LOCALLY CONVEX SPACES
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These lecture notes were developed for the topics course *locally convex spaces* held at the University of Vienna in summer term 2017. Prerequisites consist of general topology and linear algebra. Some background in functional analysis will be helpful but not strictly mandatory.

This course aims at an early and thorough development of duality theory for locally convex spaces, which allows for the systematic treatment of the most important classes of locally convex spaces. Further topics will be treated according to available time as well as the interests of the students.

Thanks for corrections of some typos go out to Benedict Schinnerl.

Contents

1. Introduction

These lecture notes are roughly based on the following texts that contain the standard material on locally convex spaces as well as more advanced topics.

- J. Horváth. *Topological vector spaces and distributions.* Vol. 1. Reading, Mass.: Addison-Wesley, 1966. ISBN: 978-0-486-48850-9
- H. H. Schaefer. *Topological Vector Spaces.* New York: Springer-Verlag, 1971. ISBN: 978-0-387-05380-6
- H. Jarchow. *Locally Convex Spaces.* Stuttgart: B. G. Teubner, 1981. ISBN: 978-3-519-02224-4
- A. Grothendieck. *Topological vector spaces.* 3rd ed. Philadelphia, PA: Gordon and Breach Science Publishers, 1992. ISBN: 0-677-30020-4/hbk
- F. Treves. *Topological Vector Spaces, Distributions and Kernels.* New York: Academic Press, 1976
- L. Narici and E. Beckenstein. *Topological Vector Spaces.* 2nd ed. Pure and applied mathematics. Boca Raton: Taylor and Francis Group, 2011. ISBN: 978-1-58488-866-6
- A. P. Robertson and W. Robertson. *Topological vector spaces.* 2nd ed. Cambridge Tracts in Mathematics and Mathematical Physics 53. Cambridge University Press, 1973. ISBN: 978-0-521-20124-7
- G. Köthe. *Topological vector spaces I.* Grundlehren der mathematischen Wissenschaften 159. New York: Springer-Verlag, 1969. ISBN: 978-0-387-04509-2
- G. Köthe. *Topological vector spaces II.* Grundlehren der Mathematischen Wissenschaften 237. New York: Springer-Verlag, 1979. ISBN: 978-0-387-90400-9

> "The principal motivation behind the general theory is the same as that of Banach himself: namely, a search for general tools which might be applied successfully to functional analysis. [...] These efforts culminated in L. Schwartz' theory of distributions (1945), which could be expressed only in the language of locally convex vector spaces."
>
> "Analysts are more interested in the properties of a space than in the way it is defined; hence the idea, first clearly formulated by L. Schwartz, of classifying topological vector spaces according to their behaviour with regard to the validity of the main theorems of functional analysis." [D]

2. Topological vector spaces

We set $\mathbb{K}$ to be either $\mathbb{R}$ or $\mathbb{C}$ and

$$\mathbb{D} := \{\lambda \in \mathbb{K} \mid |\lambda| \leq 1\}, \quad \mathbb{N} = \{1, 2, 3, \dots\}, \quad \mathbb{N}_0 = \{0, 1, 2, \dots\}.$$

Definition 2.1. *Let E be a vector space over $\mathbb{K}$. A topology $\mathscr{T}$ on E is called a* linear *topology if the mappings*

$$\begin{aligned} (x, y) &\mapsto x + y, & E \times E &\to E \\ (\lambda, x) &\mapsto \lambda x, & \mathbb{K} \times E &\to E \end{aligned}$$

are continuous. The pair $(E, \mathscr{T})$ is called a topological vector space (TVS).

We write E instead of $(E, \mathscr{T})$ if it is clear from the context which topology is used.

Examples. Normed spaces are topological vector spaces:

1. $c = \{(a_n)_n \in \mathbb{K}^{\mathbb{N}} : \lim_{n\to\infty} a_n \text{ exists}\}$ with norm $\|(a_n)_n\| = \sup_n |a_n|$,
2. $c_0 = \{(a_n)_n \in c : a_n \to 0\}$ with the induced norm,
3. $\varphi = \{(a_n)_n \in c : a_n \neq 0 \text{ only for finitely many } n\}$ with the induced norm,
4. $C(K) = \{f \colon K \to \mathbb{K} \text{ continuous}\}$ where K is a compact topological space, with norm $\|f\| = \sup_{x\in K} |f(x)|$,
5. $C_0(X) = \{f \colon X \to \mathbb{K} \text{ continuous} \mid \forall \varepsilon > 0\ \exists K \subseteq X \text{ compact} : \sup_{x\in X\setminus K} |f(x)| \leq \varepsilon\}$ where X is a topological space, with norm $\|f\| = \sup_{x\in X} |f(x)|$,
6. $l^p = \{(a_n)_n \in \mathbb{K}^{\mathbb{N}} : \sum_n |a_n|^p < \infty\}$ with norm $\|(a_n)_n\|_p = (\sum_n |a_n|^p)^{1/p}$ (for $1 \leq p < \infty$),
7. $l^\infty = \{(a_n)_n \in \mathbb{K}^{\mathbb{N}} : \sup_n |a_n| < \infty\}$ with norm $\|(a_n)_n\|_\infty = \sup_n |a_n|$,
8. $L^p(X, \mathcal{F}, \mu) = \{f \colon X \to \mathbb{K} \text{ measurable} \mid \int_X |f(x)|^p \,\mathrm{d}\mu(x) < \infty\}/\sim$ where $(X, \mathcal{F}, \mu)$ is a σ-finite measure space and $f \sim g$ if $f = g$ a.e., with norm $\|[f]\|_p = (\int_X |f(p)|^p \,\mathrm{d}\mu(x))^{1/p}$ (for $1 \leq p < \infty$),
9. $L^\infty(X, \mathcal{F}, \mu) = \{f \colon X \to \mathbb{K} \text{ measurable} \mid \exists M > 0 \text{ such that } |f(x)| < M \text{ a.e.}\}/\sim$ with norm $\|[f]\|_\infty = \inf\{M > 0 : |f(x)| < M \text{ a.e.}\}$.

But there are also non-normable topological vector spaces:

10. $C(X) = \{f \colon X \to \mathbb{K} \text{ continuous}\}$ with the coarsest topology such that all restriction mappings $C(X) \to C(K)$, $f \mapsto f|_K$ for $K \subseteq X$ compact are continuous,
11. l^p and L^p for $0 < p < 1$,
12. $\mathbb{R}^{\mathbb{R}}$ with the product topology.

Because for $z \in E$ and $\lambda \neq 0$ the map $x \mapsto z + \lambda x$ is a homeomorphism with inverse $x \mapsto \lambda^{-1}(x - z)$ we have:

Proposition 2.2. *If $\mathscr{U}$ is (a basis of) the filter of neighborhoods of 0 then for all $z \in E$ and $\lambda \neq 0$, $\{z + \lambda U \mid U \in \mathscr{U}\}$ is (a basis of) the filter of neighborhoods of z.*

We will refer to the filter of neighborhoods of 0 as the 0-filter and any basis of this filter will be called a 0-basis. Similarly, a neighborhood of 0 will be called a 0-neighborhood.

Corollary 2.3. *Let E and F be TVS. A linear map $f \colon E \to F$ is continuous if and only if it is continuous at 0.*

Corollary 2.4. *If $\mathscr{U}$ is a 0-basis of a TVS E then for every nonempty subset $A \subseteq E$ we have*

$$\bar{A} = \bigcap \{A + U \mid U \in \mathscr{U}\}.$$

Proof. Because $x - \mathscr{U}$ is a basis of the neighborhood filter of $x \in E$ we have $x \in \bar{A} \Leftrightarrow \forall U \in \mathscr{U} : (x - U) \cap A \neq \emptyset \Leftrightarrow \forall U \in \mathscr{U} : x \in A + U$. □

Corollary 2.5. *If $\mathscr{U}$ is a 0-basis of a TVS E then so is $\{\bar{U} \mid U \in \mathscr{U}\}$.*

Proof. Given $U \in \mathscr{U}$, by continuity of addition there is $V \in \mathscr{U}$ such that $\bar{V} \subseteq V + V \subseteq U$. □

Some things become simpler when we can work with a 0-basis consisting of special sets.

Definition 2.6. *Let E be a vector space. A subset $A \subseteq E$ is called*

- balanced *(or circled) if $\lambda x \in A$ for all $\lambda \in \mathbb{D}$ and $x \in A$;*
- absorbent *(or radial) if $\forall x \in E$ $\exists \lambda_0 > 0$: $x \in \lambda A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \geq \lambda_0$.*

Because arbitrary intersections of balanced sets are balanced, for every $A \subseteq E$ there exists a smallest balanced set $\breve{A}$ containing A, called its *balanced hull*; clearly $\breve{A} = \mathbb{D} \cdot A$. If A is a subset of a TVS E we call the closure $\bar{\breve{A}}$ of $\breve{A}$ the *closed balanced hull* of A. It is the smallest closed and balanced set containing A because if $A \subseteq M$ and M is closed and balanced we have $\breve{A} \subseteq \breve{M} = M$ and hence $\bar{\breve{A}} \subseteq \bar{M} = M$ by the following result.

Proposition 2.7. *Let E be a TVS and $A \subseteq E$ balanced. Then $\bar{A}$ is balanced and if $0 \in \operatorname{int}(A)$ then $\operatorname{int}(A)$ is balanced.*

We denote by $\operatorname{int}(A)$ denotes the interior of a set A.

Proof. $\mathbb{D} \cdot A \subseteq A$ implies $\mathbb{D} \cdot \bar{A} \subseteq \bar{A}$ by continuity of multiplication. For $\rho \in \mathbb{D} \setminus \{0\}$ we have $\rho \operatorname{int}(A) = \operatorname{int}(\rho A) \subseteq \operatorname{int}(A)$, which gives the second claim. □

Corollary 2.8. *If $\mathscr{U}$ is a 0-basis in E then $\{\operatorname{int}(\breve{U}) \mid U \in \mathscr{U}\}$, $\{\breve{U} \mid U \in \mathscr{U}\}$ and $\{\bar{\breve{U}} : U \in \mathscr{U}\}$ are 0-bases in E.*

Proof. Let $U \in \mathscr{U}$ and choose (Corollary 2.5) $U_0 \in \mathscr{U}$ such that $\bar{U_0} \subseteq U$. By continuity of multiplication there are $\lambda_0 > 0$ and $V \in \mathscr{U}$ such that $\lambda V \subseteq U_0$ for $|\lambda| \leq \lambda_0$. Set $V_0 := \lambda_0 V$; then $\breve{V_0} \subseteq U_0$ and for $W \in \mathscr{U}$ such that $W \subseteq \lambda_0 V$ we have

$$\operatorname{int}(W) \subseteq \operatorname{int}(\breve{W}) \subseteq \breve{W} \subseteq \bar{\breve{W}} \subseteq \bar{U_0} \subseteq U.$$ □

Proposition 2.9. *In a TVS every 0-neighborhood is absorbent.*

Proof. Let U be a 0-neighborhood in a TVS E and fix $x \in E$. Because $\lambda \mapsto \lambda x$ is continuous at 0 there is $\lambda_0 > 0$ such that $\lambda x \in U$ for $|\lambda| \leq \lambda_0$. □

Next, we identify the minimal requirements on a filter basis such that it generates a linear topology.

Theorem 2.10. *In a TVS E there exists a 0-basis $\mathscr{U}$ such that*

(i) every $U \in \mathscr{U}$ is balanced and absorbent,
(ii) for every $U \in \mathscr{U}$ there exists $V \in \mathscr{U}$ such that $V + V \subseteq U$.

Conversely, if $\mathscr{U}$ is a filter basis on a vector space E satisfying these properties then there exists a unique linear topology $\mathscr{T}$ on E which has $\mathscr{U}$ as 0-basis.

Proof. The first claim is already shown. For the second part, uniqueness follows because a linear topology is uniquely determined by its 0-neighborhood filter. Hence, we have to show that the family $\mathscr{T}$ given by all subsets $A \subseteq E$ such that $\forall x \in A\ \exists U \in \mathscr{U} : x+U \subseteq A$ is a linear topology having $\mathscr{U}$ as 0-basis.

$\mathscr{T}$ clearly is a topology.

For any $(x, y) \in E \times E$ and $U \in \mathscr{U}$ choose $V \in \mathscr{U}$ such that $V + V \subseteq U$. Then $(x + V) + (y + V) \subseteq (x + y) + U$, hence addition is continuous at (x, y).

For $(\rho, x) \in \mathbb{K}\times E$ and $U \in \mathscr{U}$ we want that for $\lambda-\rho$ and $y-x$ in suitable 0-neighborhoods in $\mathbb{K}$ and E, respectively, we have

$$\lambda y - \rho x = (\lambda - \rho)(y - x) + (\lambda - \rho)x + \rho(y - x) \in U.$$

Set $V_0 := U$ and choose a sequence $(V_k)_{k\in\mathbb{N}}$ in $\mathscr{U}$ such that $V_k + V_k \subseteq V_{k-1}$ for all $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $|\rho| \leq 2^n - 2$ and $0 < \sigma \leq 1$ such that $[0, \sigma] \cdot x \subseteq V_n$. Then for $\lambda - \rho \in \sigma\mathbb{D}$ and $y - x \in V_n$ we have

$$\begin{gathered}(\lambda - \rho)(y - x) \in V_n,\\ (\lambda - \rho)x \in V_n,\\ \rho(y - x) \in (2^n - 2)V_n \subseteq V_n + \ldots + V_n \quad (2^n - 2 \text{ summands})\end{gathered}$$

and hence

$$\lambda y - \rho x \in \underbrace{V_n + \ldots + V_n}_{2^n-2} + V_n + V_n \subseteq \ldots \subseteq V_0 = U,$$

so multiplication is continuous at (ρ, x).

Every 0-neighborhood in $\mathscr{T}$ contains an element of $\mathscr{U}$ by definition. In order to show that each $U \in \mathscr{U}$ is a 0-neighborhood in $\mathscr{T}$, set

$$M := \{x \in E \mid x + V \subseteq U \text{ for some } V \in \mathscr{U}\}.$$

Given $x \in M$ choose $V, W \in \mathscr{U}$ such that $x + V \subseteq U$ and $W + W \subseteq V$. Then $x + W + W \subseteq U$ and thus $x + W \subseteq M$, hence $M \in \mathscr{T}$. Because $0 \in M \subseteq U$, U is a 0-neighborhood in $\mathscr{T}$. □

Proposition 2.11. *A TVS E is Hausdorff if and only if for every $x \neq 0$ there exists a neighborhood U of 0 which does not contain x, i.e., if*

$$\{0\} = \bigcap\{U \mid U \text{ is a } 0\text{-neighborhood}\}.$$

Proof. Suppose E is Hausdorff and $x \neq 0$. Then there are 0-neighborhoods U, V such that $U \cap (x + V) = \emptyset$, hence $x \notin U$.

Conversely, let $x \neq y$. Then there is a 0-neighborhood U such that $x - y \notin U$. Choose a 0-neighborhood V such that $V - V \subseteq U$ and suppose that $(x + V) \cap (y + V) \neq \emptyset$; then $x + x' = y + y'$ for some $x', y' \in V$ and $x - y = y' - x' \in V - V \subseteq U$, which gives a contradiction. □

By Corollary 2.4, we see:

Proposition 2.12. *A TVS E is Hausdorff if and only if the set $\{0\}$ is closed.*

3. Locally convex spaces

Let E be a vector space. A subset $A \subseteq E$ is called *convex* if $\forall \alpha, \beta \geq 0$ with $\alpha + \beta = 1$, $\alpha A + \beta A \subseteq A$. If A is convex also $a + \lambda A$ ($x \in E, \lambda \in \mathbb{K}$) is convex. If $f\colon E \to F$ is a linear map (where F is a vector space) then $f(A)$ and $f^{-1}(B)$ are convex if A and B are so. The intersection of an arbitrary family of convex sets is convex. For any $B \subseteq E$ there is a smallest convex set A containing B, called the *convex hull* of B.

Proposition 3.1. *Let A be a convex subset of a vector space, $x_1, \dots, x_n \in A$ and $\lambda_1, \dots, \lambda_n \geq 0$ such that $\sum_i \lambda_i = 1$. Then $\sum_{i=1}^n \lambda_i x_i \in A$.*

Proof. Clear for $n = 1$. Assuming that the claim holds for $n-1$ summands, suppose that $\lambda_i > 0$ for all i. With $\alpha = \sum_{i=1}^{n-1} \lambda_i$ we have

$$\sum_{i=1}^n \lambda_i x_i = \alpha(\sum_{i=1}^{n-1} \frac{\lambda_i}{\alpha} x_i) + \lambda_n x_n \in A$$

because $\sum_{i=1}^{n-1} \frac{\lambda_i}{\alpha} = 1$ and $\sum_{i=1}^{n-1} \frac{\lambda_i}{\alpha} x_i \in A$ by assumption. □

Proposition 3.2. *Let $(A_\iota)_{\iota \in I}$ be a family of convex subsets of a vector space. Then the convex hull of $\bigcup_{\iota \in I} A_\iota$ is given by*

$$C := \left\{ \sum \lambda_\iota x_\iota \;\middle|\; x_\iota \in A_\iota, \lambda_\iota \geq 0, \sum_\iota \lambda_\iota = 1, \text{ only finitely many } \lambda_i \neq 0 \right\}.$$

Proof. Any convex set containing the A_ι contains C, so C is contained in the convex hull. Conversely, $A_\iota \subseteq C$ by taking $\lambda_\iota = 1$, so we have to show that C is convex.

Let $x = \sum \lambda_\iota x_\iota$, $y = \sum \mu_\iota y_\iota \in C$, $\alpha, \beta > 0$, $\alpha + \beta = 1$. Set $\nu_\iota = \alpha\lambda_\iota + \beta\mu_\iota$ and $J = \{\iota | \nu_\iota > 0\}$. Then

$$z_\iota := \frac{\alpha\lambda_\iota x_\iota + \beta\mu_\iota y_\iota}{\alpha\lambda_\iota + \beta\mu_\iota} \in A_\iota \qquad \forall \iota \in J,$$

and $\alpha x + \beta y = \sum_{\iota \in J} \nu_\iota z_\iota \in C$ because $\sum \nu_\iota = \alpha \sum \lambda_\iota + \beta \sum \mu_\iota = 1$. □

Corollary 3.3. *The convex hull of A is the set of all finite linear combinations $\sum \lambda_i x_i$, $x_i \in A$, $\lambda_i \geq 0$, $\sum \lambda_i = 1$.*

Proof. Apply Proposition 3.2 to $A = \bigcup_{x \in A} \{x\}$. □

Proposition 3.4. *Let E be a TVS. If $A \subseteq E$ is convex then $\bar{A}$ is convex.*

Proof. Let $x, y \in \bar{A}$, $\alpha, \beta > 0$, $\alpha + \beta = 1$ and W a neighborhood of $\alpha x + \beta y$. Because $(u, v) \mapsto (\alpha u, \beta v)$ is continuous there are neighborhoods U of x and V of y such that $\alpha U + \beta V \subseteq W$. Let $z \in U \cap A \neq \emptyset$, $w \in V \cap A \neq \emptyset$, then $\alpha z + \beta w \in W \cap A$, so $\alpha x + \beta y \in \bar{A}$. □

Definition 3.5. *A TVS is called* locally convex *if it has a 0-basis consisting of convex sets.*

We will simply call a locally convex topological vector space a locally convex space, or LCS in short.

Proposition 3.6. *A locally convex space (LCS) has a 0-basis consisting of balanced convex closed sets.*

Proof. Let W be a 0-neighborhood; by Corollary 2.5 it contains a closed 0-neighborhood V, and V contains a convex 0-neighborhood U; we have $\bar{U} \subseteq V$. Set $B := \bigcap_{|\lambda|\geq 1} \lambda \bar{U}$, which is closed.

B is a 0-neighborhood: by continuity of multiplication there are $\alpha > 0$ and a 0-neighborhood X such that $\mu x \in \bar{U}$ for $|\mu| < \alpha$, $x \in X$. αX is a 0-neighborhood and $\alpha X \subseteq B$ because for $|\lambda| \geq 1$ and $x \in X$, $\frac{\alpha}{\lambda} x \in \bar{U}$ and hence $\alpha x \in \lambda \bar{U}$.

B is balanced: let $|\mu| \leq 1$, $x \in B$. Then for $|\lambda| \geq 1$, $x \in \frac{\lambda}{\mu}\bar{U}$ because $\left|\frac{\lambda}{\mu}\right| \geq 1$, so $\mu x \in \lambda \bar{U}$. □

Conversely, we have

Proposition 3.7. *Let E be a vector space and $\mathscr{B}$ a filter basis on E consisting of absorbent, balanced, convex sets. Let $\mathscr{N} := \{\lambda V \mid \lambda > 0, V \in \mathscr{B}\}$. Then there exists a unique topology on E for which E is a locally convex space and which has $\mathscr{N}$ as a 0-basis.*

Proof. We apply Theorem 2.10; note that for $V \in \mathscr{N}$, $\frac{1}{2}V \in \mathscr{N}$ and $\frac{1}{2}V + \frac{1}{2}V \subseteq V$. □

Corollary 3.8. *Let E be a vector space and $\mathfrak{S}$ a family of absorbent, balanced, convex subsets of E. Let $\mathscr{B}$ be the collection of finite intersection of sets of the form λV, $\lambda > 0$, $V \in \mathfrak{S}$. Then there exists a unique topology on E for which E is a locally convex space and $\mathscr{B}$ is a 0-basis.*

Definition 3.9. *A seminorm on a vector space E is a mapping $q \colon E \to \mathbb{R}_+ = [0, \infty)$ such that*

$$q(\lambda x) = |\lambda|\, q(x) \qquad (\lambda \in \mathbb{K}, x \in E)$$
$$q(x + y) \leq q(x) + q(y) \qquad (x, y \in E).$$

Note that if $q(x) = 0$ implies $x = 0$ then q is a norm. [3.3.]

Remark. A balanced set A is absorbent if and only if $\forall x\ \exists \lambda_0 > 0$: $x \in \lambda_0 A$ (because for $|\lambda| \geq \lambda_0$, $\lambda_0 A \subseteq \lambda \mathbb{D} A = \lambda A$).

We denote the *open unit ball* and *closed unit ball* of a seminorm p by

$$p_{<1} := \{x \mid p(x) < 1\}, \qquad p_{\leq 1} := \{x \mid p(x) \leq 1\}.$$

Note that $\varepsilon \cdot p_{<1} = \{x \mid p(x) < \varepsilon\}$ and $\varepsilon \cdot p_{\leq 1} = \{x \mid p(x) \leq \varepsilon\}$. These are balanced ($\lambda \in \mathbb{D}$, $x \in E \Rightarrow p(\lambda x) \leq |\lambda|\, p(x) \leq p(x)$), absorbent ($x \in (\varepsilon + p(x))p_{<1}$ for any $\varepsilon > 0$) and convex ($p(\alpha x + \beta y) \leq \alpha p(x) + \beta p(y)$ for $\alpha, \beta > 0$).

Definition 3.10. *Let E be a vector space. For $A \subseteq E$, the* gauge *or* Minkowski functional *of A is the map $g_A \colon E \to \mathbb{R}_+ \cup \{\infty\}$ defined by $g_A(x) := \inf\{\rho > 0 \mid x \in \rho A\}$.*

We note:

(i) If A is absorbent then g_A is finite.
(ii) If $0 \in A$ then $g_A(0) = 0$.
(iii) If A is convex then $g_A(x + y) \leq g_A(x) + g_A(y)$. This is clear for $g_A(x)$ or $g_A(y)$ equal to ∞. Otherwise, note that $\lambda A + \mu A = (\lambda + \mu)A$ for $\lambda, \mu \geq 0$. Then $\forall \varepsilon > 0$ $\exists \rho, \sigma$ such that

$$g_A(x) \leq \rho < g_A(x) + \varepsilon, \qquad x \in \rho A,$$
$$g_A(y) \leq \sigma < g_A(y) + \varepsilon, \qquad y \in \sigma A,$$

and hence $x + y \in (\rho + \sigma)A$, which implies $g_A(x + y) < g_A(x) + g_A(y) + 2\varepsilon$.

(iv) If A is balanced then $g_A(\lambda x) = |\lambda|\, g(x)$ for $\lambda \in \mathbb{K}$ ($\lambda = 0$ clear; $\lambda \neq 0$: $\rho_A(\lambda x) = \inf\{\rho > 0 \mid |\lambda|\, x \in \rho A\} = |\lambda| \cdot \inf\{\rho\, |\lambda|^{-1} \mid \rho > 0, x \in \rho\, |\lambda|^{-1} A\} = |\lambda|\, \rho_A(x)$) and $x \in \alpha A$ for $\alpha > g_A(x)$, because then $\exists \eta$: $g_A(x) \leq \eta < \alpha$ and $x \in \eta A = \alpha \frac{\eta}{\alpha} A \subseteq \alpha \mathbb{D} A \subseteq \alpha A$.

Proposition 3.11. *In a vector space E the gauge p of an absorbent, balanced and convex set A is a seminorm with*

$$p_{<1} \subseteq A \subseteq p_{\leq 1}. \tag{1}$$

Note that if (1) holds for a seminorm p on E and a subset $A \subseteq E$, then $g_A = p$:

$$\alpha > p(x) \Rightarrow p(\frac{x}{\alpha}) < 1 \Rightarrow \frac{x}{\alpha} \in p_{<1} \Rightarrow x \in \alpha A,$$
$$0 > \alpha < p(x) \Rightarrow p(\frac{x}{\alpha}) > 1 \Rightarrow \frac{x}{\alpha} \notin p_{\leq 1} \Rightarrow x \notin \alpha A.$$

We will now relate seminorms to topologies.

Lemma 3.12. *Let p be a seminorm on a TVS E. Then*

$$\operatorname{int}(p_{\leq 1}) \subseteq p_{<1} \subseteq p_{\leq 1} \subseteq \overline{p_{<1}}.$$

Proof. $p_{<1} \subseteq p_{\leq 1}$ is clear. If $x \in p_{\leq 1}$ then $x/(1+\varepsilon) \in p_{<1}$ for all $\varepsilon > 0$, $x/(1+\varepsilon) \to x$ for $\varepsilon \to 0$, so $x \in \overline{p_{<1}}$. Similarly, one sees $E \setminus p_{<1} = p_{\geq 1} \subseteq \overline{p_{>1}} \subseteq \overline{E \setminus p_{\leq 1}}$, which is equivalent to $\operatorname{int}(p_{\leq 1}) \subseteq p_{<1}$. □

Proposition 3.13. *Let p be a continuous seminorm on a TVS. Then*

(i) *$p_{<1}$ is open,*
(ii) *$p_{\leq 1}$ is closed,*
(iii) *$\overline{p_{<1}} = p_{\leq 1}$,*
(iv) *$\operatorname{int}(p_{\leq 1}) = p_{<1}$.*

Proof. We have $p_{<1} = p^{-1}([0,1))$ and $p_{\leq 1} = p^{-1}([0,1])$, which gives (i) and (ii). Moreover, $p_{<1} \subseteq p_{\leq 1}$ implies $\overline{p_{<1}} \subseteq \overline{p_{\leq 1}} = p_{\leq 1}$ and $p_{<1} = \operatorname{int}(p_{<1}) \subseteq \operatorname{int}(p_{\leq 1})$, and the converse inclusions are given in Lemma 3.12. □

Proposition 3.14. *Let p be a seminorm on a TVS E. Then the following assertions are equivalent:*

(i) *$p_{<1}$ is open,*
(ii) *$p_{\leq 1}$ is a 0-neighborhood,*
(iii) *p is continuous at 0,*
(iv) *p is continuous.*

Proof. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii): $p^{-1}([0,\varepsilon]) = \varepsilon \cdot p_{\leq 1}$ is a 0-neighborhood for all $\varepsilon > 0$. (iii) $\Rightarrow$ (iv): We have $|p(x) - p(y)| \leq p(x-y)$, and for $x \in E$ and $\varepsilon > 0$ there is a 0-neighborhood U such that $p(U) \subseteq \varepsilon\mathbb{D}$ and hence $|p(x+U) - p(x)| \subseteq \varepsilon\mathbb{D}$. (iv) $\Rightarrow$ (i) is Proposition 3.13 (i). □

Definition 3.15. *An absorbent balanced closed convex subset of a TVS is called a* barrel.

Every LCS has a 0-basis consisting of barrels (Proposition 3.6).

Proposition 3.16. *Let A be a barrel in a TVS E. Then there is a unique seminorm p on E whose closed unit ball equals A.*

Proof. Let p be the gauge of A. Then we have $p_{<1} \subseteq A \subseteq p_{\leq 1} \subseteq \overline{p_{<1}}$ (Proposition 3.11 and Lemma 3.12). Taking the closures shows that $p_{\leq 1} = A$, and uniqueness follows from the remark after Proposition 3.11. □

We will now show how locally convex topologies are defined by seminorms.

Let $\mathscr{P}$ be a family of seminorms on a vector space E. By Corollary 3.8, the family of all finite intersections of the sets $\varepsilon \cdot p_{\leq 1}$ ($\varepsilon > 0$, $p \in \mathscr{P}$) defines a locally convex topology $\mathscr{T}$ on E, having as 0-basis the sets

$$\{x \mid p_k(x) \leq \varepsilon_k \text{ for } 1 \leq k \leq n\} \qquad (p_1, \dots, p_n \in \mathscr{P}, \varepsilon_1, \dots, \varepsilon_n > 0)$$

or, equivalently, the sets

$$\{x \mid p_k(x) \leq \varepsilon \text{ for } 1 \leq k \leq n\} \qquad (p_1, \dots, p_n \in \mathscr{P}, \varepsilon > 0).$$

We say that $\mathscr{T}$ is *defined* or *generated* by the family $\mathscr{P}$.

Theorem 3.17. *A locally convex topology can always be defined by a family of seminorms.*

Proof. By Proposition 3.6 the balanced, closed, convex 0-neighborhoods V form a 0-basis. Each gauge g_V is a continuous seminorm on E, and the locally convex topology defined by $(g_V)_V$ is the same as the original topology because $V = (\rho_V)_{\leq 1}$. □

Proposition 3.18. *A locally convex topology defined by a family $\mathscr{P}$ of seminorms is Hausdorff if and only if $\forall x \neq 0\ \exists p \in \mathscr{P}$: $p(x) \neq 0$.*

Proof. If a LCS E is Hausdorff then $x \neq 0$ implies that there is a 0-neighborhood U such that $x \notin U$, and U contains some set $\{x \mid p_k(x) \leq \varepsilon \text{ for } k = 1 \dots n\}$ for some $p_1, \dots, p_n \in \mathscr{P}$ and $\varepsilon > 0$, so $p_k(x) > \varepsilon$ for some k. Conversely, if $p(x) = \alpha > 0$ for some $p \in \mathscr{P}$ then $\{y \mid p(y) \leq \alpha/2\}$ is a 0-neighborhood not containing x, so the topology is Hausdorff by Proposition 2.11. □

Proposition 3.19. *Let E, F be LCS and $f\colon E \to F$ linear. Then f is continuous if and only if for each continuous seminorm q on F there is a continuous seminorm p on E such that $q(f(x)) \leq p(x)$ for all $x \in E$.*

Proof. Suppose f is continous and let q be a continuous seminorm on F. Then $q_{\leq 1}$ is a 0-neighborhood in F, so there is a balanced convex 0-neighborhood U in E with $f(U) \subseteq q_{\leq 1}$. Let p the gauge of U. Then for all $x \in E$ and $\varepsilon > 0$,

$$q(f(x)) = (p(x) + \varepsilon) \cdot q(f(\frac{x}{p(x) + \varepsilon})) \leq p(x) + \varepsilon$$

because $\frac{x}{p(x)+\varepsilon} \in p_{<1} \subseteq U$, so $q(f(x)) \leq p(x)$.

Conversely, let a 0-neighborhood V in F be given, which can be assumed to be balanced and convex. Its gauge q is a continuous seminorm, so there exists a continuous seminorm p on E such that $q(f(x)) \leq p(x)$ for all $x \in E$. $p_{<1}$ is a 0-neighborhood in E such that $f(p_{<1}) \subseteq q_{<1} \subseteq V$, so f is continuous. □

In practice, it is convenient if we don't have to consider all seminorms but just enough to describe the topology.

Definition 3.20. *Let $\mathscr{P}$ be a family of continuous seminorms on a LCS E. We say that $\mathscr{P}$ is a* basis of continuous seminorms *(or a* fundamental system of continuous seminorms*) on E if for every continuous seminorm p on E there exists $p' \in \mathscr{P}$ and $C > 0$ such that $p(x) \leq Cp'(x)$ for all $x \in E$.*

Proposition 3.21. *Let E, F be LCS with bases of continuous seminorms $\mathscr{P}$ and $\mathscr{Q}$, respectively, and $f\colon E \to F$ a linear mapping. Then f is continuous if and only if for every $q \in \mathscr{Q}$ there is a seminorm $p \in \mathscr{P}$ and $C > 0$ such that $q(f(x)) \le Cp(x)$ for $x \in E$.*

Proof. "$\Rightarrow$" is clear; "$\Leftarrow$": given a continuous seminorm q on F, $q \le Cq'$ for some $C > 0$ and $q' \in \mathscr{Q}$, $q'(f(x)) \le C'p(x)$ for some $p \in \mathscr{P}$ and $C' > 0$, which gives $q(f(x)) \le CC'p(x)$, and $x \mapsto CC'p(x)$ is a continuous seminorm on E. $\square$

Remark. Balanced convex sets are also called *absolutely convex* because a set A is balanced and convex and if and only if $\forall x_1, \dots, x_n \in A\ \forall \lambda_1, \dots, \lambda_n \in \mathbb{K}$ with $\sum |\lambda_i| \le 1$: $\sum \lambda_i x_i \in A$. The balanced convex hull (also: absolutely convex hull) of $\bigcup_\iota A_\iota$ (where each A_ι is absolutely convex) is

$$\left\{ \sum \lambda_\iota x_\iota \;\middle|\; \sum |\lambda_\iota| \le 1 \text{ (finite sum)}, \lambda_\iota \in \mathbb{K},\ x_\iota \in A_\iota \right\}.$$

We denote the absolutely convex hull of a set A by $\operatorname{acx}(A)$.

Examples. Let $\Omega \subseteq \mathbb{R}^n$ be open and $m \in \mathbb{N}_0 \cup \{\infty\}$. The following spaces are LCS:

1. $C(\Omega)$ with seminorms $q_K(f) = \max_{x \in K} |f(x)|$ for all $K \subseteq \Omega$ compact.
2. $\mathscr{E}^m(\Omega) := \{f\colon \Omega \to \mathbb{K} \mid \partial^\alpha f$ exists and is continuous for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m\}$ with seminorms $q_{K,\alpha}(f) = \max_{x \in K} |\partial^\alpha f(x)|$ for $K \subseteq \Omega$ compact and $|\alpha| \le m$.
3. $\mathscr{E}(\Omega) = \bigcap_{m=0}^\infty \mathscr{E}^m(\Omega)$.
4. For $K \subseteq \Omega$ compact: $\mathscr{D}^m(K) = \{f \in \mathscr{E}^m(\Omega) \mid \operatorname{supp} f \subseteq K\}$ with seminorms $q_{K,\alpha}$.
5. $\mathscr{D}(K) = \mathscr{D}^\infty(K)$.

4. Completeness

Definition 4.1. *Let E be a TVS and $A \subseteq E$. A filter (filter basis) $\mathscr{F}$ on A is called a* Cauchy filter (Cauchy filter basis) *if for every 0-neighborhood U in E there exists $X \in \mathscr{F}$ such that $X - X \subseteq U$. A sequence $(x_n)_n$ in A is called a* Cauchy sequence *if for every 0-neighborhood U in E there exists $N \in \mathbb{N}$ such that $x_n - x_m \in U$ for $n, m \ge N$.*

A filter basis is a Cauchy filter basis if and only if the filter generated by it is a Cauchy filter; moreover, any convergent filter (filter basis) is a Cauchy filter (Cauchy filter basis).

Definition 4.2. *Let E be a TVS. A subset $A \subseteq E$ is called* complete *if every Cauchy filter on A has a limit in A and* sequentially complete *if every Cauchy sequence in A has a limit in A.*

Proposition 4.3. *Let E be a Hausdorff TVS. If $A \subseteq E$ is complete then it is closed.*

Proof. For $x \in \bar{A}$ let $\mathcal{F}$ be the neighborhood filter of x. Then its trace $\mathcal{F}_A$ on A is a filter on A which is even a Cauchy filter (U 0-neighborhood in $E \Rightarrow \exists x \in \mathscr{F} : X - X \subseteq U$ $\Rightarrow X \cap A \in \mathscr{F}_A$, $(X \cap A) - (X \cap A) \subseteq U$). Hence, the filter on E generated by $\mathscr{F}_A$ is a Cauchy filter and converges to an element of A; because this filter is finer than $\mathscr{F}$ and $\mathscr{F} \to x$, $x \in A$. $\square$

Proposition 4.4. *Let E be a TVS and $A \subseteq E$ complete. Then every closed subset of A is complete.*

Proof. Let $B \subseteq A$ be closed and $\mathscr{F}$ a Cauchy filter on B. $\mathscr{F}$ generates a Cauchy filter $\mathscr{F}'$ on A, and $\mathscr{F}'$ and hence $\mathscr{F}$ converges to some $x \in A$, but x (as a limit point of $\mathscr{F}$) has to be in $\bar{B} = B$. $\square$

Definition 4.5. *Let E and F be TVS and $A \subseteq E$. A mapping $f\colon A \to F$ is called* uniformly continuous *if for every 0-neighborhood U in F there exists a 0-neighborhood V in E such that for all $x, y \in A$, $x - y \in V$ implies $f(x) - f(y) \in U$.*

If a linear map $f\colon E \to F$ between TVS is continuous at 0 it is also uniformly continuous.

Lemma 4.6. *Let E, F be TVS, $A \subseteq E$ and $f\colon A \to F$ uniformly continuous. If $\mathscr{F}$ is a Cauchy filter on A then $f(\mathscr{F})$ is a Cauchy filter basis in F.*

Proof. If U is a 0-neighborhood in F there is a 0-neighborhood V in E such that $x - y \in V$ implies $f(x) - f(y) \in U$ for $x, y \in A$. By assumption there is $F \in \mathscr{F}$ such that $F - F \subseteq V$, which implies $f(F) - f(F) \subseteq U$, and $f(F) \in f(\mathscr{F})$. □

Proposition 4.7. *If M is a linear subspace of a TVS then its closure $\bar{M}$ is so.*

Proof. Let $a, b \in \bar{M}$ and U an arbitrary neighborhood of $a + b$. There are neighborhoods V of a and W of b such that $V + W \subseteq U$, and $V \cap M \neq \emptyset$, $W \cap M \neq \emptyset$ imply $U \cap M \neq \emptyset$, so $a + b \in \bar{M}$.

Let $\lambda \in \mathbb{K}$, $a \in \bar{M}$ and U a neighborhood of λa. There is a neighborhood V of a with $\lambda V \subseteq U$. $V \cap M \neq \emptyset$ implies $\lambda V \cap M \neq \emptyset$, hence $U \cap M \neq \emptyset$ and $\lambda a \in \bar{M}$. □

[10.3.]

Lemma 4.8. *Let E, F be TVS and $A \subseteq E$. If $f\colon A \to F$ is uniformly continuous it is continuous.*

Proof. If U is a 0-neighborhood in F there is a 0-neighborhood V in E such that $x - y \in V$ implies $f(x) - f(y) \in U$. Given $x \in A$, $(x + V) \cap A$ is a neighborhood of a in A and $x + V - x \in V$ implies $f(x + V) - f(x) \subseteq U$, hence $f(x + V) \subseteq f(x) + U$. □

Proposition 4.9. *Let E be a TVS, F a complete Hausdorff TVS, and $A \subseteq E$. If $f\colon A \to F$ is uniformly continuous then there exists a unique uniformly continuous mapping $\hat{f}\colon \bar{A} \to F$ such that $\hat{f}(x) = f(x)\ \forall x \in A$.*

If A is a linear subspace of E and f is linear, $\hat{f}$ also is linear.

Proof. For $x \in \bar{A}$, let $\mathscr{F}$ be its neighborhood filter in E. Then its trace $\mathscr{F}_A$ is a Cauchy filter and by Lemma 4.6, $f(\mathscr{F}_A)$ is a Cauchy filter basis in F and hence converges to some unique $y \in F$. We define $\hat{f}(x) := y$. For $x \in A$ we have $\mathscr{F}_A \to x$ and hence $f(\mathscr{F}_A) \to f(x)$, so $\hat{f}(x) = f(x)$.

Let $\mathscr{U}$ and $\mathscr{V}$ be 0-bases in E and F, resp., consisting of balanced closed 0-neighborhoods (Corollary 2.8). Let $V_0 \in \mathscr{V}$ and choose $V \in \mathscr{V}$ such that $V + V + V \subseteq V_0$, $U \in \mathscr{U}$ such that $x_1 - x_2 \in U$ implies $f(x_1) - f(x_2) \in V$ for $x_1, x_2 \in A$, and $U_1 \in \mathscr{U}$ with $U_1 + U_1 + U_1 \subseteq U$.

For $x, y \in \bar{A}$ with $x - y \in U_1$ there are $x_1, x_2 \in A$ such that $x_1 - x, y_1 - x \in U_1$. Let us prove $\hat{f}(x) \in f(x_1) + V$: let X be a neighborhood of $\hat{f}(x)$, then $f(Y \cap A) \subseteq X$ for some neighborhood Y of x. $(x + U_1) \cap Y$ is a neighborhood of x, so there is some $u \in (x + U_1) \cap Y \cap A$. Then, $u - x_1 = (u - x) + (x - x_1) \in U_1 + U_1 \subseteq U$ so $f(u) - f(x_1) \in V$, i.e., $f(u) \in f(x_1) + V$. Moreover, $f(u) \in X$ so $f(u) \in (f(x_1) + V) \cap X$. Hence, $\hat{f}(x) \in \overline{f(x_1) + V} = f(x_1) + \bar{V} = f(x_1) + V$. Similarly, $\hat{f}(y) \in f(y_1) + V$. Then $x_1 - y_1 = (x_1 - x) + (x - y) + (y - y_1) \in U_1 + U_1 + U_1 \subseteq U$, so $f(x_1) - f(y_1) \in V$. Finally, $\hat{f}(x) - \hat{f}(y) = (\hat{f}(x) - f(x_1)) + (f(x_1) - f(y_1)) + (f(y_1) - \hat{f}(y)) \in V + V + V \subseteq V_0$, so $\hat{f}$ is uniformly continuous.

Uniqueness of $\hat{f}$ follows from the fact that F is Hausdorff, as for two mappings $\bar{A} \to F$ the set where they are equal (which contains A) is closed, hence equal to $\bar{A}$.

For the second claim let $V \in \mathscr{V}$ be arbitrary and $U \in \mathscr{U}$ such that for $x, y \in \bar{A}$, $x - y \in U + U$ implies $\bar{f}(x) - \bar{f}(y) \in V$. Choose $x_1, y_1 \in A$ such that $x - x_1, y - y_1 \in U$ and hence $x + y - (x_1 + y_1) \in U + U$. Then

$$\begin{aligned}\hat{f}(x) + \hat{f}(y) - \hat{f}(x+y) &= \hat{f}(x) - f(x_1) + \hat{f}(y) - f(y_1) - \hat{f}(x+y) + f(x_1 + y_1) \\ &\in V + V + V.\end{aligned}$$

As V is arbitrary and F is Hausdorff, $\hat{f}(x) + \hat{f}(y) = \hat{f}(x+y)$ (Proposition 2.11).

For $\lambda \in \mathbb{K} \setminus \{0\}$ (the case $\lambda = 0$ is trivial) and $V \in \mathscr{V}$ choose $V_1 \in \mathscr{V}$ with $V_1 + \lambda V_1 \subseteq V$ and $U_1 \in \mathscr{U}$ such that $x - y \in U_1$ implies $\hat{f}(x) - \hat{f}(y) \in V_1$, as well as $U \in \mathscr{U}$ such that $U \cap \lambda U \subseteq U_1$. Given $x \in \bar{A}$, choose $x_1 \in A$ such that $x - x_1 \in U$ and hence $\lambda x - \lambda x_1 \in \lambda U$. Then

$$\hat{f}(\lambda x) - \lambda \hat{f}(x) = \hat{f}(\lambda x) - f(\lambda x_1) + \lambda(f(x_1) - \hat{f}(x)) \in V_1 + \lambda V_1 \subseteq V,$$

which proves that $\hat{f}(\lambda x) = \lambda \hat{f}(x)$. □

For constructing the completion we need some terminology.

Definition 4.10. *Let E be a TVS. A Cauchy filter $\mathscr{F}$ on E is called* minimal *if for any Cauchy filter $\mathscr{G}$ on E, $\mathscr{G} \subseteq \mathscr{F}$ implies $\mathscr{G} = \mathscr{F}$.*

Proposition 4.11. *Let E be a TVS and $\mathscr{F}$ a Cauchy filter on E. There exists a unique minimal Cauchy filter $\mathscr{F}_0$ on E such that $\mathscr{F}_0 \subseteq \mathscr{F}$. If $\mathscr{B}$ is a basis of $\mathscr{F}$ and $\mathscr{U}$ a 0-basis in E, $\{F + V \mid F \in \mathscr{B}, V \in \mathscr{U}\}$ is a basis of $\mathscr{F}_0$.*

Proof. The family $\{F + V \mid F \in \mathscr{F}, V \text{ 0-neighborhood in } E\}$ is a filter basis (for $F_1, F_2 \in \mathscr{F}$, V_1, V_2 0-neighborhoods we have $F_1 \cap F_2 \in \mathscr{F}$ and $V_1 \cap V_2$ is a 0-neighborhood, and $(F_1 \cap F_2) + (V_1 \cap V_2) \subseteq (F_1 + V_1) \cap (F_2 + V_2)$) and generates a filter $\mathscr{F}_0$ on E.

$\mathscr{F}_0$ is Cauchy: given a 0-neighborhood U in E choose a 0-neighborhood V in E such that $V + V + V \subseteq U$. There is $F \in \mathscr{F}$ with $F - F \subseteq V$. We have $F + V \in \mathscr{F}_0$ and $(F + V) - (F + V) = F - F + V + V \subseteq V + V + V \subseteq U$.

$\mathscr{F}_0 \subseteq \mathscr{F}$: given $F_0 \in \mathscr{F}_0$ we have $F + V \subseteq F_0$ for some F, V, hence $F \subseteq F_0$ which means $F_0 \in \mathscr{F}$.

Finally, if $\mathscr{G} \subseteq \mathscr{F}$ is a Cauchy filter on E, let $F + V$ be a given element of the filter basis generating $\mathscr{F}_0$. There is $G \in \mathscr{G} \subseteq \mathscr{F}$ with $G - G \subseteq V$, and for any $a \in G \cap F \neq \emptyset$ we have $G \subseteq a + V \subseteq F + V$, hence $F + V \in \mathscr{G}$ which implies $\mathscr{F}_0 \subseteq \mathscr{G}$. This shows that $\mathscr{F}_0$ is minimal, and unique. □

The neighborhood filter $\mathscr{F}$ of $x \in E$ is minimal: if $\mathscr{G} \subseteq \mathscr{F}$ is a Cauchy filter and $U \in \mathscr{F}$, then $\exists V \in \mathscr{G}$ with $V - V \subseteq U - x \Rightarrow V - x \subseteq U - x \Rightarrow V \subseteq U \Rightarrow U \in \mathscr{G}$.

Lemma 4.12. *Let E be a TVS and $\mathscr{F}$ a minimal Cauchy filter on E. Then for any $B \in \mathscr{F}$, $\mathring{B} \in \mathscr{F}$.*

Proof. By Proposition 4.11, $\{F + V \mid F \in \mathscr{F}, V \text{ 0-neighborhood}\}$ is a basis of $\mathscr{F}$ because $\mathscr{F}$ is minimal. Given $B \in \mathscr{F}$, $F + V \subseteq B$ for some F, V. Taking an open 0-neighborhood $U \subseteq V$ we have $F + U \subseteq B$ and hence $F + U \subseteq \mathring{B}$ because $F + U$ is open, which implies that $\mathring{B} \in \mathscr{F}$. □

Lemma 4.13. *If E is a TVS and $A \subseteq E$ a dense subset such that every Cauchy filter on A converges in E, then E is complete.*

Proof. A Cauchy filter $\mathscr{F}$ on E contains a minimal Cauchy filter $\mathscr{F}_0$. Because $\mathring{F} \in \mathscr{F}_0$ for $F \in \mathscr{F}_0$, $\mathring{F} \cap A \neq \emptyset$ and $(\mathscr{F}_0)_A$ is a Cauchy filter on A. The filter $\mathscr{G} \supseteq \mathscr{F}_0$ it generates on E converges to some $x \in E$, so x is a cluster point of $\mathscr{F}_0$. Each Cauchy filter converges to its cluster points: if U is a closed 0-neighborhood in E then $F - F \subseteq U$ for some $F \in \mathscr{F}_0$, so $\bar{F} - \bar{F} \subseteq \overline{F - F} \subseteq \bar{U} = U$ and because $x \in \bar{F}$, $F \subseteq \bar{F} \subseteq x + U$, hence $\mathscr{F}_0 \to x$ and $\mathscr{F} \to x$. □

Theorem 4.14. *Let E be a Hausdorff TVS. There exists a complete Hausdorff TVS $\widehat{E}$, called the* completion *of E, and a mapping $\iota\colon E \to \widehat{E}$ such that*

(i) ι is a linear homeomorphism onto its image,
(ii) $\iota(E)$ is dense in $\widehat{E}$.

For any other pair $(\widehat{E}_1, \iota_1)$ such that (i) and (ii) hold there is a linear isomorphism $j\colon \widehat{E} \to \widehat{E}_1$ such that $\iota_1 = j \circ \iota$. In other words, the completion is unique up to isomorphism.

Proof. We define $\widehat{E}$ to be the set of all minimal Cauchy filters on E. For $\mathscr{F}, \mathscr{G} \in \widehat{E}$ we define $\mathscr{F} + \mathscr{G}$ to be the minimal Cauchy filter contained in the filter generated by the Cauchy filter basis $\{A + B \mid A \in \mathscr{F}, B \in \mathscr{G}\}$; similarly, define $\lambda\mathscr{F}$ ($\lambda \in \mathbb{K}$) as the minimal Cauchy filter contained in the filter generated by $\{\lambda A \mid A \in \mathscr{F}\}$. Let $\hat{0}$ be the neighborhood filter of 0 in E. $\widehat{E}$ then is a vector space over $\mathbb{K}$ (exercise!).

For any 0-neighborhood in E we set

$$\widehat{U} := \{\mathscr{F} \in \widehat{E} \mid \exists A \in \mathscr{F}\ \exists V \text{ 0-neighborhood in } E : A + V \subseteq U\}.$$

One easily verifies:

$$U_1 \subseteq U_2 \Rightarrow \widehat{U_1} \subseteq \widehat{U_2},$$
$$\widehat{U_1 \cap U_2} \subseteq \widehat{U_1} \cap \widehat{U_2},$$
$$\lambda\widehat{U} \subseteq \widehat{\lambda U} \qquad (\lambda \neq 0),$$
$$U_1 + U_1 \subseteq U_2 \Rightarrow \widehat{U_1} + \widehat{U_1} \subseteq \widehat{U_2}.$$

In fact, $\lambda\widehat{U} \subseteq \widehat{\lambda U}$ for $\lambda \neq 0$: let $\mathscr{F} \in \widehat{U}$. Then $F + V \subseteq U \Rightarrow \lambda F + \lambda V \subseteq \lambda U$, and $\lambda F \in \lambda\mathscr{F}$, λV is 0-nbhd. $U_1 + U_1 \subseteq U_2$: given $\mathscr{F}, \mathscr{G} \in \widehat{U}_1$, $F + V_1 \subseteq U_1$ and $G + V_2 \subseteq U_1$ imply $F + G + V_1 + V_2 \subseteq U_2$, and $F + G + V_1 \in \mathscr{F} + \mathscr{G}$, so $\mathscr{F} + \mathscr{G} \in \widehat{U}_2$.

Let $\mathscr{U}$ be a 0-basis in E consisting of balanced sets. By Theorem 2.10 there is a unique linear topology $\widehat{\mathscr{T}}$ on $\widehat{E}$ having $\{\widehat{U} \mid U \in \mathscr{U}\}$ as 0-basis. In fact, $\widehat{U} \neq \emptyset$ because $\hat{0} \in \widehat{U}$. $\widehat{U}$ is balanced because $\lambda\widehat{U} \subseteq \widehat{\lambda U} \subseteq \widehat{U}$ for $\lambda \neq 0$. $\widehat{U}$ is absorbent: for $\mathscr{F} \in \widehat{E}$ choose V with $V + V + V \subseteq U$. Then $\exists F \in \mathscr{F} : F - F \subseteq V$. Take $x \in F$ and $\lambda \geq 1$ such that $x \in \lambda V$. Then $F \subseteq x + V$ and $F + V \subseteq \lambda V + V + V \subseteq \lambda(V + V + V) \subseteq \lambda U$, so $\lambda^{-1}F + \lambda^{-1}V \subseteq U$, and $\lambda^{-1}\mathscr{F} \in \widehat{U}$. Given U_1, U_2 $\exists U_3$ with $U_3 \subseteq U_1 \cap U_2$, so $\widehat{U}_3 \subseteq \widehat{U_1 \cap U_2} \subseteq \widehat{U}_1 \cap \widehat{U}_2$.

$\widehat{\mathscr{T}}$ is Hausdorff: suppose $\hat{x} \in \widehat{U}$ for all $U \in \mathscr{U}$. We know that $\forall U \in \hat{0}\ \exists A \in \hat{x}$: $A \subseteq U$, so $\hat{x}$ is finer than $\hat{0}$. Because $\hat{x}$ is minimal, $\hat{0} = \hat{x}$ and $\widehat{\mathscr{T}}$ is Hausdorff.

We define $\iota\colon E \to \widehat{E}$ by mapping x to its neighborhood filter. Linearity of ι is clear. In fact, $x + V_1 + \lambda y + V_2 + V_3 \in \iota(x + \lambda y)$ implies $\iota(x + \lambda y) \subseteq \iota(x) + \lambda\iota(y)$, hence they are equal. $x + V_1$ forms a basis of $\iota(x)$, $\lambda y + V_2$ a basis of $\lambda\iota(y)$, and $x + V_1 + \lambda y + V_2 + V_3$ a basis of $\iota(x) + \lambda\iota(y)$. $\iota(x) = \hat{0}$ implies $x = 0$, so ι is injective.

Let $U \in \mathscr{U}$. For $x \in \mathring{U}$ there is a 0-neighborhood V such that $x + V + V \subseteq U$; because $x + V \in \iota(x)$, $\iota(x) \in \widehat{U}$. On the other hand, if $\iota(y) \in \widehat{U}$ for some $y \in E$ then $y + V_1 + V_2 \subseteq U$

for some 0-neighborhoods V_1 and V_2, hence $y \in U$. Consequently, $\iota(\mathring{U}) \subseteq \widehat{U} \cap \iota(E) \subseteq \iota(U)$, which shows that ι is a homeomorphism.

Finally, we show that $(\widehat{E}, \widehat{\mathscr{T}})$ is complete: let $\mathscr{F}_0$ be a Cauchy filter on $\iota(E)$. Then $\mathscr{F}' := \iota^{-1}(\mathscr{F}_0)$ is a Cauchy filter on E containing a minimal Cauchy filter $\mathscr{F}$. $\iota(\mathscr{F})$ is a Cauchy filter on $\iota(E)$ with $\mathscr{F}_0 = \iota(\mathscr{F}') \supseteq \iota(\mathscr{F})$. Because $\iota(\mathscr{F}) \to \mathscr{F}$ in $\widehat{E}$, $\mathscr{F}_0 \to \mathscr{F}$. By Lemma 4.13, completeness follows.

For uniqueness, the isomorphism $f\colon \iota(E) \to \iota_1(E)$ can be extended to a continuous linear map $\bar{f}\colon \widehat{E} \to \widehat{E}_1$, and the isomorphism $g\colon \iota_1(E) \to \iota(E)$ extends to a continuous linear map $\bar{g}\colon \widehat{E}_1 \to \widehat{E}$ (Proposition 4.9). Because $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$, $\bar{f} \circ \bar{g} = \mathrm{id}$ and $\bar{g} \circ \bar{f} = \mathrm{id}$. □

Proposition 4.15. *Let E be a dense subspace of a TVS F. If $\mathscr{U}$ is a 0-basis in E then $\{\bar{U} \mid U \in \mathscr{U}\}$ (where the closures are taken in F) is a 0-basis in F.*

Proof. Let V be an open 0-neighborhood in F. Then $U := V \cap E$ is an open 0-neighborhood in E. For $z \in V$ and any neighborhood W of z in F, $V \cap W$ is a neighborhood of z, so $\emptyset \neq V \cap W \cap E = W \cap U$, and $z \in \bar{U}$. Hence, $U \subseteq V \subseteq \bar{U} \subseteq \bar{V}$.

Given $U' \in \mathscr{U}$ choose an open 0-neighborhood U in E with $U \subseteq U'$; then $U = V \cap E$ for some open 0-neighborhood V in F and $V \subseteq \bar{U} \subseteq \bar{U}'$, so $\bar{U}'$ is a 0-neighborhood.

Given any 0-neighborhood V' in F, choose an open 0-neighborhood V with $\bar{V} \subseteq V'$. Then $U = V \cap E$ contains some $U' \in \mathscr{U}$ and $\bar{U}' \subseteq \bar{U} \subseteq \bar{V} \subseteq V'$, so the closures of elements of $\mathscr{U}$ form a 0-basis in F. □

By Proposition 3.4 the completion of a LCS is a LCS again.

Proposition 4.16. *If a family $\mathscr{P}$ defines the topology of a LCS E, the family $\{\hat{p} \mid p \in \mathscr{P}\}$ defines the topology of $\widehat{E}$.*

Note for this that a continuous seminorm p is uniformly continuous because $|p(x) - p(y)| \leq p(x-y)$.

Proof. Let $p \in \mathscr{P}$ and $x, y \in \widehat{E}$. For $\varepsilon > 0$ let U be a 0-neighborhood in $\widehat{E}$ such that for $x', y' \in \widehat{E}$ we have

$$x' - y' \in U \Rightarrow |\hat{p}(x') - \hat{p}(y')| \leq \varepsilon. \tag{2}$$

Choose a 0-neighborhood V in $\widehat{E}$ such that $V + V \subseteq U$ and elements $x', y' \in E$ such that $x - x' \in V$, $y - y' \in V$ and hence $x + y - (x' + y') \in U$. Then

$$\begin{aligned} \hat{p}(x+y) - \hat{p}(x) - \hat{p}(y) &= \hat{p}(x+y) - \hat{p}(x'+y') + p(x'+y') - \hat{p}(x) - \hat{p}(y) \\ &\leq \varepsilon + p(x') + p(y') - \hat{p}(x) - \hat{p}(y) \leq 3\varepsilon. \end{aligned}$$

This shows that $\hat{p}(x+y) \leq \hat{p}(x) + \hat{p}(y)$.

Given $x \in \widehat{E}$ and $\lambda \neq 0$ as well as $\varepsilon > 0$, choose a 0-neighborhood U such that (2) holds. Choose a 0-neighborhood V such that $\lambda V \cap V \subseteq U$. Choose any $x' \in E$ such that $x - x' \in V$. Then

$$|\hat{p}(\lambda x) - |\lambda|\,\hat{p}(x)| = |\hat{p}(\lambda x) - \hat{p}(\lambda x') + |\lambda|\,(p(x') - \hat{p}(x))| \leq (1 + |\lambda|)\varepsilon$$

shows that $\hat{p}(\lambda x) = |\lambda|\,\hat{p}(x)$.

It is clear that for $p \in \mathscr{P}$, $\hat{p}_{<\varepsilon}$ is open. Conversely, let a 0-neighborhood in $\widehat{E}$ be given; we can assume that it is of the form $\bar{U}$, where $U = (p_1)_{<\varepsilon} \cap \ldots \cap (p_n)_{<\varepsilon}$ for some $p_1, \ldots, p_n \in \mathscr{P}$

and $\varepsilon > 0$. If every $x \in (\hat{p}_1)_{<\varepsilon} \cap \ldots \cap (\hat{p}_n)_{<\varepsilon}$ is an element of $\bar{U}$, the family of all $\hat{p}$ generates the topology of $\widehat{E}$ as claimed. For each i we can choose a 0-neighborhood V_i in E such that $x + \bar{V}_i \subseteq (\hat{p}_i)_{<\varepsilon}$. With $V := V_1 \cap \ldots \cap V_n$ we have

$$x + \bar{V} \subseteq (\hat{p}_1)_{<\varepsilon} \cap \ldots \cap (\hat{p}_n)_{<\varepsilon}.$$

From this it follows that $x \in \bar{U}$. □

5. Bounded sets, normability, metrizability

Definition 5.1. *Let E be a TVS. A subset $A \subseteq E$ is called* bounded *if for each 0-neighborhood U there is $\lambda_0 > 0$ such that $A \subseteq \lambda U$ for $|\lambda| \geq \lambda_0$.*

The family of bounded sets is closed under formation of subsets, finite unions, closures, and in LCS also absolutely convex hulls. Moreover, if E, F are TVS and $f: E \to F$ is linear and continuous then $f(B)$ is bounded if B is bounded: if U is a 0-neighborhood in F and V a 0-neighborhood in E with $f(V) \subseteq U$, then $B \subseteq \lambda V$ implies $f(B) \subseteq \lambda U$.

Proposition 5.2. *A subset A of a TVS E is bounded if and only if for every sequence $(\lambda_n)_n$ in $\mathbb{K}$ with $\lambda_n \to 0$ and every sequence $(x_n)_n$ in A, $\lambda_n x_n \to 0$ in E.*

Proof. Let A be bounded, U a balanced 0-neighborhood in E and $(\lambda_n)_n$ with $\lambda_n \to 0$. There is $\lambda > 0$ with $A \subseteq \lambda U$. Because $|\lambda_n| \leq \lambda^{-1}$ for n large enough we have $\lambda_n x_n \in \lambda^{-1}\lambda U = U$ for n large enough.

Conversely, suppose A satisfies the condition but is not bounded. Then there are a 0-neighborhood U and sequences $(\lambda_n)_n$, $(x_n)_n$ with $|\lambda_n| \geq n$ and $x_n \in A \setminus \lambda_n U$. Because $\lambda_n^{-1} \to 0$ and $\lambda_n^{-1} x_n \notin U$ for all n, this gives a contradiction. □

[17.3.]

Proposition 5.3. *If a LCS E is* locally bounded *(i.e., has a bounded 0-neighborhood) its topology can be defined by a single seminorm.*

Proof. Let V be a bounded 0-neighborhood. By Proposition 3.6 there is an absolutely convex closed 0-neighborhood W contained in V. The gauge q of W is a continuous seminorm and the sets $\varepsilon q_{\leq 1}$ form a 0-basis: if U is any 0-neighborhood there is $\varepsilon > 0$ such that $\varepsilon V \subseteq U$ and hence $\varepsilon q_{\leq 1} = \varepsilon W \subseteq U$. □

In this case, q is a norm if and only if E is Hausdorff (Proposition 3.18). Hence, a TVS is *normable* if and only if it is locally convex, Hausdorff and has a bounded 0-neighborhood.

A TVS E is called *metrizable* if there is a metric on E which induces its original topology. A metrizable topological space is Hausdorff and each point has a countable neighborhood basis. In particular, the topology of a metrizable locally convex space can be defined by a countable family of seminorms, given by the gauges of a countable 0-basis consisting of absolutely convex closed sets (Proposition 3.6). The converse also holds:

Proposition 5.4. *Let E be a Hausdorff LCS whose topology $\mathscr{T}$ can be defined by a sequence $(q_n)_{n\in\mathbb{N}}$ of seminorms such that $q_n(x) \leq q_{n+1}(x)$ for all $x \in E$. Then the map*

$$|x| := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{q_n(x)}{1 + q_n(x)}$$

has the following properties:

(i) $|x| = 0 \Leftrightarrow x = 0$,
(ii) $|x| = |-x|$,

(iii) $|x+y| \le |x| + |y|$,
(iv) $|\lambda| \le 1 \Rightarrow |\lambda x| \le |x|$,
(v) $\lambda \to 0 \Rightarrow |\lambda x| \to 0 \ \forall x \in E$.

The metric $d(x,y) := |x-y|$ *defines the topology* $\mathscr{T}$ *and is translation invariant, i.e.,* $d(x+a, y+a) = d(x,y)\ \forall x,y,a \in E$.

Proof. The series converges. We have $|x| = 0 \Leftrightarrow q_n(x) = 0\ \forall n \Leftrightarrow x = 0$ by Proposition 3.18, which gives (i). (ii) is clear from $q_n(x) = q_n(-x)$.

The function $\xi \mapsto \frac{\xi}{1+\xi}$ is increasing for $\xi > -1$, hence for $|\lambda| \le 1$, $q_n(\lambda x) \le q_n(x)$ implies

$$\frac{q_n(\lambda x)}{1+q_n(\lambda x)} \le \frac{q_n(x)}{1+q_n(x)}$$

which gives (iv).

Since $q_n(x+y) \le q_n(x) + q_n(y)$, we see that

$$\frac{q_n(x+y)}{1+q_n(x+y)} \le \frac{q_n(x)+q_n(y)}{1+q_n(x)+q_n(y)} \le \frac{q_n(x)}{1+q_n(x)} + \frac{q_n(y)}{1+q_n(y)}$$

which gives (iii).

Because of (i), (ii) and (iii), $d(x,y) = |x-y|$ defines a translation-invariant metric. Denote the topology it induces on E by $\mathscr{T}'$. For any $k \in \mathbb{N}$ set

$$V = \left\{ x \mid q_{k+1}(x) \le \frac{1}{2^{k+1}} \right\}, \qquad U = \left\{ x \mid |x| \le \frac{1}{2^k} \right\}.$$

Let $x \in V$; by splitting the sum $|x|$ into two parts,

$$\sum_{n=1}^{k+1} \frac{1}{2^n} \frac{q_n(x)}{1+q_n(x)} \le \frac{1}{2^{k+1}} \sum_{n=1}^{k+1} \frac{1}{2^n} \le \frac{1}{2^{k+1}},$$

$$\sum_{n=k+2}^{\infty} \frac{1}{2^n} \frac{q_n(x)}{1+q_n(x)} \le \sum_{n=k+2}^{\infty} \frac{1}{2^n} = \frac{1}{2^{k+1}},$$

we see that $x \in U$, so $\mathscr{T}$ is finer than $\mathscr{T}'$. Conversely, for any $m \in \mathbb{N}$ set

$$X = \{x \mid |x| \le \frac{1}{2^{m+k+1}}\}, \qquad W = \{x \mid q_m(x) \le \frac{1}{2^k}\}$$

For $x \in X$ we have

$$\begin{aligned}
& \frac{1}{2^m} \frac{q_m(x)}{1+q_m(x)} \le \frac{1}{2^{m+k+1}} \\
\Longrightarrow \quad & \frac{q_m(x)}{1+q_m(x)} \le \frac{1}{2^{k+1}} \\
\Longrightarrow \quad & q_m(x)\left(1 - \frac{1}{2^{k+1}}\right) \le \frac{1}{2^{k+1}} \\
\Longrightarrow \quad & q_m(x) \le \frac{1}{2^{k+1}-1} \le \frac{1}{2^k}.
\end{aligned}$$

and hence $X \subseteq W$, so $\mathscr{T}'$ is finer than $\mathscr{T}$. □

Remark. The condition $q_n(x) \le q_{n+1}(x)$ is no restriction because for any sequence of seminorms p_n, the sequence of seminorms defined by $q_n(x) := \max_{k \le n} p_k(x)$ satisfies this condition and defines the same topology.

Combining Proposition 4.16 and Proposition 5.4, we see:

Corollary 5.5. *The completion of a metrizable LCS is metrizable.*

Definition 5.6. *A complete metrizable locally convex space is called a* Fréchet space*.*

Moreover, we can define the following notion:

Definition 5.7. *A LCS E is called* quasicomplete *if every bounded closed subset of E is complete.*

Note. If we call a filter $\mathscr{F}$ bounded if it contains a bounded set, then a LCS E is quasicomplete if and only if every bounded Cauchy filter converges.

Proposition 5.8. *A metrizable LCS E is complete if and only if it is sequentially complete.*

Proof. If E is complete and a Cauchy sequence $(x_n)_n$ is given, the filter basis consisting of all sets $S_m := \{x_n \mid n \geq m\}$ is a Cauchy filter basis and converges, which means that the sequence converges.

Conversely, if E is sequentially complete and $\widehat{E}$ denotes its completion, let $x \in \widehat{E}$ be arbitrary. Since E is dense in $\widehat{E}$ and $\widehat{E}$ is metrizable there exists a sequence $(x_n)_n$ in E converging to x. Since $(x_n)_n$ is a Cauchy sequence in E, $x \in E$. □

6. Products, subspaces, direct sums and quotients

We recall:

Proposition 6.1. *Let $(X_i)_{i \in I}$ be a family of topological spaces, X a set and $f_i \colon X \to X_i$ a mapping for each i.*

(i) There is a coarsest topology $\mathscr{T}$ on X such that all f_i are continuous.
(ii) If $\mathscr{U}_i$ is a subbasis of X_i for each i, a subbasis of $\mathscr{T}$ is given by $\{f_i^{-1}(U_i) \mid i \in I, U_i \in \mathscr{U}_i\}$.
(iii) If Y is a topological space, a mapping $g \colon Y \to (X, \mathscr{T})$ is continuous if and only if all $f_i \circ g$ are continuous, and $\mathscr{T}$ is the unique topology on X with this property.
(iv) If $\mathscr{F}$ is a filter on X then $\mathscr{F} \to x \in X$ if and only if $f_i(\mathscr{F}) \to f_i(x)$ in X_i for all $i \in I$.

Proof. (i), (ii): The family $\mathscr{B} := \{f_i^{-1}(U_i) \mid i \in I, f_i \in \mathscr{U}_i\}$ is the subbasis of a topology $\mathscr{T}$ for which all f_i are continuous. Any other such topology $\mathscr{T}'$ has to contain $\mathscr{B}$ and hence is finer than $\mathscr{T}$.

(iii): If all $f_i \circ g$ are continuous then $g^{-1}(f_i^{-1}(U_i)) = (f_i \circ g)^{-1}(U_i)$ is open and g is continuous because the $f_i^{-1}(U_i)$ form a subbasis. If $\mathscr{T}'$ is another topology with this property, continuity of $\mathrm{id} \colon (X, \mathscr{T}') \to (X, \mathscr{T}')$ gives that $f_i \colon (X, \mathscr{T}') \to X_i$ is continuous for all i, hence $\mathscr{T}$ is coarser than $\mathscr{T}$, and vice versa.

(iv): Given $\mathscr{F}$ and $x \in X$ with $f_i(\mathscr{F}) \to f_i(x)$ for all i, let U be a neighborhood of x. Then there is a finite set $J \subseteq I$ and neighborhoods $V_i \in \mathscr{U}_i$ of $f_i(x)$ $(i \in J)$ such that $x \in \bigcap_{i \in J} f_i^{-1}(V_i) \subseteq U$. For each $i \in J$ there is $F_i \in \mathscr{F}$ with $f_i(F_i) \subseteq V_i$, and $U \supseteq \bigcap_{i \in J} F_i \in \mathscr{F}$. □

The topology $\mathscr{T}$ of Proposition 6.1 is called the *projective topology* on X with respect to the family $(f_i)_i$.

Proposition 6.2. *Let $(E_i)_i$ be a family of TVS (LCS), E a vector space, $f_i\colon E \to E_i$ a linear mapping for each i, and $\mathscr{T}$ the projective topology on E with respect to $(f_i)_i$.*

(i) $\mathscr{T}$ is a linear (locally convex) topology.
(ii) If $\mathscr{U}_i$ is a 0-basis in E_i then a 0-basis of $\mathscr{T}$ is given by the family of finite intersections of sets of the form $f_i^{-1}(U_i)$ with $U_i \in \mathscr{U}_i$.
(iii) A subset $B \subseteq E$ is bounded if and only if $f_i(B)$ is bounded for all $i \in I$.

Proof. The mappings

$$(x, y) \mapsto f_i(x + y) = f_i(x) + f_i(y)$$
$$(\lambda, x) \mapsto f_i(\lambda \cdot x) = \lambda \cdot f_i(x)$$

are continuous, which gives (i) in the linear case. (ii) is clear from the definition of the projective topology, which gives (i) in the locally convex case. For (iii), $f_i(B)$ is bounded (Remark after Definition 5.1); conversely, for B to be bounded it suffices to show that $B \subseteq \lambda f_i^{-1}(U_i)$ for each i, $U \in \mathscr{U}_i$ and λ large enough, which is implied by $f_i(B) \subseteq \lambda U_i$. □

Corollary 6.3. *If $(p_i)_i$ is a family of seminorms defining the topology of a LCS E then the topology of E is the projective topology with respect to $(p_i)_i$.*

Proof. Finite intersections of sets of the form $\varepsilon(p_i)_{\le 1}$ form a 0-basis of both topologies. □

Corollary 6.4. *Let $(E_i)_{i\in I}$ be a family of LCS, E a vector space and $f_i\colon E \to E_i$ a linear map for each i. If $(p_{i,\lambda})_{\lambda \in L_i}$ is a family of seminorms defining the topology of E_i then $(p_{i,\lambda} \circ f_i)_{i\in I,\lambda\in L_i}$ is a family of seminorms on E defining the projective topology with respect to $(f_i)_i$.*

Proof. This follows from a general property of projective topologies: the projective topology with respect to $(f_i)_i$ is the projective topology with respect to $(p_{i,\lambda} \circ f_i)_{i,\lambda}$ (Proposition 6.1 (iii)). □

Examples.

(i) The *cartesian product* $\prod_{i\in I} E_i$ of a family $(E_i)_i$ of TVS with the projective topology with respect to the projections $\pi_j\colon \prod_i E_i \to E_j$. Note that each π_j is open.
(ii) A *linear subspace* E of a TVS F with the projective topology with respect to the injection $\iota\colon E \to F$.

Proposition 6.5. *A countable product of metrizable LCS is metrizable.*

Proof. A 0-basis in $\prod_i E_i$ is given by all sets $\prod_i X_i$ such that each X_i is taken from a (countable) 0-basis of E_i for finitely many i and $X_i = E_i$ for all other i. The collection of all finite subsets of a countable set is countable. □

Proposition 6.6. *Let $(E_i)_i$ be a family of TVS. The cartesian product $\prod_i E_i$ is Hausdorff if and only if all factors E_i are Hausdorff. If E is a Hausdorff TVS, every subspace of E is Hausdorff.*

We skip the proof, as this is known from topology.

Concerning completeness, we have:

Proposition 6.7. *Let $(E_i)_i$ be a family of TVS and $E = \prod_i E_i$ their product. Then for each family $(A_i)_i$ of complete subsets $A_i \subseteq E_i$, the product $A = \prod_i A_i \subseteq E$ is complete.*

In particular, if all the E_i are (quasi)complete then E is (quasi)complete.

Proof. Given a (bounded) Cauchy filter $\mathscr{F}$ on A, $\pi_i(\mathscr{F})$ is a (bounded) Cauchy filter on A_i which converges to some $x_i \in A_i$, hence $\mathscr{F} \to (x_i)_i \in A$ by Proposition 6.1 (iv). □

We will now consider the dual situation, namely that of direct sums and quotients.

Recall that given a family $(X_i)_i$ of topological spaces, a set X and mappings $f_i \colon X_i \to X$ for each i, there is a finest topology $\mathscr{T}$ on X such that all f_i are continuous. A subset $U \subseteq X$ is open for $\mathscr{T}$ if and only if all $f_i^{-1}(U)$ are open. A mapping f from $(X, \mathscr{T})$ into any topological space is continuous if and only if $f \circ f_i$ is continuous for all i, and $\mathscr{T}$ is the unique topology on X having this property. This topology is called the *inductive topology* on X with respect to $(f_i)_i$.

If the E_i are TVS or LCS and the f_i are linear, the inductive topology is not linear or locally convex anymore, in general. However, we have the following:

Proposition 6.8. *Let $(E_i)_{i\in I}$ be a family of a) TVS b) LCS, E a set and $f_i \colon E_i \to E$ a linear mapping for each i.*

- *(i) There is a finest a) linear topology, b) locally convex topology $\mathscr{T}$ on X such that all f_i are continuous.*
- *(ii) A 0-basis for $\mathscr{T}$ is given by the family of all a) balanced, absorbent b) absolutely convex, absorbent subsets $U \subseteq E$ such that $f_i^{-1}(U)$ is a 0-neighborhood in E_i for each $i \in I$.*
- *(iii) If F is an a) TVS, b) LCS and $f \colon E \to F$ is linear, then $f \colon (E, \mathscr{T}) \to F$ is continuous if and only if all compositions $f \circ f_i$ are continuous. This property uniquely characterizes $\mathscr{T}$.*

This topology is called a) the *inductive linear topology* or b) the *inductive locally convex topology* on E with respect to the family $(f_i)_i$.

Proof. (i) Let T be the class of all a) linear topologies, b) locally convex topologies on E for which all f_i are continuous. The trivial topology (having only $\emptyset$ and E as open sets) is in T. Put on E the projective topology $\mathscr{T}$ with respect to all mappings $\mathrm{id} \colon E \to (E, \mathscr{T}')$ for $\mathscr{T}' \in T$. Then the f_i are continuous into E by Proposition 6.1 (iii). Moreover, given any a) linear topology b) locally convex topology $\mathscr{T}_0$ on E such that all f_i are continuous into $\mathscr{T}_0$, $\mathrm{id} \colon (E, \mathscr{T}) \to (E, \mathscr{T}_0)$ is continuous because $\mathscr{T}_0 \in T$, so $\mathscr{T}$ is finer than $\mathscr{T}_0$.

(ii): The given family is the 0-basis of a) a linear topology, b) a locally convex topology $\mathscr{T}'$. All f_i are continuous into $(E, \mathscr{T}')$, so $\mathscr{T}$ is finer than $\mathscr{T}'$.

Conversely, if U is a 0-neighborhood in $\mathscr{T}$ there are finitely many $\mathscr{T}_1', \ldots, \mathscr{T}_n' \in T$ and a) balanced or b) absolutely convex 0-neighborhoods V_k in $\mathscr{T}_k'$ such that $x \in V := \bigcap_{k=1}^n V_k \subseteq U$. Because $f_i^{-1}(V_k)$ is a 0-neighborhood for each i, V is a 0-neighborhood in $\mathscr{T}'$ and $\mathscr{T}'$ is finer than $\mathscr{T}$.

(iii): Suppose that all $f \circ f_i$ are continuous and let U be a) a balanced b) an absolutely convex neighborhood of 0 in F. Then $f_i^{-1}(f^{-1}(U))$ is a 0-neighborhood in E_i, hence the absorbent and a) balanced, b) absolutely convex set $f^{-1}(U)$ is a 0-neighborhood in E for $\mathscr{T}$ by (ii) and f is continuous. Uniqueness is seen as in Proposition 6.1 (iii). □

Proposition 6.9. *In the case of LCS, if the linear hull of $\bigcup_i f_i(E_i)$ equals E then the absolutely convex hulls $\mathrm{acx}(\bigcup_i f_i(U_i))$ with $U_i \in \mathscr{U}_i$ (where $\mathscr{U}_i$ is a 0-basis in E_i) form a 0-basis of the inductive locally convex topology.*

Proof. Each such hull U is absolutely convex, $f_i^{-1}(U) \supseteq U_i$ is a 0-neighborhood, and U is absorbent ($x \in E \Rightarrow x = \sum \lambda_i f_i(x_i) = \alpha \sum \lambda_i \alpha^{-1} f_i(x_i) \in \alpha \cdot \operatorname{acx}(\bigcup_i f_i(U_i))$ with $\alpha = \sum |\lambda_i|$), so U is an element of the basis $\mathscr{B}$ given in Proposition 6.8 (ii). Conversely, if $V \in \mathscr{B}$ then each $f_i^{-1}(V)$ contains some $U_i \in \mathscr{U}_i$, hence $f_i(U_i) \subseteq V$, $\bigcup_{i \in I} f_i(U_i) \subseteq V$, and because V is absolutely convex the absolutely convex hull of $\bigcup_i f_i(U_i)$ is contained in V. □

Examples for the inductive locally convex topology are *quotients* and *direct sums*. We will treat those in more detail now.

Proposition 6.10. *Let E be a TVS and $M \subseteq E$ a linear subspace. Then M/E with the quotient topology is a TVS. It is Hausdorff if and only if M is closed.*

Proof. Let $\varphi\colon E \to E/M$ denote the canonical surjection; we first show that φ is open. If $U \subseteq E$ is open then $\varphi^{-1}(\varphi(U)) = U + M = \bigcup_{x \in M}(U + x)$ is open, so $\varphi(U)$ is open in E/M by definition of the quotient topology. Hence, a subset $V \subseteq E/M$ is a neighborhood of $\dot{x} = \varphi(x)$ if and only if $\varphi^{-1}(V)$ is a neighborhood of x.

If V is a neighborhood of $\dot{x} + \dot{y}$ with $\dot{x} = \varphi(x)$, $\dot{y} = \varphi(y)$, then $\varphi^{-1}(V)$ is a neighborhood of $x + y$ and there are neighborhoods U_1, U_2 of x and y such that $U_1 + U_2 \subseteq \varphi^{-1}(V)$. Then $\varphi(U_1)$ and $\varphi(U_2)$ are neighborhoods of $\dot{x}$ and $\dot{y}$ and we have $\varphi(U_1) + \varphi(U_2) \subseteq V$. [24.3.]

If $\dot{x} = \varphi(x)$ and V is a neighborhood of $\lambda\dot{x}$ in E/M then $\varphi^{-1}(V)$ is a neighborhood of λx in E, so there are neighborhoods U_1 of λ and U_2 of x such that $U_1 U_2 \subseteq \varphi^{-1}(V)$, which implies that $\varphi(U_2)$ is a neighborhood of $\dot{x}$ such that $U_1 \varphi(U_2) \subseteq V$.

If E/M is Hausdorff then $\{\dot{0}\}$ is closed, hence $M = \varphi^{-1}(\{\dot{0}\})$ is closed. Conversely, suppose that M is closed and $\dot{x} \in E/M$ is different from $\dot{0}$. Then $\dot{x} = \varphi(x)$ for some $x \neq M$, so there is a neighborhood U of x with $U \cap M = \emptyset$. Hence, $\dot{x} - \varphi(U)$ is a neighborhood of $\dot{0}$ which does not contain $\dot{x}$ because $\dot{0} \notin \varphi(U)$, so E/M is a Hausdorff space. □

Definition 6.11. *A set $\mathscr{P}$ of seminorms is called*

(i) directed *if $\forall n \in \mathbb{N}\ \forall p_1, \dots, p_n \in \mathscr{P}\ \exists q \in \mathscr{P}$: $\max_i p_i \le q$, and*
(ii) saturated *if $\forall n \in \mathbb{N}\ \forall p_1, \dots, p_n \in \mathscr{P}$: $\max_i p_i \in \mathscr{P}$.*

Lemma 6.12. *Let $\mathscr{P}$ be a set of seminorms on a vector space.*

(i) The set $\{\max_i p_i \mid p_1, \dots, p_n \in \mathscr{P}, n \in \mathbb{N}\}$ is saturated and generates the same locally convex topology as $\mathscr{P}$.
(ii) If $\mathscr{P}$ is saturated it is directed.
(iii) If $\mathscr{P}$ is directed then the locally convex topology it generates has as 0-basis the family $\{\varepsilon \cdot q_{\le 1} \mid q \in \mathscr{P}\}$ (or $\{\varepsilon \cdot q_{<1} \mid q \in \mathscr{P}\}$).

Proof. All claims follow immediately from $(\max_i p_i)_{\le 1} = (p_1)_{\le 1} \cap \dots \cap (p_n)_{\le 1}$ and $q_{\le 1} \subseteq p_{\le 1}$ for $p \le q$. □

Proposition 6.13. *If E is a LCS and $M \subseteq E$ a linear subspace then E/M is a LCS. If $\mathscr{P}$ is a directed family of seminorms defining the topology of E then the family $\{\dot{p} \mid p \in \mathscr{P}\}$ defines the topology of E/M, where for a seminorm p on E we define the quotient seminorm $\dot{p}$ on M/E by $\dot{p}(\dot{x}) = \inf_{x \in \dot{x}} p(x)$.*

Proof. If U is a 0-neighborhood in E/M, $\varphi^{-1}(U)$ (which is a 0-neighborhood in E) contains a convex 0-neighborhood V, and $\varphi(V)$ is a convex 0-neighborhood contained in U.

Next, we show that $\dot{p}$ is a seminorm. We have

$$\dot{p}(\lambda\dot{x}) = \inf_{x\in\dot{x}} p(\lambda x) = |\lambda| \inf_{x\in\dot{x}} p(x) = |\lambda|\,\dot{p}(\dot{x}).$$

Moreover, for $\dot{x}, \dot{y} \in E/M$ and any $\varepsilon > 0$ there are $x \in \dot{x}$ and $y \in \dot{y}$ such that $p(x) \le \dot{p}(\dot{x}) + \varepsilon$ and $p(y) \le \dot{p}(\dot{y}) + \varepsilon$ and hence $p(x+y) \le p(x) + p(y) \le \dot{p}(\dot{x}) + \dot{p}(\dot{y}) + 2\varepsilon$. This implies that $\dot{p}(\dot{x}+\dot{y}) \le \dot{p}(\dot{x}) + \dot{p}(\dot{y})$.

If p is a seminorm on E then $\varphi(p_{<1}) = \dot{p}_{<1}$. Hence, if p is continuous then $\dot{p}$ is continuous by Proposition 3.14.

Given a 0-neighborhood U in E/M we have $p_{<\varepsilon} \subseteq \varphi^{-1}(U)$ for some $p \in \mathscr{P}$, hence $\varphi(p_{<\varepsilon}) = \dot{p}_{<\varepsilon} \subseteq \varphi(\varphi^{-1}(U)) = U$. □

Proposition 6.14. *If E is a metrizable LCS and M a closed subspace of E, the quotient space M/E is metrizable. If in addition E is complete then M/E is complete.*

Proof. Given a sequence of seminorms defining the topology of E, the quotient seminorms define the topology of E/M and the first claim follows because E/M is Hausdorff (Proposition 5.4).

Now let $\mathscr{P} = \{p_n \mid n \in \mathbb{N}\}$ be a countable family of seminorms defining the topology of E such that $p_n \le p_{n+1}$. By Proposition 5.8 it suffices to show sequential completeness. Let $(\dot{x}_n)_n$ be a Cauchy sequence in M/E. It suffices to show that a subsequence of $(\dot{x}_n)_n$ converges, so we can assume that $\sum_{n=1}^{\infty} \dot{p}_n(\dot{x}_n - \dot{x}_{n+1}) < \infty$. In fact, $\forall i\ \exists N_i$: $\dot{p}_i(\dot{x}_n - \dot{x}_m) \le 2^{-i}$ for $n, m \ge N_i$, and we can have $N_{i+1} \ge N_i$. Choose $n_i \ge N_i$ for each i. Then $\dot{p}_i(\dot{x}_{n_i} - \dot{x}_{n_{i+1}}) \le 2^{-i}$ for all i, and we replace $(\dot{x}_n)_n$ by $(\dot{x}_{n_i})_i$.

For given $x_n \in \dot{x}_n$ there is $x_{n+1} \in \dot{x}_{n+1}$ such that $p_n(x_n - x_{n+1}) \le 2\dot{p}_n(\dot{x}_n - \dot{x}_{n+1})$. In fact, there is $z_n = x'_n - x'_{n+1} \in \dot{x}_n - \dot{x}_{n+1}$ for some $x'_n \in \dot{x}_n$, $x'_{n+1} \in \dot{x}'_{n+1}$, such that $p_n(z_n) \le 2\dot{p}_n(\dot{x}_n - \dot{x}_{n+1})$. For $x_{n+1} := x'_{n+1} + x_n - x'_n \in \dot{x}_{n+1}$ we have $p_n(x_n - x_{n+1}) = p_n(z_n) \le 2\dot{p}_n(\dot{x}_n - \dot{x}_{n+1})$. Consequently, we find a sequence $x_n \in \dot{x}_n$ such that $\sum p_n(x_n - x_{n+1}) < \infty$. This implies that $\sum p(x_n - x_{n+1}) < \infty$ for every $p \in \mathscr{P}$, so $(x_n)_n$ is a Cauchy sequence in E because

$$p(x_n - x_m) = p(\sum_{k=n}^{m-1}(x_k - x_{k+1})) \le \sum_{k=n}^{m-1} p(x_k - x_{k+1}).$$

Hence, x_n has a limit x_0 and $\dot{x}_n = \varphi(x_n) \to \varphi(x_0)$. □

Next, we consider the direct sum $\bigoplus_i E_i$ of a family $(E_i)_i$ of locally convex spaces with the inductive locally convex topology with respect to the canonical injections $\iota_i\colon E_i \to \bigoplus_i E_i$.

Proposition 6.15. *If all E_i are Hausdorff then $\bigoplus_i E_i$ is Hausdorff. In this case a subset $B \subseteq \bigoplus_i E_i$ is bounded if and only if $\pi_i(B) = \{0\}$ except for a finite subset $J \subseteq I$ and $\pi_i(B)$ is bounded for $i \in J$.*

Proof. The product topology is Hausdorff (Proposition 6.6) and the topology of $\bigoplus_i E_i$ is finer than the topology induced by $\prod_i E_i$ ($\pi_k \circ \iota_j = 0$ if $k \ne j$, $=\mathrm{id}$ if $k = j$, and is continuous in both cases).

For the second claim, let $B \subseteq \bigoplus_i E_i$ be bounded. Because the projections π_i are continuous, each $\pi_i(B)$ is bounded in E_i. Suppose $J \subseteq I$ is infinite and $\pi_j(B) \ne \{0\}$ for $j \in J$; take a sequence $(i_k)_k$ of distinct indices in J. Choose a sequence $(x_k)_k \subseteq B$ such that $\pi_{i_k}(x_k) \ne 0$ for all k, and even (as E_{i_k} is Hausdorff) $\pi_{i_k}(x_k) \notin kV_k$, where each V_k is some absolutely convex 0-neighborhood in E_{i_k}. Let $U = \mathrm{acx}(\bigcup_i \iota_i(U_i))$ be a 0-neighborhood in

$\bigoplus_i E_i$ (Proposition 6.9) with $U_{i_k} = V_k$ for all k. Then $\pi_{i_k}(U) = V_k$ but $\pi_{i_k}(k^{-1}x_k) \notin V_k$, so $k^{-1}x_k \notin U$ for all k, contradicting the boundedness of B by Proposition 5.2.

Conversely, given a 0-neighborhood $U = \operatorname{acx}(\bigcup_i \iota_i(U_i))$ there is $\lambda > 0$ such that $\pi_i(B) \subseteq \lambda U_i \subseteq \pi_i(\lambda U)$ for $i \in J$ and even all $i \in I$, which implies $B \subseteq \lambda U$. □

Proposition 6.16. *If all the E_i are complete then so is $\bigoplus E_i$.*

Proof. We consider on E two topologies: $\mathscr{T}_0$, the trace topology of $\prod_i E_i$, and $\mathscr{T}_i$, the inductive locally convex topology with respect to the injections $\iota_j \colon E_j \to \bigoplus_i E_i$ ($\mathscr{T}_i$ is finer than $\mathscr{T}_0$).

Let $\mathscr{F}$ be a Cauchy filter on E with respect to $\mathscr{T}_i$. Then $\pi_i(\mathscr{F})$ converges to some $x_i \in E_i$. Set $a_i := x_i$ if $x_i \notin \overline{\{0\}}$, and $a_i := 0$ if $x_i \in \overline{\{0\}}$. Set $M := \{i \in I : a_i \neq 0\}$. Note that also for $i \in I \setminus M$, $\pi_i(\mathscr{F}) \to a_i = 0$: let U be an arbitrary 0-neighborhood in E_i, and V one with $V - V \subseteq U$. Then $\exists F \in \mathscr{F} : \pi_i(F) \subseteq x_i + V$, and $0 \in x_i + V$, i.e., $x_i \in -V$, so we have $\pi_i(F) \subseteq V - V \subseteq U$, which means that $\pi_i(\mathscr{F}) \to 0$.

For each $i \in M$ choose an absolutely convex 0-neighborhood W_i in E_i such that $a_i \notin \overline{W_i}$, and for $i \in I \setminus M$ let W_i be an arbitrary 0-neighborhood. Let $A \in \mathscr{F}$ be such that $A - A \subseteq \operatorname{acx}(\bigcup_i \iota_i(W_i))$, then for $i \in M$, $\pi_i(A) - \pi_i(A) \subseteq W_i$, which implies $a_i - \pi_i(A) \subseteq \overline{W_i}$. For $(y_i)_i \in A$ and $i \in M$, $y_i \in a_i - \overline{W_i} \neq 0$, so M is finite and $(a_i)_i \in E$. Because $\pi_i(\mathscr{F}) \to a_i$ we have $\mathscr{F} \to a := (a_i)_i$ in $\mathscr{T}_0$.

$\mathscr{T}_i$ has a 0-basis $\mathscr{U}$ of $\mathscr{T}_0$-closed sets: let $U = \operatorname{acx}(\bigcup_i \iota_i(V_i))$ be a 0-neighborhood in $\mathscr{T}_i$ and $x = (x_i)_i \in E$ in the $\mathscr{T}_0$-closure of U. As $H := \{i \in I : x_i \neq 0\}$ is finite, $\mathscr{T}_0$ and $\mathscr{T}_i$ induce the same topology on $\bigoplus_{i \in H} E_i$: we had $\mathscr{T}_0 \leq \mathscr{T}_i$ above, and $\mathscr{T}_i \leq \mathscr{T}_0$ follows from $n^{-1}(U_1 \times \ldots \times U_n) \subseteq \operatorname{acx}(\iota_1(U_1) \cup \ldots \cup \iota_n(U_n))$. Define $P \colon \bigoplus_{i \in I} E_i \to \bigoplus_{i \in H} E_i$ as the canonical projection, and $Q \colon \bigoplus_{i \in H} E_i \to \bigoplus_{i \in I} E_i$ by $Q(\iota_i(y_i)) = \iota_i(y_i)$. Both P and Q are continuous with respect to $\mathscr{T}_i$, and $Q(P(U)) \subseteq U$ by the form U has. Then

$$x \in \overline{U}^{\mathscr{T}_0} \Rightarrow P(x) \in \overline{P(U)} \Rightarrow x = Q(P(x)) \in \overline{Q(P(U))}^{\mathscr{T}_i} \subseteq \overline{U}^{\mathscr{T}_i}.$$

Hence, the $\mathscr{T}_i$ and $\mathscr{T}_0$-closure of U are the same. Apply Corollary 2.5 to obtain the 0-basis $\mathscr{U}$.

Now given $U \in \mathscr{U}$, let $A \in \mathscr{F}$ be such that $A - A \subseteq U$, and fix $y_0 \in A$. $y_0 - \mathscr{F}$ is a $\mathscr{T}_i$-Cauchy filter and, by the above, $\mathscr{T}_0$-convergent to $y_0 - a$; it contains the set $y_0 - A \subseteq U$, and because U is $\mathscr{T}_0$-closed, $y_0 - a \in U$. Then $A \subseteq a + U$, which gives $\mathscr{F} \to a$ with respect to $\mathscr{T}_i$. □

We summarize the permanence properties of being complete, Hausdorff or metrizable in the following table.

	complete	Hausdorff	metrizable
product	+ (Proposition 6.7)	+ (Proposition 6.6)	+ if countable (Proposition 6.5)
subspace	+ if closed (Proposition 4.4)	+ (Proposition 6.6)	+ (Corollary 6.3)
direct sum	+ (Proposition 6.16)	+ (Proposition 6.15)	− ([S, Ex. 11, p. 70])
quotient	+ if E metrizable and M closed (Proposition 6.14)	+ if M closed (Proposition 6.10)	+ if M closed (Proposition 6.14)

7. Projective and inductive limits

Recall: A set I is said to be directed by an order relation $\leq$ if $\forall i, j \in I\ \exists k \in I$: $i \leq k, j \leq k$.

Definition 7.1. *Let $(I, \leq)$ be a directed set, $(E_i)_{i \in I}$ a family of TVS (LCS), and $g_{ij}\colon E_j \to E_i$ a family of continuous linear mappings for each pair (i, j) with $i \leq j$ such that $g_{ii} = \mathrm{id}$ for all i and $g_{ij} \circ g_{jk} = g_{ik}$ whenever $i \leq j \leq k$. Then $(E_i, g_{ij})_{(I,\leq)}$ is called a* projective system *of TVS (LCS).*

The projective limit $\varprojlim g_{ij}(E_j) := \varprojlim E_j$ *of such a system is defined to be the subspace*

$$\{(x_i)_i \in \prod_i E_i \mid x_i = g_{ij}(x_j) \text{ whenever } i \leq j\}$$

endowed with the projective topology with respect to the restrictions T_k of the projections $\pi_k\colon \prod_i E_i \to E_k$.

Proposition 7.2. *For any family of continuous linear mappings $f_i\colon F \to E_i$ (where F is a TVS (LCS)) ($i \in I$) such that $f_i = g_{ij} \circ f_j$ there is a unique continuous linear map $f\colon F \to \varprojlim_i E_i$ such that $f_i = T_i \circ f$.*

$\varprojlim_i E_i$ together with the T_i is, up to linear homeomorphisms, unique with this property.

Proof. There is a unique linear continuous map $f\colon F \to \prod_j E_j$ such that $\pi_i \circ f = f_i$. Then, f is linear and continuous into $\varprojlim_j E_j$ because $\pi_i(f(x)) = f_i(x) = g_{ij}(f_j(x)) = g_{ij}(\pi_j(f(x)))$.

For uniqueness: we have

$$\left.\begin{aligned} \exists| f\colon E \to \widetilde{E} : \widetilde{T}_i \circ f = T_i\ \forall i \\ \exists| g\colon \widetilde{E} \to E : T_i \circ g = \widetilde{T}_i\ \forall i \end{aligned}\right\} \Rightarrow \begin{cases} T_i \circ g \circ f = \widetilde{T}_i \circ f = T_i,\ T_i \circ \mathrm{id} = T_i \Rightarrow g \circ f = \mathrm{id} \\ \widetilde{T}_i \circ f \circ g = T_i \circ g = \widetilde{T}_i,\ \widetilde{T}_i \circ \mathrm{id} = \widetilde{T}_i \Rightarrow f \circ g = \mathrm{id}. \end{cases}$$

□

Remarks.

1. The topology of a projective limit equals the subspace topology induced by the product topology on $\prod_i E_i$.
2. If all E_i are Hausdorff, $\varprojlim E_j = \bigcap_{i \leq j} (\pi_i - g_{ij} \circ \pi_j)^{-1}(0)$ is closed in $\prod_i E_i$.
3. If $\mathscr{U}_i$ is a 0-basis in E_i, the sets $T_i^{-1}(U_i)$ with $U_i \in \mathscr{U}_j$ already form a 0-basis in $\varprojlim E_j$: let $U = \bigcap_{i \in J} T_i^{-1}(U_i)$ be a 0-neighborhood, J finite, $U_j \subseteq E_j$ a 0-neighborhood. There is $k \in I$ with $k \geq i$ for all $i \in J$. Let $V_k \subseteq E_k$ be a 0-neighborhood such that $g_{ik}(V_k) \subseteq U_i$ for all $i \in J$. Then for $i \in J$, $T_i(T_k^{-1}(V_k)) = g_{ik}(\pi_j(T_k^{-1}(V_k))) \subseteq U_i$, so $T_k^{-1}(V_k) \subseteq U$.

Corollary 7.3. *The projective limit of a system of complete Hausdorff TVS is complete.*

Proposition 7.4. *Every Hausdorff LCS is linearly homeomorphic to a dense subspace of the projective limit of Banach spaces.*

Proof. Let $\mathscr{U}$ be a 0-basis of E consisting of absolutely convex sets. Then the corresponding family of gauges $\mathscr{P} = \{g_U \mid U \in \mathscr{U}\}$ is directed by $\leq$ because $(U, \supseteq)$ is directed and $U' \supseteq U$ if and only if $g_{U'} \leq g_U$.

For $p \in \mathscr{P}$ set $N_p := p^{-1}(0)$, which is a linear subspace of E. Let

$$\Phi_p\colon E \to E_p := E/N_p$$

be the quotient map. For the quotient seminorm $\dot{p}$ we have

$$\dot{p}(\Phi_p(x)) = \inf_{y \in N_p} p(x+y) = p(x)$$

because $z - z' \in N_p$ implies $|p(z) - p(z')| \le p(z - z') = 0$ and $p(z) = p(z')$. Clearly $\dot{p}$ is a norm on E_p and Φ_p is continuous. For $p \le q$, $N_q \subseteq N_p$ so the mapping

$$\Phi_{pq} \colon E_q \to E_p, \quad \Phi_q(x) \mapsto \Phi_p(x)$$

is well-defined, linear and continuous ($\dot{p}(\Phi_{pq}(\Phi_q(x))) = \dot{q}(\Phi_q(x))$) and extends to a linear continuous map

$$\widehat{\Phi}_{pq} : \widehat{E}_q \to \widehat{E}_p$$

on the completions. Then $(\widehat{E}_p, \widehat{\Phi}_{pq})$ is a projective system of LCS and we can form the projective limit $F := \varprojlim_p \widehat{E}_p$.

We define $f \colon E \to F$ by $f(x) := (\Phi_p(x))_p$, noting that for $p \le q$ we have $\Phi_p(x) = \Phi_{pq}(\Phi_q(x))$. Because each $\pi_p \circ f = \Phi_p$ is continuous, f is continuous. Conversely, let [31.3.] $U = p_{\le 1}$ be a 0-neighborhood in E and set $V = f(E) \cap \prod_{q \in \mathscr{P}} V_q$ with $V_q = \widehat{E}_q$ for $q \ne p$ and $V_p = \widehat{\dot{p}}_{\le 1}$. Then $f(x) \in V$ implies $p(x) \le 1$, so $f^{-1}(V) \subseteq U$ and f is a homeomorphism.

Moreover, f is injective because $\Phi_p(x) = 0$ for all p implies that $p(x) = 0$ for all p, hence $x = 0$ as E is Hausdorff.

Finally, to see that $f(E)$ is dense in F, let $(z_p)_p \in F$ and a 0-neighborhood U in F be given. U contains a set $V = \prod_q V_q$ with V_q a 0-neighborhood in $\widehat{E}_q$, and $V_q \ne \widehat{E}_q$ only for q equal to some p. With any $x \in E$ such that $\Phi_p(x) - z_p \in V_p$ we have $\Phi_q(x) - z_q \in V_q$ for all q and hence $f(x) \in (z_p)_p + U$. □

Corollary 7.5. *A LCS is complete if and only if it is linearly homeomorphic to the projective limit of Banach spaces.*

Every Fréchet space is linearly homeomorphic to the projective limit of a sequence of Banach spaces.

Examples.

1. $(\mathbb{N}, \le)$ is directed, for $n \le m$ the inclusion $C^m(\Omega) \to C^n(\Omega)$ is continuous. We have
$$\varprojlim_m C^m(\Omega) = \bigcap_m C^m(\Omega) = C^\infty(\Omega).$$
2. Let $\mathscr{K} := \{K \subseteq X \mid K \text{ compact}\}$ for X a locally compact topological space. Then $(\mathscr{K}, \subseteq)$ is directed, as $K_1 \cup K_2$ is compact. For $K \subseteq L$ the restriction $\rho_{KL} \colon C(L) \to C(K)$ is continuous and
$$\varprojlim_{K \in \mathscr{K}} C(K) \cong C(X).$$

The concept which is dual to projective limits is the following:

Definition 7.6. *Let $(I, \le)$ be a directed set, $(E_i)_i$ a family of LCS, and $h_{ji} \colon E_i \to E_j$ a continuous linear mapping for each pair (i, j) with $i \le j$, such that $h_{ii} = \mathrm{id}$ for all i and $h_{ki} = h_{kj} \circ h_{ji}$ whenever $i \le j \le k$. Then $(E_i, h_{ji})_{(I,\le)}$ is called an* inductive system *of LCS. The* inductive limit *$\varinjlim E_i$ of such a system is defined to be the quotient space of $\bigoplus_{i \in I} E_i$ by the subspace*

$$H := \operatorname{span}\{\iota_i(x_i) - \iota_j(h_{ji}(x_i)) \mid i \le j, x_i \in E_i\},$$

endowed with the quotient topology induced by the locally convex direct sum topology.

Remark. By Proposition 6.8 (iii) the topology of the inductive limit equals the inductive locally convex topology with respect to the maps $h_j := \varphi \circ \iota_j$, where φ is the quotient mapping and $\iota_j \colon E_j \to \bigoplus_i E_i$ the canonical inclusion.

The inductive limit has the following universal property:

Proposition 7.7. *For any family of continuous linear mappings $f_i \colon E_i \to F$ into a LCS F $(i \in I)$ such that $f_i = f_j \circ h_{ji}$ there is a unique continuous linear map $f \colon \varinjlim_i E_i \to F$ such that $f_i = f \circ h_i$.*

$\varinjlim_i E_i$ together with the h_i is unique up to linear homeomorphisms in the following sense: given any LCS $\widetilde{E}$ and mappings $\tilde{h}_i \colon E_i \to \widetilde{E}$ with the same property, there is a linear homeomorphism $f \colon \varinjlim_i E_i \to \widetilde{E}$ such that $h_i \circ f = \tilde{h}_i$.

Proof. There is a unique linear continuous map $\tilde{f} \colon \bigoplus_i E_i \to F$ such that $\tilde{f} \circ \iota_i = f_i$. Because $H \subseteq \ker \tilde{f}$ there is a unique linear continuous map $f \colon \varinjlim_i E_i \to F$ such that $f \circ \varphi = \tilde{f}$, so $f \circ h_i = f \circ \varphi \circ \iota_i = \tilde{f} \circ \iota_i = f_i$. Now if $g \circ h_i = f_i$ for some g, $g \circ \varphi \circ \iota_i = f_i \Rightarrow g \circ \varphi = \tilde{f} \Rightarrow g = f$.

Uniqueness follows as in Proposition 7.2. □

For general inductive limits only few good results hold. We introduce the following:

Definition 7.8. *An inductive system is called* reduced *if all $h_j \colon E_j \to \varinjlim_i E_i$ are injective.*

In this case, $h_i = h_j \circ h_{ji}$ implies that also all h_{ji} are injective.

Given a reduced inductive system, consider the space $F_i := h_i(E_i)$ with the direct image topology (i.e., h_i is a homeomorphism). For $i \leq j$ and $h_i(x_i) \in F_i$, $h_i(x_i) = h_j(h_{ji}(x_i)) \in F_j$, so $F_i \subseteq F_j$ and this inclusion $\iota_{ji} := h_j \circ h_{ij} \circ h_i^{-1}$ is continuous. The family $(F_i, \iota_{ji})_i$ with these inclusion mappings forms an inductive system.

By Proposition 7.7, $(\varinjlim_i E_i, h_i) \cong (\varinjlim_i F_i, \tilde{h}_i \circ h_i)$ (where $\tilde{h}_i \colon F_i \to \varinjlim_j F_j$) and as $\tilde{h}_i \circ h_i$ is injective, $(F_i, \iota_{ji})_i$ is reduced. However, because of its simpler structure $\varinjlim_i F_i$ can be described in a nice way as follows:

Proposition 7.9. *Let E be a vector space and $(E_i)_i$ a family of LCS which are subspaces of E such that $\bigcup_i E_i = E$. Suppose that for $i \leq j$, $E_i \subseteq E_j$ and the inclusion mapping $h_{ji} \colon E_i \to E_j$ is continuous. Then $\varinjlim E_i \cong (E, \mathscr{T}_i)$, where $\mathscr{T}_i$ is the inductive locally convex topology w.r.t. the inclusions $E_i \to E$, and the inductive system is reduced.*

Proof. We show that E has the universal property of Proposition 7.7. Given mappings $f_i \colon E_i \to F$ compatible with the inclusions, define $f \colon E \to F$ by $f|_{E_i} := f_i$. Then f is well-defined, continuous and the unique linear continuous map whose restriction to E_i is f_i. □

In practice one starts with a space $(\bigcup_i E_i, \mathscr{T}_i)$ and establishes its properties (completeness, ...) via results about inductive limits.

If the index set of a reduced inductive system is $\mathbb{N}$ with its natural order we call it an *inductive sequence*. An inductive sequence and its inductive limit are called *strict* if each E_n is a topological subspace of E_k whenever $n \leq k$.

Definition 7.10. *A LCS is called (LB)-space or (LF)-space, respectively, if it can be given by a strict inductive limit of a strictly increasing sequence of Banach spaces or Fréchet spaces, respectively.*

Lemma 7.11. *Let E be a LCS, M a subspace of E and U an absolutely convex 0-neighborhood in M. Then there is an absolutely convex 0-neighborhood V in E with $U = V \cap M$. For $x_0 \in E \setminus \bar{M}$, V can be chosen such that $x_0 \notin V$.*

Proof. Let W be an absolutely convex 0-neighborhood in E such that $W \cap M \subseteq U$ and set $V := \operatorname{acx}(W \cup U)$. Then $V \cap M = U$: $U \subseteq V \cap M$ is clear; for $V \cap M \subseteq U$, take some $z \in V \cap M$, which is of the form $z = \alpha w + \beta u$ with $|\alpha| + |\beta| \le 1$, $w \in W$ and $u \in U$. Then $\alpha w = z - \beta u \in M$ means that either $\alpha = 0$ or $w \in M$, which both implies $z \in U$.

For $x_0 \notin \bar{M}$, W can be chosen such that $(x_0 + W) \cap M = \emptyset$. Then $x_0 \notin V$ because $x_0 = \alpha w + \beta u \in V$ would imply $x_0 - \alpha w = \beta u \in M$ and $x_0 - \alpha w \in x_0 + W$, contradiction. □

For the remainder of this section let $(E, \mathscr{T}) = \varinjlim E_n$ be the strict inductive limit of an inductive sequence $(E_n, \mathscr{T}_n)_n$ of LCS.

Theorem 7.12. *$\mathscr{T}$ induces $\mathscr{T}_n$ on each E_n.*

Proof. The inclusion $E_n \to E$ is continuous, so $\mathscr{T}_n$ is finer than the topology $\mathscr{T}_n'$ induced by $\mathscr{T}$ on E_n. Let $V_n \subseteq E_n$ now be an absolutely convex 0-neighborhood in $\mathscr{T}_n$. By Lemma 7.11 we can construct a sequence of absolutely convex 0-neighborhoods V_{n+k} in E_{n+k} such that $V_{n+k+1} \cap E_{n+k} = V_{n+k}$ $(k = 0, 1, 2, \dots,)$. Then $V := \bigcup_{k \ge 0} V_{n+k}$ is absolutely convex and absorbent, $V \cap E_i$ is a 0-neighborhood in E_i for all i, so V is a 0-neighborhood in $\varinjlim E_i$ (Proposition 6.8 (ii)). $V \cap E_n = V_n$ is a 0-neighborhood in $\mathscr{T}_n'$, so $\mathscr{T}_n'$ is finer than $\mathscr{T}_n$. □

Corollary 7.13. *If each E_n is Hausdorff, $\varinjlim E_n$ is Hausdorff.*

Proof. For $x \neq 0$ there is some n with $x \in E_n$ and hence there is an absolutely convex 0-neighborhood V_n in E_n with $x \notin V_n$. By Theorem 7.12 there is a 0-neighborhood V in E such that $V \cap E_n = V_n$, $x \notin V$, so E is Hausdorff. □

Definition 7.14. *An inductive limit is called* regular *if every bounded set in $\varinjlim E_i$ is contained and bounded in some E_i.*

Theorem 7.15. *If E_n is closed in E_{n+1}, the inductive limit is regular.*

Proof. Suppose a set B is not contained in any E_n and choose a sequence $(x_n)_n$ in B with $x_n \notin E_n$. Choose a subsequence $(y_k)_k$ of x_k and a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that $y_k \in E_{n_{k+1}} \setminus E_{n_k}$. Take an increasing sequence $(V_k)_k$ of absolutely convex sets such that V_k is a 0-neighborhood in E_{n_k}, $V_{k+1} \cap E_{n_k} = V_k$ and $k^{-1} y_k \notin V_{k+1}$. Then $V = \bigcup_k V_k$ is a 0-neighborhood in $\varinjlim E_i$ and $k^{-1} y_k \notin V$ for all k, so $k^{-1} y_k$ does not converge to 0 and B cannot be bounded by Proposition 5.2.

Hence, a bounded set B is countained in some E_n. If V_n is an absolutely convex 0-neighborhood in E_n, by Theorem 7.12 and Lemma 7.11 there is an absolutely convex 0-neighborhood V in E with $V \cap E_n = V_n$, and $B \subseteq \lambda V$ for some λ implies $B \subseteq \lambda V_n$. □

Theorem 7.16. *If all E_n are complete then so is $E = \varinjlim_n E_n$.*

Proof. If $\mathscr{F}$ is a Cauchy filter in E and $\mathscr{U}$ the neighborhood filter of 0, $\mathscr{F} + \mathscr{U}$ is a Cauchy filter basis of $\mathscr{F}$ (cf. the proof of Proposition 4.11). We claim that the trace of $\mathscr{F} + \mathscr{U}$ on some E_{n_0} is a filter basis. Otherwise, we would have a decreasing sequence $F_n \in \mathscr{F}$ and a decreasing sequence of absolutely convex 0-neighborhoods W_n in E such that $(F_n + W_n) \cap E_n = \emptyset$ for all n. Then $U = \operatorname{acx}(\bigcup_n (W_n \cap E_n))$ is a 0-neighborhood in E (Proposition 6.9 and Theorem 7.12), and $(F_n + U) \cap E_n = \emptyset$ for all n: supposing there is some $y \in (F_n + U) \cap E_n$, $y = z_n + \sum \lambda_i x_i$, with $x_i \in W_i \cap E_i$ $(i = 1 \dots p)$ and $z_n \in F_n$, which gives

$$y - \sum_{i \le n} \lambda_i x_i = z_n + \sum_{i > n} \lambda_i x_i.$$

Because $W_i \subseteq W_n$ for $i > n$ and W_n is absolutely convex, $z_n + \sum_{i \ge n} \lambda_i x_i \in F_n + W_n$, but $y - \sum_{i \le n} \lambda_i x_i \in E_n$, which contradicts $(F_n + W_n) \cap E_n = \emptyset$, so $(F_n + U) \cap E_n = \emptyset$ for all n. Because $\mathscr{F}$ is a Cauchy filter, $F - F \subseteq U$ for some $F \in \mathscr{F}$. Take $w \in F$, then $w \in E_k$ for some k. For $v \in F_k \cap F$, $w = v + (w - v) \in v + (F - F) \subseteq F_k + U$, which gives a contradiction.

Hence, the trace of $\mathscr{F} + \mathscr{U}$ on E_{n_0} is a Cauchy filter basis and converges, and $\mathscr{F}$ converges in E. $\square$

Corollary 7.17. *Every (LF)-space is complete.*

Example. For $\Omega \subseteq \mathbb{R}^n$ open, $\mathcal{D}(K) := \{ f \in C^\infty(\Omega) \mid \operatorname{supp} f \subseteq K \}$. If $(K_n)_n$ is a sequence of compact sets with $\bigcup_n K_n = \Omega$ and $K_n \subseteq K_{n+1}^\circ$, we set $\mathcal{D}(\Omega) := \varinjlim \mathcal{D}(K_n)$, which is an (LF)-space.

8. Finite-dimensional and locally compact TVS

Lemma 8.1. *If a TVS E is the algebraic direct sum of subspaces M and N, then $E = M \oplus N$ topologically if and only if the canonical mapping $v \colon E/M \to N$ is an isomorphism of TVS.*

Proof. Note that $M \oplus N \to E$ is always continuous; its inverse is continuous if and only if the projections π_M and π_N are continuous.

With $\varphi \colon E \to E/M$ and $\pi_N \colon E \to N$ the canonical mappings, $\pi_N = v \circ \varphi$.

"$\Rightarrow$": π_N continuous, φ open $\Rightarrow$ v continuous ($U \subseteq N$ open $\Rightarrow \pi_N^{-1}(U)$ open $\Rightarrow \varphi(\pi_N^{-1}(U))$ open and $v(\varphi(\pi_N^{-1}(U))) = \pi_N(\pi_N^{-1}(U)) = U$). π_N open, φ continuous $\Rightarrow$ v open ($U \subseteq E/M$ open $\Rightarrow v(U) = v(\varphi(\varphi^{-1}(U))) = \pi_N(\varphi^{-1}(U))$ open).

"$\Leftarrow$": v and φ continuous $\Rightarrow$ π_N and $\pi_M = (\mathrm{id} - \pi_N)$ continuous. $\square$

Theorem 8.2. *If E is a Hausdorff TVS with $\dim E = n$ then $E \cong \mathbb{K}^n$.*

For each basis $\{x_1, \dots, x_n\}$ of E, the map $(\lambda_1, \dots, \lambda_n) \mapsto \lambda_1 x_1 + \dots + \lambda_n x_n$ is a linear homeomorphism of $\mathbb{K}^n$ onto E, and all linear homeomorphisms are of this form.

Proof. We perform induction over n.

$n = 1$: $\lambda \mapsto \lambda x_1$ is continuous and an algebraic isomorphism of $\mathbb{K}$ onto E. For continuity of $\lambda x_1 \mapsto \lambda$ at 0, let $0 < \varepsilon < 1$ and choose a balanced 0-neighborhood V such that $\varepsilon x_1 \notin V$. Then $\lambda x_1 \in V$ implies $|\lambda| < \varepsilon$.

Now suppose the claim holds for dimension $n-1$. If $\{x_1, \dots, x_n\}$ is any basis of E, let $M = \operatorname{span}\{x_1, \dots, x_{n-1}\}$ and $N = \operatorname{span}\{x_n\}$. By induction hypothesis,

$$(\lambda_1, \dots, \lambda_{n-1}) \mapsto \sum_{i=1}^{n-1} \lambda_i x_i$$

is a linear homeomorphism $\mathbb{K}^{n-1} \to M$. Because $\mathbb{K}^{n-1}$ is complete, M is complete and hence closed in E, so E/M is Hausdorff and has dimension 1, and it is isomorphic to $\mathbb{K}$. The canonical map $E/M \to N$ is the composition

$$E/M \to \mathbb{K} \to N$$

and hence an isomorphism, so by Lemma 8.1, $\mathbb{K}^n \cong \mathbb{K}^{n-1} \oplus \mathbb{K} \cong M \oplus N = E$.

It is clear that every linear homeomorphism is of this form. □ [7.4.]

Definition 8.3. *A Hausdorff TVS E is called* locally compact *if it has a compact* 0-*neighborhood*

The class of locally compact TVS is rather restrictive, as the following result shows:

Theorem 8.4. *A Hausdorff TVS E is* locally compact *if and only if it is finite-dimensional.*

Proof. If $\dim E = n \in \mathbb{N}$, $E \cong \mathbb{K}^n$. A 0-basis of $\mathbb{K}^n$ is given by products of closed bounded intervals, which are compact.

Conversely, assume that E is locally compact and let V be a balanced compact neighborhood of 0. We show that $\{2^{-n}V : n \in \mathbb{N}\}$ is a 0-basis. Let B be any 0-neighborhood and choose a balanced 0-neighborhood U such that $U + U \subseteq B$. Since V is compact, there are $x_i \in V$ $(i = 1 \dots k)$ such that $V \subseteq \bigcup_{i=1}^k (x_i + U)$, and there is $\lambda \in \mathbb{K}$ with $0 < |\lambda| < 1$ such that $\lambda x_i \in U$ for all i. There exists $n \in \mathbb{N}$ such that $2^{-n} \leq |\lambda|$, and

$$2^{-n}V \subseteq \lambda V \subseteq \bigcup_{i=1}^{k} (\lambda x_i + \lambda U) \subseteq U + U \subseteq B.$$

Since V is compact there is a finite subset $D = \{x_1, \dots, x_n\} \subseteq E$ such that $V \subseteq D + \frac{1}{2}V$. Let $M := \operatorname{span} D$, which is closed because it is finite dimensional and hence complete. Since $V \subseteq M + \frac{1}{2}V$ and $tM = M$ for all $t \in \mathbb{K} \setminus \{0\}$, we have

$$\frac{1}{2}V \subseteq M + \frac{1}{4}V,$$
$$V \subseteq M + \frac{1}{2}V \subseteq M + (M + \frac{1}{4}V) = M + \frac{1}{4}V,$$
$$V \subseteq \bigcap_{m \in \mathbb{N}} (M + 2^{-m}V) = \bar{M} = M$$

by induction, and $M = E$ because V is absorbent. □

9. The theorem of Hahn-Banach

We recall some facts about hyperplanes from linear algebra. If E is a vector space, a linear subspace $H_0 \subseteq E$ with $\operatorname{codim} H_0 = \dim E/H_0 = 1$ is called a *hyperplane*. If $f : E \to \mathbb{K}$ is linear and $f \neq 0$ then $f^{-1}(0)$ is a hyperplane: $\dim E/\ker f = \dim \operatorname{im} f = 1$. Conversely, if H_0 is a hyperplane it is given by $f^{-1}(0)$ for some nontrivial linear $f : E \to \mathbb{K}$: let $v \in E/H_0$ be nonzero and $\ell : E/H_0 \to \mathbb{K}$, $\lambda v \mapsto \lambda$ the coefficient map. Then for $f := \ell \circ \varphi$, where $\varphi : E \to E/H_0$ is the quotient map, we have $\ker f = \ker \varphi = H_0$.

If $x_0 \in E$ and H_0 is a hyperplane in E then $H = x_0 + H_0$ is called an *affine hyperplane*. For a nontrivial linear $f\colon E \to \mathbb{K}$ and $\alpha \in \mathbb{K}$, $f^{-1}(\alpha)$ is an affine hyperplane: given any $x_0 \in f^{-1}(\alpha)$ we have $f^{-1}(\alpha) = x_0 + \ker f$. Conversely, if $H = x_0 + H_0$ is an affine hyperplane then $H_0 = f^{-1}(0)$ for some f and $H = f^{-1}(0) + x_0 = f^{-1}(f(x_0))$.

Lemma 9.1. *Let E be a TVS and f a nontrivial linear form on E. Then the following are equivalent:*

(i) f is continuous.
(ii) $\ker f$ is closed in E.
(iii) $\ker f$ is not dense in E.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is clear. For (iii) $\Rightarrow$ (i), let $x \in \operatorname{int}(E \setminus \ker f)$ and choose a balanced 0-neighborhood U in E such that $(x + U) \cap \ker f = \emptyset$. Then $f(x) \notin f(U)$, which implies $f(U) \subseteq |f(x)| \cdot \mathring{\mathbb{D}}$; in fact, $|\lambda| \geq |f(x)| \neq 0$ for some $\lambda \in f(U)$ would imply $f(x) = \frac{f(x)}{\lambda} \cdot \lambda \in \mathcal{D} \cdot f(U) \subseteq f(U)$ because $f(U)$ is balanced. This implies continuity of f. ☐

Let E be a vector space, $A \subseteq E$ convex and nonempty, and $H = f^{-1}(\alpha)$ a hyperplane such that $A \cap H = \emptyset$. Then either $A \subseteq \{x\colon f(x) < \alpha\}$ or $A \subseteq \{x\colon f(x) > \alpha\}$ because $f(A)$ is convex and does not contain α. The *geometric form* of the Hahn-Banach theorem states the following.

Theorem 9.2. *Let E be a TVS, $A \subseteq E$ open, convex and nonempty, and M an affine subspace of E with $M \cap A = \emptyset$. Then there is a closed affine hyperplane H in E such that $M \subseteq H$ and $H \cap A = \emptyset$.*

Proof. By applying a translation we can assume that $0 \in M$. It suffices to show the case $M = \{0\}$. In fact, supposing the theorem is known for this case, let $\varphi\colon E \to E/M$ be the quotient map. Then $\varphi(A)$ is open, convex, and $0 \notin \varphi(A)$. By assumption there exists a closed hyperplane $H_1 \subseteq E/M$ such that $H_1 \cap \varphi(A) = \emptyset$. Then $H := \varphi^{-1}(H_1)$ is a closed hyperplane in E ($H_1 = \ker f \Rightarrow H = \ker(f \circ \varphi)$, and $f \circ \varphi$ is nontrivial because φ is surjective and f is nontrivial). Because $\varphi(H \cap A) \subseteq \varphi(H) \cap \varphi(A) = H_1 \cap \varphi(A) = \emptyset$ we have $H \cap A = \emptyset$.

For the case $M = \{0\}$ we first assume that $\mathbb{K} = \mathbb{R}$ and introduce some terminology. A subset $A \subseteq E$ is called a *cone with vertex at x_0* if $\forall x \in C : \{x_0 + \lambda(x - x_0) \mid \lambda > 0\} \subseteq C$. It is called *pointed* if $x_0 \in C$ and *blunt* if $x_0 \notin C$. A subset $C \subseteq E$ is a convex cone with vertex at 0 if and only if $\lambda C \subseteq C$ for all $\lambda > 0$ and $C + C \subseteq C$. In fact, the first condition is the definition of a cone with vertex at 0. For the second, if C is convex then $C+C = 2(\frac{1}{2}C+\frac{1}{2}C) = 2C \subseteq C$. Conversely, if $C+C \subseteq C$ holds then $\alpha x+\beta y \in C+C \subseteq C$ for a convex combination of $x, y \in C$.

Fix any interior point a of A. Let $\mathscr{C}$ be the set of all convex pointed cones C with vertex at 0 such that $A \subseteq C$ and $-a \notin C$. Then $\mathscr{C}$ contains $B := \bigcup_{\lambda \geq 0} \lambda A$, so it is not empty. Order $\mathscr{C}$ by inclusion. Then every nonempty totally ordered subset $\mathscr{C}_0$ of $\mathscr{C}$ has an upper bound, obtained as $C_0 := \bigcup\{C : C \in \mathscr{C}_0\}$. In fact, $\lambda C_0 \subseteq C_0$ is clear, and for $x_1 \in C_1$ and $x_2 \in C_2$ we have either $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$, which in both cases gives $C_0 + C_0 \subseteq C_0$. $A \subseteq C_0$ and $-a \notin C_0$ is clear. By Zorn's Lemma there is a maximal element C of $\mathscr{C}$.

We will show that $H := C \cap (-C)$ is a closed hyperplane such that $H \cap A = \emptyset$.

First, we show that $C \cup (-C) = E$. If $x \in E$ were such that $x \notin C$ and $x \notin -C$, then $\lambda x \notin C$ for all $\lambda \neq 0$. Set $C_1 := \{\lambda x + y \mid \lambda \geq 0, y \in C\}$. Then we have $\lambda C_1 \subseteq C_1$ $\forall \lambda > 0$, $C_1 + C_1 \subseteq C_1$, $A \subseteq C$ and $-a \notin C_1$ because $\lambda x + y = -a$ would imply $C \ni y = -a \notin C$ for

$\lambda = 0$ and $C \not\ni -\lambda x = y + a \in C$ for $\lambda > 0$, both a contradiction. Hence, $C_1 \in \mathscr{C}$. Because $C \subseteq C_1$ and $x \in C_1 \setminus C$ this contradicts maximality of C, so necessarily $C \cup (-C) = E$ holds.

We now call a point $x \in C$ an *internal* point of C if $\forall y \in E$ there is some $\tau_0 > 0$ such that $x + \tau y \in C$ for all $\tau \in \mathbb{R}$ with $|\tau| < \tau_0$.

We claim that if $x \in C$ is not internal then $x \in -C$. Assume to the contrary that $x \notin -C$, or $-x \notin C$. If x is not internal there is $y \in E$ such that $x + \tau y \notin C$ for a 0-sequence of $\tau \neq 0$, and (by convexity of C) for all $\tau > 0$. We now set

$$C_1 := \{w - \lambda x \mid w \in C, \lambda \geq 0\}.$$

$\lambda C_1 \subseteq C_1$, $C_1 + C_1 \subseteq C_1$ and $A \subseteq C$ are clear. We have $y \notin C$ because $y \in C$ would imply $x + \tau y \in C$ for all $\tau > 0$. Moreover, $y \in C_1$ would imply $y = w - \lambda x$ with $\lambda > 0$ and $x + \lambda^{-1} y = \lambda^{-1} w \in C$, which contradicts the choice of y, so necessarily we have $y \notin C_1$ and $C_1 \neq E$. Now in order to see that $-a \notin C_1$, take any $b \in E$. Then $a + \lambda(b - a) \in A \subseteq C_1$ for $|\lambda| \leq \lambda_0$ (A is open), and the assumption $-a \in C_1$ would imply $\lambda(b - a) \in C_1$ and hence $b \in C_1$ for all $b \in E$, which contradicts $C_1 \neq E$, so we have $-a \notin C_1$. Hence, $C_1 \in \mathscr{C}$.

Now $C \subset C_1 \in \mathscr{C}$ and $-x \in C_1 \setminus C$ contradicts maximality of C, so $x \in -C$ has to hold for every $x \in C$ which is not internal.

Let now $y \in C$, $z \in -C$ and set

$$x(\tau) := (1 - \tau) y + \tau z \qquad (0 \leq \tau \leq 1).$$

We then claim that $x(\sigma) \in H$ for some σ. Set $\sigma := \sup\{\tau : x(\tau) \in C\}$. Then $x(\tau) \in C$ for $\tau < \sigma$ and $x(\tau) \notin C$ for $\tau > \sigma$. If now $x(\sigma) \in C$ then x is not an internal point of C and hence $x \in -C$ by what was shown above. If, on the other hand, we have $x(\sigma) \in -C$ then we have either $x(\tau) \in -C$ for some $\tau < \sigma$ or $x(\tau) \notin -C$ for all $\tau < \sigma$. In the first case we have $x(\tau) \in H$; in the second case, $x(\sigma)$ is not an internal point of $-C$, i.e., $-x(\sigma)$ is not an internal point of C, which means that $x(\sigma) \in H$.

In order to see that H is a hyperplane, we will show that $H \oplus \mathbb{R}a = E$. H is a linear subspace because it is closed under addition, reflection $x \mapsto -x$ and multiplication by nonnegative reals. $H \cap \mathbb{R}a = \{0\}$ is clear from $-a \notin C$. Finally, if $x \in C$ then the line segment

$$x(\tau) = (1 - \tau) x + \tau(-a)$$

contains a point $y = x(\sigma) \in H$ with $\sigma \neq 1$ because $-a \notin C$, which gives

$$x = \frac{y}{1 - \sigma} + \frac{\sigma}{1 - \sigma} a.$$

Similarly, for $x \in -C$ the line segment

$$x(\tau) = (1 - \tau) a + \tau x$$

contains a point $y = x(\sigma) \in H$ with $\sigma \neq 0$, which gives

$$x = \frac{y}{\sigma} + \frac{\sigma - 1}{\sigma} a.$$

The complement of H contains the open set A, so H cannot be dense and hence is closed by Lemma 9.1. To see that $A \cap H = \emptyset$, we first note that every point of A is an internal point of C. It hence suffices to prove that H contains no internal points of C. For any $x \in H$ we have $x - \lambda a \notin C$ for all $\lambda > 0$ because $x - \lambda a = y \in C$ would give

$$-a = \lambda^{-1}(y - x) \in C.$$

This complete the proof for the case of real scalars.

For complex scalars, let E_0 be the real vector space underlying the complex vector space E, with the trace topology. Then E_0 is a TVS and M a linear subspace of E_0. Let H_0 be a closed hyperplane in E_0 such that $M \subseteq H_0$ and $H_0 \cap A = \emptyset$. We set $H = H_0 \cap iH_0$. Then by Lemma 9.3, H is a closed hyperplane in E such that we have $M = M \cap iM \subseteq H$ and $H \cap A = \emptyset$. □

Lemma 9.3. *Let E be a complex vector space and E_0 the underlying real vector space. If H_0 is a hyperplane of E_0 with equation $f_0(x) = 0$ then $H = H_0 \cap iH_0$ is a hyperplane of E with equation $f(x) = f_0(x) - if_0(ix) = 0$.*

Proof. We show that $H = \ker f$. For $x \in H_0 \cap iH_0$, we have $x = iy$ for some $y \in H_0$ and hence $f(x) = f_0(x) - if_0(-y) = 0$. Conversely, if $f_0(x) - if_0(ix) = 0$ then $f_0(x) = 0$, which implies $x \in H_0$, and $-f_0(ix) = f_0(-ix) = 0$, which implies $-ix \in H_0$ and hence $i(-ix) = x \in iH_0$. □

The *analytic form* of the Hahn-Banach theorem is the following.

Theorem 9.4. *Let E be a vector space, q a seminorm on E and M a linear subspace of E. If f is a linear form on M satisfying $|f(x)| \leq q(x)$ for all $x \in M$ then there exists a linear form g on E such that $g(x) = f(x)$ for all $x \in M$ and $|g(x)| \leq q(x)$ for all $x \in E$.*

Proof. First suppose that $\mathbb{K} = \mathbb{R}$. Then q defines a locally convex topology on E such that $A := q_{<1}$ is open, convex and nonempty. The case $f = 0$ is trivial, so we assume $f \neq 0$. The set $N := f^{-1}(1)$ is an affine hyperplane in M (and an affine subspace of E) such that $N \cap A = \emptyset$. By Theorem 9.2 there is an affine hyperplane H in E containing N such that $H \cap A = \emptyset$, and we have $H = g^{-1}(1)$ for some linear form $g\colon E \to \mathbb{R}$.

For $y \in N$, $f(y) = g(y) = 1$; for $z \in M$ and any $x_0 \in N$ we have the implication

$$f(z) = 0 \Rightarrow z + x_0 \in N \subseteq H \Rightarrow g(z) = 0.$$

For any $x \in M$ and $y \in N$ we have $x = f(x)y + (x - f(x)y)$ with $x - f(x)y \in \ker f \subseteq \ker g$, which implies $g(x) = f(x)$.

If $x \in A$ then $g(x) < 1$ because $g(x) = 1$ would mean $x \in H$, but $H \cap A = \emptyset$. Hence, $g(x) = 1$ implies $q(x) \geq 1$. For $g(x) = 0$, $|g(x)| \leq q(x)$. For $g(x) \neq 0$, $x = g(x) \cdot \frac{x}{g(x)}$ and $g(x) = g(x)g(\frac{x}{g(x)}) \leq |g(x)|\, q(\frac{x}{g(x)}) = q(x)$. Moreover, $-g(x) = g(-x) \leq q(-x) = q(x)$, so $|g(x)| \leq q(x)$ for all $x \in E$.

In the complex case, let E_0 be the real vector space underlying E and M_0 the set M, as a subspace of E_0. Set $f_1(x) = \Re f(x)$, where $\Re$ denotes the real part; then $f_1\colon M_0 \to \mathbb{R}$ is a linear form and for $x \in E$ we have $f(x) = f_1(x) - if_1(ix)$. For $x \in M_0$, $|f_1(x)| \leq |f(x)| \leq q(x)$. By the real case there is a linear form $g_1\colon E_0 \to \mathbb{R}$ with $g_1(x) = f_1(x)$ on M_0 and $|g_1(x)| \leq q(x)$ for $x \in E_0$. With $g(x) := g_1(x) - ig_1(ix)$ for $x \in E$ we then have $g(x) = f(x)$ on M and $g(ix) = ig(x)$, so g is a linear form on E. For each $x \in E$ we then have $g(x) = \rho e^{i\theta}$ for some $\rho \geq 0$, which gives

$$|g(x)| = \rho = e^{-i\theta}g(x) = g(e^{-i\theta}x) = g_1(e^{-i\theta}x) \leq q(e^{-i\theta}x) = q(x).$$ □

We list some consequences.

Proposition 9.5. *Let M be a linear subspace of a LCS E and f a continuous linear form on M. Then there is a continuous linear form g on E such that $g(x) = f(x)$ for $x \in M$.*

Proof. There is a continuous seminorm q on E such that $|f(x)| \leq q(x)$ for all $x \in M$. By Theorem 9.4 there is a linear form g on E with $g(x) = f(x)$ for $x \in M$ and $|g(x)| \leq q(x)$ for $x \in E$, i.e., g is continuous. □

Proposition 9.6. *Let M be a closed linear subspace of a LCS E.*

(i) For $z \in E \setminus M$ there is a continuous linear form f on E such that $f(z) = 1$ and $f(x) = 0$ for $x \in M$.
(ii) M is the intersection of all closed hyperplanes containing it.

Proof. (i): Let $\varphi\colon E \to E/M$ be the canonical surjection. Then $\dot{z} := \varphi(z) \neq 0$ and there is a continuous seminorm $\dot{q}$ on E/M such that $\dot{q}(\dot{z}) \neq \emptyset$. Define $g\colon \mathbb{K}\dot{z} \to \mathbb{K}$ by $g(\lambda\dot{z}) = \lambda$. Then $|g(\lambda\dot{z})| = |\lambda| = \frac{\dot{q}(\lambda\dot{z})}{\dot{q}(\dot{z})}$, so by Theorem 9.4 there is a linear form h on E/M such that $|h(\dot{x})| \leq \frac{1}{\dot{q}(\dot{z})}\dot{q}(\dot{x})$ and $h(\dot{z}) = 1$; then $f := h \circ \varphi$ is a linear form on E such that $f(z) = h(\dot{z}) = 1$ and $f(x) = h(0)$ for $x \in M$. [5.5.]

(ii): Let $\mathcal{H}$ be the collection of all closed hyperplanes in E containing M. Then $M \subseteq \bigcap_{H \in \mathcal{H}} H$ and for $z \in E \setminus M$ there is, by (i), a linear form $f \in E'$ with $f(z) = 1$ and $M \subseteq \ker f$. Hence, $\ker f \in \mathcal{H}$ but $z \notin \ker f$. □

Definition 9.7. *If E is a TVS then*

$$E' := \{f\colon E \to \mathbb{K} \mid f \text{ linear and continuous}\}$$

is called the (topological) dual *of E.*

Corollary 9.8. *A LCS E is Hausdorff if and only if $\forall x \in E \setminus \{0\}$ there is $u \in E'$ such that $u(x) \neq 0$.*

Proof. If E is Hausdorff then $\{0\} = \overline{\{0\}} = \bigcap\{\ker u \mid u \in E'\}$ by Proposition 2.12 and Proposition 9.6 (ii), so for $x \neq 0$ there is $u \in E'$ such that $u(x) \neq 0$.

Conversely, if $u(x) \neq 0$ there is a balanced 0-neighborhood V such that $0 \notin u(x + V)$, i.e., $x \notin V$, so E is Hausdorff by Proposition 2.11. □

Lemma 9.9. *If p is a continuous seminorm on a TVS E and $x \in E$, then $\exists u \in E'$ such that $|u(y)| \leq p(y)$ for all $y \in E$ and $u(x) = p(x)$.*

Proof. If $p(x) = 0$ set $u = 0$. If $p(x) \neq 0$ set $G := \ker p + \operatorname{span}\{x\}$ and choose $v \in G^*$ such that $\ker v = \ker p$ and $v(x) = p(x)$. Then for $y \in G$, $|v(y)| \leq |p(y)|$ and Theorem 9.4 gives the claim. □

Proposition 9.10. *Let E be a TVS. Then $E' \neq \{0\}$ if and only if there is a convex 0-neighborhood $U \neq E$.*

Proof. If $U \in E' \setminus \{0\}$ then $U := u^{-1}(\mathbb{D}) = \{x \in E : |u(x)| \leq 1\}$ is a convex 0-neighborhood and $U \neq E$.

Conversely, if a convex 0-neighborhood U is given, U contains an absolutely convex 0-neighborhood V. The gauge q_V of V is a continuous seminorm (Proposition 3.11). By Lemma 9.9, for $x \in E \setminus V$ there is $u \in E'$ with $u(x) = q_V(x)$. Because $q_V(x) = 0$ would imply $x \in V$ we have $q_V(x) > 0$ and hence $u \neq 0$. □

Finally, we have the following separation theorem:

Proposition 9.11. *Let E be a LCS over $\mathbb{R}$, $A, B \subset E$ disjoint, convex and nonempty. If A is open there is $u \in E'$ and $\alpha \in \mathbb{R}$ such that $u(x) < \alpha \leq u(y)$ for $(x, y) \in A \times B$. If B is open as well, one can have $u(x) < \alpha < u(y)$ for $(x, y) \in A \times B$.*

Proof. $A - B$ is open, convex, nonempty and does not contain 0, so by Theorem 9.2 there is $v \in E' \setminus \{0\}$ such that $\ker v \cap (A - B) = \emptyset$. Because $v(A)$ and $v(B)$ are nonempty, disjoint and convex subsets of $\mathbb{R}$ (i.e., intervals) we can find $\alpha \in \mathbb{R}$ such that $u(x) \le \alpha \le u(y)$ for $(x, y) \in A \times B$, where u is v or $-v$. u is surjective and open (u induces a map $E/\ker u \to \mathbb{R}$ which is a linear homeomorphism by Theorem 8.2). Because $u(A)$ is open we have $u(x) < \alpha$ for $x \in A$. Similarly, if B is open then $\alpha < u(y)$ for $y \in B$. □

Corollary 9.12. *In a real locally convex space, every closed convex subset A is the intersection of the closed half-spaces $f(x) \le \alpha$ containing it ($f \in E'$, $\alpha \in \mathbb{R}$).*

Proof. For $x \notin A$, there is a closed affine hyperplane H such that x and A lie on different sides of H. □

10. Dual Pairings

Definition 10.1. *If E, F are vector spaces and $B\colon E \times F \to \mathbb{K}$ a bilinear form we say that E and F are* paired *or* form a pairing *with respect to B. B is called* nondegenerate *if it separates points of E and F, i.e.,*

$$B(x, y) = 0 \ \forall y \in F \Rightarrow x = 0,$$
$$B(x, y) = 0 \ \forall x \in E \Rightarrow y = 0.$$

In this case we call the pairing separated *and (E, F) a* dual system *with respect to B.*

Examples.

1. The *canonical bilinear form* $E \times E^* \to \mathbb{K}$, $(x, x') \mapsto \langle x, x'\rangle = \langle x', x\rangle = x'(x)$ is nondegenerate: $x'(x) = 0$ for all x means that $x' = 0$; if $x'(x) = 0$ for all x', let $(b_i)_i$ be a basis of E and write $x = \sum x_i b_i$; defining $x'_j(b_i) := \delta_{ij}$ we have that $x'_j(x) = x_j = 0$ for all j and hence $x = 0$.
2. If E is a TVS, $E' := \{x' \colon E \to \mathbb{K} \mid x' \text{ is linear and continuous}\}$ is the *topological dual* of E. The restriction of $\langle ., .\rangle\colon E \times E^* \to \mathbb{K}$ to $E \times E'$ separates points of E'. If E is locally convex and Hausdorff, $\langle ., .\rangle$ separates points of E (Proposition 9.6 (i) with $M = \{0\}$).

Suppose that E and F are paired with respect to B and consider the mappings

$$\Psi\colon F \to E^*, \quad y \mapsto [x \mapsto B(x, y)],$$
$$\Phi\colon E \to F^*, \quad x \mapsto [y \mapsto B(x, y)].$$

If B separates points of F, Ψ is injective and we can identify F with the linear subspace $\mathrm{im}(F)$ of E^*. In this case, B is the restriction of the canonical bilinear form $(x, y) \mapsto \langle x, y\rangle$ on $E \times E^*$ and we write $\langle x, y\rangle$ instead of $B(x, y)$.

Similarly, if B separates points of E then Φ is injective and $E \subseteq F^*$.

We have seen that if E is a Hausdorff LCS then (E, E') is a dual system. We will see that for every dual system (E, F) one can find a topology $\mathscr{T}$ on E such that $(E, \mathscr{T})' = F$.

Definition 10.2. *Let E, F be paired with respect to a bilinear form B. The* weak topology *$\sigma(E, F)$ on E is the locally convex topology defined by the family of seminorms*

$$\{x \mapsto |B(x, y)| : y \in F\}.$$

We see that $\sigma(E, F)$ is Hausdorff if and only if $\forall x \in E \setminus \{0\}\ \exists y \in F\colon |B(x, y)| \neq 0$ (Proposition 3.18), i.e., if and only if B separates points of E.

Lemma 10.3. *Let E, F be paired w.r.t. B such that B separates points of E. Considering E as a subspace of F^* and the latter as a subspace of $\mathbb{K}^F$, $\sigma(E, F)$ is the topology inherited from the product topology on $\mathbb{K}^F$.*

Proof. A subbasis of the product topology is given by the collection of all sets of the form $\{f : F \to \mathbb{K} \mid f(y) \in \varepsilon\mathbb{D}\}$ for $y \in F$, $\varepsilon > 0$, which in E correspond to the sets $\{x \in E : |B(x, y)| < \varepsilon\}$. □

Lemma 10.4. *Let E be a vector space and $u, u_1, \ldots, u_n \in E^*$. Then $u \in \operatorname{span}\{u_1, \ldots, u_n\} \Leftrightarrow \ker u \supseteq \bigcap_{i=1}^n \ker u_i$.*

Proof. "$\Rightarrow$" is clear. Conversely, define $T : E \to \mathbb{K}^n$ by $T(x) := (\langle u_i, x\rangle)_{i=1,\ldots,n}$ and $\tilde{v} : \operatorname{im} T \to \mathbb{K}$ by $\tilde{v}(T(x)) := u(x)$, which is well-defined by assumption. Extend $\tilde{v}$ to a linear form $v \in (\mathbb{K}^n)^*$ such that $v \circ T = u$ and choose $(\nu_i)_i \in \mathbb{K}^n$ such that $v(\alpha_1, \ldots, \alpha_n) = \sum \nu_i \alpha_i$. Then $u(x) = v(T(x)) = \sum \nu_i \langle u_i, x\rangle = \langle \sum \nu_i u_i, x\rangle$, so $u = \sum \nu_i u_i$. □

Proposition 10.5. *Let E, F be paired with respect to B. Then for all $y \in F$, $\Psi(y) = B(., y) \in (E, \sigma(E, F))'$; conversely, for each $u \in (E, \sigma(E, F))'$ there is $y \in F$ such that $u(x) = B(x, y)\ \forall x \in E$.*

In particular, $\Psi : F \to (E, \sigma(E, F))'$ is surjective and $F/\ker\Psi \cong (E, \sigma(E, F))'$. If B separates points of F then $F = (E, \sigma(E, F))'$.

Proof. Continuity of $B(., y)$ is clear. For $u \in (E, \sigma(E, F))'$ there are $y_1, \ldots, y_n \in F$ such that $|u(x)| \le \max_i |B(x, y_i)|$. With $u_i := \Psi(y_i) \in E^*$ we have that $u_i(x) = 0$ for all i implies $u(x) = 0$, so by Lemma 10.4 $u = \sum_i \nu_i u_i$ for some $\nu_i \in \mathbb{K}$, which gives $u(x) = B(x, \sum_i \nu_i y_i)$. □

Proposition 10.6. *Let E, F form a pairing which separates points of E (i.e., $E \subseteq F^*$). Then the pairing separates points of F if and only if E is dense in $(F^*, \sigma(F^*, F))$.*

Proof. If E is dense and $y \in F$ satisfies $\langle x, y\rangle = 0$ for all $x \in E \subseteq F^*$ then this also is the case for all $x \in F^*$ because $\langle ., y\rangle$ is $\sigma(F^*, F)$-continuous. Conversely, if E is not dense take $x^* \in F^* \setminus \bar{E}$. Then by Proposition 9.6 (i) there is an element $f \in (F^*, \sigma(F^*, F))'$ given by $f(y^*) = \langle y^*, y\rangle$ for some $y \in F$ and such that $\langle x, y\rangle = f(x) = 0$ for $x \in E$ but $\langle x^*, y\rangle = f(x^*) \neq 0$, i.e., $y \neq 0$, which implies that the pairing does not separate points of F. □

Proposition 10.7. *Let E, F form a pairing with respect to B which separates points of F. If N is a proper subspace of F then $\sigma(E, F)$ is strictly finer than $\sigma(E, N)$.*

Proof. For $y \in F \setminus N$, $B(., y)$ is $\sigma(E, F)$-continuous. If $B(., y)$ was $\sigma(E, N)$-continuous, Proposition 10.5 would give some $z \in N$ such that $B(., y) = B(., z)$, i.e., $B(x, y - z) = 0$ for all $x \in E$, which would imply $y = z$. □

Proposition 10.8. *Let (E, F) be a dual system. Then $(E, \sigma(E, F))$ is complete if and only if $E = F^*$.*

Proof. If we consider F^* as a subspace of $\mathbb{K}^F$ then $\sigma(F^*, F)$ is the topology inherited from the product topology on $\mathbb{K}^F$; the latter is complete.

Take f in the closure of F^* in $\mathbb{K}^F$. For $x, y \in F$, $\rho \in \mathbb{K}$ and $\varepsilon > 0$ there is $x^* \in F^*$ such that

$$|f(z) - x^*(z)| \le \min\{\frac{\varepsilon}{3}, \frac{\varepsilon}{3\,|\rho| + 1}\}$$

for $z \in \{x, y, \rho x + y\}$. Then we have

$$|f(\rho x + y) - \rho f(x) - f(y)| \leq |f(\rho x + y) - \langle x^*, \rho x + y\rangle| \\ + |\rho\langle x^*, x\rangle - \rho f(x)| + |\langle x^*, y\rangle - f(y)| \leq \varepsilon$$

so $f \in F^*$, and F^* is closed and hence complete.

If $(E, \sigma(E, F))$ is complete then $E = F^*$ follows from Proposition 10.6. $\square$

11. Polarity

Definition 11.1. *Let E, F be vector spaces paired with respect to B. The* polar *of a subset $A \subseteq E$ is the set*

$$A^\circ = \{y \in F : |B(x, y)| \leq 1 \ \forall x \in A\}.$$

Proposition 11.2. *(i) $A_1 \subseteq A_2 \Rightarrow A_1^\circ \supseteq A_2^\circ$,*
(ii) $A^\circ = \breve{A}^\circ$,
(iii) $A \subseteq A^{\circ\circ} := (A^\circ)^\circ$,
(iv) $A^\circ = A^{\circ\circ\circ}$,
(v) A° is absolutely convex and $\sigma(F, E)$-closed,
(vi) $(\lambda A)^\circ = \frac{1}{\lambda} A^\circ$ for $\lambda \in \mathbb{K} \setminus \{0\}$,
(vii) A° is absorbent if and only if A is $\sigma(E, F)$-bounded, and
(viii) $\left(\bigcup_{i \in I} A_i\right)^\circ = \bigcap_{i \in I} A_i^\circ$.

Proof. (i): $y \in A_2^\circ \Rightarrow |B(x, y)| \leq 1 \ \forall x \in A_2 \Rightarrow |B(x, y)| \leq 1 \ \forall x \in A_1 \Rightarrow y \in A_1^\circ$.

(ii): $A \subseteq \mathbb{D} \cdot A \Rightarrow (\mathbb{D} \cdot A)^\circ \subseteq A^\circ$. Let $y \in A^\circ$; for all $\lambda \in \mathbb{D}$ and $x \in A$, $|B(\lambda x, y)| \leq |\lambda| \cdot |B(x, y)| \leq 1$, so $y \in (\mathbb{D} \cdot A)^\circ$.

(iii): $x \in A \Rightarrow |B(x, y)| \leq 1 \ \forall y \in A^\circ \Rightarrow x \in A^{\circ\circ}$.

(iv): $A \subseteq A^{\circ\circ} \Rightarrow A^{\circ\circ\circ} \subseteq A^\circ$ by (i) and $A^\circ \subseteq A^{\circ\circ\circ}$ by (iii).

(v): Let $y, z \in A^\circ$ and $\lambda, \mu \in \mathbb{K}$ with $|\lambda| + |\mu| \leq 1$. Then for all $x \in A$: $|B(x, \lambda y + \mu z)| \leq 1$, so $\lambda y + \mu z \in A^\circ$. For all $x \in E$, $B(x, .)$ is $\sigma(F, E)$-continuous (Proposition 10.5), so $A^\circ = \bigcap_{x \in A} B(x, .)^{-1}(\mathbb{D})$ is $\sigma(F, E)$-closed.

(vi): $y \in (\lambda A)^\circ \Leftrightarrow \forall x \in \lambda A : |B(x, y)| \leq 1 \Leftrightarrow \forall x \in A : |B(\lambda x, y)| = |B(x, \lambda y)| \leq 1 \Leftrightarrow \lambda y \in A^\circ \Leftrightarrow y \in \frac{1}{\lambda} A^\circ$.

(vii): A° is absorbent if and only if $\forall y \in F \ \exists \mu > 0$ such that $\mu y \in A^\circ$, which is equivalent to $\forall x \in A : |B(x, y)| \leq \frac{1}{\mu}$; using Proposition 6.2 (iii) and Corollary 6.3, this is equivalent to A being $\sigma(E, F)$-bounded.

(viii): $y \in \left(\bigcup_{i \in I} A_i\right)^\circ \Leftrightarrow \forall x \in \bigcup_i A_i : |B(x, y)| \leq 1 \Leftrightarrow \forall i \ \forall x \in A_i$: $|B(x, y)| \leq 1 \Leftrightarrow \forall i : y \in A_i^\circ \Leftrightarrow y \in \bigcap_i A_i^\circ$. $\square$

Theorem 11.3 (Bipolar theorem)**.** *For $A \neq \emptyset$, $A^{\circ\circ}$ is the absolutely convex $\sigma(E, F)$-closed hull of A.*

Proof. By Proposition 11.2, $A^{\circ\circ}$ is absolutely convex, $\sigma(E, F)$-closed and contains A. We need to show that $A^{\circ\circ} \subseteq D$ for each set D of this kind.

Take $a \notin D$ and let E_0 be the real vector space underlying E. If U is a convex open neighborhood of a such that $(a + U) \cap D = \emptyset$ then by Proposition 9.11 there is $\alpha \in \mathbb{R}$ and a $\sigma(E_0, F)$-continuous linear form $u \colon E_0 \to \mathbb{R}$ such that $u(x) \leq \alpha$ for $x \in D$ and $u(a) > \alpha$. Replacing α by $(\alpha + u(a))/2$, we also have that $u(x) < \alpha$ for $x \in D$. Because $0 \in D$, $0 < \alpha$ and by replacing u with u/α we can assume that $\alpha = 1$.

In the real case, set $v = u$; in the complex case, set $v(x) = u(x) - iu(ix)$. Then $v \in E'$, such that $v(x) = B(x, y)$ for some $y \in F$ and $u(x) = \Re v(x) = \Re B(x, y)$ for all $x \in E$. Now for each $x \in D$ there is some $\lambda \in \mathbb{K}$ with $|\lambda| = 1$ such that

$$|B(x, y)| = \lambda B(x, y) = B(\lambda x, y) = u(\lambda x) \leq 1$$

because $\lambda x \in D$, so $y \in D^\circ \subseteq A^\circ$. However, we have

$$|B(a, y)| \geq |u(a)| > 1$$

so $a \notin A^{\circ\circ}$. ☐

We denote the absolutely convex $\sigma(E, F)$-closed hull of a subset $A \subseteq E$ by $\overline{\operatorname{acx}} A$.

Proposition 11.4. *Let $(A_i)_i$ be a nonempty family of absolutely convex nonempty $\sigma(E, F)$-closed subsets of E. Then*

$$\left(\bigcap_i A_i\right)^\circ = \overline{\operatorname{acx}}\left(\bigcup_i A_i^\circ\right).$$

Proof. We have

$$\bigcap_i A_i = \bigcap_i A_i^{\circ\circ} = \left(\bigcup_i A_i^\circ\right)^\circ$$

and hence

$$\left(\bigcap_i A_i\right)^\circ = \left(\bigcup_i A_i^\circ\right)^{\circ\circ} = \overline{\operatorname{acx}}\left(\bigcup_i A_i^\circ\right).$$ ☐

If $M \subseteq E$ is a linear subspace then $M^\circ = M^\perp$, defined as follows:

Definition 11.5. *Let E, F be paired with respect to B. Given any subset $M \subseteq E$, the linear subspace of F defined by $M^\perp := \{y \in F : B(x, y) = 0\ \forall x \in M\}$ is called the subspace* orthogonal *to M. Similarly, one defines $N^\perp \subseteq E$ for $N \subseteq F$.*

If $(M_i)_i$ is a family of subsets of E we denote by $\bigvee_i M_i$ the smallest $\sigma(E, F)$-closed linear subspace of E containing all sets M_i, equal to the $\sigma(E, F)$-closure of the linear span of $\bigcup_i M_i$.

Proposition 11.6. *Let E, F be paired with respect to B.*

(i) $M_1 \subseteq M_2 \Rightarrow M_1^\perp \supseteq M_2^\perp$,
(ii) $M \subseteq M^{\perp\perp} := (M^\perp)^\perp$,
(iii) $M^\perp = M^{\perp\perp\perp}$,
(iv) $M^\perp$ *is* $\sigma(F, E)$*-closed,*
(v) $M^{\perp\perp}$ *is the* $\sigma(E, F)$*-closed subspace of* E *generated by* M*,*
(vi) $(\bigcup_i M_i)^\perp = (\bigvee_i M_i)^\perp = \bigcap_i M_i^\perp$, *and*
(vii) *if each* M_i *is a* $\sigma(E, F)$*-closed subspace of* E *then* $(\bigcap_i M_i)^\perp = \bigvee_i M_i^\perp$.

Proof. (i)–(iv) are clear (as in Proposition 11.2).

(v): by (iv) $M^{\perp\perp}$ is a $\sigma(E, F)$-closed subspace of E and $M \subseteq M^{\perp\perp}$ by (ii). Let $L \subseteq M^{\perp\perp}$ be the $\sigma(E, F)$-closed subspace generated by M. Supposing that there exists some $a \in M^{\perp\perp} \setminus L$, Proposition 9.6 (i) gives some $u \in E'$ which is of the form $u(x) = B(x, y)$ for some $y \in F$, such that $u(x) = B(x, y) = 0$ for $x \in L \supseteq M$ but $B(a, y) = 1$, which would imply $y \in M^\perp$ but $y \notin M^{\perp\perp\perp}$, contradicting (iii).

(vi): this is seen from

$$\left(\bigcup_i M_i\right)^{\perp} = \bigcap_i M_i^{\perp},$$

$$\left(\bigcup_i M_i\right)^{\perp} = \left(\bigcup_i M_i\right)^{\perp\perp\perp},$$

$$\left(\bigcup_i M_i\right)^{\perp\perp} = \bigvee_i M_i.$$

(vii): we have

$$\bigcap_i M_i = \bigcap_i M_i^{\perp\perp} = (\bigvee_i M_i^{\perp})^{\perp},$$

$$\left(\bigcap_i M_i\right)^{\perp} = \bigvee_i M_i^{\perp}.$$

□

12. $\mathfrak{S}$-TOPOLOGIES

Let E, F be vector spaces paired with respect to a bilinear form B and $\mathfrak{S}$ be a collection of $\sigma(E,F)$-bounded subsets of E. Then the polars A° ($A \in \mathfrak{S}$), which are absorbent and absolutely convex, by Corollary 3.8 define a locally convex topology on F, called the $\mathfrak{S}$-topology or the topology of uniform convergence on sets of $\mathfrak{S}$. A 0-basis of this topology is given by finite intersections of sets of the form λA° ($\lambda > 0$, $A \in \mathfrak{S}$), and it is defined by the family of seminorms $q_A(y) = \sup_{x \in A} |B(x,y)|$ for $A \in \mathfrak{S}$ because $y \in \lambda A^\circ \Leftrightarrow q_A(y) \leq \lambda$.

Proposition 12.1. *The $\mathfrak{S}$-topology is Hausdorff if and only if $\bigcup\{A : A \in \mathfrak{S}\}$ is total in E for $\sigma(E,F)$ (i.e., if its linear span is dense) and the pairing separates points of F.*

Proof. Suppose that $q_A(y) = 0$ for all $A \in \mathfrak{S}$, which is equivalent to

$$y \in \bigcap_{A \in \mathfrak{S}} A^{\perp} = \left(\bigvee_{A \in \mathfrak{S}} A\right)^{\perp}.$$

If now $\bigcup_{A \in \mathfrak{S}} A$ is total then $\bigvee_{A \in \mathfrak{S}} A = E$ and $B(x,y) = 0$ for all $x \in E$, which implies that $y = 0$, so the $\mathfrak{S}$-topology is Hausdorff.

Conversely, if $\bigvee_{A \in \mathfrak{S}} A \neq E$ or B does not separate points of F, then there is $y \in F \setminus \{0\}$ such that $y \in A^{\perp}$ for all $A \in \mathfrak{S}$, i.e., such that $q_A(y) = 0$ for all $A \in \mathfrak{S}$, which means that the topology is not Hausdorff. □

Proposition 12.2. *The $\mathfrak{S}$-topology on F is unchanged if we replace $\mathfrak{S}$ by any of the following:*

(i) all subsets of sets in $\mathfrak{S}$,
(ii) finite unions of sets in $\mathfrak{S}$,
(iii) sets λA with $\lambda \in \mathbb{K}$ and $A \in \mathfrak{S}$,
(iv) balanced hulls of sets in $\mathfrak{S}$,
(v) $\sigma(E,F)$-closures of sets in $\mathfrak{S}$,

(vi) absolutely convex $\sigma(E,F)$-closed hulls of sets in $\mathfrak{S}$.

Proof. (i): If $A_1 \subseteq A \in \mathfrak{S}$ then $A^\circ \subseteq A_1^\circ$, so A_1° is a 0-neighborhood.

(ii): $(\bigcup_{i=1}^n A_i)^\circ = \bigcap_{i=1}^n A_i^\circ$ is a 0-neighborhood.

(iii): For $\lambda \neq 0$, $(\lambda A)^\circ = \lambda^{-1} A^\circ$ is a 0-neighborhood; for $\lambda = 0$, $(\lambda A)^\circ = F$.

(iv): $(\mathbb{D} \cdot A)^\circ = A^\circ$.

(v): $A \subseteq \bar{A} \subseteq A^{\circ\circ}$, so $A^\circ \supseteq (\bar{A})^\circ \supseteq A^{\circ\circ\circ} = A^\circ$, so $(\bar{A})^\circ = A^\circ$.

(vi): for $A \in \mathfrak{S}$, its absolutely convex $\sigma(E,F)$-closed hull is $A^{\circ\circ}$, but $A^{\circ\circ\circ} = A^\circ$. □

Example. The weak topology $\sigma(E,F)$ is the $\mathfrak{S}$-topology for $\mathfrak{S}$ the collection of all singletons of F.

Definition 12.3. *If E, F form a pairing separating points of F, then a locally convex topology $\mathscr{T}$ on E is said to be* compatible *with the pairing if $(E, \mathscr{T})' = F$.*

Proposition 12.4. *If E, F form a pairing separating points of F, then the closed convex sets of E are the same for all locally convex topologies on E compatible with the pairing.*

Proof. We may assume that E, F are real vector spaces. A closed convex set is given by the intersection of all half-spaces $f(x) \leq \alpha$ containing it ($f \in E'$, $\alpha \in \mathbb{R}$), and the continuous linear forms are the same for all topologies compatible with the pairing. □

Definition 12.5. *Let E be a topological space, F a TVS and H a set of mappings $E \to F$. H is called* equicontinuous *at $a \in E$ if for every 0-neighborhood V in F there is a neighborhood U of a such that $f(U) \subseteq f(a) + V$ for all $f \in H$. H is called* equicontinuous *on E if it is equicontinuous at every point of E.*

If also E is a TVS then H is called uniformly equicontinuous *if for every 0-neighborhood V in F there is a 0-neighborhood U in E such that $x - y \in U$ implies $f(x) - f(y) \in V$ for all $f \in H$.*

If H is uniformly equicontinuous then it is equicontinuous and each $f \in H$ is uniformly continuous.

Proposition 12.6. *If E, F are TVS and H a set of linear maps $E \to F$ then H is uniformly equicontinuous if it equicontinuous at the origin.*

Proof. If V is a 0-neighborhood in F there is a 0-neighborhood U in E with $H(U) \subseteq V$. Then, $x - y \in U$ implies $f(x) - f(y) = f(x - y) \in V$ for all $f \in H$. □

Proposition 12.7. *If E is a TVS then a subset $A \subseteq E'$ is equicontinuous if and only if $A \subseteq U^\circ$ for some 0-neighborhood U in E.*

Proof. If A is equicontinuous there is a 0-neighborhood U in E such that $|\langle x, u \rangle| \leq 1$ for $x \in U$ and $u \in A$, so $A \subseteq U^\circ$. Conversely, if $A \subseteq U^\circ$ for some 0-neighborhood U then $|\langle x, u \rangle| \leq \varepsilon$ for $x \in \varepsilon U$ and $u \in A$, so A is equicontinuous at 0 and hence equicontinuous. □

Proposition 12.8. *The topology $\mathscr{T}$ of any LCS E coincides with the topology of uniform convergence on the family of equicontinuous subsets of E'.*

Proof. There is a 0-basis of $\mathscr{T}$ consisting of absolutely convex closed sets V, which by Proposition 12.4 are also $\sigma(E, E')$-closed, so by Theorem 11.3 we have $V = V^{\circ\circ}$. By Proposition 12.7 V° is equicontinuous, so V is a 0-neighborhood in the $\mathfrak{S}$-topology, which hence is finer than $\mathscr{T}$.

Conversely, if $M \subseteq E'$ is equicontinuous then $M \subseteq V^\circ$ where V is an absolutely convex $\sigma(E, E')$-closed 0-neighborhood in $\mathscr{T}$. Because $M^\circ \supseteq V^{\circ\circ} = V$, M° is a 0-neighborhood in $\mathscr{T}$ and the topologies are equal. $\square$ [12.5.]

Theorem 12.9 (Alaoglu-Bourbaki)**.** *Let E be a TVS. Any equicontinuous subset of E' is relatively $\sigma(E', E)$-compact.*

Proof. Let $H \subseteq E'$ be equicontinuous. By Proposition 12.7, $H \subseteq U^\circ$ for some 0-neighborhood U in E and it suffices to show that U° is $\sigma(E', E)$-compact.

Note that

$$(E', \sigma(E', E)) \to (E^*, \sigma(E^*, E)) \to \mathbb{K}^E$$

are topological embeddings.

Denoting by $U^\bullet$ the polar of U in E^*, $U^\bullet$ is $\sigma(E^*, E)$-closed (Proposition 11.2 (v)) and $\sigma(E^*, E)$-bounded (U absorbent $\Rightarrow U^{\bullet\bullet}$ absorbent $\Rightarrow U^\bullet$ $\sigma(E^*, E)$-bounded by Proposition 11.2 (vii)).

$U^\bullet$ is bounded in $\mathbb{K}^E$, so each projection $\pi_x(U^\bullet) \subseteq \mathbb{K}$ ($x \in E$) is bounded and $U^\bullet \subseteq \prod_{x \in E} \overline{\pi_x(U^\bullet)}$ is relatively compact.

For $u \in U^\bullet$, $u(\varepsilon U) \subseteq \varepsilon\mathbb{D}$ gives $u \in E'$, so $U^\bullet = U^\circ \subseteq E'$.

We now have that $\overline{V^\circ} \subseteq \mathbb{K}^E$ is compact. By Lemma Lemma 12.10, the closure of V° in $\mathbb{K}^E$ is contained in E' and hence equal to its $\sigma(E', E)$-closure, which therefore is compact. $\square$

Lemma 12.10. *Let E be a TVS and $H \subseteq E'$ equicontinuous. Then the closure of H in $\mathbb{K}^E$ is contained in E' and equicontinuous.*

Proof. Let $u \in \overline{H}'$, $x, y \in E$, $\rho \in \mathbb{K}$, $\varepsilon > 0$. Then $\exists f \in H$ such that

$$|u(z) - f(z)| \leq \min\{\frac{\varepsilon}{3}, \frac{\varepsilon}{3(|\rho| + 1)}\} \qquad (z \in \{x, y, \rho x + y\}),$$

so

$$\begin{aligned} |u(\rho x + y) - \rho u(x) - \rho(y)| &\leq |u(\rho x + y) - f(\rho x + y)| \\ &+ |\rho u(x) - \rho f(x)| + |u(y) - f(y)| \leq \varepsilon \end{aligned}$$

so u is linear. Moreover, $H(U) \subseteq \mathbb{D}$ for some 0-neighborhood U in E, so $H \subseteq \bigcap_{x \in U} \pi_x^{-1}(\mathbb{D})$, which is a closed subset of $\mathbb{K}^E$ mapping u to $\mathbb{D}$. Because $\overline{H}(U) \subseteq \mathbb{D}$, $\overline{H} \subseteq E'$ and it is equicontinuous. $\square$

Corollary 12.11. *If E is a normed space, the closed unit ball $B_1' := \{x' \in E' \mid \|x'\| \leq 1\}$ is $\sigma(E', E)$-compact.*

Proof. $\|x'\| = \sup_{\|x\| \leq 1} |\langle x, x' \rangle|$, so B_1' is the polar of $B_1 := \{x \in E \mid \|x\| \leq 1\}$, hence equicontinuous, and the claim follows from Theorem 12.9. $\square$

Definition 12.12. *Let E, F be paired vector spaces. If $\mathfrak{S}$ is the set of all $\sigma(E, F)$-bounded subsets of E then the corresponding $\mathfrak{S}$-topology on F is denoted by $\beta(F, E)$ and called the* strong topology *or* topology of uniform convergence on bounded subsets of E.

Example. Let E be a normed space and E' its dual.

- $A \subseteq E$ is $\sigma(E, E')$-bounded if for all $x' \in E'$ there is $\lambda > 0$ such that $|\langle x, x' \rangle| \leq \lambda$ for all $x \in A$.
- if $A \subseteq E$ is bounded (w.r.t. the norm) it is bounded for $\sigma(E, E')$: $\|x\| \leq M$ for all $x \in A$ implies $|\langle x, x' \rangle| \leq M\|x'\|$ for all $x \in A$.
- Recall: $x \mapsto [x' \mapsto \langle x, x' \rangle]$ is an isometric embedding of E into $(E')'$, i.e., $\|x\| = \sup\{|\langle x, x' \rangle| / \|x\| \mid x' \in E \setminus \{0\}\}$.
- If $A \subseteq E$ is $\sigma(E, E')$-bounded then for all $x \in E'$, the set $\{|\langle x, x' \rangle| \mid x \in A\}$ is bounded, i.e., A is pointwise bounded in $(E')'$. By the Banach-Steinhaus theorem it is uniformly bounded, i.e., there is M such that for all $x \in A$ we have $|\langle x, x' \rangle| \leq M\|x'\|$, so $\|x\| \leq M$ for all $x \in A$ and A is norm-bounded.

  Hence, the $\sigma(E, E')$-bounded subsets of E are exactly the norm-bounded ones.
- $A \subseteq E$ is norm-bounded if and only if there is $\lambda > 0$ such that $A \subseteq \lambda B_1$, where B_1 is the closed unit ball in E. Hence, $\beta(E', E)$ is the $\mathfrak{S}$-topology with $\mathfrak{S} = \{B_1\}$. Because $(B_1)^\circ = B_1'$ (unit ball in E'), the $\beta(E', E)$-topology is the norm-topology on E'.
- In particular, $\beta(E', E)$ is *not* compatible with the duality of E is not reflexive.

13. The Mackey Topology

Let E, F form a pairing separating points of E (i.e., we have $E \subseteq F^*$). For which topologies $\mathscr{T}$ on F do we have $(F, \mathscr{T})' = E$ – which topologies on F are compatible with the pairing?

Proposition 13.1. *Let $\mathfrak{S}$ be a collection of absolutely convex $\sigma(E, F)$-closed $\sigma(E, F)$-bounded subsets of E such that*

(i) $(A_i)_i$ finite family in $\mathfrak{S} \Rightarrow \overline{\operatorname{acx}} \bigcup_i A_i \in \mathfrak{S}$,
(ii) $A \in \mathfrak{S}$, $\lambda \neq 0 \Rightarrow \lambda A \in \mathfrak{S}$.

Then, if $\mathscr{T}_\mathfrak{S}$ is the corresponding $\mathfrak{S}$-topology on F, we have

$$(F, \mathscr{T}_\mathfrak{S})' = \bigcup \{\operatorname{cl}_{\sigma(F^*, F)} A \mid A \in \mathfrak{S}\}$$

where $\operatorname{cl}_\mathscr{T}$ denote the closure with respect to a topology $\mathscr{T}$.

Proof. Let $u \in (F, \mathscr{T}_\mathfrak{S})'$. Then $u(U) \subseteq \mathbb{D}$ for some 0-neighborhood U in F which is of the form

$$U = \bigcap_{i=1}^n \lambda_i A_i^\circ \qquad (\lambda_i \in \mathbb{K} \setminus \{0\}, A_i \in \mathfrak{S})$$

$$= \bigcap_{i=1}^n (\lambda_i^{-1} A_i)^\circ = (\bigcup_{i=1}^n \lambda_i^{-1} A_i)^\circ = (\bigcup_{i=1}^n \lambda_i^{-1} A_i)^{\circ\circ\circ} = (\overline{\operatorname{acx}} \bigcup_{i=1}^n \lambda_i^{-1} A_i)^\circ = A^\circ$$

with $A := \overline{\operatorname{acx}} \bigcup_{i=1}^n \lambda_i^{-1} A_i \in \mathfrak{S}$. Now $u(A^\circ) \subseteq \mathbb{D}$, so $u \in A^{\circ\circ} = \operatorname{cl}_{\sigma(F^*, F)} A$.

Conversely, let $u \in \operatorname{cl}_{\sigma(F^*, F)} A = A^{\circ\circ}$ for some $A \in \mathfrak{S}$, then $u(\varepsilon A^\circ) \subseteq \varepsilon \mathbb{D}$ and $\varepsilon A^\circ = (\varepsilon^{-1} A)^\circ$ is a 0-neighborhood in the $\mathfrak{S}$-topology. □

Lemma 13.2. *Let E be a LCS and $A_1, \ldots, A_n \subseteq E$ convex and compact. Then the convex and the absolutely convex hulls of $\bigcup_{i=1}^n A_i$ are compact.*

Proof. $L := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{K}^n \mid \lambda_i \geq 0, \sum \lambda_i = 1\}$ is compact, so $K := L \times A_1 \cdots A_n$ is compact in $\mathbb{K}^n \times E^n$. The mapping $f\colon K \to E$, $(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_n) \mapsto \sum \lambda_i x_i$ is continuous, so $f(K)$ (which is the convex hull of $\bigcup_i A_i$) is compact. For the absolutely convex hull use $L = \{(\lambda_1, \ldots, \lambda_n) \mid \sum |\lambda_i| \leq 1\}$. □

Proposition 13.3. *Let $\mathfrak{S}_0$ be a family of $\sigma(E,F)$-bounded subsets of E. Then*
$$(F,\mathscr{T}_{\mathfrak{S}_0})' = \operatorname{span}\{\overline{\operatorname{acx}}^{\sigma(F^*,F)} B \mid B\in\mathfrak{S}_0) =: H.$$

Proof. The set
$$\mathfrak{S} := \{\overline{\operatorname{acx}}^{\sigma(e,F)} \bigcup_{i=1}^n \lambda_i B_i \mid n\in\mathbb{N}, \lambda_i\in\mathbb{K}, B_i\in\mathfrak{S}_0\}$$
satisfies (i) and (ii) of Proposition 13.1; to see this, set
$$\begin{aligned}\sigma_1 &:= \{\lambda B \mid \lambda\neq 0, B\in\mathfrak{S}_0\}\\ \sigma_2 &:= \{B_1\cup\ldots\cup B_n \mid n\in\mathbb{N}, B_i\in\mathfrak{S}_1\}\\ \sigma &= \{\overline{\operatorname{acx}}\, B \mid B\in\mathfrak{S}_2\}.\end{aligned}$$
These define the same topologies as $\mathfrak{S}_0$.

For (i), take $A_i = (\bigcup_j \lambda_{ij} B_{ij}^{\circ\circ}\in\mathfrak{S}$. Then we have
$$\begin{aligned}(\bigcup_i A_i)^{\circ\circ} &= (\bigcup_i(\bigcup_j \lambda_{ij}B_{ij})^{\circ\circ})^{\circ\circ} = (\bigcup_i(\bigcap_j \lambda_{ij}B_{ij})^{\circ\circ\circ})^{\circ}\\ &= (\bigcap_i(\bigcup_j \lambda_{ij}B_{ij})^{\circ})^{\circ} = (\bigcup_{ij}\lambda_{ij}B_{ij})^{\circ\circ}\in\mathfrak{S}.\end{aligned}$$
For (ii), we have $\lambda(\bigcup_j \lambda_j B_j)^{\circ\circ} = (\bigcup_j \lambda\lambda_j B_j)^{\circ\circ}\in\mathfrak{S}$.

Each $A\in\mathfrak{S}$ is $\sigma(E,F)$-bounded. We then have
$$(F,\mathscr{T}_{\mathfrak{S}_0})' = (F,\mathscr{T}_\sigma)' = \bigcup_{A\in\mathfrak{S}} \overline{A}^{\sigma(F^*,F)}.$$
For $B\in\mathfrak{S}_0$, $\overline{\operatorname{acx}}^{\sigma(F^*,F)} B = \operatorname{cl}_{\sigma(F^*,F)} \overline{\operatorname{acx}}\, B \subseteq (F,\mathscr{T}_{\sigma_0})'$, so $H\subseteq (F,\mathscr{T}_{\mathfrak{S}_0})'$.

Conversely, take $A = \overline{\operatorname{acx}}(\bigcup_i \lambda_i B_i)\in\mathfrak{S}$ ($B_i\in\mathfrak{S}_0$) and set $A_i := \overline{\operatorname{acx}}^{\sigma(F^*,F)} B_i = B_i^{\circ\circ} = (B_i^\circ)^\circ \subseteq H$ (polars taken with respect to (F^*,F)), then B_i° is a 0-neighborhood in $(F,\mathscr{T}_{\sigma_0})$, so $(B_i^\circ)^\circ \subseteq (F,\mathscr{T}_{\sigma_0})'$ is equicontinuous, by Theorem 12.9 $\sigma((F,\mathscr{T}_{\sigma_0})',F)$-compact and hence $\sigma(F^*,F)$-compact. By Lemma 13.2, $A_0 := \operatorname{acx}\bigcup \lambda_i A_i \subseteq H$ is $\sigma(F^*,F)$-compact, hence $\sigma(F^*,F)$-closed in F^*. Because $\bigcup_i \lambda_i B_i\subseteq A_0$ we have
$$\operatorname{cl}_{\sigma(F^*,F)} \overline{\operatorname{acx}} \bigcup_i \lambda_i B_i \subseteq A_0 \subseteq H,$$
so $(F,\mathscr{T}_{\sigma_0})'\subseteq H$. ∎

Corollary 13.4. *Let E,F form a pairing separating points of E and let $\mathfrak{S}$ be a collection of $\sigma(E,F)$-bounded subsets of E. Then*

(i) $(F,\mathscr{T}_{\mathfrak{S}})'\subseteq E \iff \forall B\in\mathfrak{S}$*:* $\overline{\operatorname{acx}}\, B$ *is $\sigma(E,F)$-compact.*
(ii) $(F,\mathscr{T}_{\mathfrak{S}})'\supseteq E \iff \operatorname{span}\{\overline{\operatorname{acx}}\, B \mid B\in\mathfrak{S}\} = E$.

Proof. (i) "$\Leftarrow$": $\overline{\operatorname{acx}}\, B$ $\sigma(E,F)$-compact $\Rightarrow$ $\sigma(F^*,F)$-compact $\Rightarrow$ $\sigma(F^*,F)$-closed $\Rightarrow$ $\overline{\operatorname{acx}}^{\sigma(F^*,F)} B\subseteq E \Rightarrow (F,\mathscr{T}_{\mathfrak{S}})'\subseteq E$ by Proposition 13.3.

"$\Rightarrow$": for $B\in\mathfrak{S}$, $\overline{\operatorname{acx}}^{\sigma(F^*,F)} B\subseteq (F,\mathscr{T}_f S)'\subseteq E$ by Proposition 13.3, so $\overline{\operatorname{acx}}\, B$ is $\sigma(F^*,F)$-closed. As $\overline{\operatorname{acx}}\, B = (B^\circ)^\circ$ and B° is a 0-neighborhood in $\mathscr{T}_{\mathfrak{S}}$, $(B^\circ)^\circ$ is relatively $\sigma(F',F)$-compact (Theorem 12.9), hence relatively $\sigma(F^*,F)$-compact $\Rightarrow$ $\sigma(F^*,F)$-compact $\Rightarrow$ $\sigma(E,F)$-compact.

(ii) "$\Leftarrow$": $E\subseteq \operatorname{span}\{\overline{\operatorname{acx}}\, B\mid B\in\mathfrak{S}\}\subseteq \operatorname{span}\{\overline{\operatorname{acx}}^{\sigma(F^*,F)} B \mid B\in\mathfrak{S}\}\subseteq (F,\mathscr{T}_{\mathfrak{S}})'$.

"⇒": $E \subseteq E \cap \operatorname{span}\{\overline{\operatorname{acx}}^{\sigma(F^*,F)} B \mid B \in \mathfrak{S}\} = \operatorname{span}\{(\overline{\operatorname{acx}}^{\sigma(F^*,F)} B) \cap E \mid B \in \mathfrak{S}\} = \operatorname{span}\{\overline{\operatorname{acx}}\, B \mid B \in \mathfrak{S}\}$. To see this, take $x \in \operatorname{cl}_{\sigma(F^*,F)A} \cap E = \operatorname{cl}_{\sigma(E,F)} A$ with $A = \overline{\operatorname{acx}} \bigcup_i \lambda_i B_i$. As above, we have that $A_i := \overline{\operatorname{acx}}\, B_i$ is $\sigma(E,F)$-compact, so $A_0 := \operatorname{acx} \bigcup_i \lambda_i A_i$ is $\sigma(E,F)$-compact, hence $\sigma(E,F)$-closed, and $\overline{\operatorname{acx}} \bigcup_i \lambda_i B_i \subseteq A_0 \subseteq \operatorname{span}\{\overline{\operatorname{acx}}\, B \mid B \in \mathfrak{S}\}$. □

Theorem 13.5 (Mackey-Arens)**.** *Let E, F form a pairing separating points of F. A topology $\mathscr{T}$ on E is compatible with the pairing if and only if $\mathscr{T}$ is a $\mathfrak{S}$-topology for a collection of absolutely convex $\sigma(F,E)$-compact subsets of F which cover F.*

Proof. Let $\mathcal{N}$ be the set of all absolutely convex closed 0-neighborhoods in E; these are also $\sigma(E,F)$-closed. Set $\mathfrak{S} := \{V^\circ \mid V \in \mathcal{N}\}$. Because $V = V^{\circ\circ}$ for $V \in \mathcal{N}$, $\mathscr{T} = \mathscr{T}_{\mathfrak{S}}$. By Corollary 13.4 (i), $\overline{\operatorname{acx}}\, V^\circ = V^\circ$ is $\sigma(F,E)$-compact. For $y \in F$, $\langle ., y\rangle \in E'$, so $\exists V \in \mathcal{N}$: $|\langle x, y\rangle| \leq 1$ for all $x \in V$, and $y \in V^\circ$.

The converse follows directly from Corollary 13.4. □

Definition 13.6. *Let E, F form a pairing separating points of F. For $\mathfrak{S}$ the collection of all absolutely convex, $\sigma(F,E)$-compact subsets of F, the $\mathfrak{S}$-topology on E is called the Mackey topology and denoted by $\tau(E,F)$.*

Proposition 13.7. *Let E, F form a pairing separating points of F. A locally convex topology $\mathscr{T}$ on E is compatible with the pairing if and only if it is finer than $\sigma(E,F)$ and coarser than $\tau(E,F)$.*

Proof. "⇒": By Theorem 13.5, $\mathscr{T} = \mathscr{T}_{\mathfrak{S}}$ for some collection $\mathfrak{S}$ of absolutely convex $\sigma(F,E)$-compact subsets of F, so $\mathscr{T} \leq \tau(E,F)$. A 0-basis in $\sigma(E,F)$ is given by sets of the form

$$U_{y_1,\dots,y_k,\varepsilon} := \{x \in E \mid |\langle x, y_i\rangle| \leq \varepsilon \ \forall i = 1 \dots k\}$$

for $y_1, \dots, y_k \in F$ and $\varepsilon > 0$. Because $(E, \mathscr{T})' = F$, there is a 0-neighborhood V in $\mathscr{T}$ such that $|\langle V, y_i\rangle| \in \varepsilon\mathbb{D}$ for $i = 1 \dots k$, i.e., $V \subseteq U_{y_1,\dots,y_k,\varepsilon}$ and $\sigma(E,F) \leq \mathscr{T}$.

"⇐": $\sigma(E,F) \leq \mathscr{T}$ means that $F \subseteq (E, \sigma(E,F))' \subseteq (E,\mathscr{T})'$, and $\mathscr{T} \leq \tau(E,F)$ means that $(E,\mathscr{T})' \leq (E, \tau(E,F))' = F$. □

Let now E be a LCS, $A \subseteq E$ nonempty and absolutely convex, and set

$$E_A := \operatorname{span} A = \bigcup_{i=1}^{n} nA.$$

The gauge q_A of A is a seminorm on this space.

Lemma 13.8. *Let $A \neq \emptyset$ be an absolutely convex and bounded subset of the LCS E.*

(i) *If E is Hausdorff, E_A is a normed space.*
(ii) *If in addition A is complete, E_A is a Banach space.*

Proof. (i): $x \in E_A$, $x \neq 0 \Rightarrow \exists U$ 0-neighborhood in E such that $x \notin U$. With λ such that $\lambda A \subseteq U$ we have $x \notin \lambda$, hence $q_A(x) \geq \lambda$.

(ii): Let $\{x_k\}$ be Cauchy sequence in E_A. Because it is bounded it is contained in some λA, which is complete for the topology induced by E, so $x_k \to x \in \lambda A$ in this topology, and by Lemma 13.9 also in E_A, so E_A is complete. Note that the topology of E_A is finer than the induced topology because give a 0-neighborhood U, we have $\lambda B \subseteq U$ for some λ. Moreover, each λA is closed with respect to the induced topology because A is complete. □

Lemma 13.9. *Let $\mathscr{T}, \mathscr{T}'$ be linear topologies on a vector space E such that $\mathscr{T}'$ is finer than $\mathscr{T}$. Suppose that $\mathscr{T}'$ has a 0-basis consisting of $\mathscr{T}$-closed sets. If $A \subseteq E$ and $\mathscr{F}$ is a $\mathscr{T}'$-Cauchy filter on A which $\mathscr{T}$-converges to $x \in A$, then $\mathscr{F} \to x$ in $\mathscr{T}'$.*

Proof. This is contained in the proof of Proposition 6.16. □

Recall that a *barrel* in a locally convex space is an absolutely convex, absorbent and closed set.

Proposition 13.10. *Let E be a LCS. A subset $M \subset E'$ is $\sigma(E', E)$-bounded if and only if $M \subseteq T^\circ$ for a barrel T in E.*

Proof. "⇒": $T := M^\circ$ is absolutely convex, $\sigma(E, E')$-closed, absorbent, hence also closed for the topology of E and a barrel; we have $T^\circ = M^{\circ\circ} \supseteq M$.

"⇐": if T is a barrel, $T^{\circ\circ} = T$, so T° is $\sigma(E', E)$-bounded because its polar is absorbent. □

Theorem 13.11 (Banach-Mackey)**.** *Let E be a Hausdorff LCS and T a barrel in E. Then T absorbs every absolutely convex, bounded, complete subset A of E.*

Proof. E_A is a Banach space; $S = T \cap E_A$ is a barrel in E_A because the topology of E_A is finer, so S° is $\sigma((E_A)', E_A)$-bounded in $(E_A)'$: for $x \in E_A$ there is $\lambda > 0$ such that $|\langle x, x'\rangle| < \lambda$ for $x' \in S^\circ$, so by the Banach-Steinhaus theorem for normed spaces there is $\mu > 0$ such that $|\langle x, x'\rangle| \leq \mu\|x\|$ for $x \in E_A$, $x' \in S^\circ$, hence $\{x \mid \|x\| \leq \frac{1}{\mu}\} \subseteq S^{\circ\circ} = S$. Because $A \subseteq (q_A)_{\leq 1}$ we have $\frac{1}{\mu}A \subseteq S \subseteq T$ and T absorbs A. □

[19.5.]

Theorem 13.12 (Mackey)**.** *Let E, F form a dual system (i.e., separating points of both E and F). The bounded subsets of E are the same for all locally convex topologies on E compatible with the pairing of E and F.*

Proof. Because a bounded subset of E is also bounded for a coarser topology we only have to show thach each $\sigma(E, F)$-bounded subset B of E is $\tau(E, F)$-bounded. If we equip F with the topology $\sigma(F, E)$ it is clear that B is also $\sigma(F', F)$-bounded in F, hence by Proposition 13.10 contained in T° for a barrel T in $(F, \sigma(F, E))$.

Let A be an absolutely convex $\sigma(F, E)$-compact subset of F. The polars of all such A form a 0-basis of the Mackey topology $\tau(E, F)$, so we need to show that A° absorbs T°. Because A is also $\sigma(F, E)$-complete and $\sigma(F, E)$-bounded and $\sigma(F, E)$ is Hausdorff, Theorem 13.11 gives that $\lambda A \subseteq T$ for some $\lambda > 0$ and hence $B \subseteq T^\circ \subseteq \lambda^{-1} A^\circ$. □

Consequently, if E is a LCS then its bounded subsets are exactly the $\sigma(E, E')$-bounded ones and the topology $\beta(E, E')$ is the topology of uniform convergence on bounded sets. The $\sigma(E', E)$-bounded subsets of E' are furthermore characterized as follows:

Proposition 13.13. *Let E be a LCS. Then $M \subseteq E'$ is $\beta(E', E)$-bounded if and only if $M \subseteq T^\circ$ for a bornivorous barrel T in E (i.e., a barrel absorbing every bounded set).*

Proof. If M is $\beta(E', E)$-bounded, its polar M° is absolutely convex and $\sigma(E, E')$-closed, hence also closed. Because M is also $\sigma(E', E)$-bounded, M° is absorbent, so M° is a barrel in E. Given a bounded subset $A \subseteq E$, A° is a 0-neighborhood in $\beta(E', E)$ so there is $\lambda > 0$ with $M \subseteq \lambda A^\circ$ and hence $A \subseteq A^{\circ\circ} \subseteq \lambda M^\circ$, so M° is a bornivorous barrel and $M \subseteq (M^\circ)^\circ$.

Conversely, let T be a bornivorous barrel and $A \subseteq E$ bounded. Then $A \subseteq \lambda T$ for some $\lambda > 0$, i.e., $T^\circ \subseteq \lambda A^\circ$ which means that T° is $\beta(E', E)$-bounded. □

Theorem 13.14. *If a LCS E is quasicomplete then every $\sigma(E', E)$-bounded subset of E' is $\beta(E', E)$-bounded.*

Proof. Using Proposition 13.10 and Proposition 13.13 we only have to show that every barrel T is bornivorous. Give a bounded set B in E, also its absolutely convex closed hull is bounded and by assumption complete, so T absorbs this hull and also B. □

14. Barrelled spaces

Proposition 14.1. *Let E be a LCS and $A \subseteq E'$. Then*

$$A \text{ equicontinuous } \Rightarrow A\ \beta(E', E)\text{-bounded } \Rightarrow A\ \sigma(E', E)\text{-bounded}\,.$$

Proof. If A is equicontinuous then $A \subseteq V^\circ$ for an absolutely convex closed 0-neighborhood V in E. V is a bornivorous barrel, so Proposition 13.13 gives the first claim. The second claim is clear because $\beta(E', E) \geq \sigma(E', E)$. □

Example. Let E be a normed vector space and E' its dual. Then $\{\rho B_1 \mid \rho > 0\}$ is a fundamental system of 0-neighborhoods and also of bounded sets.

- $M \subseteq E'$ is equicontinuous if and only if $M \subseteq (\rho B_1)^\circ = \rho^{-1} B_1^\circ$, i.e., if and only if it is bounded in E' (B_1° is the unit ball in E').
- M is bounded in $(E', \beta(E', E))$ if and only if it is norm-bounded (the $\beta(E', E)$-topology is the norm topology).
- Hence, in E' the equicontinuous sets are exactly the strongly bounded sets.
- If E is a Banach space and $M \subseteq E'$ is $\sigma(E', E)$-bounded, then by the Banach-Steinhaus theorem M is norm-bounded. Hence, in E' the equicontinuous, the $\beta(E', E)$-bounded and the $\sigma(E', E)$-bounded sets coincide.

Definition 14.2. *A LCS E is called* barrelled *if every barrel in E is a 0-neighborhood.*

Proposition 14.3. *A LCS E is barrelled if and only if every weakly bounded subset of E' if equicontinuous.*

Proof. Suppose that E is a barrelled and let $A \subseteq E'$ be $\sigma(E', E)$-bounded. By Proposition 13.10, $A \subseteq T^\circ$ for a barrel in E which by assumption is also a 0-neighborhood, so A is equicontinuous by Proposition 12.7.

Conversely, if T is a barrel then T° is $\sigma(E', E)$-bounded and hence equicontinous by assumption, so by Proposition 12.8 $T = T^{\circ\circ}$ is a 0-neighborhood. □

Corollary 14.4. *If E is barrelled then for $A \subseteq E'$ we have the equivalences*

$$\begin{aligned} & A \text{ equicontinuous} \\ \iff & A \text{ relatively } \sigma(E', E)\text{-compact} \\ \iff & A\ \beta(E', E)\text{-bounded} \\ \iff & A\ \sigma(E', E)\text{-bounded.} \end{aligned}$$

Examples of barrelled LCS are obtained by Baire's theorem. Recall that if a Baire space is the union of a sequence of closed sets, then at least one of those sets has an interior point.

Proposition 14.5. *If a LCS E is a Baire space, it is barrelled.*

Proof. If T is a barrel in E then $E = \bigcup_{n=1}^{\infty} nT$, so one of the closed sets nT and hence also T has to have an interior point $x_0 \in T$. If U is a balanced 0-neighborhood such that $x_0 + U \subseteq T$ we have $-x_0 + U \subseteq T$ and $U \ni x = \frac{1}{2}(x_0 + x) + \frac{1}{2}(-x_0 + x) \in T$ because T is convex, so T is a 0-neighborhood. □

By Baire's theorem, every complete metrizable topological space is a Baire space, so we have:

Corollary 14.6. *Every Fréchet space is barrelled.*

Proposition 14.7 (Banach-Steinhaus Theorem)**.** *Let E be a barrelled LCS and $\mathscr{F}$ a filter basis on E' containing a $\sigma(E', E)$-bounded set. If for every $x \in E$ $\mathscr{F}(x)$ converges to some $f(x)$ then $[f : x \mapsto f(x)] \in E'$.*

Proof. Let $A \in \mathscr{F}$ be $\sigma(E', E)$-bounded and hence equicontinuous. Then for each $x \in E$, $\mathscr{F}_A(x) \to f(x)$, which means that $\mathscr{F}_A \to f$ in $\mathbb{K}^E$, so f is in the closure of A in $\mathbb{K}^E$ and hence continuous by Lemma 12.10. □

Corollary 14.8. *If E is a barrelled LCS and $(f_n)_n$ is a sequence in E' converging to some $f \in \mathbb{K}^E$, then $f \in E'$.*

A slightly weaker notion is the following:

Definition 14.9. *A LCS E is called* infrabarrelled *or* quasibarrelled *if every bornivorous barrel in E is a 0-neighborhood.*

Proposition 14.10. *A LCS E is infrabarrelled if and only if every $\beta(E', E)$-bounded subset of E' is equicontinuous.*

Proof. "⇒": if M if $\beta(E', E)$-bounded then $M \subseteq T^\circ$ for a bornivorous barrel T, which is a 0-neighborhood by assumption, so M is equicontinuous.

"⇐": if T is a bornivorous barrel in E, then T° is $\beta(E', E)$-bounded and hence equicontinuous, so by Proposition 12.8 $T = T^{\circ\circ}$ is a 0-neighborhood in E. □

Theorem 13.14 implies:

Corollary 14.11. *A quasicomplete Hausdorff LCS is infrabarrelled if and only if it is barrelled.*

Proposition 14.12. *Let E be a Hausdorff LCS. If a convex subset $A \subseteq E'$ is relatively $\sigma(E', E)$-compact it is $\beta(E', E)$-bounded.*

Proof. $N := \operatorname{acx}(\bar{A})$ is absolutely convex and weakly compact by Lemma 13.2, so N° is a 0-neighborhood in the Mackey topology. If $B \subseteq E$ is bounded then it is so for the Mackey topology, so $B \subseteq \lambda N^\circ$ for some $\lambda > 0$, which implies $A \subseteq N = N^{\circ\circ} \subseteq \lambda B^\circ$, so A is $\beta(E', E)$-bounded. □

Proposition 14.13. *If E is an infrabarrelled Hausdorff LCS, its topology coincides with the Mackey topology $\tau(E, E')$.*

Proof. Let $U = A^\circ$ be a 0-neighborhood in E, where $A \subseteq E'$ is equicontinuous. Then $\operatorname{acx} A$ and hence also $\overline{\operatorname{acx}}\, A$ are equicontinuous (Lemma 12.10) and by Theorem 12.9 $\sigma(E', E)$-compact, hence $(\overline{\operatorname{acx}}\, A)^\circ \subseteq U$, which shows that the Mackey topology is finer.

Conversely, let $U = B^\circ$ be a 0-neighborhood in the Mackey topology, i.e., B is absolutely convex and $\sigma(E', E)$-compact. Then it is $\beta(E', E)$-bounded by Proposition 14.12 and hence equicontinuous by Proposition 14.10, so U is a 0-neighborhood in the topology of E and the topologies are equal. □

Proposition 14.14. *Let E be a vector space, $(E_\iota)_\iota$ a family of (infra-)barrelled locally convex spaces, and $(f_\iota)_\iota$ a family of linear mappings $E_\iota \to E$. The inductive locally convex topology on E with respect to $(f_\iota)_\iota$ is (infra-)barrelled.*

Proof. If T is a barrel in E then each $f_\iota^{-1}(T)$ is a barrel in E_i: absolute convexity and closedness are clear; for $x \in E_\iota$, $\lambda f_\iota(x) \in T$ for some λ and hence $\lambda x \in f_\iota^{-1}(T)$ and $f_\iota^{-1}(T)$ is absorbent. If T is bornivorous, then for a bounded set $B \subseteq E_\iota$, $\lambda f_\iota(B) \subseteq T$ for some $\lambda > 0$ and $\lambda B \subseteq f_\iota^{-1}(T)$, so $f_\iota^{-1}(T)$ is bornivorous. Because $f_\iota^{-1}(T)$ then is a 0-neighborhood of E_ι, T is a 0-neighborhood in E by Proposition 6.8 (ii). □

Corollary 14.15. *(i) If E is an (infra-)barrelled LCS and $M \subseteq E$ a subspace then E/M is (infra-)barrelled.*
(ii) If $(E_\iota)_\iota$ is a family of (infra-)barrelled LCS, the locally convex direct sum $\bigoplus_\iota E_\iota$ is (infra-)barrelled.
(iii) Inductive limits of (infra-)barrelled LCS are (infra-)barrelled.

Examples. Normed spaces are infrabarrelled, Banach spaces are barrelled. The spaces $\mathscr{E}^m(\Omega)$, $\mathscr{D}^m(\Omega)$ are barrelled because they are Fréchet spaces. (LF)-spaces (e.g., $\mathscr{D}(K)$) are barrelled.

15. Bornological Spaces

Definition 15.1. *A LCS E is called* bornological *if every absolutely convex bornivorous subset of E is a 0-neighborhood.*

Proposition 15.2. *A LCS E is bornological if and only if for all locally convex spaces F, every linear map $E \to F$ which maps bounded sets to bounded sets is continuous. It suffices to know this for normed spaces F.*

Proof. "$\Rightarrow$": let V be an absolutely convex 0-neighborhood in F, then $f^{-1}(V)$ is absolutely convex and bornivorous: given $A \subseteq E$ bounded, $f(A)$ is bounded and $\exists \lambda > 0$ s.t. $f(A) \subseteq \lambda V$ and hence $A \subseteq \lambda f^{-1}(V)$, so $f^{-1}(V)$ is a 0-neighborhood.

"$\Leftarrow$": If $U \subseteq E$ is absolutely convex and bornivorous and q_U the gauge of U, consider the quotient space $E_U := E/q_U^{-1}(0)$ with canonical surjection φ. Then $\|\varphi(x)\| := q_U(x)$ is a norm on E_U. If $A \subseteq E$ is bounded, $A \subseteq \lambda U$ for some $\lambda > 0$, so $\|\varphi(x)\| \le \lambda$ for $x \in A$ which means that $\varphi(A)$ is bounded and hence φ is continuous by assumption. Consequently, $\varphi^{-1}(\{\dot{x} \mid \|\dot{x}\| < 1\}) = \{x \in E \mid \|\varphi(x)\| < 1\} = (q_U)_{<1} \subseteq U$ and U is a 0-neighborhood. □

Proposition 15.3. *A LCS with a countable 0-basis is bornological.*

Proof. Let $(U_n)_n$ be a decreasing sequence of balanced 0-neighborhoods forming a 0-basis and let $U \subseteq E$ be absolutely convex and bornivorous. If U contains a set of the form $n^{-1}U_n$ it is a 0-neighborhood. Assuming the contrary we would have a sequence $(x_n)_n$ with $x_n \in n^{-1}U_n$ but $x_n \notin U$. Then $nx_n \to 0$, hence $A := \{nx_n \mid n\}$ is bounded. U cannot absorb A because $nx_n \in \lambda U$ would imply $x_n \in \frac{\lambda}{n}U \subseteq U$ for $n \ge \lambda$, which gives a contradiction. □

Examples. The spaces $\mathscr{E}^m(\Omega)$, $\mathscr{D}^m(K)$ are bornological.

Proposition 15.4. *Let E be a vector space, $(E_\iota)_\iota$ a family of bornological locally convex spaces, and $(f_\iota)_\iota$ a family of linear mappings $E_\iota \to E$. The inductive locally convex topology on E with respect to $(f_\iota)_\iota$ is bornological.*

Proof. If $U \subseteq E$ is absolutely convex and bornivorous, each $f_\iota^{-1}(U)$ is absolutely convex. If $B \subseteq E_\iota$ is bounded, $f_\iota(B)$ is bounded and hence contained in some λU, so $B \subseteq \lambda f_\iota^{-1}(U)$ and $f_\iota^{-1}(U)$ is bornivorous, hence a 0-neighborhood, which proves that U is a 0-neighborhood. □

Corollary 15.5. *(i) If E is a bornological LCS and $M \subseteq E$ a subspace, then E/M is bornological.*
(ii) The direct sum of a family of bornological LCS is bornological.
(iii) The inductive limit of a family of bornological LCS is bornological.

Example. The space $\mathscr{D}^m(\Omega)$ is bornological.

We finish this section with two important results. The first one is an analogoue to Proposition 7.4:

Proposition 15.6. *Every (complete) Hausdorff bornological LCS can be represented as the inductive limit of a family of normed (Banach) spaces.*

Proof. For each absolutely convex closed bounded subset $A \subseteq E$, E_A with the seminorm q_A is a normed (Banach) space by Lemma 13.8. Let $f_A \colon E_A \to E$ be the canonical injection. Denoting by $\mathscr{T}$ the original topology of E and by $\mathscr{T}'$ the inductive locally convex topology with respect to the mappings f_A, we will show that $\mathscr{T} = \mathscr{T}'$.

If U is an absolutely convex 0-neighborhood in $\mathscr{T}$ then $\lambda A \subseteq U$ for some $\lambda > 0$, i.e., $\lambda A \subseteq f_A^{-1}(U)$, so each $f_A^{-1}(U)$ is a 0-neighborhood in E_A and U is a 0-neighborhood in $\mathscr{T}'$ by Proposition 6.8 (ii).

If U is an absolutely convex 0-neighborhood in $\mathscr{T}'$ then $\lambda A \subseteq f_A^{-1}(U)$ for some λ, so $\lambda A \subseteq U$ which means that U is bornivorous and hence a 0-neighborhood in $\mathscr{T}$. Proposition 7.9 now gives the claim because $E = \bigcup\{E_A \mid A \subseteq E$ absolutely convex, closed, bounded$\}$ and for $A \subseteq B$, $E_A \subseteq E_B$ and $q_B \le q_A$, so $(E, \mathscr{T}) = (E, \mathscr{T}_i) = \varinjlim E_A$. □

Proposition 15.7. *If E is bornological then $(E', \beta(E', E))$ is complete.*

Proof. Let $\mathscr{F}$ be a Cauchy filter on E', the it is also a Cauchy filter with respect to $\sigma(E', E)$ and hence converges to some f in $\mathbb{K}^E$, and even in E^* as seen in the proof of Lemma 12.10. To show continuity of f let $B \subseteq E$ be bounded; then for some $A \in \mathscr{F}$, $A - A \subseteq B^\circ$. Fixing any $v \in A$ there is $\alpha > 0$ such that for all $u \in A$ and $x \in B$, $|u(x)| \le |u(x) - v(x)| + |v(x)| \le \alpha$. Because also $\mathscr{F}_A \to f$ we can for each $x \in B$ find some $u_x \in A$ such that $|u_x(x) - f(x)| \le \alpha$. Then $|f(x)| \le |f(x) - u_x(x)| + |u_x(x)| \le 2\alpha$, so f is bounded and hence continuous. Because $\beta(E', E)$ has a 0-basis of $\sigma(E', E)$-closed sets, Lemma 13.9 gives the claim. □

Example. The space $\mathscr{D}'(\Omega) := (\mathscr{D}(\Omega)', \beta(\mathscr{D}(\Omega)', \mathscr{D}(\Omega)))$ of distributions on Ω is complete.

16. Reflexivity

Definition 16.1. *Let E be a Hausdorff LCS. $E'' := (E', \beta(E', E))'$ is called the* bidual *of E.*

Definition 16.2. *A Hausdorff LCS E is called* semireflexive *if the canonical embedding $E \to E''$ is surjective, and* reflexive *if it is an isomorphism onto $(E'', \beta(E'', E'))$.*

Lemma 16.3. *The following statements are equivalent:*

(i) E *is semireflexive.*
(ii) *Every element of* $(E', \beta(E', E))'$ *is of the form* $x' \mapsto \langle x, x'\rangle$ *for some* $x \in E$.
(iii) *Every* $\beta(E', E)$*-continuous linear form on* E' *is* $\sigma(E', E)$*-continuous.*
(iv) $(E', \beta(E', E))' = E$.

The proof is clear. Moreover, by (iv) and Theorem 13.5 E is semireflexive if and only if $\beta(E', E) = \tau(E', E)$.

Proposition 16.4. *A LCS* E *is semireflexive if and only if every bounded* $\sigma(E, E')$*-closed subset of* E *is* $\sigma(E, E')$*-compact.*

Proof. If E is semireflexive we have $E'' = (E', \mathscr{T}_{\mathfrak{S}})' \subseteq E$ where $\mathfrak{S}$ is the family of all $\sigma(E, E')$-bounded subsets of E, hence by Corollary 13.4 (i), for every $B \in \mathfrak{S}$ we have that $\overline{\operatorname{acx}}\, B$ is $\sigma(E, E')$-compact. If now A is bounded and $\sigma(E, E')$-closed it is $\sigma(E, E')$-bounded, so $\overline{\operatorname{acx}}\, A$ is $\sigma(E, E')$-compact, hence A is $\sigma(E, E')$-compact.

Conversely, $\beta(E', E)$ is the topology of uniform convergence on all absolutely convex $\sigma(E, E')$-closed $\sigma(E, E')$-bounded subsets of E, which by assumption are exactly the absolutely convex $\sigma(E, E')$-compact subsets of E, so $\beta(E', E)$ is the Mackey topology. ☐

Proposition 16.5. *The strong dual of a semireflexive Hausdorff LCS is barrelled.*

Proof. If T is a barrel in E' then T° is bounded in $E'' = E$ for $\sigma(E'', E') = \sigma(E, E')$ by Proposition 13.10, so $T^{\circ\circ} = T$ is a 0-neighborhood in $\beta(E', E)$. ☐

Proposition 16.6. *A Hausdorff LCS* E *is reflexive if and only if it is semireflexive and infrabarrelled.*

Proof. If E is reflexive then it clearly is semireflexive. Moreover, if $B \subseteq E'$ is $\beta(E', E)$-bounded its polar $B^\bullet$ is a 0-neighborhood in E''. As $\iota \colon E \to E''$ is a homeomorphism, $\iota^{-1}(B^\bullet) = B^\circ$ is a 0-neighborhood in E, and $B \subseteq (B^\circ)^\circ$, shows that B is equicontinuous and by Proposition 14.10 E is infrabarrelled.

If E is semireflexive then ι is surjective. Given a 0-neighborhood $U = B^\bullet$ in E'' for some $\beta(E', E)$-bounded subset $B \subseteq E'$, B is equicontinuous because E is infrabarrelled and hence $B \subseteq V^\circ$ for some 0-neighborhood V in E. Hence, we have

$$U = B^\bullet \supseteq (V^\circ)^\bullet = \iota(V^{\circ\circ}) \supseteq \iota(V)$$

so ι is continuous. Conversely, if $V = V^{\circ\circ}$ is a 0-neighborhood in E then $\iota(V) = \iota(V^{\circ\circ}) = (V^\circ)^\bullet$ is a 0-neighborhood in E'' because V° is equicontinuous and hence $\beta(E', E)$-bounded in E'. ☐

[2.6]

Corollary 16.7. *A reflexive space is barrelled.*

Proof. Let $M \subseteq E'$ be $\sigma(E', E)$-bounded. If E is semireflexive then M is also $\beta(E', E)$-bounded[1], hence equicontinuous because E is infrabarrelled, so E is barrelled. ☐

Proposition 16.8. *The strong dual of a reflexive space is reflexive.*

Proof. E' is barrelled by Proposition 16.5. If $M \subseteq E'$ is bounded and $\sigma(E', E'')$-closed it is also $\sigma(E', E)$-closed because E is reflexive, and hence it is $\sigma(E', E)$-compact by Corollary 14.4, which shows that E' is also semireflexive by Proposition 16.4. ☐

[1]the β-topology is compatible with the pairing

Proposition 16.9. *A normed vector space is reflexive if and only if its closed unit ball is weakly compact.*

Proof. Suppose E is reflexive. The closed unit ball is bounded and convex, hence $\sigma(E, E')$-closed by Proposition 12.4 and $\sigma(E, E')$-compact by Proposition 16.4.

Conversely, each bounded $\sigma(E, E')$-closed subset of E is contained in some closed ball $B_\rho = \rho B_1$ and therefore is $\sigma(E, E')$-compact, so E is reflexive by Proposition 16.4. □

17. Montel spaces

Recall: semireflexive spaces are those where every bounded set is weakly relatively compact (Proposition 16.4). When is this the case for the original topology instead of the weak one?

Definition 17.1. *A Hausdorff LCS E is called* semi-Montel *if every bounded subset is relatively compact. An infrabarrelled semi-Montel space is called a* Montel *space.*

By Theorem 8.4, if a normed space if semi-Montel it is finite-dimensional.

Proposition 17.2. *Every semi-Montel space is quasicomplete and semireflexive.*

Proof. If $B \subseteq E$ is bounded and $\mathscr{F}$ a Cauchy filter on B, $\mathscr{F}$ has a cluster point in E and hence converges. Moreover, if B is $\sigma(E, E')$-closed it is also $\sigma(E, E')$-compact and E is semi-reflexive by Proposition 16.4. □

Proposition 16.6 gives:

Corollary 17.3. *Every Montel space is reflexive.*

In particular, taking into account Corollary 16.7 one sees that for a semi-Montel space the properties of being infrabarrelled, reflexive or barrelled are equivalent.

Definition 17.4. *A subset A of a Hausdorff LCS E is called* precompact *or* totally bounded *if for every 0-neighborhood U in E there is a finite subset $H \subseteq E$ such that $A \subseteq H + U$.*

Lemma 17.5. *Let E be a Hausdorff LCS. Then for a subset $A \subseteq E$, the following assertions are equivalent:*

(i) Every ultrafilter on A is a Cauchy filter.
(ii) A is relatively compact in the completion $\widehat{E}$.
(iii) A is totally bounded.

Proof. (i) $\Rightarrow$ (ii): suppose that $B := \overline{i(A)}$ is not compact, where $i \colon E \to \widehat{E}$ is the canonical injection. Then there is a filter $\mathscr{F}$ on B which has no adherent point. The collection $\{X + V \mid X \in \mathscr{F}, V \text{ balanced 0-neighborhood in } \widehat{E}\}$ is the basis of a filter $\mathscr{G}$ on $\widehat{E}$. Because $(X + V) \cap A \neq \emptyset$ for all X, V, the trace $\mathscr{G}_A$ is a filter on A and hence contained in an ultrafilter $\mathscr{U}$ on A. Because $\mathscr{U}$ is a Cauchy filter, the filter generated by it on B would be a Cauchy filter which would converge to some $x \in B$; x would be an adherent point of $\mathscr{U}$, hence of $\mathscr{G}_A$, and of $\mathscr{G}$. For $F \in \mathscr{F}$ and a 0-neighborhood U, let V be a balanced 0-neighborhood with $V + V \subseteq U$. Then $(F + V) \cap (x + V) \neq \emptyset$, so $f + v_1 = x + v_2$ for some f, v_1, x, v_2, and $f = x + v_2 - v_1$ and $F \cap (x + U) \neq \emptyset$, so x would be an adherent point of $\mathscr{F}$.

(ii) $\Rightarrow$ (iii): Given a 0-neighborhood U, let $V \subseteq U$ be a closed 0-neighborhood and $\widehat{V}$ its closure in $\widehat{E}$. Because B is compact, $B \subseteq \bigcup_{i=1}^n (i(x_i) + \widehat{V})$ for a finite family $(x_i)_i$ in A and $A \subseteq \overline{\iota(A)} \cap E \subseteq \bigcup_i (x_i + V)$ because $\widehat{V} \cap E = V$.

(iii) $\Rightarrow$ (i): Let $\mathscr{U}$ be an ultrafilter on A, U a 0-neighborhood and V a balanced 0-neighborhood such that $V + V \subseteq U$. For $A \subseteq \bigcup (x_i + V)$, $A = \bigcup (x_i + V) \cap A \in \mathscr{U}$, so for some j, $X := (x_j + V) \cap A \in \mathscr{U}$, and we have $X - X \subseteq V - V \subseteq U$. □

Definition 17.6. *If E is a Hausdorff LCS, $\lambda(E', E)$ denotes the topology of uniform convergence on precompact subsets of E and $\kappa(E', E)$ the topology of uniform convergence on absolutely convex compact subsets of E.*

Consider the following picture:

$$\begin{array}{ccccccc} & & & & \tau & & \\ & & & \rotatebox{45}{$\leq$} & & \rotatebox{-45}{$\leq$} & \\ \sigma & \leq & \kappa & & & & \beta \\ & & & \rotatebox{-45}{$\leq$} & & \rotatebox{45}{$\leq$} & \\ & & & & \lambda & & \end{array}$$

Every precompact set is bounded: $A \subseteq H + U \subseteq \lambda U + U = (1 + \lambda)U$ if U is a convex 0-neighborhood and λ chosen such that $H \subseteq \lambda U$. Consequently, $\lambda(E', E) \leq \beta(E', E)$.

Lemma 17.7. *If E is quasicomplete then $\kappa = \lambda$. If E is semi-Montel then $\lambda = \beta$ and hence $\kappa = \tau = \lambda = \beta$.*

Proof. If E is quasicomplete and a 0-neighborhood A° in λ is given with A precompact, $\operatorname{acx} A$ is precompact by Lemma 17.8 and hence relatively compact because it is bounded, so $\overline{\operatorname{acx}} A$ is absolutely convex, compact and we have $(\overline{\operatorname{acx}} A)^\circ \subseteq A^\circ$.

For the second claim, by Lemma 17.5 in a semi-Montel space each bounded set is relatively compact and hence precompact. □

Lemma 17.8. *If E is a LCS and $A \subseteq E$ precompact, then $\operatorname{acx} A$ is precompact.*

Proof. Set $B := \operatorname{acx} A = \{x = \sum_{i=1}^n \lambda_i x_i \mid \sum_{i=1}^n |\lambda_i| \leq 1, x_i \in A, n \in \mathbb{N}\}$. If U is an absolutely convex 0-neighborhood in E we have to show that $B \subseteq \bigcup_{j=1}^p (a_j + U)$ for some $a_j \in A$. Because A is precompact we know that $A \subseteq \bigcup_{k=1}^q (b_k + \frac{1}{2}U)$. Set $L := \{\lambda = (\lambda_1, \dots, \lambda_q) \mid \sum_{k=1}^q |\lambda_k| \leq 1\} \subseteq \mathbb{K}^q$ compact. Then for each $\delta > 0$ there are $\mu^1, \dots, \mu^p \in \mathbb{K}^q$ such that for all $\lambda \in L$ one can find j with $\sum_{k=1}^q \left|\lambda_k - \mu_k^j\right| < \delta$. Choose δ with $\delta A \subseteq \frac{1}{2}U$; this implies $\delta B \subseteq \frac{1}{2}U$ because U is absolutely convex. We then claim that $B \subseteq \bigcup_{j=1}^p (\sum_{k=1}^q \mu_k^{(j)} b_k + U)$. In fact, take $x = \sum_{i=1}^n \lambda_i x_i \in B$. Then $x_i = b_{k_i} + y_i$ with $y_i \in \frac{1}{2}U$, and $x = \sum_{i=1}^n \lambda_i b_{k_i} + \sum_{i=1}^n \lambda_i y_i = \sum_{k=1}^q \nu_k b_k + y$ with $\sum |\nu_k| \leq 1$ and $y \in \frac{1}{2}U$. Consequently, $x = \sum_{k=1}^q \mu_k^j b_k + \sum_{k=1}^q (\nu_k - \mu_k^j) b_k + y \in \sum_{k=1}^q \mu_k^j b_k + U$. □

Proposition 17.9. *If E a is Hausdorff LCS and $H \subseteq E'$ is equicontinuous, the topologies $\lambda(E', E)$ and $\sigma(E', E)$ coincide on H.*

Proof. $\sigma(E', E)$ is coarser than $\lambda(E', E)$, so we only have to show the converse. Let $H \subseteq E'$ be equicontinuous. It suffices to show that for u_0 in H and $A \subseteq E$ precompact there is a finite subset $M \subseteq E$ such that $(u_0 + M^\circ) \cap H \subseteq u_0 + A^\circ$. Choose a 0-neighborhood U such that $H(U) \subseteq \frac{1}{2}\mathbb{D}$, then there is a finite subset $K \subseteq E$ such that $A \subseteq K + \frac{1}{2}U$; set

$M := 2K$. Then for $u \in (u_0 + M^\circ) \cap H$, $u(U) \in \frac{1}{2}\mathbb{D}$, and $u - u_0 \in M^\circ = \frac{1}{2}K^\circ$ implies $(u - u_0)(K) \subseteq \frac{1}{2}\mathbb{D}$. Any $x \in A$ can be written as $x = y + \frac{1}{2}z$ with $y \in K$ and $z \in U$, which gives $|(u - u_0)(x)| \leq 1$ and hence $u - u_0 \in A^\circ$. □

Corollary 17.10. *If $(E, \mathscr{T})$ is a semi-Montel space and $B \subseteq E$ bounded, then the topology induced on B by $\mathscr{T}$ coincides with the topology induced by $\sigma(E, E')$.*

Corollary 17.11. *Let E be semi-Montel. If $(x_n)_n$ converges weakly to $x \in E$ then $x_n \to x$ in E.*

Proof. The set of all x_n together with x is weakly bounded and hence bounded, so the topologies induced by $\mathscr{T}$ and $\sigma(E, E')$ coincide. □

Proposition 17.12. *Closed subspaces and products of semi-Montel spaces are semi-Montel. Projective limits of semi-Montel spaces are (semi-)Montel. Regular inductive limits of (semi-)Montel spaces are (semi-)Montel.*

Proof. Let $M \subseteq E$ be closed and $A \subseteq M$ bounded. Then A is bounded in E and hence relatively compact, because M is closed also relatively compact in M.

Let a family $(E_\iota)_\iota$ of semi-Montel spaces be given and $A \subseteq E$ be bounded. Each $\pi_\iota(A)$ is bounded and hence relatively compact, then the product $\prod_\iota \pi_\iota(A)$ is relatively compact. Because it contains A, A also is relatively compact. □

Proposition 17.13. *The strong dual of a Montel space is a Montel space.*

Proof. By Corollary 17.3 E is reflexive, so E' is barrelled by Proposition 16.5.

Let $M \subseteq E'$ be bounded for $\beta(E', E)$, then M is contained in an absolutely convex $\beta(E', E)$-closed $\beta(E', E)$-bounded set N, which is equicontinuous by Proposition 14.10 because E is infrabarrelled. N is $\sigma(E', E)$-closed because $\beta(E', E)$ is compatible with the duality, and hence $\sigma(E', E)$-compact by Theorem 12.9. On N, the topologies $\sigma(E', E)$ and $\lambda(E', E)$ coincide, so N is $\lambda(E', E)$-compact and hence $\beta(E', E)$-compact (Lemma 17.7), and M is relatively compact. □

Examples. One can show that the spaces $\mathscr{E}(\Omega)$, $\mathscr{S}(\mathbb{R}^n)$, $\mathscr{D}(\Omega)$ and their strong duals are Montel spaces.

18. The transpose of a linear map

We recall that the *transpose* (or adjoint) of a linear map $u\colon E_1 \to E_2$ is defined as the mapping $u'\colon E_2^* \to E_1^*, y' \mapsto y' \circ u$.

Proposition 18.1. *Let (E_1, F_1) and (E_2, F_2) be two dual systems and $u\colon E_1 \to E_2$ linear. Then $u'(F_2) \subseteq F_1$ if and only if u is $(\sigma(E_1, F_1), \sigma(E_2, F_2))$-continuous. In this case, $u'\colon F_2 \to F_1$ is $(\sigma(F_2, E_2), \sigma(F_1, E_1))$-continuous and $u = (u')'$.*

Proof. Using Proposition 10.5, the first claim is a direct consequence of $\langle x, u'(y)\rangle = \langle u(x), y\rangle$.

The second claim is equivalent to $\sigma(F_2, E_1)$-continuity of $y \mapsto \langle x, u'(y)\rangle = \langle u(x), y\rangle$ for all $x \in E_1$, which is clear from $E_2 = (F_2, \sigma(F_2, E_2))'$, and $T = (T')'$ is clear from the definition. □

Proposition 18.2. *Let (E_1, F_1) and (E_2, F_2) be two dual systems and $u\colon E_1 \to E_2$ weakly continuous, $A \subseteq E_1$ and $B \subseteq E_2$. Then:*

(i) $(u(A))^\circ = (u')^{-1}(A^\circ)$.
(ii) $u(A) \subseteq B \Rightarrow u'(B^\circ) \subseteq A^\circ$
(iii) *If* A, B *are nonempty, absolutely convex and weakly closed then* $u(A) \subseteq B \Leftrightarrow u'(B^\circ) \subseteq A^\circ$.
(iv) u' *is injective if and only if the image of* u *is* $\sigma(E_2, F_2)$*-dense.*

Proof. (i): $y \in (u(A))^\circ \Leftrightarrow \forall x \in u(A)$: $|\langle x, y\rangle| \leq 1 \Leftrightarrow \forall x \in A : |\langle u(x), y\rangle| = |\langle x, u'(y)\rangle| \leq 1 \Leftrightarrow u'(y) \in A^\circ \Leftrightarrow y \in (u')^{-1}(A^\circ)$.

(ii): for $y \in B^\circ$ and $x \in A$, $|\langle x, u'(y)\rangle| = |\langle u(x), y\rangle| \leq 1$.

(iii): $u(A) = u''(A^{\circ\circ}) \subseteq B^{\circ\circ} = B$.

(iv): $\ker u' = (u')^{-1}(\{0\}) = (u')^{-1}(E_1^\circ) = (u(E_1))^\circ = u(E_1)^\perp$. Now $u(E_1)^\perp = \{0\}$ if and only if $\overline{u(E_1)} = E_2$ (weak closure) ("$\Rightarrow$": $u(E_1)^{\perp\perp} = \overline{u(E_1)} = \{0\}^\perp = E_2$; "$\Leftarrow$": $u(E_1)^\perp = u(E_1)^{\perp\perp\perp} = (\overline{u(E_1)})^\perp = E_2^\perp = \{0\}$). □

Proposition 18.3. *Let* E, F *be LCS and* $u \colon E \to F$ *linear.*

(i) *If* u *is continuous it is weakly continuous.*
(ii) u *is weakly continuous if and only if it is* $((\tau(E, E'), \tau(F, F'))$*-continuous.*
(iii) *If* u *is weakly continuous then it is* $(\beta(E, E'), \beta(F, F'))$*-continuous.*

Proof. (i) If $u \colon E \to F$ is continuous, $y \circ u \in E'$ for all $y \in F'$, so $u'(F') \subseteq E'$ and u is weakly continuous by Proposition 18.1.

By (i), continuity w.r.t. the Mackey topologies implies weak continuity. Now suppose that u is weakly continuous. If $B \subseteq F'$ is absolutely convex and weakly compact, then also $u'(B) \subset E'$ is so, and we have $u((u'(B))^\circ) = u((u'')^{-1}(B^\circ)) = u(u^{-1}(B^\circ)) \subseteq B^\circ$, so u is continuous w.r.t. the Mackey topologies, which shows (ii). The same argument shows (iii). □

Corollary 18.4. *If* E *has the Mackey topology* $\tau(E, E')$ *then every weakly continuous linear map* $E \to F$ *is continuous.*

This applies in particular to Fréchet spaces.

19. Topological tensor products

Let $L(E, F; G)$ be the vector space of all bilinear mappings $E \times F \to G$ (where E, F, G are vector spaces) and set $B(E, F) := L(E, F; \mathbb{K})$ (the space of bilinear forms). For $x, y \in E \times F$ define $u_{x,y} \in B(E, F)^*$ by $u_{x,y}(b) := b(x, y)$ for all $b \in B(E, F)$. Then the mapping

$$\otimes \colon E \times F \to B(E, F)^*,$$
$$(x, y) \mapsto u_{x,y}$$

is bilinear and $E \otimes F := \operatorname{span}(\otimes(E \times F))$ is called the *tensor product* of E and F. We write $x \otimes y$ instead of $x \otimes y$.

Proposition 19.1. *If* $(e_i)_i$ *is a basis for* E *and* $(f_j)_j$ *a basis for* F *then* $(e_i \otimes f_j)_{i,j}$ *is a basis for* $E \otimes F$.

Proof. We only have to show linear independence. Let $z = \sum z_{ij} e_i \otimes f_j = 0$ (finite sum). Given i_0, j_0 there is $b \in B(E, F)$ such that $b(e_{i_0}, f_{j_0}) = 1$ and $b(e_i, f_j) = 0$ for $i \neq i_0, j \neq j_0$, and $z(b) = 0$ gives $z_{i_0 j_0} = 0$. □

Hence, every $z \in (E \otimes F) \setminus \{0\}$ can be written as $z = \sum_{i=1}^n x_i \otimes y_i$ with $x_i \in E$ linearly independent and $y_i \in F$ linearly independent. In fact, let $z = \sum_{i=1}^r x_i \otimes y_i$ with r minimal. For $r = 1$, clearly $x_i \neq 0$ and $y_i \neq 0$. For $r \geq 2$, assume $x_r = \sum_{i=1}^{r-1} \lambda^i x_i$. Then

$$z = \sum_{i=1}^{r-1} x_i \otimes (y_i + \lambda^i y_r) = \sum_{i=1}^{r-1} x_i \otimes y_i,$$

contradicting minimality of r.

The map

$$\theta \colon L(E \otimes F, G) \to L(E, F; G), \quad T \mapsto T \circ \otimes$$

is linear and injective. Given $b \in L(E, F; G)$ we define $T \in L(E \otimes F, G)$ by $T(e_i \otimes f_j) := b(e_i, f_j)$; then $(\Theta T)(e_i, f_j) = T(e_i \otimes f_j) = b(e_i, f_j)$, so Θ is an isomorphism; T is called the *linearization* of b.

Theorem 19.2. *$L(E, F; G) \cong L(E \otimes F, G)$.*

In particular, $B(E, F) \cong (E \otimes F)^*$.

Let E_i, F_i be vector spaces and $T_i \in L(E_i, F_i)$. The linearization $T_1 \otimes T_2 \colon E_1 \otimes E_2 \to F_1 \otimes F_2$ of the bilinear map

$$E_1 \times E_2 \to F_1 \otimes F_2, \quad (x_1, x_2) \mapsto T_1(x_1) \otimes T_2(x_2)$$

is called the *tensor product* of T_1 and T_2.

Proposition 19.3. *If T_1 and T_2 are injective (surjective), then so is $T_1 \otimes T_2$.*

Proof. Clear from Proposition 19.1 because a linear map is injective (surjective) if and only if the image of a given basis is linearly independent (spans the codomain). □

For vector spaces E and F, define $\chi \colon E^* \otimes F \to L(E, F)$ as the linearization of

$$E^* \times F \to L(E, F), \quad (u, y) \mapsto [x \mapsto u(x) \cdot y].$$

Proposition 19.4. *χ is injective and $\operatorname{im} \chi = \{T \in L(E, F) \mid \dim \operatorname{im} T < \infty\}$.*

Proof. Let $z = \sum_{u_i \otimes y_i}$ with y_i linearly independent. Then $\chi(z) = 0$ implies that $\sum u_i(x) y_i = 0$ for all x, hence $u_i = 0$ and $z = 0$.

$T \in \operatorname{im} \chi$ has finite rank. Conversely, if $T \in L(E, F)$ has finite rank, choose a basis $(y_1, \dots, y_n)$ of $\operatorname{im} T$ and extend it to a basis C of F. Define $v_i \in F^*$ by $v_i(y_i) = 1$, $v_i(y) = 0$ for $y \in C \setminus \{y_i\}$. With $u_i := v_i \circ T \in E^*$, $\chi(\sum_{i=1}^n u_i \otimes y_i) = \sum v_i(T(x)) y_i = T(x)$. □

Let E, F now be locally convex spaces. Let $\mathscr{T}_\pi$ be the finest locally convex topology on $E \otimes F$ such that $\otimes \colon E \times F \to E \otimes F$ is continuous; it is given by the projective topology with respect to all mappings

$$\mathrm{id} \colon E \otimes F \to (E \otimes F, \mathscr{T})$$

where $\mathscr{T}$ is any locally convex topology such that $\otimes \colon E \times F \to (E \otimes F, \mathscr{T})$ is continuous.

Let $\mathscr{U}, \mathscr{V}$ be 0-basis in E and F, respectively, consisting of absolutely convex sets. For $A \subseteq E$ and $B \subseteq F$ we set $A \otimes B = \otimes(A, B)$. We claim that

$$\{\operatorname{acx}(U \otimes V) \mid U \in \mathscr{U}, V \in \mathscr{V}\}$$

is a 0-basis of $\mathscr{T}_\pi$.

First, we show that $\operatorname{acx}(U \otimes V)$ is absorbent. For $z = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F$ with $x_i \in E$, $y_i \in F$, we choose $\rho_i > 0$ and $\sigma_i > 0$ with $x_i \in \rho_i U$, $y_i \in \sigma_i V$, and obtain

$$z = \sum \rho_i \sigma_i \frac{x_i}{\rho_i} \otimes \frac{y_i}{\sigma_i} \in \left(\sum \rho_i \sigma_i\right) \cdot \operatorname{acx}(U \otimes V). \tag{3}$$

Obvisouly $\{\operatorname{acx}(U \otimes V) \| \ U \in \mathscr{U}, V \in \mathscr{V}\}$ is a filter basis on $E \otimes F$, and for $\rho > 0$ and given U, V there are U_1, U_2 such that $\operatorname{acx}(U_1 \otimes U_2) \subseteq \rho \cdot \operatorname{acx}(U \otimes V)$ for some U_1, V_1. By Theorem 2.10 there is a unique locally convex topology $\mathscr{T}$ on $E \otimes F$ having this family as a 0-basis. [9.6.]

(1) $\mathscr{T} \leq \mathscr{T}_\pi$: we need that $\otimes \colon E \times F \to (E \otimes F, \mathscr{T})$ is continuous. This is the case because $U \otimes V \subseteq \operatorname{acx}(U \otimes V)$.
(2) $\mathscr{T}_\pi \leq \mathscr{T}$: every 0-neighborhood in $\mathscr{T}_\pi$ contains a set of the form $W_1 \cap \ldots \cap W_k$, where each W_i is a 0-neighborhood in $E \otimes F$ for some topology into which $\otimes$ is continuous. Hence, $U_i \otimes V_i \subseteq W_i$ for some U_i, V_i, and as we can assume W_i to be absolutely convex, $\operatorname{acx}(U_i \otimes V_i) \subseteq W_i$ so W_i and hence $W_1 \cap \ldots \cap W_k$ is a 0-neighborhood in $\mathscr{T}$.

Definition 19.5. *$\mathscr{T}_\pi$ is called the* projective tensor topology *or* π-topology *on $E \otimes F$, and $E \otimes_\pi F := (E \otimes F, \mathscr{T}_\pi)$ is called the* projective tensor product *or* π-tensor product *of E and F. In case it is Hausdorff, its completion is denoted by $E \widehat{\otimes}_\pi F$.*

For LCS E, F, G let $\mathcal{L}(E, F; G)$ be the set of all continuous bilinear mappings $E \otimes F \to G$ and $\mathcal{L}(E, F)$ the set of all linear continuous mappings $E \to F$.

Proposition 19.6. *Let E, F, G be LCS. Then the map $\theta \colon L(E \otimes F, G) \to L(E, F; G)$, $\theta T = T \circ \otimes$, induces an isomorphism*

$$\mathcal{L}(E \otimes_\pi F, G) \to \mathcal{L}(E, F; G). \tag{4}$$

Proof. As $\otimes$ is continuous, we have an injection (4). Let $b \in \mathcal{L}(E, F; G)$ and an absolutely convex 0-neighborhood W in G be given. Choose U, V such that $b(U, V) \subseteq W$. Then $T := \Theta^{-1}(b) \in L(E \otimes F, G)$ satisfies $T(\operatorname{acx}(U \times V)) \subseteq \operatorname{acx} T(U \otimes V) = \operatorname{acx} b(U, V) \subseteq \operatorname{acx} W$, so $T \in \mathcal{L}(E \otimes_\pi F, G)$. ☐

In particular, we have $(E \otimes_\pi F)' = \mathcal{B}(E, F) := \mathcal{L}(E, F; \mathbb{K})$.

We will now show that for Hausdorff LCS E, F we have $E' \otimes F' \subseteq (E \otimes_\pi F)'$. In fact, for $u \in E'$, $v \in F'$ define $b_{u,v} \in \mathcal{B}(E, F)$ by $b_{u,v}(x, y) := \langle u, x \rangle \cdot \langle v, y \rangle$ for $x, y \in E \times F$. Let $\psi \colon E' \otimes F' \to \mathcal{B}(E, F)$ be the linearization of the biliner map

$$E' \times F' \to \mathcal{B}(E, F)$$
$$(u, v) \mapsto b_{u,v}.$$

Suppose now that $\psi(z) = 0$, where $z = \sum u_i \otimes v_i$; we can assume the v_i to be linearly independent. Choose $y_j \in F$ such that $\langle v_i, y_j \rangle = \delta_{ij}$. Then for all $x \in E$, $\psi(z)(x, y_j) = \sum \langle u_i, x \rangle \langle v_i, y_j \rangle = \langle u_j, x \rangle = 0$, so $u_j = 0$ for all j and $z = 0$, so ψ is injective.

Identifying $E' \otimes F'$ with a subspace of $(E \otimes_\pi F)'$, for u, v we denote by $u \otimes v$ also the corresponding linear or bilinear form.

Proposition 19.7. *If E and F are Hausdorff LCS then $E \otimes_\pi F$ is Hausdorff.*

Proof. Let $z \neq 0$, then $z = \sum x_i \otimes y_i$ with $(x_i)_i$ and $(y_i)_i$ linearly independent. Let $u \in E'$, $v \in F'$ be such that $u(x_i) = \delta_{1i} = v(y_i)$. Then $u \otimes v \in (E \otimes_\pi F)'$ and $(u \otimes v)(z) = 1$. ☐

We will now examine a family of seminorms defining the projective tensor topology.

Definition 19.8. *Given two vector spaces E and F and seminorms p on E and q on F, we define a seminorm $p \otimes q$ on $E \otimes F$ by*

$$(p \otimes q)(z) := \inf\{\sum_{i=1}^{n} p(x_i)q(y_i) \mid z = \sum_{i=1}^{n} x_i \otimes y_i\} \qquad (z \in E \otimes F).$$

Proposition 19.9. *Let $U \subseteq E$ and $V \subseteq F$ be absolutely convex and absorbent, and q_U, q_V their gauges. If $\pi_{U,V}$ is the gauge of $\operatorname{acx}(U \otimes V)$ then $\pi_{U,V} = q_U \otimes q_V$. Moreover, for $x \in E$ and $y \in F$ we have $\pi_{U,V}(x \otimes y) = q_U(x)q_V(y)$.*

Proof. Recall (3): we can write any $z = \sum_{i=1}^{n} x_i \otimes y_i$ as

$$z = \sum_{i=1}^{n} \rho_i \sigma_i \frac{x_i}{\rho_i} \otimes \frac{y_i}{\sigma_i} \in (\sum \rho_i \sigma_i) \operatorname{acx}(U \otimes V)$$

for any $\rho_i > 0$, $\sigma_i > 0$ with $x_i \in \rho_i U$, $y_i \in \sigma_i V$. For any $\varepsilon > 0$ it is possible to choose $\rho_i < q_U(x_i) + \varepsilon$, $\sigma_i < q_V(y_i) + \varepsilon$, so $\pi_{U,V}(z) \leq \sum (q_U(x_i) + \varepsilon)(q_V(y_i) + \varepsilon)$ and $\pi_{U,V}(z) \leq q_U \otimes q_V$.

Conversely, let $z \in \rho \cdot \operatorname{acx}(U \otimes V)$ for some $\rho > 0$. Then $z = \sum \lambda_i x_i \otimes y_i$ with $\sum |\lambda_i| \leq \rho$, $x_i \in U$, $y_i \in V$, so we have $(q_U \otimes q_V)(z) \leq \rho$ and hence $(q_U \otimes q_V)(z) \leq \inf\{\rho > 0 \mid z \in \operatorname{acx}(U \otimes V)\} = \pi_{U,V}$.

For the last claim let $z = x \otimes y$. Choose $u \in E'$, $v \in F'$ such that $|u| \leq q_U$, $|v| \leq q_V$, $\langle u, x \rangle = q_U(x)$ and $\langle v, y \rangle = q_V(y)$ (Theorem 9.4). For any representation $z = \sum x_i \otimes y_i$ then

$$\begin{aligned} \pi_{U,V}(z) &\leq q_U(x) \cdot q_V(y) = \langle u, x \rangle \cdot \langle v, y \rangle = \langle u \otimes v, x \otimes y \rangle \\ &= \langle u \otimes v, z \rangle \leq \sum |\langle u, x_i \rangle| \cdot |\langle v, y_i \rangle| \leq \sum q_U(x_i) \cdot q_V(y_i). \end{aligned}$$

Taking the infimum over all representations of z gives the claim. □

Corollary 19.10. *Let $\mathscr{P}$ and $\mathscr{Q}$ be directed families of seminorms generating the topologies of E and F; then $\{p \otimes q \mid p \in \mathscr{P}, q \in \mathscr{Q}\}$ is a directed family of seminorms generating the topology of $E \otimes_\pi F$.*

Corollary 19.11. *If E, F are metrizable (normable) LCS then $E \otimes_\pi F$ is metrizable (normable).*

Proposition 19.12. *Let E_i, F_i be LCS and $T_i \in \mathcal{L}(E_i, F_i)$. Then $T_1 \otimes T_2 \in \mathcal{L}(E_1 \otimes_\pi E_2, F_1 \otimes_\pi F_2)$.*

Proof. Given absolutely convex 0-neighborhoods U_i in F_i, choose absolutely convex 0-neighborhoods V_i in E_i such that $T_i(V_i) \subseteq U_i$; then $(T_1 \otimes T_2)(\operatorname{acx}(V_1 \otimes V_2)) \subseteq \operatorname{acx}(T_1 \otimes T_2)(V_1 \otimes V_2) \subseteq \operatorname{acx}(U_1 \otimes U_2)$. □

To state an important result about projective tensor products of Fréchet spaces we need the following terminology.

Definition 19.13. *Let E be a LCS.*

A sequence $(x_n)_n$ in E is called summable *if for each 0-neighborhood V in E there is $N \in \mathbb{N}$ such that for all finite subsets $J \subseteq \{n \in \mathbb{N} \mid n \geq N\}$ we have $\sum_{n \in J} x_n \in V$.*

A sequence $(x_n)_n$ is called absolutely summable *if for each continuous seminorm p on E the sequence $(p(x_n))_n$ is summable.*

Remark 19.14. *An absolutely summable sequence is summable. If $(x_n)_n$ is summable, the sequence of partial sums $(\sum_{n=1}^N x_n)_N$ is a Cauchy sequence. If the limit exists it is denoted by $\sum_{n=1}^\infty x_n$ is called an (absolutely) convergent series.*

Theorem 19.15. *Let E, F be Fréchet spaces. Any element $u \in E \widehat{\otimes}_\pi F$ can be represented by an absolutely convergent series*

$$u = \sum_{n=1}^\infty \lambda_n x_n \otimes y_n$$

with $(\lambda_n)_n \in \ell^1$, $x_n \to 0$ in E and $y_n \to 0$ in F.

Proof. Let $(p_n)_n$ and $(q_n)_n$ be increasing sequences of seminorms defining the topologies of E and F, respectively, and set $r_n := p_n \otimes q_n$. Denote by $\tilde r_n$ the continuous extension of r_n to $E \widehat{\otimes}_\pi F$. There is a sequence $(u_n)_n$ in $E \otimes F$ such that $\tilde r_n(u - u_n) < n^{-2} 2^{-(n+1)}$. Set $v_n = u_{n+1} - u_n$ for $n \in \mathbb{N}$ and let $u_1 = \sum_{i=1}^{i_1} \lambda_i x_i \otimes y_i$ be any representative of u_1. Then

$$\begin{aligned} r_n(v_n) &\le \tilde r_n(u - u_n) + \tilde r_n(u - u_{n+1}) \le \tilde r_n(u - u_n) + \tilde r_{n+1}(u - u_{n+1}) \\ &< n^{-2} 2^{-(n+1)} + (n+1)^{-2} 2^{-(n+2)} \le n^{-2} 2^{-n}. \end{aligned}$$

Because $r_n = \pi_{U_n, V_n}$ where U_n and V_n are the unit balls of p_n and q_n Proposition 19.9 this means that $v_n \in n^{-2} 2^{-n} \operatorname{acx}(U_n \otimes V_n) = 2^{-n} \operatorname{acx}(\frac{U_n}{n} \otimes \frac{V_n}{n})$, so there is a representation $v_n = \sum_{i=i_n+1}^{i_{n+1}} \lambda_i x_i \otimes y_i$ such that $p_n(x_i) \le n^{-1}$, $q_n(x_i) \le n^{-1}$ for $i_n < i \le i_{n+1}$ and $\sum_{i=i_n+1}^{i_{n+1}} |\lambda_i| \le 2^{-n}$. Thus, we can write

$$u = u_1 + \sum_{n=1}^\infty v_n = \sum_{i=1}^\infty \lambda_i x_i \otimes y_i$$

as desired. This series is absolutely convergent because

$$r_n(\lambda_i x_i \otimes y_i) \le |\lambda_i|$$

for i large enough. □

We now move to the other important topology on the tensor product. Because $E' \otimes F' \subseteq \mathscr{B}(E, F)$ we have

$$E \otimes F = (E'_\sigma)' \otimes (F'_\sigma)' \subseteq \mathscr{B}(E'_\sigma, F'_\sigma) \subseteq \mathscr{B}^s(E'_\sigma, F'_\sigma)$$

where $\mathscr{B}^s(E'_\sigma, F'_\sigma)$ denotes the set of separately continous bilinear mappings $E'_\sigma \times F'_\sigma \to \mathbb{K}$, and $E'_\sigma = (E', \sigma(E', E))$.

Definition 19.16. *The topology of bi-equicontinuous convergence on $\mathscr{B}^s(E'_\sigma, F'_\sigma)$ is the unique locally convex topology having as 0-basis all sets of the form*

$$W_{A,B} := \{\phi \in \mathscr{B}^s(E'_\sigma, F'_\sigma) \mid \phi(A, B) \subseteq \mathbb{D}\}$$

for $A \subseteq E'$ and $B \subseteq F'$ equicontinous.

For this to be well-defined by Theorem 2.10 we have to verify:

(1) $W_{A,B}$ is absolutely convex (this is clear) and absorbent (see below).
(2) $W_{A \cup A', B \cup B'} \subseteq W_{A,B} \cap W_{A',B'}$, so these sets form a filter basis.
(3) $\forall \rho, A, B\ \exists A', B' : W_{A',B'} \subseteq \rho W_{A,B}$: in fact, $W_{\rho A, B} \subseteq \rho W_{A,B}$.

For $W_{A,B}$ to be absorbent we note that $\phi \in \lambda W_{A,B} \Leftrightarrow \phi(A,B) \subseteq \lambda\mathbb{D}$, so we need that each $\phi \in \mathscr{B}^s(E'_\sigma, F'_\sigma)$ is bounded on products of equicontinuous sets. We can assume that A, B are absolutely convex and weakly closed (Lemma 12.10), hence weakly compact (Theorem 12.9) and thus bounded and complete. The following result then gives the claim:

Lemma 19.17. *Let E, F be LCS and $\phi\colon E \times F \to \mathbb{K}$ bilinear and separately continuous. If $A \subseteq E$ and $B \subseteq F$ are absolutely convex, bounded and complete then $\phi(A \times B)$ is bounded.*

Proof. Set $\phi_x := \phi(x, .) \in F'$. The family $\{\phi_x\colon x \in A\}$ is $\sigma(F', F)$-bounded, so $U := \bigcap_{x\in A} \phi_x^{-1}(\mathbb{D}) \subseteq F$ is absolutely convex, closed and absorbent ($y \in F \Rightarrow \phi(A, y) \subseteq \lambda\mathbb{D} \Rightarrow \lambda^{-1}y \in U$), i.e., a barrel. By Theorem 13.11, U absorbs B, and $B \subseteq \lambda U$ means that $\phi(A, B) \subseteq \lambda\mathbb{D}$. □

We denote by $\mathscr{B}^s_\varepsilon(E'_\sigma, F'_\sigma)$ the space $\mathscr{B}^s(E'_\sigma, F'_\sigma)$ with the topology of bi-equicontinuous convergence, and by $E \otimes_\varepsilon F$ the space $E \otimes F$ with the induced topology. $E \otimes_\varepsilon F$ is called the ε-tensor product of E and F, its topology the ε-tensor topology (or: injective tensor product, injective tensor topology).

Proposition 19.18. *$E \otimes_\pi F \to E \otimes_\varepsilon F$ is continuous.*

Proof. Let $W_{A,B}$ be given, then $A \subseteq U^\circ$, $B \subseteq V^\circ$ for some 0-neighborhoods U, V. For $x \in U$, $y \in V$ we have $(x \otimes y)(a, b) = \langle a, x\rangle \cdot \langle b, y\rangle \in \mathbb{D}$, so $x \otimes y \in W_{A,B}$ and $\operatorname{acx}(U \otimes V) \subseteq W_{A,B}$. □

Proposition 19.19. *For $T_i \in \mathcal{L}(E_i, F_i)$, $T_1 \otimes T_2 \in \mathcal{L}(E_1 \otimes_\varepsilon E_2, F_1 \otimes_\varepsilon F_2)$.*

Proof. Given $W_{A,B}$, $T_1'(A)$ and $T_2'(B)$ are equicontinuous and $(T_1 \otimes T_2)(W_{T_1'(A),T_2'(B)}) \subseteq W_{A,B}$. □

We now move to nuclear mappings. Note that the canonical mapping $\chi\colon E^* \otimes F \to L(E, F)$ maps $E' \otimes F$ into $\mathcal{L}(E, F)$, as $\|\chi(x' \otimes y)(x)\| = \|x'(x) \cdot y\| \le \|x'\| \cdot \|y\| \cdot \|x\|$.

Lemma 19.20. *If E, F are Banach spaces and $\mathcal{L}(E, F)$ is endowed with the operator norm, then $\chi\colon E' \otimes_\pi F \to \mathcal{L}(E, F)$ is continous.*

Proof. For $u \in E' \otimes F$, given any representation $u = \sum x_i' \otimes y_i$ we have

$$\|\chi(u)\| = \sup_{\|x\|\le 1} \sum |x_i'(x)| \cdot \|y_i\| \le \sum \|x_i\|\|y_i\|$$

and the infimum over all representations of u gives $\|u\|$ on the right side. □

Definition 19.21. *Let E, F be Banach spaces. A linear map $T \in \mathcal{L}(E, F)$ is called* nuclear *$T = \tilde\chi(u)$ for some $u \in E'\widehat{\otimes}_\pi F$, where $\tilde\chi$ is the extension of χ to the completion.*

Using Theorem 19.15 we obtain the following characterization:

Proposition 19.22. *Let E, F be Banach spaces and $T \in \mathcal{L}(E, F)$. The following are equivalent:*

(i) T is nuclear.
(ii) T is given by

$$T(x) = \sum_{n=1}^\infty \lambda_n x_n'(x) y_n$$

with $\sum |\lambda_n| < \infty$, $x_n' \to 0$ and $y_n \to 0$.

(iii) T is given by

$$T(x) = \sum_{n=1}^{\infty} x'_n(x) y_n$$

with $\sum \|x'_n\| \|y_n\| < \infty$.

Proof. (i) $\Rightarrow$ (ii): if T is nuclear then $T = \hat{\chi}(u)$ for some $u \in E' \widehat{\otimes}_\pi F$, and $u = \sum_{n=1}^\infty \lambda_n x'_n \otimes y_n$ with $(\lambda_n) \in \ell^1$, $x_n \to 0$ in E and $y_n \to 0$ in F. Because $\hat{\chi}$ is continuous and $u = \lim_{N\to\infty} u_N$ with $u_N := \sum_{i=1}^N \lambda_n x_n \otimes y_n$ we have $\chi(u)(x) = \lim \chi(u_N)(x) = \sum_{n=1}^\infty \lambda_n x'_n(x) y_n$.

(ii) $\Rightarrow$ (iii): set $\tilde{x}'_n := \lambda_n x'_n$, and we have $\sum \|\tilde{\chi}'_n\| \cdot \|y_n\| = \sum |\lambda_n| \, \|x'_n\| \|y_n\| < \infty$.

(iii) $\Rightarrow$ (i): define $u := \sum_{n=1}^\infty x'_n \otimes y_n \in E' \widehat{\otimes}_\pi F$, which is absolutely convergent by assumption. Then $\hat{\chi}(u)(x) = \hat{\chi}(\lim_{N\to\infty} u_N)(x) = \lim_{N\to\infty} \chi(u_N)(x) = \sum_{n=1}^\infty x'_n(x) y_n = T(x)$. □

We now generalize the definition of nuclearity to arbitrary LCS E, F, which we suppose to be Hausdorff. We call a linear map $u \in L(E, F)$ *bounded* if it maps some 0-neighborhood U to a bounded set, i.e., $u(U)$ is bounded in F. In this case it also is continuous. We can suppose that U and B are absolutely convex. First, we note that $u(E) \subseteq F_B$: $x \in E$ implies $x \in \lambda U$ for some $\lambda > 0$, so $u(x) \in \lambda B$ and $u(x) \in F_B$. If $p_U(x) = 0$ then $u(x) = 0$ by Lemma 13.8 (i), so the map $v \in L(E, F_B)$ defined by u factors as $v = u_0 \circ \phi_U$ for some $u_0 \in L(E_u, F_B)$. Because $u_0(\phi_U(U)) = u(U) \subseteq B$ and $\phi_U(U)$ is a 0-neighborhood in E_U, u_0 is continuous. Hence, we can write $u = \psi_B \circ u_0 \circ \phi_U$, where $\phi_U \in \mathcal{L}(E, E_U)$ and $\psi_B \in \mathcal{L}(F_B, F)$ are the canonical mappings and $u_0 \in \mathcal{L}(E_u, F_B)$. If F_B is complete, u_0 has a continuous extension $\hat{u}_0 \in \mathcal{L}(\widehat{E}_U, F_B)$ for which $u = \psi_B \circ \hat{u}_0 \circ \phi_V$.

Definition 19.23. *Let E, F be Hausdorff LCS. $u \in L(E, F)$ is called* nuclear *if there is an absolutely convex 0-neighborhood U in E and an absolutely convex bounded subset $B \subseteq F$ which is infracomplete (i.e., F_B is complete) and such that the induced mapping $\hat{u}_0 \in \mathcal{L}(\hat{E}_U, F_B)$ is nuclear.*

The following characterization is very useful:

Theorem 19.24. *$u \in \mathcal{L}(E, F)$ is nuclear if and only if it is of the form*

$$u(x) = \sum_{n=1}^\infty \lambda_n f_n(x) y_n$$

where $\sum_{n=1}^\infty |\lambda_n| < \infty$, $\{f_n : n \in \mathbb{N}\} \subseteq E'$ is equicontinuous, and $\{y_n : n \in \mathbb{N}\}$ is bounded in F_B for some absolutely convex bounded infracomplete set $B \subseteq F$.

If u is given like this we write $u = \sum_{n=1}^\infty \lambda_n f_n \otimes y_n$.

Proof. If u is nuclear then $u = \psi_B \circ \hat{u}_0 \circ \phi_U$ with U an absolutely convex 0-neighborhood, B absolutely convex bounded infrabarrelled, and $\hat{u}_0 \in \mathcal{L}(\widehat{E}_U, F_B)$ nuclear. We know (Proposition 19.22) that $\hat{u}_0(x) = \sum_{n=1}^\infty \lambda_n g_n(x) y_n$ with $\sum |\lambda_n| < \infty$, $g_n \in (\widehat{E}_U)'$, $g_n \to 0$ and $y_n \in F_B$ with $y_n \to 0$. Define $f_n := g_n \circ \phi_U \in E'$. Because $(g_n)_n$ is bounded in $(E_U)'$ it is equicontinuous and $(f_n)_n$ is equicontinuous. Hence, we have

$$u(x) = \sum_{n=1}^\infty \lambda_n f_n(x) y_n$$

which is of the desired form.

Conversely, if $u(x) = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n$, set $U := \bigcap_n f_n^{-1}(\mathbb{D})$. The series converges absolutely in F_B, which is complete:

$$\sum \|\lambda f_n(x) y_n\| \leq \sum |\lambda_n| \cdot \sup_n |f_n(x)| \cdot \|y_n\|$$

and in fact $u(x) = \psi_B(\sum_{n=1}^{\infty} \lambda_n f_n(x) y_n)$.

As $u(U)$ is bounded in F_B we have $p_U(x) = 0$, hence $u(x) = 0$, so we can factor $u \colon E \to f_B$ as $u = \psi_B \circ \hat{u}_0 \circ \phi_U$, with $u_0(\phi_U(x)) = u(x)$ for all $x \in E$; Because $f_n \equiv 0$ on $p_U^{-1}(0)$, there is $h_n \in (E_U)'$ with $f_n = h_n \circ \phi_U$. We have $\|h_n\| = \sup_{x \in E_U, \|x\| \leq 1} |h_n(x)| = \sup_{p(x) \leq 1, x \in E} |h_n(\phi_U(x))| = \sup_{p(x) \leq 1, x \in E} |f_n(x)| \leq 1$. Hene, $u_0(\phi_U(x)) = \sum_{n=1}^{\infty} \lambda_n h_n(\phi_U(x)) y_n$, and because ϕ_U is surjective, $u_0(x) = \sum_{n=1}^{\infty} \lambda_n h_n(x) y_n$. Because $\hat{u}_0(x) = \sum_{n=1}^{\infty} \lambda_n \hat{h}_n(x) y_n$, we are finished.

Thus,

$$u_0(x) = \sum_{n=1}^{\infty} \lambda_n h_n(x) y_n$$

and because

$$\hat{u}_0(x) = \sum_{n=1}^{\infty} \lambda_n \hat{h}_n(x) y_n$$

(use absolute convergence of the series) we are finished (using Proposition 19.22). □

We have seen in the proof that we can have $f_n \to 0$ in $(\widehat{E_U})'$ and $y_n \to 0$ in F_B. Hence, assuming that $\sum |\lambda_n| \leq 1$, $f_n(U)$ with $U = \bigcap_n f_n^{-1}(\mathbb{D})$ is contained in the absolutely convex closed hull C of $\{y_n\}$ in F_B; C is compact in F_B and hence also in F. This proves:

Corollary 19.25. *Every nuclear map is compact (i.e., maps a 0-neighborhood to a relatively compact set).*

We note the following ideal property:

Proposition 19.26. *Let $u \in \mathcal{L}(E, F)$, $v \in \mathcal{L}(F, G)$ and $w \in \mathcal{L}(G, H)$. If v is nuclear then $v \circ u$ and $w \circ v$ are nuclear.*

Proof. First, we have $(v \circ u)(x) = \sum \lambda_n (f_n \circ u)(x) y_n$ with $f_n \circ u$ equicontinuous.

Second, there is an absolutely convex 0-neighborhood V such that $\overline{v(V)} = B$ is compact. Hence, $B_1 = w(B)$ is compact and H_{B_1} is complete, so we have

$$(w \circ u)(x) = \sum \lambda_n f_n(x) w(y_n)$$

which shows nuclearity. □

Definition 19.27. *A LCS E is called* nuclear *if it has a 0-basis $\mathscr{U}$ consisting of absolutely convex 0-neighborhoods such that for each $U \in \mathscr{U}$ the canonical mapping $E \to \widetilde{E}_U$ is nuclear.*

Proposition 19.28. *If a normed space E is nuclear then it is finite dimensional.*

Proof. If U is an absolutely convex bounded 0-neighborhood then $E \to E_V$ is a homeomorphism; if $E \to \widetilde{E}_V$ is nuclear it is compact and hence E is locally compact and finite dimensional by Theorem 8.4. □

We give the following characterization:

Proposition 19.29. *For a LCS E, the following assertions are equivalent:*

(i) *E is nuclear*
(ii) *Every continuous linear map of E into any Banach space is nuclear.*
(iii) *For $U \subseteq E$ an absolutely convex 0-neighborhood there is $V \subseteq U$ absolutely convex 0-neighborhood such that $\widetilde{E}_V \to \widetilde{E}_U$ is nuclear.*

Proof. (i) $\Rightarrow$ (ii): if F is any Banach space and $u \in \mathcal{L}(E, F)$, there is V such that $\phi_V : E \to \widetilde{E}_V$ is nuclear and $u(V)$ is bounded in F. As u vanishes on $p_V^{-1}(0)$ and $\phi_V(E) = E_V$ there is a map $v \in \mathcal{L}(\widehat{E}_V, F)$ such that $u = v \circ \phi_V$, hence u is nuclear.

(ii) $\Rightarrow$ (iii): given U, $\phi_U : E \to \widetilde{E}_U$ is nuclear, hence given by $\phi_U = \sum \lambda_n f_n \otimes y_n$. We set $V := U \cap \bigcap_n f_n^{-1}(\mathbb{D})$, which is an absolutely convex 0-neighborhood, and each f_n induces a a continuous linear form of norm ≤ 1 on E_V, which has a continuous extension h_n to $\widehat{E}_V$. As $V \subseteq U$ there is a canonical map $\phi_{U,V} : \widetilde{E}_V \to \widetilde{E}_U$ given by (the continuous extension of) $\phi_{U,V}(\phi_V(x)) = \phi_U(x)$. Now $\phi_{U,V}(\phi_V(x)) = \phi_U(x) = \sum \lambda_n f_n(x) y_n = \sum \lambda_n h_n(\phi_V(x)) y_n$, hence $\phi_{U,V} = \sum \lambda_n h_n \otimes y_n$, which is nuclear.

(iii) $\Rightarrow$ (i): given U there is V with $\phi_{U,V}$ nuclear; because $\phi_U = \phi_{U,V} \circ \phi_V$, ϕ_U is nuclear. □

[23.6.]

Lemma 19.30. *If E is nuclear, every bounded subset of E is precompact.*

Proof. We use the representation $\widehat{E} = \varprojlim \widehat{E}_U$, where U runs through a 0-basis of absolutely convex 0-neighborhoods. Each $\phi_U : E \to \widehat{E}_U$ is nuclear, so if $B \subseteq E$ is bounded, $K_U := \overline{\phi_U(B)}$ is compact (Corollary 19.25). The injection $f : E \to \varprojlim \widehat{E}_U$ is given by $f(x) = (\phi_U(x))_U$, so $\overline{f(B)} = (K_U)_U \subseteq f(\widehat{E})$ is compact because $f(\widehat{E})$ is closed in $\prod \widehat{E}_U$. □

Corollary 19.31. *If E is nuclear and quasicomplete, $\tau(E', E) = \beta(E', E)$.*

We come back to the injective tensor product in more detail now.

Definition 19.32. *Let E, F be vector spaces. We call a topology $\mathscr{T}$ on $E \otimes F$ compatible with the tensor product if the following conditions are satisfied:*

(a) *The canonical mapping $E \times F \to E \otimes_\tau F := (E \otimes F, \mathscr{T}_\tau)$ is separately continuous.*
(b) *$u \in E'$, $v \in F' \Rightarrow u \otimes v \in (E \otimes_\tau F)'$.*
(c) *If $G_1 \subseteq E'$ is equicontinuous and $G_2 \subseteq F'$ is equicontinuous then $G_1 \otimes G_2 \subseteq E' \otimes F' \subseteq (E \otimes_\tau F)'$ is equicontinuous.*

We remark that (a) implies $(E \otimes_\tau F)' \subseteq \mathscr{B}^s(E, F)$: $u \in (E \otimes_\tau F)'$, $\varepsilon > 0$ and W a 0-neighborhood in $E \otimes_\tau F$ with $u(W) \subseteq \varepsilon\mathbb{D}$. Given $x_0 \in E$ there is a 0-neighborhood V in F with $x_0 \otimes V \subseteq W$, so $(\theta u)(x_0, V) = u(x_0 \otimes V) \subseteq u(W) \subseteq \varepsilon\mathbb{D}$, so θu is continuous in y; similarly one sees continuity in x.

Moreover, (a) and (b) imply that $E' \otimes F' \subseteq (E \otimes_\tau F)' \subseteq \mathscr{B}^s(E, F)$.

Lemma 19.33. *$\mathscr{T}_\tau$ is compatible with the tensor product if and only if $\mathscr{T}$ is the $\mathfrak{S}$-topology for a class $\mathfrak{S}$ of subsets of $\mathscr{B}^s(E, F) \subset (E \otimes F)^*$ such that*

(1) *each $A \in \mathfrak{S}$ is separately equicontinuous,*
(2) *for all equicontinuous subsets $G_1 \subseteq E'$ and $G_2 \subseteq F'$, $G_1 \otimes G_2 \in \mathscr{M}$.*

Proof. Suppose $\mathscr{T}_\tau$ is compatible and let $\mathfrak{S}$ be the class of all equicontinuous subsets of $(E \otimes_\tau F)'$. If G_1, G_2 are equicontinuous then $G_1 \otimes G_2$ is equicontinuous by Definition 19.32 (c), hence an element of $\mathfrak{S}$. Moreover, for $A \in \mathfrak{S}$, then $A(W) \subseteq \mathbb{D}$ for some 0-neighborhood W in $E \otimes_\tau F$. For given $x_0 \in E$ there is V with $x_0 \otimes V \subseteq W$, hence $A(x_0 \otimes V) \subseteq \mathbb{D}$ and (as the same works for the second variable) A is separately equicontinuous.

Conversely, if these conditions hold, let $u \in E'$ and $v' \in F'$. Then $A := \{u \otimes v\}$ is in $\mathfrak{S}$, hence A° is a 0-neighborhood in $E \otimes_\tau F$ with $(u \otimes v)(A^\circ) \subseteq \mathbb{D}$, so $u \otimes v \in (E \otimes_\tau F)'$. This implies Definition 19.32 (b), and (c) is seen the same way.

Given W and x_0 we now want V with $x_0 \otimes V \subseteq W$. We can assume $W = M^\circ$ for some M which is absolutely convex and separately equicontinuous, hence $M(x_0, V) \subseteq \mathbb{D}$ for some V, which means $x_0 \otimes V \subseteq M^\circ = W$. $\square$

This motivates to take for $\mathfrak{S}$ the family of all $G_1 \otimes G_2$ where $G_1 \subseteq E'$ and $G_2 \subseteq F'$ are equicontinuous; this gives the coarsest topology compatible with the tensor product. Is is evident that this topology coincides with the ε-tensor topology introduced in Definition 19.16.

For the seminorms we recall the following result (see also Section 12).

Lemma 19.34. *Let (E, F) be paired vector spaces and $A \subseteq F$ be $\sigma(F, E)$-bounded. Then the gauge of A° is given by*

$$q_{A^\circ}(x) = \sup_{y \in A} |\langle x, y\rangle|$$

Proof. Suppose $r = \sup_{y \in A} |\langle x, y\rangle| > 0$. Then $|\langle x, y\rangle| \leq r$ for all $y \in A$, hence $x/r \in A^\circ$, $x \in rA^\circ$ and $q_{A^\circ}(x) \leq r$. For any $t < r$, $x/t \notin A^\circ$ and hence $x \notin tA^\circ$, which gives the claim. For $r = 0$, $x \in \lambda A^\circ$ for all $\lambda > 0$ and hence $q_{A^\circ}(x) = 0$. $\square$

The seminorms of the ε-tensor topology hence are given by

$$\varepsilon_{G_1,G_2}(z) = \sup_{u \in G_1, v \in G_2} |\langle u \otimes v, z\rangle| \sup_{(u,v) \in G_1 \times G_2} \left| \sum_{i=1}^n \langle u, x_i\rangle \langle v, y_i\rangle \right|$$

for $z = \sum_{i=1}^n x_i \otimes y_i$, where $G_1 \subseteq E'$ and $G_2 \subseteq F'$ are equicontinuous.

If E and F are normed spaces then $E \otimes_\varepsilon F$ is a normed space with norm

$$\varepsilon(z) = \|z\|_\varepsilon = \sup_{\|u\| \leq 1, \|v\| \leq 1} |(u \otimes v)(z)| = \sup_{\|u\| \leq 1, \|v\| \leq 1} \left| \sum \langle u, x_i\rangle \langle v, y_i\rangle \right|.$$

With this norm, if U and V are the closed unit balls of E and F, then $(U^\circ \otimes V^\circ)^\circ$ is the closed unit ball of $E \otimes_\varepsilon F$.

Proposition 19.35. *If E, F are LCS and E is nuclear, $E \otimes F$ is dense in $\mathscr{B}^s_\varepsilon(E'_\sigma, F'_\sigma)$.*

For this, we need some preparations.

Let F be a topological vector space and T any set, $\mathfrak{S}$ a family of subsets of T directed under inclusion. Then the family

$$W_{A,V} := \{f : T \to F \mid f(A) \subseteq V\}$$

with $A \in \mathfrak{S}$ and V running through all 0-neighborhoods in F, is a 0-basis of a unique translation-invariant topology on F^T, called the $\mathfrak{S}$-Topology or topology of uniform convergence on sets $A \in \mathfrak{S}$. In fact, if $V_3 \subseteq V_1 \cap V_2$ and $A_1 \cup A_2 \subseteq A_3$, then $W_{A_3,V_3} \subseteq W_{A_1,V_1} \cap W_{A_2,V_2}$, so these sets form a filter basis. Moreover, we have

$$V + V \subseteq U \Rightarrow W_{A,V} + W_{A,V} \subseteq W_{A,U}.$$

Note that we can replcae $\mathfrak{S}$ with a fundamental subfamily $\mathfrak{S}_1$, i.a., a family such that each $A \in \mathfrak{S}$ is contained in some $A_1 \in \mathfrak{S}_1$, an we can take the 0-neighborhoods of V only from some 0-basis, without changing the $\mathfrak{S}$-topology.

Lemma 19.36. *A linear subspace $G \subseteq F^T$ is a topological vector space with respect to the $\mathfrak{S}$-topology if and only if $\forall f \in G\ \forall A \in \mathfrak{S}$, $f(A)$ is bounded in F.*

Proof. One only has to show that $W_{A,V}$ is absorbent, but this onle is the case if the stated condition holds. $\square$

If $\{p_\alpha\}$ is a family of seminorms generating the topology of F, the seminorms

$$p_{A,\alpha}(u) := \sup_{x \in A} p_\alpha(u)$$

generate the $\mathfrak{S}$-topology on $\mathcal{L}(E, F)$.

Lemma 19.37. *$\mathscr{B}^s_\varepsilon(E'_\sigma, F'_\sigma) \cong \mathcal{L}_\varepsilon(E'_\tau, F)$, where $E'_\tau = (E', \tau(E', E))$ is E' with the Mackey topology.*

Proof. Algebraically we have $B(E, F) \cong L(E, F^*)$, where $b \in B(E, F)$ corresponds to $\tilde{b} \in L(E, F^*)$ via $b(x, y) = \tilde{b}(x)(y)$. Topologically, if $b \in \mathscr{B}^s(E, F)$ then obviously $\tilde{b} \in L(E, F')$. Moreover, if M° with $M \subseteq F$ finite is a 0-neighborhood in F'_σ, there is a 0-neighborhood U in E with $b(U, M) \subseteq \mathbb{D}$ and hence $\tilde{b}(U) \subseteq M^\circ$, which means that $\tilde{b} \in \mathcal{L}(E, F'_\sigma)$. Conversely, if $\tilde{b} \in \mathcal{L}(E, F'_\sigma)$ then $b \in B(E, F)$ and $B(x.,) \in F'$ for fixed x. For $y \in F$, $\{y\}^\circ$ is a 0-neighborhood in F'_σ, so there is a 0-neighborhood U in E with $\tilde{b}(U) \subseteq \{y\}^\circ$, which means that $B(U, y) \subseteq \mathbb{D}$. This shows that $\mathscr{B}^s(E, F) \cong \mathcal{L}(E, F'_\sigma)$, so

$$\mathscr{B}^s(E'_\sigma, F'_\sigma) \cong \mathcal{L}(E'_\sigma, (F'_\sigma)'_\sigma) \cong \mathcal{L}(E'_\sigma, F_\sigma).$$

Moreover, if $u \in \mathcal{L}(E'_\sigma, F_\sigma)$ then $u \in \mathcal{L}(E'_\tau, F_\tau)$ by Proposition 18.3 (ii), hence $u \in \mathcal{L}(E'_\tau, F)$, which in turn implies $u \in \mathcal{L}(E'_\sigma, F_\sigma)$ by Proposition 18.3 (i).

For the topological isomorphism we note that for equicontinuous subsets $A \subseteq E'$ and $B \subseteq F'$,

$$b \in W_{A,B} \Leftrightarrow b(A, B) \subseteq \mathbb{D} \Leftrightarrow \tilde{b}(A)(B) \subseteq \mathbb{D} \Leftrightarrow \tilde{b}(A) \subseteq B^\circ \Leftrightarrow \tilde{b} \in W_{A,B^\circ}. \qquad \square$$

Proposition 19.38. *$\mathcal{L}_\varepsilon(E'_\tau, F)$ is complete if E and F are complete.*

Proof. Let $\mathscr{F}$ be a Cauchy filter on $\mathcal{L}_\varepsilon(E'_\tau, F)$. The evaluation map $\mathcal{L}_\varepsilon(E'_\mu, F) \to F$ is continuous, hence for each $x \in E'$, $\mathscr{F}(x)$ is a Cauchy filter basis converging to some $u(x) \in F$, which defines a map $u\colon E'_\tau \to F$. u is linear because $\mathscr{F}(x) + \mathscr{F}(y) \subseteq \mathscr{F}(x+y)$ (for $F, G \in \mathscr{F}$ there is $H \in \mathscr{F}$ with $H(x) \subseteq F(x) + G(x)$, hence $H(x+y) \subseteq F(x) + G(y)$) implies that $\mathscr{F}(x+y) \to u(x) + u(y)$; similarly, $\mathscr{F}(\lambda x) = \lambda \mathscr{F}(x) \to \lambda u(x)$. So, $\mathscr{F} \to u$ in $F^{E'}$.

Now because the 0-neighborhoods $W_{A,V}$ of $\mathcal{L}_\varepsilon(E'_\mu, F)$ are closed w.r.t. the topology of pointwise convergence ($W_{A,V} = \bigcap_{x \in A} \mathrm{ev}_x^{-1}(V)$ is closed in $\mathcal{L}_\sigma(E'_\tau, F)$), Lemma 13.9 (or rather: its variant for topological groups) applies and the convergence of $\mathscr{F}$ is uniform on equicontinuous sets. It suffices to show that $u \in L(E', F)$ is weakly continuous because then it is continuous w.r.t. the Mackey topologies and hence in $\mathcal{L}(E'_\tau, F_\tau) \subseteq \mathcal{L}(E'_\tau, F)$. Now u is weakly continuous if and only if its transpose $u' \in L(F', (E')*)$ maps F' into $(E'_\tau)' = E$. Transposition is a mapping

$$\mathrm{ad}\colon L(E', F) \to L(F*, (E')*),$$
$$\mathrm{ad}\colon \mathcal{L}(E'_\tau, F) \to \mathcal{L}(F'_\sigma, E_\sigma)$$

$\mathscr{F}$ is a filter in $\mathcal{L}(E'_\tau, F)$, so $\operatorname{ad}(\mathscr{F})$ is a filter in $\mathcal{L}(F'_\sigma, E_\sigma)$, and $\operatorname{ad}(\mathscr{F})(v)$ a filter in $E = (E'_\tau)'$. $\operatorname{ad}(u)(v)$ is an element of $(E')^*$. Let $\varepsilon > 0$, $M \subseteq E'$ equicontinuous and $V \subseteq F$ a 0-neighborhood such that $v(V) \subseteq \varepsilon\mathbb{D}$. Then there is $F \in \mathscr{F}$ with $F(M) - u(M) \subseteq V$, so $(\operatorname{ad}(\mathscr{F})(u) - \operatorname{ad}(u)(v))(M) = v(F(M) - u(M)) \subseteq v(V) \subseteq \varepsilon\mathbb{D}$, hence $\operatorname{ad}(\mathscr{F})(v) \to \operatorname{ad}(u)(v)$ uniformly on equicontinuous subsets of E', which implies that $\operatorname{ad}(u)(v) = u'(v) \in E$ because $(E'_\tau)'_\varepsilon = E$ is complete. □

Moreover, we need the following result.

Lemma 19.39. *Let $U \subseteq E$ be an absolutely convex 0-neighborhood and set $A = U^\circ$. Then the adjoint $\phi'_U \colon (\widehat{E}_U)' \to E'$ of the quotient mapping $\phi_U \colon E \to \widehat{E}_U$ maps into E'_A and defines a topological isomorphism $(\widehat{E}_U)' \cong E'_A$.*

Proof. Let $u \in (\widehat{E}_U)'$ and set $v := u \circ \phi_U$; then clearly $v \in E'$ and because $v(U)$ is bounded, v is absorbed by A.

Conversely, let $v \in E'_A$ be given. Then $v(U)$ is bounded, hence v vanishes on $q_U^{-1}(0)$ and there is $u \in (E_U)'$ such that $u \circ \phi_U = v$; extending u to the completion gives the desired element of $(\widehat{E}_U)'$. □

Proof of Proposition 19.35. Let $f \in \mathcal{L}(E'_\tau, F)$, $A \subseteq E'$ absolutely convex and equicontinuous, and $V \subseteq F$ an absolutely convex 0-neighborhood. We have to find $f_0 \in E \otimes F$ such that $(f - f_0)(A) \subseteq V$.

Set $U := A^\circ$, which is an absolutely convex 0-neighborhood in E. By Proposition 19.29 (iii) there is an absolutely convex 0-neighborhood $W \subseteq U$ such that the canonical mapping $\phi_{U,W} \colon \widehat{E}_W \to \widehat{E}_U$ is nuclear, hence given by

$$\phi_{U,W} = \sum \lambda_i y_i \otimes z_i$$

for $(\lambda_i) \in \ell^1$, (y_i) bounded in $(\widehat{E}_W)'$, and (z_i) bounded in $\widehat{E}_U$. Set $B := W^\circ$. Because $W \subseteq U$ we have $A \subseteq B$, so there is a canonical inclusion $\phi_{B,A} \colon E'_A \to E'_B$. We now claim that the following diagram commutes:

$$\begin{tikzcd} E'_A \arrow[r, "\phi_{B,A}"] \arrow[d, "(\phi'_U)^{-1}"'] & E'_B \\ (\widehat{E}_U)' \arrow[r, "\phi'_{U,W}"'] & (\widehat{E}_W)' \arrow[u, "\phi'_W"'] \end{tikzcd}$$

In fact, for $u \in E'_A$ we have

$$\phi'_W(\phi'_{U,W}((\phi'_U)^{-1}(u)))(x) = (\phi'_U)^{-1}(u)(\phi_{U,W}\phi_W(x)) = (\phi'_U)^{-1}(u)(\phi_U(x)) = u(x)$$

for all $x \in E$. It follows that we can write (with $z_i \in \widehat{E}_u \subseteq (E'_W)'$)

$$\phi_{BA} = \sum_{i=1}^{\infty} \lambda(z_i \circ (\phi'_U)^{-1}) \otimes \phi'_W(y_i).$$

and given any $\varepsilon > 0$ we can choose n such that

$$\|\sum_{i=1}^{n} \lambda_i(z_i \circ (\phi'_U)^{-1}) \otimes \phi'_W(y_i) - \phi_{BA}\| \leq \varepsilon/2.$$

where the norm is the operator norm in $\mathcal{L}(E'_A, E'_B)$. Moreover, for any given $\kappa_i > 0$ we can choose $\tilde{z}_i \in E_U$ such that $\|\tilde{z}_i - z_i\| \leq \kappa_i$ in $((\widehat{E}_U)')'$. It follows that for $x' \in A$,

$$\|\sum_{i=1}^n (\tilde{z}_i \circ (\phi'_U)^{-1})(x')\phi'_W(y_i) - x'\| \leq \|\sum_{i=1}^n ((\tilde{z}_i - z_i) \circ (\phi'_U)^{-1})(x')\phi'_W(y_i)\| + \varepsilon/2 \leq \varepsilon$$

(norm in E'_B) for the κ_i chosen small enough, i.e.,

$$\sum_{i=1}^n (\tilde{z}_i \circ (\phi'_U)^{-1})(x')\phi'_W(y_i) - x' \in \varepsilon B.$$

Choose now ε such that $\varepsilon f(B) \subseteq V$. Then

$$\sum_{i=1}^n (\tilde{z}_i \circ (\phi'_U)^{-1})(x') f(\phi'_W(y_i)) - f(x') \in V$$

and if we take $z'_i \in E$ such that $\tilde{z}_i = \phi_U(z'_i) = z'_i \circ \phi'_U$ then

$$f_0 := \sum_{i=1}^n z_i \otimes f(\phi'_W(y_i))$$

is the desired element of $E \otimes F$. $\square$

Proposition 19.40. *If E, F are LCS and E is nuclear then $E \otimes_\pi F = E \otimes_\varepsilon F$.*

Proof. The map $E \otimes_\pi F \to \mathscr{B}^s_\varepsilon(E'_\sigma, F'_\sigma)$ is continuous and has dense image, hence by Proposition 18.2 its adjoint $(\mathscr{B}^s_\varepsilon(E'_\sigma, F'_\sigma))' \to \mathscr{B}(E, F)$ is injective. In order to show that the identity $\mathrm{id}\colon E \otimes_\pi F \to E \otimes_\varepsilon E$ is open, let $U \subseteq E \otimes_\pi F$ be an absolutely convex closed 0-neighborhood. We need to find a 0-neighborhood W in $\mathscr{B}^s_\varepsilon(E'_\sigma, F'_\sigma)$ such that $W \cap (E \otimes F) \subseteq U$. The polar $A := U^\circ$ is an equicontinuous subset of $\mathscr{B}(E, F)$, and we can assume that $A = \{u \in \mathscr{B}(E, F) \mid u(U, V) \in \mathbb{D}\}$ for some absolutely convex 0-neighborhoods U in E and V in F. We claim that A is contained and equicontinuous in $(\mathscr{B}^s_\varepsilon(E'_\sigma, F'_\sigma))'$, because then its polar W is a 0-neighborhood in $\mathscr{B}^s_\varepsilon(E'_\sigma, F'_\sigma)$ and $W \cap (E \otimes F)$ is the polar in $E \otimes F$, which equals $U^{\circ\circ} = U$.

Because E is nuclear the map $\phi_U\colon E \to \widehat{E}_U$ is nuclear, hence of the form

$$\phi_U = \sum \lambda_i f_i \otimes x_i$$

with (x_i) bounded in $\widehat{E}_U$ and $(f_i) \subseteq E'$ equicontinous as well as $\sum |\lambda_i| \leq 1$. Let $u \in A$; because $u(U, V)$ is bounded there is $v \in \mathscr{B}(\widehat{E}_U, F)$ such that $u(x, y) = v(\phi_U(x), y)$, and $v(B, V) \subseteq \mathbb{D}$ if B is the unit ball in $\widehat{E}_U$. So, we have

$$u(x, y) = \sum_{i=1}^\infty \lambda_i f_i(x) v(x_i, y) = \sum \lambda_i f_i(x) g_i(y)$$

if we set $g_i := v(x_i, .)$. The family $(g_i)_i$ is equicontinuous, hence $u \in \overline{\mathrm{acx}}(G_1 \otimes G_2)$ for equicontinuous sets $G_1 \subseteq E'$, $G_2 \subseteq F'$. Because $(\overline{\mathrm{acx}}(G_1 \otimes G_2))^\circ$ is a 0-neighborhood in $\mathscr{B}^s_\varepsilon(E'_\sigma, F'_\sigma)$, and A is contained in its polar, A is equicontinuous as desired. $\square$

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