. By shifting the Hamiltonian by a multiple of the identity if necessary, we may assume that

$$\inf_{k \in K} E_k = 0, \quad \sup_{k \in K} E_k = \infty. \quad (16)$$

As special cases, we consider for a fixed frequency $\omega > 0$ the **harmonic oscillator spectrum**

$$E_k := k\omega, \quad k \in K := \{0, 1, 2, 3, \dots\}, \quad (17)$$

which has a ground state with energy $E_0 = 0$, and for a fixed number $q > 1$ the **geometric spectrum**

$$E_k := q^k \omega, \quad k \in K := \{\dots, -2, -1, 0, 1, 2, \dots\} \quad (18)$$

(cf. [25]), which has no ground state as infinitely many eigenvalues cluster at $E = 0$.

Type B models have an interacting Hamiltonian of the form

$$H := H_0 - gee^*, \quad (19)$$

where g is some *nonnegative* parameter indicating the strength of an attractive force, $e \in \mathbb{H}$ is a fixed vector with (for simplicity) real e_k , and $e^* \in \mathbb{H}^*$ is the linear functional which maps

ψ to $e6 * (\psi) := e^* \psi$. (In Dirac notation, one would write $\psi = |\psi\rangle$, $e = |e\rangle$ and $e^* = \langle e|$; however, we use the Dirac notation only to label basis vectors. The present notation is more in the spirit of Heisenberg's matrix quantum mechanics, where vectors are considered as infinite column vectors, and their adjoints as infinite row vectors.)

Ultimately, for arbitrary values of g , the renormalized eigenvalues and eigenvectors will be in one-to-one correspondence with the bare eigenvalues, and hence can be indexed uniquely by an index k . In the traditional perturbative procedure where g is small, this is assumed from the start, given that the spectrum is nondegenerate for $g = 0$ and depends continuously on g . However, we aim at a nonperturbative solution for general $g \geq 0$, hence we cannot make this assumption but have to prove it during the course of the analysis. (In finite dimensions, one could settle this by counting the total number of eigenvalues. But in infinite dimension, such an argument does not guarantee that all eigenvalues have been found.)

Therefore we begin our analysis for $g \neq 0$ with an arbitrary unlabelled renormalized eigenvector ψ of H with renormalized eigenvalue E . Then $H\psi = E\psi$, giving

$$(H_0 - E)\psi = \gamma_E e,$$

where

$$\gamma_E = g e^* \psi. \quad (20)$$

Decomposing with respect to the basis vectors $|k\rangle$ we find

$$(E_k - E)\psi_k = \gamma_E e_k \quad \text{for all } k \in K. \quad (21)$$

To analyze this equation, we distinguish two cases.

CASE 1. For all ℓ with $e_\ell = 0$, we see immediately that $|\ell\rangle$ is an eigenvector with eigenvalue $E = E_\ell$ and $\gamma_E = 0$, and the corresponding eigenspace is 1-dimensional since all E_k are distinct. Conversely, suppose that $E = E_\ell$ for some ℓ . Then $\gamma_E e_\ell = 0$. If $e_\ell \neq 0$ then $\gamma_E = 0$, and we find $(E_k - E_\ell)\psi_k = 0$ for all k , so that ψ_k is only nonzero for $k = \ell$, and ψ is a multiple of $|\ell\rangle$. But then (20) implies $\psi_\ell = 0$, so that $\psi = 0$, contradicting the assumption that ψ is an eigenvector. Thus

$$E = E_\ell \quad \text{iff} \quad e_\ell = 0.$$

Thus the eigenvalue structure of H agrees with that of H_0 whenever $e_\ell = 0$, i.e., for all but finitely many values of ℓ . In particular, for large energies and near accumulation points of the bare spectrum that do not belong to the bare spectrum, the Hamiltonian H behaves like the free Hamiltonian. Thus we see that the interaction term $-gee^*$ in the Hamiltonian only influences states whose energy is bounded and bounded away from the accumulation points of the bare spectrum (unless these belong to the bare spectrum).

CASE 2. If $E_k - E \neq 0$ for all k , then

$$\psi_k = \frac{\gamma_E e_k}{E_k - E}. \quad (22)$$

The value of γ_E is fixed (up to an irrelevant phase) by the condition that ψ is normalized, giving

$$\gamma_E = \left(\sum_k \frac{e_k^2}{(E_k - E)^2} \right)^{-1/2}. \quad (23)$$

In particular, this implies that the eigenvalue E is simple. Inserting (22) into (20) gives the **characteristic equation**

$$g \sum_k \frac{e_k^2}{E_k - E} = 1. \quad (24)$$

By the derivation, E is a renormalized eigenvalue if and only if it satisfies this equation; then its associated renormalized eigenstate is given by (22). Note that since $e \in \mathbb{H}$, all but finitely many e_k vanish; thus the above sums are finite.

Let $k = k_0$ be the index with smallest E_k among those with $e_k \neq 0$. For $E < E_{k_0}$, the left hand side of (24) is monotone increasing in E , and varies from 0 to ∞ ; therefore there is a unique solution $E = E_{k_0}(g) < E_{k_0}$ of the characteristic equation. Between any two adjacent values of E_k with $e_k \neq 0$, the left hand side is also monotone increasing in E , and varies from $-\infty$ to ∞ ; between these bounds, there is therefore also a unique eigenvalue of the form considered in Case 2. Moreover, by monitoring signs, this eigenvalue can easily be determined by a bisection procedure. We call the eigenvalues covered by Case 2 the **sliding eigenvalues**.

By counting eigenvalues for small g , it is easily seen that the effect of the interaction is to move the bare eigenvalues E_k for all k with $e_k \neq 0$ to slightly smaller, physical energies given by the corresponding renormalized eigenvalues $E = E_k(g)$, consistent with the assumption of an attractive potential.

Thus we have a fairly complete picture of how the renormalized spectrum looks like for a general model of Type B.

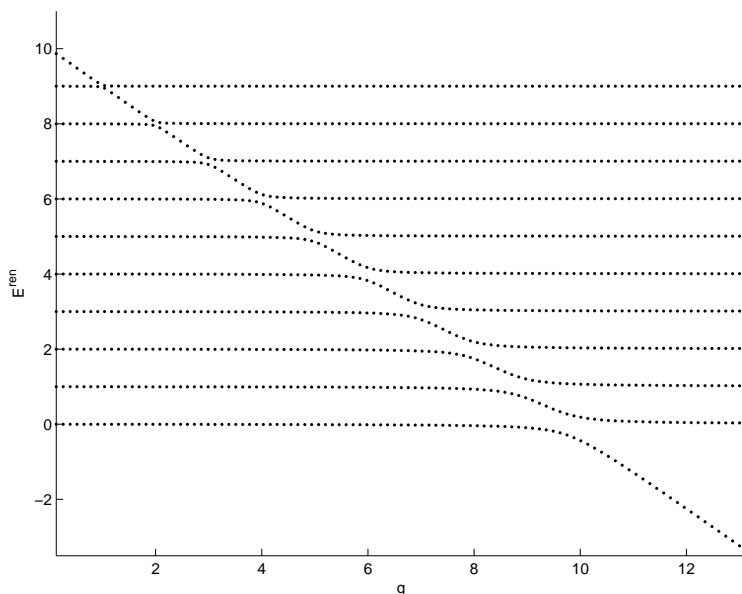


Figure 1: A renormalized spectrum in dependence on the bare coupling constant g .

Figure 1 shows a renormalized spectrum in dependence on the bare coupling constant g for the case of the harmonic spectrum (17) with $\omega = 1$ and $e_{10} = 1$, $e_k = 0$ for $k > 10$, and

$e_k = \kappa$ for $k = 0 : 9$. For $\kappa = 0$, only the 10th eigenvalue would slide; for the case $\kappa = 0.03$ shown in the figure, this is still visible approximately as a sequence of avoided crossings, forced by the properties derived above.

8 Explicit renormalized nonperturbative formulas

We now consider the case when g is tiny and $\sum_{E_k \neq 0} e_k^2/E_k$ is large. In this case, the left hand side of (24) changes extremely slowly with $E < E_{k_0}$ when the equation is approximately satisfied. As a result, the smallest sliding eigenvalue $E_{k_0}(g)$ is extremely sensitive to the choice of g . (For tiny g , this is also the case for the other sliding eigenvalues, though this is not so easy to see.)

Thus an even slightly inaccurate bare coupling constant contains no longer any relevant information about the smallest sliding eigenvalue, and renormalization is called for. Given the form of the characteristic equation, we therefore express the bare coupling constant g in terms of a number Δ , and choose as reparameterization

$$g = g(\Delta) := \left(\sum_k \frac{e_k^2}{E_k + \Delta} \right)^{-1}. \quad (25)$$

For $\Delta > -E_{k_0}$, the right-hand side of (25) is strictly increasing in Δ , covering the range from zero to infinity. Therefore, the equation $g = g(\Delta)$ has a unique solution $\Delta = \Delta(g)$ in this range, which can easily be found by a bisection procedure. Thus the parametrization of our family of Hamiltonians by $g > 0$ or by $\Delta > -E_{k_0}$ are equivalent. Note that the bare case now corresponds to the limit $\Delta \rightarrow -E_{k_0}$.

In terms of the renormalized shift parameter Δ rather than the bare coupling constant g , the Hamiltonian takes the renormalized form

$$H = H_{\text{ren}}(\Delta) := H_0 - g(\Delta)ee^*.$$

Inserting (25) into (24), multiplied by g^{-1} gives

$$\sum_k \frac{e_k^2}{E_k + \Delta} = \sum_k \frac{e_k^2}{E_k - E}. \quad (26)$$

In particular, we see that this holds for $E = -\Delta$. In particular, if $\Delta > 0$, this must be the smallest renormalized eigenvalue. Thus, in this case, Δ has the physical interpretation as the shift in the ground state energy. We therefore refer to Δ as the **shift parameter**. In a continuum limit, where the bare Hamiltonian describes a free particle with a continuous spectrum, Δ would be the binding energy of the unique bound state created by the potential. Thus in this case, the shift parameter has a direct physical interpretation. (This would not be the case in situations such as QED or QCD where the bare particles cannot be observed at all, and only energy differences are physical.)

In general, taking differences and simplifying a little, we find

$$\Sigma_{\Delta}(E) := \sum_k \frac{e_k^2(E + \Delta)}{(E_k + \Delta)(E_k - E)} = 0 \quad (27)$$

as **renormalized characteristic equation** for the nontrivial eigenvalues. The bare coupling constant g has been eliminated from the characteristic equation in favor of the renormalized shift parameter, which has a direct interpretation in terms of the spectrum. We stress that the result is **nonperturbative**, i.e., the renormalized formulas work for an arbitrary shift parameter Δ , not – as perturbative results – only in the case of weak coupling.

With this reparameterization, our problem has become computationally much more robust. The sensitive eigenvalue $E = -\Delta$ is found immediately as the zero of the numerator. As before, one can deduce from (27) that between any two adjacent values of E_{ℓ} with $e_{\ell} \neq 0$, there is a unique eigenvalue $E_{\ell}(\Delta)$, which can be easily determined by a bisection procedure. They are squeezed in between two bare eigenvalues, and not very sensitive to changes in Δ .

From (22) and (23), we find that the renormalized eigenvectors take the form

$$|\ell\rangle_{\text{ren}} := \begin{cases} \psi(E_{\ell}(\Delta)) & \text{if } e_{\ell} \neq 0, \\ |\ell\rangle & \text{otherwise,} \end{cases} \quad (28)$$

where $\psi(E)$ is the normalized vector with components

$$\psi_k(E) = \frac{e_k \gamma_E}{E_k - E}, \quad \gamma_E = \left(\sum_k \frac{e_k^2}{(E_k - E)^2} \right)^{-1/2}. \quad (29)$$

Of course, the eigenvectors form a complete orthonormal system and hence a basis of the Hilbert space \mathbb{H}_{phys} . Their orthogonality can be checked directly by noting that for renormalized eigenvalues E and E' ,

$$\begin{aligned} \frac{(E - E')\psi(E')^* \psi(E)}{\gamma_E \gamma_{E'}} &= \sum_k \frac{e_k^2(E - E')}{(E_k - E)(E_k - E')} \\ &= \sum_k \frac{e_k^2}{E_k + \Delta} \left(\frac{E + \Delta}{E_k - E} - \frac{E' + \Delta}{E_k - E'} \right) \\ &= \Sigma_{\Delta}(E) - \Sigma_{\Delta}(E') = 0. \end{aligned}$$

In terms of the adjoints $\langle \ell|_{\text{ren}}$ of the renormalized eigenvectors, the renormalized Hamiltonian takes the form

$$H = H_{\text{ren}}(\Delta) := \sum_{\ell} E_{\ell}(\Delta) |\ell\rangle_{\text{ren}} \langle \ell|_{\text{ren}}. \quad (30)$$

We now give closed formulas for the **resolvent**

$$R = R(E) := (E - H)^{-1},$$

where E is not an eigenvalue of H , in terms of the shift parameter. Introducing

$$r^T := g e^T R \quad (31)$$

and the reference resolvent

$$R_0 := R_0(E) := (E - H_0)^{-1},$$

we have

$$\begin{aligned} R_0 &= R_0(E - H)R = R_0(E - H_0 + gee^T)R \\ &= R_0(E - H_0)R + R_0er^T = R + R_0er^T, \end{aligned}$$

giving

$$R = R_0 - R_0er^T. \quad (32)$$

Now (31) gives $g^{-1}r^T = e^T R = e^T R_0 - e^T R_0er^T$. Noting that

$$g^{-1} + e^T R_0e = \sum_k \frac{e_k^2}{E_k + \Delta} - \sum_k \frac{e_k^2}{E_k - E} = -\Sigma_\Delta(E) \neq 0,$$

we find that $\Sigma_\Delta(E)r^T = -e^T R_0 = -(R_0e)^T$. Solving for r^T and inserting the result into (32), we find the closed formula

$$(E - H)^{-1} = R(E) = R_0 + R_0e\Sigma_\Delta(E)^{-1}(R_0e)^T. \quad (33)$$

This formula nicely exhibits the fact that the eigenvalues of H , i.e., the poles of the resolvent, are the zeros of the characteristic function $\Sigma_\Delta(E)$. These were excluded from the argument since $R(E)$ is not defined at the eigenvalues. Nevertheless, corresponding (unnormalized) eigenvectors are visible in the resolvent formula (33) as the vectors $R_0(E)e = (E - H_0)^{-1}e$ for $E = E_\ell(\Delta)$.

Indeed, this is related to a feature of resolvents of arbitrary Hamiltonians. In general, if

$$(E - H)^{-1} = (E - E_k)^{-1}P_k + R_k(E)$$

with $R_k(E)$ regular at $E = E_k$ then, taking the limit $E \rightarrow E_k$ in the relation

$$(E - H)P_k = (E - E_k)(1 - (E - H)R_k(E))$$

gives $(E_k - H)P_k = 0$, so that all vectors in the image of P_k are eigenvectors of H belonging to the eigenvalue E_k . (The spectral theorem implies that the P_k are the projectors to the eigenspace defined by the eigenvalue E_k and that $P_kR_k(E) = R_k(E)P_k = 0$.)

9 Nonperturbative singular renormalization

We now extend the class of Type B models to incorporate more singular cases, which in naive perturbation theory would lead to divergences. We want to give the model with Hamiltonian (19) an interpretation in cases where e is a vector with infinitely many nonzero components. For example, if one takes as reference Hamiltonian that of a harmonic oscillator

$$H_0 = \frac{p^2}{2m} + \frac{k_s}{2}q^2$$

with positive mass m and stiffness constant k_s where q is multiplication by x and $p = -i\hbar\partial_x$, and a nonlocal interaction V defined on sufficiently fast decaying wave functions ψ by

$$(V\psi)(x) := -gu(x) \int dx' u(x')\psi(x')$$

for some given continuous real-valued function u , then the Schrödinger equation with Hamiltonian $H = H_0 + V$ is an integro-differential equation. The problem with $k_s > 0$ has the Type B form when expanded in the bare eigenstates, but typically infinitely many entries in e are nonzero and significant. (In the limiting case $k_s \rightarrow 0$ of vanishing stiffness, the bare spectrum becomes continuous, and we recover models discussed by FRIEDRICHS [20], YAMAGUCHI [63, 64]; cf. also THIRRING [54, Example 3.4.13].) An even more special limiting case, often treated in the literature, is the case $k_s = 0$, $u(x) \rightarrow \delta(x)$, where V tends to multiplication by $-g\delta(x)$ since, in the limit, $(V\psi)(x) = -g\delta(x)\psi(0) = -g\delta(x)\psi(x)$, corresponding to a free particle subject to an attractive force with an infinitesimally short range.

Thus we shall consider the more general case where e is a vector with infinitely many nonzero components, corresponding to an interaction that significantly affects all energies, not only a few ones. Then our treatment of the regular case needs to be modified. This can be seen from the fact that the sum in the naively accepted expression (25) diverges to $+\infty$ if the decay law

$$\sum \frac{e_k^2}{\max(M, E_k)} < \infty \quad (34)$$

(where $M > 0$ is large enough that the bare spectrum has no accumulation point $\geq M$) is violated, so that $g(\Delta) = 0$ independent of the value of Δ . One speaks of **ultraviolet (resp. infrared) divergence** if the part of the sum corresponding to huge (resp. tiny) energies, is divergent. In terms of our sensitivity analysis above, the spectrum has become infinitely sensitive with respect to the bare parameter; the only sensible value for the bare coupling constant is $g = 0$, and the information about the shift parameter has been lost. As we shall see in Section 11, this causes the breakdown of naive perturbation theory; so the latter is not applicable. Instead, the lost degree of freedom must be recovered by a more careful procedure. It turns out that renormalization (which reduced the sensitivity already in the regular case) achieves this without any difficulties.

In rigorous terms, the situation is essentially the same as before only as long as $e \in \mathbb{H}_{\text{phys}}$. But when $e \notin \mathbb{H}_{\text{phys}}$, we leave the realm of our previously well-defined situation. For example, if $\psi \in \mathbb{H}$ and $e^*\psi \neq 0$ then the vector $H\psi = H_0\psi - g(e^*\psi)e$ is not normalizable, so $H\psi \notin \mathbb{H}_{\text{phys}}$. Thus H maps only the subspace \mathbb{H}' of ψ with $e^*\psi = 0$ (where it agrees with H_0) into the Hilbert space \mathbb{H}_{phys} . But \mathbb{H}' is no longer dense in \mathbb{H}_{phys} . Thus, when $e \notin \mathbb{H}_{\text{phys}}$, we have a singular situation not covered by the standard theory, and the manipulations we did in the regular case are no longer valid.

We therefore need to extend our formal basis. To give a rigorous meaning to the formal expression defining the Hamiltonian (19) in the case $e \in \mathbb{H}_{\text{phys}}$, we note that the product

$$e^*\psi := \sum \bar{e}_k^* \psi_k = \sum_{\psi_k \neq 0} \bar{e}_k^* \psi_k$$

is still well-defined for any $\psi \in \mathbb{H}$ since the sum is finite no matter how many components of e vanish. Therefore $e^* \in \mathbb{H}^*$, and the matrix elements

$$\phi^* H(g) \psi = \phi^* H_0 \psi - g \phi^* e e^* \psi$$

are well-defined for all $\phi, \psi \in \mathbb{H}$. Thus one can make rigorous sense of the Hamiltonian as a symmetric bilinear form or, in the traditional terminology, a **quadratic form** on \mathbb{H} . This is precisely the situation that arises naturally in quantum field theory, though with more complex Hamiltonians.

Equivalently, we may consider the Hamiltonian as a Hermitian linear operator $H(g) \in \text{Lin}(\mathbb{H}, \mathbb{H}^*)$ mapping each $\phi \in \mathbb{H}$ to a ϕ -dependent linear functional $\lambda := H(g)\phi$ in the dual space \mathbb{H}^* . This is consistently done by requiring that λ maps $\psi \in \mathbb{H}$ to the complex number $\lambda^* \psi := \phi^* H(g) \psi$.

For general $e \in \mathbb{H}^*$, the Hamiltonians $H(g)$ are well-defined mathematical objects, but they are too singular to be regarded as self-adjoint operators. However, using a limiting procedure, we can associate with the family $H(g)$ a family of renormalized self-adjoint Hamiltonians.

To motivate how we proceed, we first reason heuristically in terms of a physicist, who observes that huge energies and tiny energy differences are not observable anyway. Thus the model should be trusted only below some large energy threshold $\Lambda > 0$, called an **ultraviolet cutoff**, and for energy differences above some tiny **infrared cutoff** $\lambda > 0$ (which, for simplicity, we take to be of order $O(\Lambda^{-1})$). Thus we can as well approximate the singular Hamiltonian by a regularized one which has in place of e a regularized $e_\Lambda \in \mathbb{H}$ defined by

$$(e_\Lambda)_k = \begin{cases} e_k & \text{if } k \in K_\Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where K_Λ is a finite subset of the set K of eigenvalue labels that approaches K as the cutoff is removed ($\Lambda \rightarrow \infty$). Since one expects that the coupling constant g best fitting experimental data depends on the chosen cutoff, we use a cutoff-dependent **running coupling constant** $g = g_\Lambda$. Such a running coupling constant cannot have a physical meaning since it depends on the artificial cutoff Λ ; hence it is regarded as a bare, unphysical parameter. For a useful model, the results that can be compared with experiments must be essentially independent of the precise behavior at very large energies. Thus all results, and in particular the energy differences, should be insensitive to changes in Λ once this is large enough. This is the case if they have a well-defined limit for $\Lambda \rightarrow \infty$. We should therefore choose the dependence of g_Λ on the cutoff in a way that ensures this behavior. This is completely analogous to the reasoning in quantum field theory, where one calculates the effective physical coupling constants as a function of the cut-off energy.

With these heuristic considerations in mind, we treat our Hamiltonian rigorously as the limit of the family of **regularized Hamiltonians**

$$H_\Lambda := H_0 - g_\Lambda e_\Lambda e_\Lambda^*.$$

and define the spectrum of H as the limit of the corresponding regularized spectra. To have a well-defined limit, it turns out that we need to assume the decay condition

$$\sum \frac{e_k^2}{\max(M, E_k)^2} < \infty. \quad (35)$$

We say that the family of Hamiltonians (19) is **renormalizable** if (35) holds. This condition still demands the absence of infrared divergences but is considerably less demanding than the condition (34) for the regular case, permitting a large class of models with ultraviolet divergences to be handled. Stronger ultraviolet divergences cannot be tamed by a limiting procedure and therefore lead to **non-renormalizable** models.

Since each H_Λ is regular, our old analysis applies with e_Λ in place of e . In particular, by construction, our regularized Hamiltonians behave for energies above Λ like the free Hamiltonian. As in the regular case, we have a choice in what we regard to be the independent constant parameterizing our family of theories. So far, the shift parameter Δ is a function of g and Λ , defining a nonlinear consistency relation between Δ , g , and Λ . But since it lead to more robust formulas in the regular case, we shall treat instead Δ (which, unlike g , has a physical meaning) as relevant parameter, and use this relation to express the coupling constant g in terms of Δ and Λ . Thus the coupling constant becomes both cutoff-dependent and dependent on the energy shift. As before, we find the coupling constant as a function of Δ ,

$$g = g_\Lambda(\Delta) := \left(\sum_{k=1}^{\Lambda} \frac{e_k^2}{E_k + \Delta} \right)^{-1}. \quad (36)$$

Our regularized Hamiltonian thus becomes

$$H_\Lambda(\Delta) = H_0 - g_\Lambda(\Delta) e_\Lambda e_\Lambda^*. \quad (37)$$

Note that a natural dependence on the cutoff Λ appeared; so we can hope to get away without a cutoff dependence of Δ . We are interested in the limit $\Lambda \rightarrow \infty$.

For example, in case of the harmonic spectrum (17), we find from (36) that

$$g_\Lambda(\Delta) \approx \left(\int_0^\Lambda \frac{dk}{k\omega + \Delta} \right)^{-1} = \omega / \log \frac{\Lambda\omega + \Delta}{\Delta},$$

$g_\Lambda(\Delta)$ goes to zero as the cutoff is removed ($\Lambda \rightarrow \infty$). In case of the geometric spectrum (18), we find similarly (using $\alpha = \log q$ and the substitution $x = q^k = e^{\alpha k}$) that

$$g_\Lambda(\Delta) \approx \left(\int_{\log \lambda / \alpha}^{\log \Lambda / \alpha} \frac{dk}{e^{\alpha k} \omega + \Delta} \right)^{-1} = \left(\int_\lambda^\Lambda \frac{dx}{\alpha x(x\omega + \Delta)} \right)^{-1} = \alpha \Delta / \log \frac{\omega + \Delta / \lambda}{\omega + \Delta / \Lambda}.$$

Again, $g_\Lambda(\Delta)$ goes to zero as the cutoff is removed ($\Lambda \rightarrow \infty$ and $\lambda \rightarrow 0$). Note that other families of Hamiltonians may have more complex asymptotic behavior for $\Lambda \rightarrow \infty$. Indeed g_Λ may approach asymptotically a periodic or chaotic function of Λ ; see, e.g. GLAZEK & WILSON [26].

We now consider the regularized version of the renormalized characteristic equation (27),

$$\Sigma_{\Lambda,\Delta}(E) := \sum_{k \in K_\Lambda} \frac{e_k^2(E + \Delta)}{(E_k + \Delta)(E_k - E)} = 0.$$

If $\Delta > 0$ and E is different from the bare eigenvalues and their accumulation points, this equation has a well-defined limit as the cutoff is removed. Indeed, for $\Lambda \rightarrow \infty$, we get

$$\Sigma_\Delta(E) := \sum_{k \in K} \frac{e_k^2(E + \Delta)}{(E_k + \Delta)(E_k - E)} = 0; \quad (38)$$

the sum is dominated by a multiple of $\sum e_k^2/E_k^2$, hence is absolutely convergent under our decay assumption (35). Analyzing the equation as before, we find that apart from the ground state energy $E_0(\Delta) = -\Delta$, there is for each $\ell > 0$ a unique eigenvalue $E_\ell(\Delta)$ between the bare eigenvalue E_ℓ and the bare eigenvalue immediately above it, determined as the solution of the renormalized characteristic equation $\Sigma_\Delta(E) = 0$ in this interval.

Singular Type B models illustrate an interesting phenomenon typical for renormalization in relativistic quantum field theories. This is the fact that no matter which dimension the bare coupling constant has, the renormalized theory is always parameterized by a parameter with the dimension of energy. If (as in massless QCD) all coupling constants are dimensionless, this phenomenon – called **dimensional transmutation** – is puzzling at first sight since an energy scale appears through renormalization although the bare Hamiltonians are scale invariant.

In type B models, dimensional transmutation finds its natural expression in the fact that, by construction, the shift parameter Δ always has the dimension of energy, whereas the dimension of the bare coupling constant depends on the dimension of e . In particular, g is dimensionless if we take e to have components $e_k = \sqrt{E_k}$; this is consistent with our decay condition (35) provided that $\sum \min(M^{-2}E_k, E_k^{-1}) < \infty$. E.g., this holds for the geometric spectrum (18). Indeed, this particular case of a Type B model was discussed by GLAZEK & WILSON [25] from a purely numerical point of view to illustrate the technique of similarity renormalization.

The eigenvectors are still given by the formulas (28)–(29), which survive the limit. Assuming (35), the sums are absolutely convergent, so that $|\ell\rangle_{\text{ren}} \in \mathbb{H}_{\text{phys}}$. Their orthogonality and completeness follows by taking the limit in the completeness relations for the regularized eigenvectors. Thus even in the singular case, the renormalized eigenvectors $|\ell\rangle_{\text{ren}}$ form an orthonormal basis, in terms of which we can define the self-adjoint **renormalized Hamiltonian** $H_{\text{ren}}(\Delta)$ by

$$H_{\text{ren}}(\Delta) := \sum_{\ell} E_\ell(\Delta) \psi_\ell^{\text{ren}} \langle \ell |_{\text{ren}}.$$

Note that one can also regularize the problem by replacing e with an arbitrary family of $e_\Lambda \in \mathbb{H}$ whose weak limit for $\Lambda \rightarrow \infty$ is e , and take the limit after all renormalizations have been performed. It is straightforward to verify that the results are independent of the details of the regularization.

We conclude that *all aspects of the singular case have been successfully renormalized, without any approximation and with full mathematical rigor*. We simply take the renormalized formulas from the regular case but interpret the sums now as infinite series. Under our decay condition (35), these sums are absolutely convergent, so the problems from a naive approach using the bare coupling constant have disappeared.

As an alternative to our procedure one could have invoked the Trotter–Kato theorem from functional analysis (see, e.g., REED & SIMON [44, Theorem VIII.22]) to reconstruct a self-adjoint renormalized Hamiltonian from the renormalized resolvent (33). Indeed, the latter process generalizes to much more complex systems than Type B models; see DIMOCK & RAJEEV [14, 43].

Our treatment leaves open the question whether in a suitable sense

$$H_{\text{ren}}(\Delta) = \lim_{\Lambda \rightarrow \infty} H_{\Lambda}(\Delta).$$

We may try to take the formal limit in (37) naively, using $e_{\Lambda} \rightarrow e$ and asymptotic freedom ($g_{\Lambda} \rightarrow 0$). But this gives the free Hamiltonian $H = H_0$, and cannot be valid since its spectrum is different. Indeed, rigor is violated: One cannot simply take a naive limit of the term $g_{\Lambda}(\Delta)e_{\Lambda}e_{\Lambda}^T$, since e_{Λ} goes to e only weakly and the formula $\lim a_{\Lambda}b_{\Lambda} = \lim a_{\Lambda} \lim b_{\Lambda}$ only holds if one of the two limits is strongly converging. In general, to recover the renormalized Hamiltonian in a limit, one needs to do appropriate similarity transformations *before* taking the limit, in the spirit of the similarity renormalization approach.

10 The renormalization group

The renormalization group makes its appearance whenever there are multiple ways of performing the renormalization, depending on one or more additional parameters that can be freely chosen. In the conventional treatment, the free parameter is an energy scale which defines the renormalization prescription used. The freedom in the choice of this parameter is analogous to the free choice of the bare frequency μ in the variational perturbation theory for the anharmonic oscillator mentioned in the introduction. Choosing it correctly may make the difference between a useless and a successful approximation scheme.

So far, we discussed renormalization in terms of physical parameters of the theory. In our models, the shift parameter Δ corresponds to a binding energy that must be obtained from experiment; in quantum field theories basic physical parameters are masses and charges of elementary or composite particles. In theories more complex than our toy examples, it is not always convenient to renormalize directly in terms of physical parameters. Instead, one often introduces other parameterizations through appropriate renormalization conditions that are more convenient computationally. These parameterizations, and therefore the renormalized coupling constants, typically depend on an additional parameter, and lead to so-called running coupling constants.

To see how these can arise in our models of Type B, we write the unrenormalized charac-

teristic equation (24) in the form $\Sigma(E) = 0$, where

$$\Sigma(E) := g^{-1} + e^*(E - H_0)^{-1}e$$

Like in the transition from (24) to (27), the sensitive part of $\Sigma(E)$ can be cancelled if we shift the constant g^{-1} by a constant of the form $\beta e^*(1 + \beta H_0)^{-1}e$, where $\beta > 0$ is an arbitrary **renormalization scale** with the units of inverse energy. Indeed, let us introduce the **running coupling constant**

$$g_\beta := \frac{g}{1 - \beta g e^*(1 + \beta H_0)^{-1}e}$$

depending on the renormalization scale β , chosen such that the denominator is nonzero. Since

$$g_\beta^{-1} = g^{-1} - \beta e^*(1 + \beta H_0)^{-1}e, \quad (39)$$

we find

$$\begin{aligned} \Sigma(E) &= g_\beta^{-1} + \beta e^*(1 + \beta H_0)^{-1}e + e^*(E - H_0)^{-1}e \\ &= g_\beta^{-1} + e^*(1 + \beta H_0)^{-1}(\beta(E - H_0) + (1 + \beta H_0))(E - H_0)^{-1}e \\ &= g_\beta^{-1} + (1 + \beta E)e^*(1 + \beta H_0)^{-1}(E - H_0)^{-1}e = g_\beta^{-1}\Sigma_\beta(E), \end{aligned}$$

with

$$\Sigma_\beta(E) := 1 + (1 + \beta E)g_\beta e^*(1 + \beta H_0)^{-1}(E - H_0)^{-1}e.$$

Therefore the characteristic equation is equivalent to the **renormalized characteristic equation**

$$\Sigma_\beta(E) = 0 \quad (40)$$

for any nonnegative value of β . For $\beta > 0$, (40) is much more robust than the original characteristic equation (with $\beta = 0$) since large bare eigenvalues appear squared in the denominators of $H_{\text{red}}^{\text{ren}}(E)$ when expressed in the bare eigenbasis. The bare coupling constant can be recovered by solving (39) for g , giving

$$g = \frac{g_\beta}{1 + \beta g_\beta e^*(1 + \beta H_0)^{-1}e}. \quad (41)$$

When the renormalization scale β is tiny, the theory becomes almost free; indeed, the limit $\beta \rightarrow 0$ gives the "free" reference case. This is called **asymptotic freedom**, a feature shared with QCD.

Note that renormalizable singular cases can be handled by the same formula, using regularization and the limit removing the cutoff to derive it in the singular case, except that the bare coupling constant (41) now becomes infinitesimally small. In particular, the renormalized formulas make sense under the same decay condition as our previous renormalized results in terms of the shift parameter Δ .

The fact that the renormalization scale β is arbitrary leads to an associated renormalization group, which specifies how the running coupling constant changes when β is changed. By taking differences in (39), we find

$$\begin{aligned} g_\beta^{-1}g_{\beta'}^{-1}(g_\beta - g_{\beta'}) &= g_{\beta'}^{-1} - g_\beta^{-1} = \beta e^*(1 + \beta H_0)^{-1}e - \beta' e^*(1 + \beta' H_0)^{-1}e \\ &= (\beta - \beta')e^*(1 + \beta H_0)^{-1}(1 + \beta' H_0)^{-1}e = -(\beta' - \beta)e_\beta^*e_{\beta'}, \end{aligned}$$

with

$$e_\beta := (1 + \beta H_0)^{-1} e. \quad (42)$$

If we multiply by $g_\beta g_{\beta'}/(\beta' - \beta)$ and take the limit $\beta' \rightarrow \beta$, we get the so-called **renormalization group equation**

$$\frac{dg_\beta}{d\beta} = g_\beta^2 e_\beta^* e_\beta. \quad (43)$$

The word "group" just signals the well-known fact that any sufficiently regular autonomous differential equation generates a flow and with it a one-parameter group of transformations. Multiple free parameters would give a multiparameter renormalization group.

As our derivation shows, the renormalization group equation essentially expresses the fact – equivalent to (39) – that

$$g^{-1} = g_\beta^{-1} + \beta e^* (1 + \beta H_0)^{-1} e \quad (44)$$

is independent of β . However, in the singular case, (44) is meaningless, while the renormalization group equation still makes sense and is fully justified by regularization followed by the limit that removes the cutoff.

In the analogous situation in QED and QCD, one has in place of μ an energy (or temperature) scale T which defines (together with the physical constants) the renormalization prescription used. As in the present case, the exact results (which are now inaccessible, unlike in our models) should be independent of T . But the results computable with low order perturbation theory depend on T in an essential way, and choosing T in the order of the energies of interest usually gives much better results than other choices of T . The dependence on T is far more implicit than for our models, but one can still construct perturbatively a renormalization group equation, providing a differential equation for how the terms of interest change with T .

In the case of asymptotic freedom, it turns out that the dependence on T dies out as $T \rightarrow \infty$ (i.e., $\beta \rightarrow 0$), where the theory becomes free. This provides natural boundary conditions for the renormalization group equation and allows the approximate derivation of formulas suitable for actual prediction. Thus asymptotic freedom has the consequence that at high energies, the results of low order renormalized perturbation theory at the corresponding energy scale are more accurate. Details in an elementary – Feynman diagram free – setting not very far from the present discussion can be found in the paper by DELAMOTTE [12].

In the context of renormalized perturbation theory, the infinite accuracy results are independent of z and the choice of the calibration terms, once as many independent renormalization conditions are prescribed as there are variable coefficients in (15). Again, we can get by differentiation of the resulting identities corresponding renormalization group equations. Note that the choices made may strongly affect the results at every order of perturbation theory!

11 Perturbation theory for type B models

In this section, we use type B models to compare our nonperturbative results with those of second-order perturbation theory. This relates the discussion so far to the standard perturbative approach to renormalization.

For type B models, where $H = H_0 - gee^*$, there is only one independent coupling constant g , and we choose the reference Hamiltonian as H_0 . Thus, in the notation of Section 5, we have

$$B^{(0)} = H_0, \quad B^{(1)} = -ee^*, \quad c^{(0)}(z) = 1, \quad c^{(1)} = g(z) = \sum_{j=1}^{\infty} z^j \gamma_j,$$

so that

$$c_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad c_j = \begin{pmatrix} 0 \\ \gamma_j \end{pmatrix}, \quad H_j = -\gamma_j ee^* \quad \text{for } j > 0.$$

Therefore, the ℓ th perturbed eigenvalue is

$$\epsilon(z) = \epsilon_\ell(z) = \sum_{j=0}^{\infty} z^j \epsilon_j,$$

where by (10), (12), and (14),

$$\epsilon_0 = E_\ell, \quad \epsilon_1 = \langle \ell | H_1 | \ell \rangle = -\gamma_1 e_\ell^2, \quad (45)$$

$$\epsilon_2 = \alpha_\ell = \langle \ell | H_2 | \ell \rangle + \sum_{j \neq \ell} \frac{|\langle j | H_1 | \ell \rangle|^2}{E_\ell - E_j} = -\gamma_2 e_\ell^2 + \gamma_1^2 e_\ell^2 \sum_{j \neq \ell} \frac{e_j^2}{E_\ell - E_j}. \quad (46)$$

Renormalized perturbation theory. If we expand in the physical shift parameter $z = \Delta$, i.e., minus the lowest renormalized eigenvalue, we know (without invoking our closed formulas) that, for $\ell = 0$, we have $\epsilon(z) = -\Delta = -z$, hence

$$\epsilon_0 = E_0 = 0, \quad \epsilon_1 = -1, \quad \epsilon_j = 0 \quad \text{for } j > 1.$$

This fixes the expansion coefficients, giving

$$\gamma_1 = \frac{1}{e_0^2}, \quad \gamma_2 = \frac{1}{e_0^4} \sum_{j \neq 0} \frac{e_j^2}{E_0 - E_j}, \quad \dots \quad (47)$$

Using this in the formulas for $\ell > 0$, we find for the ℓ th eigenvalue the expansion coefficients

$$\begin{aligned} \epsilon_1 &= -\frac{e_\ell^2}{e_0^2}, \\ \epsilon_2 &= -\frac{e_\ell^2}{e_0^4} \sum_{j \neq 0} \frac{e_j^2}{E_0 - E_j} + \frac{e_\ell^2}{e_0^4} \sum_{j \neq \ell} \frac{e_j^2}{E_\ell - E_j} \\ &= \frac{e_\ell^2}{e_0^4} \left(\frac{e_0^2 + e_\ell^2}{E_\ell - E_0} + \sum_{j \neq 0, \ell} \frac{e_j^2 (E_0 - E_\ell)}{(E_0 - E_j)(E_\ell - E_j)} \right). \end{aligned}$$

This is convergent under the same conditions (35) as the nonperturbative formulas, and shows that renormalized perturbation theory indeed gives a good perturbative interpretation to Hamiltonians with singular interactions, at least for Type B models. The fact that (47) diverges has no observable consequence.

Thus the renormalized perturbation theory is well-defined, and the approximation makes sense, at least in the regime where z is small enough that the higher order terms can be safely neglected. This is the situation that holds in QED, when z is taken to be the renormalized (rather than bare) electron charge; the above renormalized sums correspond to (after renormalization) convergent integrals in scattering calculations for QED. In QCD, the renormalized coupling constant is small only for very large energies, and a nonperturbative approach is needed to get useful renormalized results at low energies.

Renormalized perturbation theory using $z = -g_\beta$. The running coupling constant approach can easily be combined with renormalized perturbation theory by using $z = -g_\beta$ as the expansion parameter. From (39), we find

$$g^{-1} - z^{-1} = -g_\beta^{-1}g^{-1} = \beta e^*(1 + \beta H_0)^{-1}e = \sum_j \frac{\beta e_j^2}{1 + \beta E_j} =: \sigma_\beta,$$

hence $g^{-1} = z^{-1} + \sigma_\beta$. Therefore

$$g = \frac{z}{1 + \sigma_\beta z} = z - \sigma_\beta z^2 + O(z^3),$$

so that

$$\gamma_1 = 1, \quad \gamma_2 = -\sigma_\beta.$$

Now (45)–(46) become $\epsilon_0 = E_\ell$, $\epsilon_1 = -e_\ell^2$, and

$$\begin{aligned} \epsilon_2 = \epsilon_2(\beta) &= e_\ell^2 \left(\sum_j \frac{\beta e_j^2}{1 + \beta E_j} + \sum_{j \neq \ell} \frac{e_j^2}{E_\ell - E_j} \right) \\ &= e_\ell^2 \left(\frac{\beta e_\ell^2}{1 + \beta E_\ell} + \sum_{j \neq \ell} \frac{e_j^2(1 + \beta E_\ell)}{(E_\ell - E_j)(1 + \beta E_j)} \right). \end{aligned}$$

The second formula has a good limit in the renormalizable case. This shows that successful renormalization is not tied to a parameterization in terms of parameters with a *direct* physical meaning.

Naive perturbation theory. On the other hand, using the bare expansion parameter $z = g$, we have $\gamma_1 = 1$, $\gamma_j = 0$ for $j > 1$, hence

$$\epsilon(z) = E_\ell - g e_\ell^2 + g^2 \sum_{j \neq \ell} \frac{e_\ell^2 e_j^2}{E_\ell - E_j} + O(g^3).$$

In this expansion, the coefficient of g^2 diverges for many combinations of bare energies and e_j 's for which the renormalized version converges; in particular, this happens for the harmonic oscillator $E_k = k\omega$ perturbed using $e_k = 1$ for all k .

Thus renormalized perturbation theory is far superior. The divergences resulting for singular interactions from a naive (bare) perturbation theory in second (and higher) order are simply due to a poor choice of the expansion parameter, which creates a representation of the quantities of interest as a sum of divergent coefficients. It is like trying to take the limit

$$\lim_{\Lambda \rightarrow \infty} \frac{2\Lambda^2}{\Lambda^2 - 1} = 2$$

by rewriting it in the divergent form

$$\lim_{\Lambda \rightarrow \infty} \frac{\Lambda^2}{\Lambda - 1} - \lim_{\Lambda \rightarrow \infty} \frac{\Lambda^2}{\Lambda + 1} = \infty - \infty.$$

12 Type C: Mass renormalization

This section discusses another family of exactly solvable problems, which require an additional phenomenon common to QED and QCD: mass renormalization.

Type C models have a Hamiltonian of the form

$$H = \begin{pmatrix} \omega_0 & ge^* \\ ge & H' \end{pmatrix}, \quad (48)$$

where the ground state energy $\omega_0 > 0$ plays the role of a mass, and H' is a self-adjoint Hamiltonian with spectrum $> \omega_0$, diagonal in a basis where e only has finitely many nonzero components. A continuum limit of the above problem arises naturally in an exactly solvable sector of the Lee model (LEE [38]), a simplified nonrelativistic model for a quantum field theory.

Writing $\psi = \begin{pmatrix} \psi_0 \\ \psi' \end{pmatrix}$, we can rewrite the eigenvalue problem $H\psi = E\psi$ as

$$(E - \omega_0)\psi_0 - ge^*\psi' = 0, \quad (49)$$

$$-ge\psi_0 + (E - H')\psi' = 0. \quad (50)$$

Assuming that $\psi_0 \neq 0$, we can normalize the eigenvector such that $\psi_0 = 1$, and find that

$$\psi' = g(E - H')^{-1}e, \quad \psi = \begin{pmatrix} 1 \\ g(E - H')^{-1}e \end{pmatrix}. \quad (51)$$

Substituting this into (49) gives the characteristic equation

$$E - \omega_0 - g^2e^*(E - H')^{-1}e = 0 \quad (52)$$

as the condition for E being an eigenvalue. Thus we found an explicit solution as for type B models. Again, we may express the coupling constant g in terms of the eigenvalue shift $\Delta := \omega_0 - \omega$, where ω is the smallest renormalized eigenvalue. Replacing E in (52) by $\omega = \omega_0 - \Delta$ and assuming that $\Delta > 0$, we may solve this for g , and find

$$g = \pm \sqrt{\frac{\Delta}{e^*(\Delta - \omega_0 + H')^{-1}e}}.$$

But when H' has a harmonic spectrum and all $e_j \rightarrow 1$ then $g \rightarrow 0$ independent of Δ , and $\psi/\|\psi\| \rightarrow \binom{1}{0}$ independent of the eigenvalue. Thus, in the singular case, the simple renormalization procedure for models of Type B collapses the eigenvalue problem in an unphysical way.

This is characteristic of many problems with a Hamiltonian more complex than Type C and occurs whenever the original problem formulation has not enough parameters to embed the family of Hamiltonians into one with a good singular limit. Thus our previous recipe for the nonperturbative case was incomplete, as it had assumed that there was only a single relevant constant to be renormalized. The argument just given revealed that this is indeed too restrictive, and realistic theories need more general recipes. For example, in QED, it is not enough to renormalize the coupling constant (usually called the “bare electron charge”), but one also needs to renormalize the “bare mass” (**mass renormalization**), and also factors arising from scaling the electron field and the photon field (**wave function renormalization**), before one arrives at a renormalizable family.

In the case of (48), we realize that H depends naturally not only on the parameter g but also on ω_0 , and thus is part of a natural 2-parameter family of Hamiltonians. In this particular case, it turns out that renormalizing *only* the bare mass ω_0 is already sufficient. Thus we solve (52) for ω_0 , with ω in place of E , keeping g fixed, and find

$$\omega_0 = \omega - g^2 e^* (\omega - H')^{-1} e, \quad \psi = \begin{pmatrix} 1 \\ g(E - H')^{-1} e \end{pmatrix}.$$

When H' has a harmonic spectrum and all $e_j \rightarrow 1$ (but also in many other cases where the original problem becomes singular), the bare mass ω_0 diverges, but ψ has a good, normalizable limit for all E not in the spectrum of H' . In terms of the renormalized mass, the characteristic equation becomes

$$\begin{aligned} 0 &= E - \omega + g^2 e^* (\omega - H')^{-1} e - g^2 e^* (E - H')^{-1} e \\ &= E - \omega + g^2 e^* ((\omega - H')^{-1} - (E - H')^{-1}) e \\ &= (E - \omega)(1 + g^2 e^* (E - H')^{-1} (\omega - H')^{-1} e). \end{aligned}$$

Therefore, the renormalized eigenvalues distinct from ω are the solutions of the renormalized characteristic equation

$$g^{-2} + e^* (E - H')^{-1} (\omega - H')^{-1} e = 0, \tag{53}$$

which again makes sense for many singular problems, including the case of a harmonic spectrum and all $e_j = 1$. Thus we performed a successful mass renormalization.

13 Conclusions

We have shown that one can understand renormalization and the need for it at a very low level of sophistication (hence much earlier in the physics education), and that it is useful even in nonsingular situations.

Our description of renormalized perturbation theory is a systematic way of doing what is commonly done on an ad hoc basis. It may drastically improve standard perturbation theory, not only in cases where it is indispensable since interactions are singular, but in many other contexts, even for the anharmonic oscillator.

We have shown that one can do renormalization nonperturbatively with very simple models, and sees there everything (including asymptotic freedom and the renormalization group) in closed form – while one usually sees it for the first time in a course on quantum field theory, where it is perceived as very puzzling. In particular, one can see that a singular renormalizable theory is as respectable as a theory not needing renormalization, in that it has a well-defined spectrum with an associated complete system of eigenvectors.

Thus the paper shows that the need for renormalization is not a defect of a theory but a legitimate way to construct quantum theories that are not easily constructed by giving an explicit Hamiltonian.

We also provided explicit, closed-form examples of renormalization groups and running coupling constants, giving more insight into their meaning. The running coupling constant plays a similar role as a gauge field: One needs it for a valid description, but one can change its value by applying a transformation from the renormalization group without altering the physics, in the same way as one can change the value of a gauge field by applying a transformation from the gauge group.

Finally, we have shown how the failure of bare perturbation theory is overcome by renormalized perturbation theory, without any of the usual nonsense, where one has to pretend that expansions with infinite coefficients make sense asymptotically after resumming infinitely many infinities.

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