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Hausdorff dimension for some hyperbolic attractors with overlaps and without finite Markov partition

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Abstract. In this paper some families of skew product self maps F on the square are considered. The main example is a family forming a two dimensional analogue of the tent map family. According to the assumptions made in the paper these maps are almost injective. This means that the points of the attractor having more than one inverse image form a set of measure zero for all interesting measures. It may be that F does not have a finite Markov partition. The Hausdorff dimension of the attractor is computed.

1. Introduction

We investigate skew product maps $F : Q \to Q$ with $Q := [a, b] \times [0, 1]$, defined by

$$F(x, y) = \big(T(x), g(x, y)\big)\,, \tag{1}$$

where $T : [a, b] \to [a, b]$ is piecewise monotonic and the map $y \mapsto g(x, y)$ is a contraction on $[0, 1]$ for all $x \in [a, b]$. A map $T : [a, b] \to [a, b]$ is called piecewise monotonic, if there exists a finite partition of $[a, b]$ into intervals,

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such that on each of these intervals T is continuous and strictly monotonic. The set

$$\Lambda := \bigcap_{n=0}^{\infty} F^n(Q) \tag{2}$$

is the attractor of F. In order to describe the "thickness" of this attractor we like to determine the Hausdorff dimension $\dim_{\mathrm{H}}(\Lambda)$ of Λ. Our motivating example is a two dimensional analogue of the tent map family (see Figure 1).

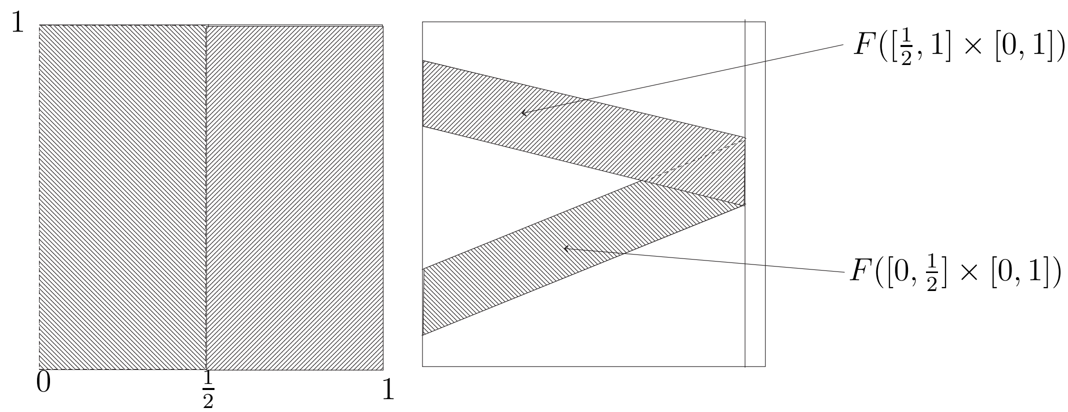

Figure 1: $F([0, 1]^2)$.

Theorem 1. *Let* $\sqrt{2} \le \alpha \le 2$, *and define* $T_\alpha : [0, 1] \to [0, 1]$ *by*

$$T_\alpha(x) := \begin{cases} \alpha x\,, & \text{if } 0 \le x \le \frac{1}{2}, \\ \alpha - \alpha x\,, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

If $0 < \lambda < \frac{1}{\alpha^2}$ *and* $\varphi : [0, 1] \to \left[\frac{\lambda}{2}, 1 - \frac{\lambda}{2}\right]$ *is linear with nonzero slope, then for*

$$F(x, y) := \big(T_\alpha(x), \varphi(x) + \lambda(y - \tfrac{1}{2})\big)$$

the dimension of the attractor of F *is*

$$\dim_{\mathrm{H}}\left(\bigcap_{n=0}^{\infty} F^n\big([0, 1] \times [0, 1]\big)\right) = 1 + \frac{\log \alpha}{-\log \lambda}\,. \tag{3}$$

The question of determining the Hausdorff dimension of invariant subsets has been studied in several papers for different classes of dynamical system (a good overview can be found in [9]). For example, in [1], [7] and [10] versions of the Bowen-McCluskey-Manning formula relating the Hausdorff dimension to a zero of a certain pressure function are proved. The situation considered in these papers is different from our situation. However, also for different classes of skew product maps F some versions of the Bowen-McCluskey-Manning formula have been obtained. We describe these results for skew product maps below.

In [13] T. Steinberger obtained a result on the Hausdorff dimension of the attractor under the assumption that F is injective. Such maps F are used as geometric models for the Lorenz attractor (see [3]). The result in [13] gives an implicit formula for $\dim_{\mathrm{H}}(\Lambda)$ using the topological pressure $p(F,h)$ of a function $h : Q \to \mathbb{R}$. Under certain assumptions on T and g, but not assuming that T admits a Markov partition, it is shown that $\dim_{\mathrm{H}}(\Lambda) = 1 + z$, where z is the unique zero of $t \mapsto p\left(F, t\log\left|\frac{\partial}{\partial y} g\right|\right)$. One obtains an explicit formula, if $\frac{\partial}{\partial y} g$ is a constant, say λ, which means $g(x,y) := \varphi(x) + \lambda\left(y - \frac{1}{2}\right)$ for some function $\varphi : [a,b] \to \left[\frac{\lambda}{2}, 1 - \frac{\lambda}{2}\right]$. In this case we have $p\left(F, t\log\left|\frac{\partial}{\partial y} g\right|\right) = h_{\mathrm{top}}(F) + t\log\lambda$, and as $h_{\mathrm{top}}(F)$ equals $h_{\mathrm{top}}(T)$ it follows that $\dim_{\mathrm{H}}(\Lambda) = 1 + \frac{h_{\mathrm{top}}(T)}{-\log\lambda}$.

Noninjective skew product maps F of the above kind were studied in [5] by M. V. Jacobson from the point of the existence of absolutely continuous invariant measure. Especially, he considered the case, where T is conjugate to the map $x \mapsto 4x(1-x)$. This implies that there is a $c \in (a,b)$ such that T is strictly monotone on the intervals $[a,c]$ and $[c,b]$, and both of these intervals are mapped onto $[a,b]$. In particular, T has a Markov partition. Set $Q_1 := [a,c] \times [0,1]$ und $Q_2 := [c,b] \times [0,1]$. Then F is injective on Q_1 and on Q_2, and the images $F(Q_1)$ and $F(Q_2)$ are nearly horizontal strips from the left to the right margin of the rectangle $Q = [a,b] \times [0,1]$. If these two strips do not overlap, then F is injective. Additionally, in [5] the map g is chosen as $g(x,y) := \varphi(x) + \lambda\left(y - \frac{1}{2}\right)$ with an increasing continuous function φ. This gives rise to overlaps of the horizontal strips $F(Q_1)$ and $F(Q_2)$ near the right margin of Q. For these maps F in [12] conditions are given by K. Simon (including the existence of a Markov partition for T), under which the formula for $\dim_{\mathrm{H}}(\Lambda)$ described above still holds.

In this paper we combine both. Neither we assume that T admits a Markov partition nor that F is injective. We are able to do this for piecewise monotonic maps T with two monotonic pieces and suitable linear contractions $y \mapsto g(x,y)$. To be more specific, for $c \in (a,b)$ let $\mathcal{D}_c([a,b])$ be the set of all functions $f : [a,b] \to \mathbb{R}$, which are continuous on $[a,c]$ and on $(c,b]$ and have a continuous derivative f' on $(a,c) \cup (c,b)$ satisfying $\inf|f'| > 0$ and $\sup|f'| < \infty$. In particular, each $f \in \mathcal{D}_c([a,b])$ is strictly monotonic on the two intervals $[a,c]$ and $(c,b]$, since its derivative cannot change its sign on these intervals. (One could also choose continuity from the right instead of continuity from the left in c.) A map $T \in \mathcal{D}_c([a,b])$ is called expanding, if the one-sided limits $\lim_{x\to a^+} T'(x)$, $\lim_{x\to c^-} T'(x)$, $\lim_{x\to c^+} T'(x)$, $\lim_{x\to b^-} T'(x)$ exist, and if $\inf |T'| > 1$ (note that this coincides with the definition of expanding which will be given in Section 2 for general piecewise monotonic maps).

Now we can state the main theorem of this paper, which gives the formula for the Hausdorff dimension of the attractor.

Theorem 2. *Suppose that $T : [a,b] \to [a,b]$ is topologically transitive, in $\mathcal{D}_c([a,b])$, and expanding. Assume that $0 < \lambda < \min\left\{\frac{1}{2}, \inf \frac{1}{(T')^2}\right\}$ and that $\varphi : [a,b] \to \left[\frac{\lambda}{2}, 1-\frac{\lambda}{2}\right]$ is also in $\mathcal{D}_c([a,b])$ and satisfies $\frac{\sup|\varphi'|}{\inf|\varphi'|} < \frac{\inf|T'|}{\lambda} - 1$. Set $Q := [a,b]\times[0,1]$ and define $F : Q \to Q$ by $F(x,y) := \big(T(x), \varphi(x) + \lambda(y - \frac{1}{2})\big)$. If the product $T' \cdot \varphi'$ has different signs on the two intervals (a,c) and (c,b), then the attractor $\Lambda := \bigcap_{n=0}^{\infty} F^n(Q)$ has Hausdorff dimension*

$$\dim_{\mathrm{H}}(\Lambda) = 1 + \frac{h_{\mathrm{top}}(T)}{-\log\lambda}\ .$$

Next we show how Theorem 1 follows from Theorem 2.

Proof of Theorem 1. Set $b := T_\alpha(\frac{1}{2}) = \frac{\alpha}{2}$ and $a := T_\alpha^2(\frac{1}{2}) = \alpha - \frac{\alpha^2}{2}$. Then $T_\alpha : [a,b] \to [a,b]$ is in $\mathcal{D}_c([a,b])$ with $c := \frac{1}{2}$. Moreover, T_α is topologically transitive, as $\alpha \geq \sqrt{2}$. Since $\alpha \geq \sqrt{2}$, $0 < \lambda < \frac{1}{\alpha^2}$ and $\varphi : [a,b] \to \left[\frac{\lambda}{2}, 1-\frac{\lambda}{2}\right]$ is linear with nonzero slope, the assumptions of Theorem 2 are fulfilled for $F(x,y) := \big(T_\alpha(x), \varphi(x) + \lambda(y-\frac{1}{2})\big)$. We have $h_{\mathrm{top}}(T_\alpha) = \log\alpha$, and therefore we get

$$\dim_{\mathrm{H}}(\Lambda) = 1 + \frac{\log\alpha}{-\log\lambda}\ , \tag{4}$$

where Λ is the attractor defined in (2).

Observe that in this case $\bigcap_{n=0}^{\infty} F^n\big([0,1]\times[0,1]\big)$ differs from Λ by an at most countable union of segments of the form $\big\{\big(x, f(x)\big) : x \in I\big\}$, where I is an interval, and $f : I \to [0,1]$ is affine. As the Hausdorff dimension of each of these segments equals 1 we get by (4) that $\dim_{\mathrm{H}}\left(\bigcap_{n=0}^{\infty} F^n\big([0,1]\times[0,1]\big)\right) = \dim_{\mathrm{H}}(\Lambda) = 1 + \frac{\log\alpha}{-\log\lambda}$. □

Parts of the proof of Theorem 2 are given in a more general situation, since this causes no additional difficulties. In particular, some results are stated and proved for general piecewise monotonic maps.

In Section 4 a nonoverlapping version G of F is constructed just by changing φ in a suitable way. Using the result of [13] for G and the fact that F is a factor of G by a Lipschitz continuous map, we get $\dim_{\mathrm{H}}(\Lambda) \leq 1 + \frac{h_{\mathrm{top}}(T)}{-\log\lambda}$.

The difficulty for proving the inequality in the opposite direction comes from two sources. The first one is that we do not have a Markov partition for T. On the other hand, F may not be injective, which may cause a drop of the dimension of the attractor of F in comparison with the dimension of the attractor of G.

We overcome the problem that there may be no finite Markov partition for T by an approximation of T by Markov maps defined on Cantor sets. This construction relies on the results in [4] and [10], and is similar to the horseshoes in [8] (see also Section 15 of [6]). In Section 2, for each $\varepsilon > 0$ we construct a closed invariant subset D of $[a, b]$, on which T has a finite Markov partition and is topologically transitive, such that $\dim_{\mathrm{H}}(D) > 1 - \varepsilon$ and $h_{\mathrm{top}}\left(T\big|_D\right) > h_{\mathrm{top}}(T) - \varepsilon$. Then we work with $T\big|_D$ instead of T.

In Section 3 we deal with the noninjectivity of F. The socalled transversality condition is introduced. Under this condition, it can be shown that the overlapping parts of $F^n(Q)$ are in some sense small enough. We show in Proposition 2 that this transversality condition holds under the assumptions of Theorem 2. This is the only part of the proof of Theorem 2, where we need, that the piecewise monotonic map T is in $\mathcal{D}_c([a, b])$.

A measure μ on Q is constructed in Section 5 in such a way that it attributes the in some sense "correct measure" to cylinder sets. Then, in Section 6, the fibres $l_p := \{p\} \times [0, 1]$ over a point $p \in [a, b]$ are considered. Let E be the set of all $p \in D$ such that the fibre over p does not contain points in overlapping parts of Λ. By the results implied by the transversality condition one gets that $E \times [0, 1]$ is a set of full μ-measure, and that $\dim_{\mathrm{H}}(D \setminus E) < 1$. For all $p \in D$ a measure ν_p on l_p is derived from μ, which is then used to estimate the Hausdorff dimension of $\Lambda \cap l_p$. Using the pointwise dimension of the measure ν_p one gets $\dim_{\mathrm{H}}(\Lambda \cap l_p) \geq \frac{h_{\mathrm{top}}\left(T\big|_D\right)}{-\log\lambda}$ for all $p \in E$. This implies $\dim_{\mathrm{H}}(\Lambda) \geq \dim_{\mathrm{H}}(D) + \frac{h_{\mathrm{top}}\left(T\big|_D\right)}{-\log\lambda} > 1 - \varepsilon + \frac{h_{\mathrm{top}}(T)-\varepsilon}{-\log\lambda}$, and letting $\varepsilon \to 0$ gives $\dim_{\mathrm{H}}(\Lambda) \geq 1 + \frac{h_{\mathrm{top}}(T)}{-\log\lambda}$.

2. Approximation of the one-dimensional map by Markov maps

We introduce a "big" Cantor-set $D_k \subset [a, b]$ having a Markov partition, on which T is topologically transitive. Later in this paper we will consider the restriction of Λ to $D_k \times [0, 1]$, which will be called Λ_k. This will be a

Cantor set in Q. It will be proved that as k tends to infinity the dimension of Λ_k tends to $1 + \frac{h_{\text{top}}(T)}{-\log \lambda}$. To this end we have to construct D_k with appropiate properties.

Let $T : [a,b] \to [a,b]$ be a piecewise monotonic map, this means that there exist $c_0 := a < c_1 < c_2 < \cdots < c_{m_0} := b$, such that $T\big|_{(c_{j-1},c_j)}$ is continuous and strictly monotonic for every $j \in \{1,2,\dots,m_0\}$ (note that T may have a discontinuity at c_j). For $j \in \{1,2,\dots,m_0\}$ define $V_j := (c_{j-1}, c_j)$, and set $\mathcal{V} := \{V_1,\dots,V_{m_0}\}$. We call $c_0, c_1, \dots, c_{m_0}$ the critical points of T, and set $\mathcal{C} := \{c_0, c_1, \dots, c_{m_0}\}$.

A piecewise monotonic map T is called expanding, if T is continuously differentiable on $(a,b) \setminus \mathcal{C}$, if the one-sided limits $\lim_{x\to c_j^+} T'(x)$ for $j \in \{0,1,\dots,m_0-1\}$ and $\lim_{x\to c_j^-} T'(x)$ for $j \in \{1,2,\dots,m_0\}$ exist, and if $\inf |T'| > 1$. For a map $T \in \mathcal{D}_c([a,b])$ this definition coincides with the definition given in the introduction.

Denote the partition of $[a,b]$ into the intervals of monotonicity of T^{n+1} by $\mathcal{V}_n$. This means $\mathcal{V}_n = \bigvee_{i=0}^{n} T^{-i}(\mathcal{V})$. The elements of $\mathcal{V}_{n-1}$ are called n-cylinders. For $x \in \bigcup_{V\in\mathcal{V}_n} V$ we write $V_n(x)$ for the n-cylinder containing x. There are finitely many $x \in [a,b]$, which are not contained in an n-cylinder. This implies that the set $\mathcal{C}_\infty$ of all $x \in [a,b]$ which are not contained in an n-cylinder for some n is at most countable. Obviously we have $\mathcal{C}_\infty = \bigcup_{j=0}^{\infty} T^{-j}\mathcal{C}$.

The interior of a set A is denoted by $\operatorname{int} A$, and the closure of A is denoted by $\overline{A}$.

For $k \geq 1$ set

$$N_k := \bigcup \left\{\overline{V} : V \in \mathcal{V}_{k-1}, \exists i \in \{0,\dots,m_0\}, c_i \in \overline{V}\right\}. \tag{5}$$

Note that the assumption $c_i \in \overline{V}$ above means that c_i is an endpoint of V. The set N_k is by definition the union of the closures of all k-cylinders having a critical point as an endpoint.

Proposition 1. *Let $T : [a,b] \to [a,b]$ be a piecewise monotonic map, which is expanding and topologically transitive. Then for every $\varepsilon > 0$ there exists a $k \geq 2$ such that there exists a perfect set $D_k \subset [a,b]$ having the following properties:*

(a) *We have $T(D_k) = D_k$.*

(b) *The map $T\big|_{D_k}$ is topologically transitive on D_k.*

(c) *We have $h_{\text{top}}\left(T\big|_{D_k}\right) > h_{\text{top}}(T) - \varepsilon$.*

(d) *The property* $\dim_{\mathrm{H}}(D_k) > 1 - \varepsilon$ *holds.*

(e) *For every* $\beta \in \left(0, \frac{1}{\sup|T'|}\right)$ *there is a* $d > 0$ *such that for every* $x \in D_k$ *and for every* $n \geq 1$ *we have*

$$|V_n(x)| \geq d \cdot \beta^n \ . \tag{6}$$

(f) *We can find a set* $N_k' \subset [a,b]$ *which is the union of closures of* k*-cylinders such that* $N_k \subset N_k'$ *and*

$$D_k = \{x \in [a,b] : T^n(x) \notin \operatorname{int} N_k', \forall n \geq 0\} \ . \tag{7}$$

Moreover, for every k*-cylinder* V *satisfying* $V \cap D_k \neq \emptyset$ *there are* k*-cylinders* $W_1, \ldots, W_l$ *such that*

$$T(V \cap D_k) = T(V) \cap D_k = \bigcup_{j=1}^{l} (W_j \cap D_k). \tag{8}$$

In the rest of this section we prove this proposition via a number of lemmas.

Lemma 1. *Let* T *be a piecewise monotonic map, and let* $k \geq 1$*. For every* $n \geq k$ *with* $n \geq 2$ *we have for* $x \in [a,b] \setminus \mathcal{C}_\infty$ *that*

$$V_n(x) \cap N_k = \emptyset \ \text{ implies that } \ T\big(V_n(x)\big) = V_{n-1}\big(T(x)\big) \ . \tag{9}$$

Proof. Given $x \notin \mathcal{C}_\infty$, $x \notin N_k$ and $n \geq k$, $n \geq 2$. Then there exist $V_{i_0}, \ldots, V_{i_{n-1}} \in \mathcal{V}$ such that

$$V_n(x) = \bigcap_{l=0}^{n-1} T^{-l} V_{i_l} = V_{i_0} \cap T^{-1}(\widetilde{V}) \ , \tag{10}$$

where $\widetilde{V} := \bigcap_{l=0}^{n-2} T^{-l} V_{i_{l+1}}$. Then

$$T(V_n(x)) = TV_{i_0} \cap \widetilde{V} \ . \tag{11}$$

Since $\widetilde{V}$ is an $(n-1)$-cylinder containing Tx it is enough to prove that

$$\widetilde{V} \subset TV_{i_0} \ . \tag{12}$$

To verify this, we argue with contradiction. Assume that $\widetilde{V} \not\subset TV_{i_0}$. By (11) $TV_{i_0} \cap \widetilde{V} \neq \emptyset$. Since TV_{i_0} and $\widetilde{V}$ are intervals, $\widetilde{V}$ must contain an endpoint of TV_{i_0}. Therefore $T^{-1}\widetilde{V}$ must contain an endpoint c of V_{i_0}, which is (by definition) a critical point. Then by (10) c must be an endpoint of $V_n(x)$. As $V_n(x) \subset V_k(x) \subset V_{i_0}$, the critical point c must be an endpoint of $V_k(x)$. This is a contradiction to $x \notin N_k$. $\square$

Now we define

$$C_k := \{x \in [a,b] : T^n(x) \notin \operatorname{int} N_k, \forall n \geq 0\} . \tag{13}$$

Note that $T(C_k) \subset C_k$. Moreover, observe that $x \in C_k$ implies $x \notin \mathcal{C}_\infty$, and therefore $V_n(x)$ is defined for all $x \in C_k$ and all $n \geq 1$. The set C_k could be empty. However, the proof of Proposition 1 will show that under the assumptions of Proposition 1 $C_k \neq \emptyset$ for all sufficiently large k.

Lemma 2. *Let T be a piecewise monotonic map, let $k \geq 2$, and assume that $C_k \neq \emptyset$. Then the set of the k-cylinders intersecting C_k forms a Markov partition for C_k. This means that for every k-cylinder V with $V \cap C_k \neq \emptyset$ there are k-cylinders $W_1, \ldots, W_l$ such that*

$$T(V \cap C_k) = T(V) \cap C_k = \bigcup_{j=1}^{l} (W_j \cap C_k) .$$

Proof. Let V be a k-cylinder with $V \cap C_k \neq \emptyset$. Then there is an $x \in C_k$ with $V = V_k(x)$. As $x \in C_k$, (5) and (13) give that $V_k(x) \cap N_k = \emptyset$. By Lemma 1 $TV = TV_k(x) = V_{k-1}(Tx)$, hence TV is a $(k-1)$-cylinder. Since every $(k-1)$-cylinder is a union of k-cylinders, there are k-cylinders $W_1, \ldots, W_l$ with $TV = \bigcup_{j=1}^{l} W_j$. Now (13) implies $T(V \cap C_k) = T(V) \cap C_k = \bigcup_{j=1}^{l} (W_j \cap C_k)$, since $V \cap N_k = \emptyset$. $\square$

Lemma 3. *Suppose that T is a piecewise monotonic map which is differentiable on $[a,b] \setminus \mathcal{C}$, and $|T'|$ is bounded. Let $k \geq 1$, and assume that $C_k \neq \emptyset$. Then for every $\beta \in \left(0, \frac{1}{\sup |T'|}\right)$ there is a $d > 0$ such that for every $x \in C_k$ and for every $n \geq 1$ we have $|V_n(x)| \geq d \cdot \beta^n$.*

Proof. Set $d_4 := \min\left\{\frac{|V_l(x)|}{\beta^l} : l \in \{0, 1, \ldots, k\}\right\}$. Let $x \in C_k$. If $n \geq k$ then (5) and (13) imply $V_{n-j}(T^j x) \cap N_k = \emptyset$ for $j \in \{0, 1, \ldots, n-k\}$. Therefore Lemma 1 gives $T^j(V_n(x)) = V_{n-j}(T^j x)$ for $j \in \{0, 1, \ldots, n-k\}$, in particular $T^{n-k}(V_n(x)) = V_k(T^{n-k}x)$. Set $\tilde{d} := \inf\{|V| : V \in \mathcal{V}_k\}$. The mean value theorem implies that

$$\begin{aligned}
|V_n(x)| &\geq |V_k(T^{n-k}(x))| \frac{1}{\sup_{y \in [a,b]} |(T^{n-k})'(y)|} \\
&\geq \tilde{d} \frac{1}{(\sup |T'|)^{n-k}} \geq \tilde{d}\beta^{n-k} = (\tilde{d}\beta^{-k})\beta^n.
\end{aligned}$$

Choosing $d := \min\left\{\tilde{d}\beta^{-k}, d_4\right\}$ completes the proof of the lemma. $\square$

Proof of Proposition 1. If V is an n-cylinder, then the mean value theorem gives $1 \geq |T^n V| \geq (\inf |(T^n)'|)|V| \geq (\inf |T'|)^n |V|$, hence

$$|V| \leq \frac{1}{(\inf |T'|)^n} \,. \tag{14}$$

As $\inf |T'| > 1$, (14) implies that there is a k_0, such that for every $k \geq k_0$ the set N_k is a union of (m_0+1) pairwise disjoint intervals. Moreover, using again $\inf |T'| > 1$ and (14), for any $\eta > 0$ there is an $l_\eta \geq k_0$ such that for any $k \geq l_\eta$, and for any critical point c the endpoints of the maximal subinterval of N_k containing c differ at most by η from c. Hence for any $\delta > 0$ there is a $k_\delta \geq k_0$ such that for any $k \geq k_\delta$, $T\big|_{[a,b]\setminus N_k}$ and T have the same number of intervals of monotonicity, the graphs of $T\big|_{[a,b]\setminus N_k}$ and T are δ-close in the Hausdorff metric, and also the graphs of their derivatives are δ-close in the Hausdorff metric. This means that, as k tends to infinity, $T\big|_{[a,b]\setminus N_k}$ converges to T in the R^1-topology described in [11].

Let $\varepsilon \in (0,1)$. As $\inf |T'| > 1$ we obtain $h_{\text{top}}(T) > 0$. Since T is topologically transitive and $h_{\text{top}}(T) > 0$ we obtain by Theorem 3 in [11] that there exists a $k \geq 2$ and a topologically transitive subset $L_k \subset C_k$ such that $h_{\text{top}}\left(T\big|_{L_k}\right) > h_{\text{top}}(T) - \varepsilon$ and $\dim_{\text{H}}(L_k) > 1 - \varepsilon$. The partition of k-cylinders intersecting C_k forms a Markov partition on C_k by Lemma 2. Define $\mathcal{J}$ as the set of all k-cylinders V with $V \cap L_k \neq \emptyset$. The Markov matrix corresponding to $\mathcal{J}$ is irreducible, since L_k is topologically transitive. Define D_k as the set of all x satisfying $T^n x \in \bigcup_{V \in \mathcal{J}} V$ for all n. As $L_k \subset D_k \subset C_k$ by the definitions of $\mathcal{J}$ and D_k we obtain that $h_{\text{top}}\left(T\big|_{D_k}\right) \geq h_{\text{top}}\left(T\big|_{L_k}\right) > h_{\text{top}}(T) - \varepsilon$ and $\dim_{\text{H}}(D_k) \geq \dim_{\text{H}}(L_k) > 1 - \varepsilon$. This shows (c) and (d).

Using that the Markov matrix corresponding to $\mathcal{J}$ is irreducible it follows from the proof of Theorem 4 in [4] that $T\big|_{D_k}$ is topologically transitive. Hence (b) holds. Moreover, this also implies $T(D_k) = D_k$, showing (a). From the definition of D_k we get that $T(V \cap D_k) = TV \cap D_k$ for every k-cylinder V with $V \cap D_k \neq \emptyset$. Hence Lemma 2 implies (f), as the first part of (f) follows immediately from the definition of D_k. By Lemma 3 we obtain (e). $\square$

3. Transversality condition

In this section we begin the investigation of the two-dimensional transformation $F : Q \to Q$, where $Q := [a,b] \times [0,1]$. To this end we need some notation. We denote the projection of $(x,y) \in Q$ to $x \in \mathbb{R}$ by π_1. That is, $\pi_1(x,y) = x$. Similarly $\pi_2(x,y) := y$. Using the partition $\mathcal{V}$ of $[a,b]$ of intervals of monotonicity of T introduced in the previous section, we define $\mathcal{Z} := \pi_1^{-1}(\mathcal{V})$, which is a partition of Q into rectangles. For $n \geq 1$ we set $\mathcal{Z}_n := \bigvee_{i=0}^{n} F^{-i}\mathcal{Z} = \pi_1^{-1}\mathcal{V}_n$ and call the sets in $\mathcal{Z}_n$ the vertical n-cylinders. Furthermore, for $n \geq 1$ we set $\mathcal{I}_n^F := F^n\mathcal{Z}_{n-1}$ and call the sets in $\mathcal{I}_n^F$ the horizontal n-cylinders. See Figure 2 for an illustration of $\mathcal{Z}_1$ and $\mathcal{I}_2^F$. Finally define $\mathcal{I}_\infty^F := \left\{\bigcap_{n\geq 0} I_n^F \neq \emptyset : I_n^F \in \mathcal{I}_n^F\right\}$. The elements of $\mathcal{I}_\infty^F$ are curves on which π_1 is one-to-one. We call them unstable curves of Λ. Clearly $\Lambda = \bigcup\left\{I : I \in \mathcal{I}_\infty^F\right\}$.

To handle the overlapping between the horizontal cylinders we need the so-called transversality condition. In order to state it we need a symbolic representation of the horizontal cylinders.

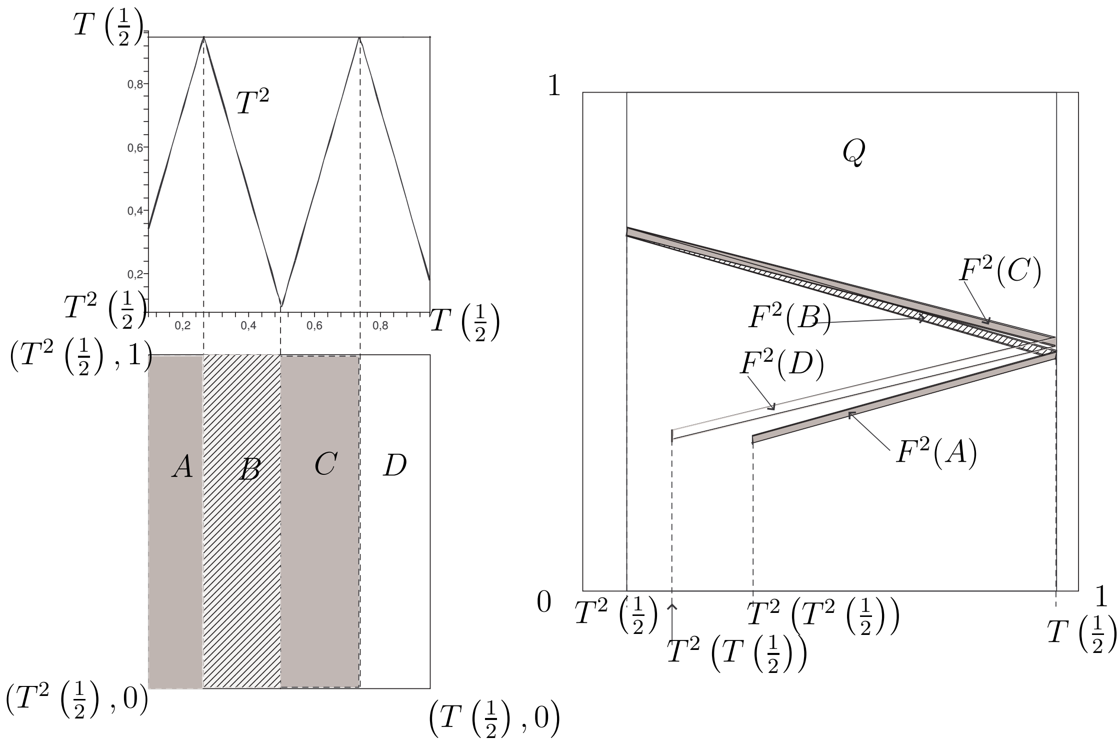

Figure 2: $A, B, C, D \in \mathcal{Z}_1$ and $F^2(A), F^2(B), F^2(C), F^2(D) \in \mathcal{I}_2^F$.

Define

(15)
$$\mathcal{H} := \left\{(\ldots, i_{-2}, i_{-1}, i_0) : V_{i_{-n}} \cap T^{-1}V_{i_{-(n-1)}} \cap \cdots \cap T^{-n}V_{i_0} \neq \emptyset \ \forall n \geq 0\right\}.$$

For every $\mathbf{i} = (\ldots, i_{-2}, i_{-1}, i_0) \in \mathcal{H}$ we write

$$p(\mathbf{i}) := \bigcap_{k=0}^{\infty} F^{k+1} Z_{i_{-k}}, \tag{16}$$

where $Z_l := V_l \times [0,1]$ for $l \in \{1, \ldots, m_0\}$. Then $p : \mathcal{H} \to \mathcal{I}_\infty^F$ is surjective. Set

$$S_n^\lambda(x) := \varphi(T^n(x)) + \lambda\varphi(T^{n-1}x) + \cdots + \lambda^{n-1}\varphi(Tx) + \lambda^n\varphi(x)\ . \tag{17}$$

Now we define the shift $\sigma : \mathcal{H} \to \mathcal{H}$ by

$$\sigma(\ldots, i_{-2}, i_{-1}, i_0) := (\ldots, i_{-3}, i_{-2}, i_{-1})\ . \tag{18}$$

From the definition of F we get

$$F^{n+1}(x,y) = \left(T^{n+1}(x), S_n^\lambda(x) + \lambda^{n+1}y + q_n\right)\ , \tag{19}$$

where $q_n := -\frac{1}{2}\left(\lambda^{n+1} + \cdots + \lambda\right)$. Let $I = p(\mathbf{i})$ and for $N \in \mathbb{N}$ we call $I_N := \bigcap_{k=0}^N F^{k+1} Z_{i_{-k}}$ the horizontal N-cylinder containing I. Then

$$I_N = F^{N+1}\left(\left(V_{i_{-N}} \cap T^{-1}V_{i_{-(N-1)}} \cap \cdots \cap T^{-N}V_{i_0}\right) \times [0,1]\right)\ . \tag{20}$$

That is I_N is a nearly horizontal strip of height λ^{N+1}.

For any $n \geq 0$ denote by $T_{i_0 \ldots i_{-n}}^{-(n+1)} : T^{(n+1)}W \to W$ the inverse of $T^{(n+1)}\big|_W : W \to T^{(n+1)}W$, where $W = V_{i_{-n}} \cap T^{-1}V_{i_{-(n-1)}} \cap \cdots \cap T^{-n}V_{i_0}$. We introduce the new variable $t = T^{N+1}x$ for $x \in V_{i_{-N}} \cap T^{-1}V_{i_{-(N-1)}} \cap \cdots \cap T^{-N}V_{i_0}$. For

$$t \in T^{N+1}\left(V_{i_{-N}} \cap T^{-1}V_{i_{-(N-1)}} \cap \cdots \cap T^{-N}V_{i_0}\right)$$

put

$$f_{\mathbf{i}}^N(t) := \varphi(T_{i_0}^{-1}t) + \lambda\varphi(T_{i_0 i_{-1}}^{-2}t) + \cdots + \lambda^N\varphi(T_{i_0 \ldots i_{-N}}^{-(N+1)}t) + q_N\ . \tag{21}$$

Then we get that I_N is bounded from below by the curve

$$\gamma_N^l(\mathbf{i})(t) := \left(t, f_{\mathbf{i}}^N(t)\right) \tag{22}$$

and I_N is bounded from above by the curve

$$\gamma_N^u(\mathbf{i})(t) := \left(t, f_{\mathbf{i}}^N(t) + \lambda^{n+1}\right)\ . \tag{23}$$

Definition 1. Let $I, I' \in \mathcal{I}_\infty^F$ and $I = p(\mathbf{i}), I' = p(\mathbf{j})$. We say that I and I' have different n-cylinders if $(i_{-(n-1)}, \ldots, i_0) \neq (j_{-(n-1)}, \ldots, j_0)$.

Definition 2 (Transversality condition)**.** We say that the transversality condition holds if there exists a $d_1 > 0$ such that for all $I, I' \in \mathcal{I}_\infty^F$ having different 1-cylinders, and for every N we have

$$|\pi_1(I_N \cap I'_N)| \le d_1 \cdot \lambda^N \ , \tag{24}$$

where I_N and I'_N are the horizontal N-cylinders containing I and I' respectively.

Now we can give properties of T and φ, under which the transversality condition holds.

Proposition 2. *Suppose that the map $T : [a, b] \to [a, b]$ is in $\mathcal{D}_c([a, b])$ and satisfies $\inf |T'| > 1$. Assume that $0 < \lambda < \frac{1}{2}$ and that $\varphi : [a, b] \to \left[\frac{\lambda}{2}, 1 - \frac{\lambda}{2}\right]$ is also in $\mathcal{D}_c([a, b])$ and satisfies $\frac{\sup|\varphi'|}{\inf|\varphi'|} < \frac{\inf|T'|}{\lambda} - 1$. If the product $T' \cdot \varphi'$ has different signs on the two intervals (a, c) and (c, b), then the transversality condition holds for $F(x, y) = \big(T(x), \varphi(x) + \lambda(y - \frac{1}{2})\big)$.*

Proof. By differentiation of (21) we get

$$\left(f_{\mathbf{i}}^N\right)'(t) = \sum_{k=0}^{N} \lambda^k \varphi'(T_{i_0 \dots i_{-k}}^{-(k+1)} t)(T_{i_{-k}}^{-1})'(T_{i_0 \dots i_{-(k-1)}}^{-k} t) \cdots (T_{i_0}^{-1})'(t) \ . \tag{25}$$

Set $\gamma(t) := 1 + \sum_{k=1}^{N} \lambda^k \frac{\varphi'(T_{i_0 \dots i_{-k}}^{-(k+1)} t)}{\varphi'(T_{i_0}^{-1} t)} (T_{i_{-k}}^{-1})'(T_{i_0 \dots i_{-(k-1)}}^{-k} t) \cdots (T_{i_{-1}}^{-1})'(T_{i_0}^{-1} t)$. We have then $\gamma(t) \ge d_5$ for all t with $d_5 := 1 - \sum_{k=1}^{\infty} \frac{\sup|\varphi'|}{\inf|\varphi'|} \left(\frac{\lambda}{\inf|T'|}\right)^k$, and $d_5 > 0$ by the assumptions. Furthermore,

$$\left(f_{\mathbf{i}}^N\right)'(t) = \varphi'(T_{i_0}^{-1} t) \cdot (T_{i_0}^{-1})'(t) \cdot \gamma(t) \ .$$

If ε_1 is the sign of $T' \cdot \varphi'$ on (a, c) and ε_2 that on (c, b), we get

$$\varepsilon_{i_0} \left(f_{\mathbf{i}}^N\right)'(t) \ge d_6 \ ,$$

where $d_6 := \frac{\inf|\varphi'|}{\sup|T'|} d_5$ is a positive constant by the assumptions.

Now let I and I' be in $\mathcal{I}_\infty^F$ with $I = p(\mathbf{i}), I' = p(\mathbf{j})$ and $i_0 \ne j_0$, which means they have different 1-cylinders. Let I_N and I'_N be the horizontal N-cylinders containing I and I' respectively. By assumption we have $\varepsilon_{i_0} \ne \varepsilon_{j_0}$. We have shown above, that the slopes of the boundaries of I and I' have different signs and that these slopes are in absolute value greater or equal to d_6. Since the vertical diameter of I_N and I'_N is λ^N, we get

$$|\pi_1(I_N \cap I'_N)| \le \frac{1}{d_6} \lambda^N \ .$$

This shows that the transversality condition holds. $\square$

From the transversality condition we get the following more general estimate.

Lemma 4. *Suppose that $T : [a,b] \to [a,b]$ is a piecewise monotonic map which is differentiable on the interiors of the intervals of monotonicity and satisfies* $\sup|T'| < \infty$. *Let $\lambda \in (0,1)$ and let $\varphi : [a,b] \to \left[\frac{\lambda}{2}, 1-\frac{\lambda}{2}\right]$. Furthermore, assume that the transversality condition holds for $F(x,y) = \big(T(x), \varphi(x) + \lambda(y - \frac{1}{2})\big)$. Then for every $k \geq 1$ there exists $d(k) > 0$ such that for all $I, I' \in \mathcal{I}_\infty^F$ having different k-cylinders, and for every $N \geq k$ we have*

$$|\pi_1(I_N \cap I'_N)| \leq d(k) \cdot \lambda^N \ , \tag{26}$$

where I_N and I'_N are the horizontal N-cylinders containing I and I' respectively.

Proof. Let $I = p(\mathbf{i})$ and $I' = p(\mathbf{j})$. As I and I' have different k-cylinders we get $(i_{-(k-1)}, \dots, i_0) \neq (j_{-(k-1)}, \dots, j_0)$. Set $l := \min\{n : i_{-n} \neq j_{-n}\}$. We know from (20) that

$$\begin{aligned} I_N &= F^{N+1}\Big((V_{i_{-N}} \cap T^{-1}V_{i_{-(N-1)}} \cap \cdots \cap T^{-N}V_{i_0}) \times [0,1]\Big) \quad \text{and} \\ I'_N &= F^{N+1}\Big((V_{j_{-N}} \cap T^{-1}V_{j_{-(N-1)}} \cap \cdots \cap T^{-N}V_{j_0}) \times [0,1]\Big) \ . \end{aligned}$$

Using that F^{N+1} is injective on the domains on the right hand side of the previous equalities, we obtain that

$$\begin{aligned} I_N &\subset F^l \underbrace{\left(F^{N+1-l}\left((V_{i_{-N}} \cap \cdots \cap T^{-(N-l)}V_{i_{-l}}) \times [0,1]\right)\right)}_{(N-l)\text{-horizontal cylinder for } p(\sigma^l \mathbf{i})} \quad \text{and} \\ I'_N &\subset F^l \underbrace{\left(F^{N+1-l}\left((V_{j_{-N}} \cap \cdots \cap T^{-(N-l)}V_{j_{-l}}) \times [0,1]\right)\right)}_{(N-l)\text{-horizontal cylinder for } p(\sigma^l \mathbf{j})} \ . \end{aligned}$$

Now using the transversality condition for the two $(N-l)$-horizontal cylinders in the previous two equations we obtain $|\pi_1(I_N \cap I'_N)| \leq \left(\frac{\sup|T'|}{\lambda}\right)^l \cdot d_1 \cdot \lambda^N$. As $l \leq k$ this immediately implies the desired result with the choice of $d(k) = d_1 \cdot \max\left\{1, \frac{\sup|T'|}{\lambda}, \left(\frac{\sup|T'|}{\lambda}\right)^2, \dots, \left(\frac{\sup|T'|}{\lambda}\right)^k\right\}$. □

4. Overlapping

In this section we always assume that T is an expanding piecewise monotonic map which is topologically transitive, and that $0 < \lambda < \frac{1}{m_0}$, where m_0 is the number of intervals of monotonicity for T. Now we construct an injective self map G of Q which is practically the natural extension of T.

Choose the numbers $\frac{\lambda}{2} < q_1 < \cdots < q_{m_0} < 1 - \frac{\lambda}{2}$ such that $q_i - q_{i-1} > \lambda$ for $i = 1, \ldots, m_0$. This is possible by the choice of λ. For $x \in V_i$ let

$$G(x, y) := \left(T(x), q_i + \lambda \left(y - \frac{1}{2} \right) \right). \tag{27}$$

Obviously, for every $(x, y) \in Q$ we have

$$\pi_1(F(x, y)) = \pi_1(G(x, y)). \tag{28}$$

Let $\Lambda^G := \bigcap_{n=0}^{\infty} G^n Q$ be the attractor of G. Clearly Λ^G consists of uncountably many horizontal segments. We can define $\mathcal{I}_n^G, \mathcal{I}_\infty^G$ in a similar way for G as $\mathcal{I}_n^F, \mathcal{I}_\infty^F$ was defined for F. Observe that geometrically the elements of $\mathcal{I}_n^G$ are rectangles with sides parallel to coordinate axes and the elements of $\mathcal{I}_\infty^G$ are horizontal segments.

Next we define a natural projection from Λ^G onto Λ which will be called Φ.

Definition 3 (Definition of Φ)**.** For any $I^G \in \mathcal{I}_n^G$ there exists a $Z \in \mathcal{Z}_{n-1}$ such that $I^G = G^n(Z)$. The corresponding horizontal $(n-1)$-cylinder for the map F is $I^F = F^n(Z)$. Note that $\pi_1(F^n(Z)) = \pi_1(G^n(Z))$. For every $I^F \in \mathcal{I}_\infty^F$ we can find a sequence $(Z^n)_{n=1}^{\infty}$, $Z^n \in \mathcal{Z}_{n-1}$ such that $I^F = \bigcap_{n=1}^{\infty} F^n(Z^n)$. In this way there is a corresponding element $I^G \in \mathcal{I}_\infty^G$ defined by $I^G = \bigcap_{n=1}^{\infty} G^n(Z^n)$. Let $(x, y) \in \Lambda^G$ and let $I^G \in \mathcal{I}_\infty^G$ such that $\{(x, y)\} = \Lambda^G \cap I^G \cap \{(x, t) : t \in [0, 1]\}$. Furthermore, let $\{(x, y')\} := \Lambda \cap I^F \cap \{(x, t) : t \in [0, 1]\}$. The natural projection $\Phi : \Lambda^G \to \Lambda$ is defined by $\Phi(x, y) = (x, y')$.

It is easy to see that Φ is a Lipschitz map and the following diagram is commutative.

$$\begin{tikzcd} \Lambda^G \arrow[r, "G"] \arrow[d, "\Phi"'] & \Lambda^G \arrow[d, "\Phi"] \\ \Lambda^F \arrow[r, "F"'] & \Lambda^F \end{tikzcd} \tag{29}$$

We say $I, I' \in \mathcal{I}_\infty^G$ *have different* 1*-cylinders* if a statement analogous to Definition 1 holds with replacing F with G everywhere. The main result of Steinberger's paper [13] immediately implies that

$$\dim_{\mathrm{H}}(\Lambda^G) = 1 + \frac{h_{\mathrm{top}}(T)}{-\log \lambda}\,. \tag{30}$$

Since the map Φ is Lipschitz, we obtain the following result.

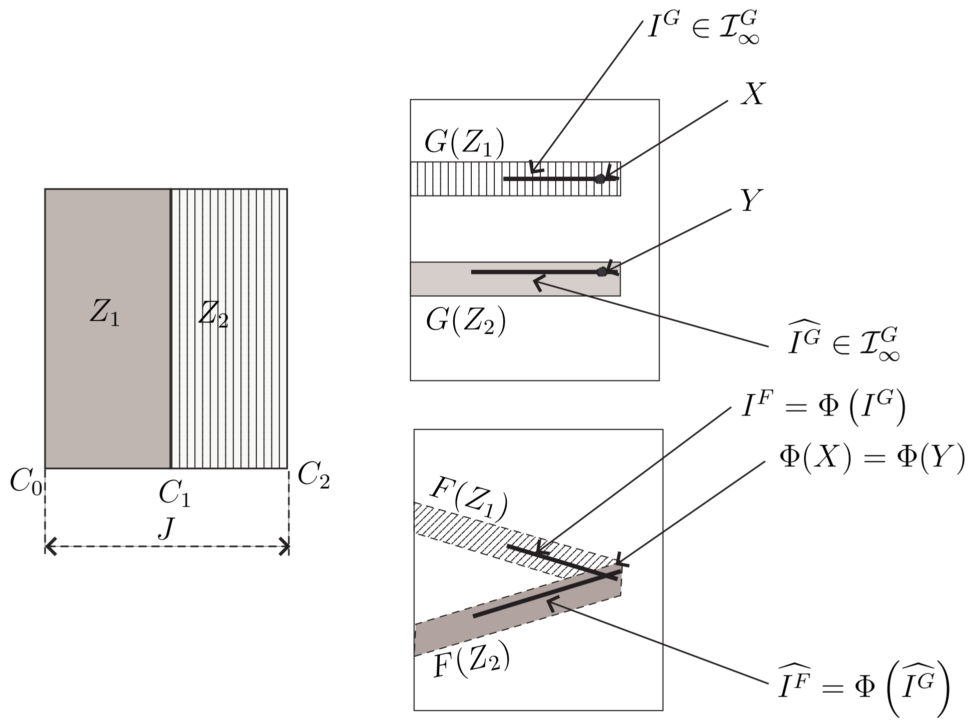

Figure 3: The definition of Φ.

Proposition 3. *Suppose that $T : [a,b] \to [a,b]$ is a piecewise monotonic map with critical points $c_0 := a < c_1 < c_2 < \cdots < c_{m_0} := b$. Furthermore assume that T is expanding and topologically transitive. Let $\lambda \in \left(0, \frac{1}{m_0}\right)$, and let $\varphi : [a,b] \to \left[\frac{\lambda}{2}, 1 - \frac{\lambda}{2}\right]$ be a function, which is continuously differentiable on $(a,b) \setminus \{c_0, c_1, \ldots, c_{m_0}\}$ and satisfies $\inf|\varphi'| > 0$ and $\sup|\varphi'| < \infty$. Define $Q := [a,b] \times [0,1]$ and define $F : Q \to Q$ by $F(x,y) := \left(T(x), \varphi(x) + \lambda(y - \frac{1}{2})\right)$. Then the attractor $\Lambda := \bigcap_{n=0}^{\infty} F^n(Q)$ satisfies*

$$\dim_{\mathrm{H}}(\Lambda) \leq 1 + \frac{h_{\mathrm{top}}(T)}{-\log\lambda} .$$

In the rest of the paper we will prove that this upper bound is in fact the Hausdorff dimension of Λ, if the transversality condition is satisfied.

To this end we need to prove that on the one hand the set of those points of Λ which are covered by at least two elements of $\mathcal{I}_\infty^F$ ("set of bad points" defined in (31) below) is small. On the other hand, we need to prove something about the size of the small neighborhood of these points. We can prove the first in Lemma 5 below. The second is much more difficult and requires a proposition which uses a number of results proved earlier.

Let us denote the "bad" points of Λ^G, i.e. those whose image under Φ in Λ have at least two preimages by

$$B := \left\{ x \in \Lambda^G : \#\Phi^{-1}(\Phi x) > 1 \right\} . \tag{31}$$

Lemma 5. *If the assumptions of Proposition* 3 *are satisfied, if the transversality condition holds, and if* $\lambda < \inf \frac{1}{(T')^2}$, *then we have*

$$\dim_{\mathrm{H}}\big(\pi_1(\Phi(B))\big) = \dim_{\mathrm{H}}(\pi_1(B)) \leq 2\frac{h_{\mathrm{top}}(T)}{-\log \lambda} < 1 .$$

Proof. Let $I, I' \in \mathcal{I}_\infty^F$ with different 1-cylinders. We define

$$B_{I,I'} := \left\{ x \in \Lambda^G : \Phi(x) \in I \cap I' \right\} .$$

Furthermore let

$$B_1 := \bigcup B_{I,I'},$$

where the union is taken over all $I, I' \in \mathcal{I}_\infty^F$ with different 1-cylinders. It follows from the transversality condition that for such I, I' and their horizontal N-cylinders I_N, I'_N we have

$$|\pi_1(I_N \cap I'_N)| \leq d_1 \cdot \lambda^N. \tag{32}$$

Note that

$$\pi_1(B_1) = \bigcap_{N \geq 1} \; \bigcup_{I,I' \text{ with different 1-cylinders}} \pi_1(I_N \cap I'_N) \tag{33}$$

Let $s > 2\frac{h_{\mathrm{top}}(T)}{-\log \lambda}$ be arbitrary. Then $\lambda^s < \exp\{-2h_{\mathrm{top}}(T)\}$, and hence

$$\lim_{N \to \infty} d_1 \left(\#\mathcal{V}_{N-1}\right)^2 \left(\lambda^s\right)^N = 0 .$$

Therefore it follows from (32) and (33) that $\dim_{\mathrm{H}}(\pi_1(B_1)) \leq s$. Using that $B = \bigcup_{n=0}^\infty G^n(B_1)$ we obtain $\dim_{\mathrm{H}}(\pi_1(B)) \leq s$. As $s > 2\frac{h_{\mathrm{top}}(T)}{-\log \lambda}$ was arbitrary, this implies $\dim_{\mathrm{H}}(\pi_1(B)) \leq 2\frac{h_{\mathrm{top}}(T)}{-\log \lambda}$.

Finally we get that $\lambda < \exp(-2h_{\mathrm{top}}(T))$, since $\lambda < \inf \frac{1}{(T')^2}$ and $h_{\mathrm{top}}(T) \leq \log \sup |T'|$. Therefore $\dim_{\mathrm{H}}(\pi_1(B)) \leq 2\frac{h_{\mathrm{top}}(T)}{-\log \lambda} < 1$. $\square$

5. An invariant subset Λ_{k_0} of Λ

To prove that in some sense the n-th level neighborhood of the "bad" points is small we need a result which provides us with a "large" subset of $[a,b]$ on which T is conjugated to a Markov shift and all k-cylinders of this subset are "long".

Fix $k_0 \in \mathbb{N}$, $k_0 \geq 2$. Let D_{k_0} be as in Proposition 1 for some $\varepsilon > 0$. Then there are closed intervals $J_1, \ldots, J_h \in \mathcal{V}_{k_0-1}$ such that

$$D_{k_0} = \left\{ x \in [a,b] : T^n(x) \in \bigcup_{k=1}^{h} J_k, \ \forall n \geq 0 \right\} .$$

Set

$$s := s_{k_0} = \frac{h_{\mathrm{top}}(T|_{D_{k_0}})}{-\log \lambda} . \tag{34}$$

We define a $h \times h$-matrix $A = (a_{i,j})_{1 \leq i,j \leq h}$ by

$$a_{i,j} := \begin{cases} 1\,, & \text{if } T(J_i) \supset J_j, \\ 0\,, & \text{otherwise.} \end{cases}$$

Then by Theorem 11 in [4] the spectral radius satisfies

$$\rho(A) = \exp\left(h_{\mathrm{top}}(T\big|_{D_{k_0}}) \right) = \lambda^{-s} . \tag{35}$$

Define

$$\Sigma_A := \left\{ \mathbf{i} \in \{1, \ldots, h\}^{\mathbb{Z}} : a_{i_l, i_{l+1}} = 1 \, \forall l \in \mathbb{Z} \right\}$$

and let σ be the left shift on Σ_A. We call $(i_0, i_1, \ldots, i_r) \in \{1, \ldots, h\}^{r+1}$ an admissible word, if $a_{i_{l-1}, i_l} = 1$ for all $l \in \{1, 2, \ldots, r\}$. Given $l \in \mathbb{Z}$, $j \geq 1$ and an admissible word $(i_l, i_{l+1}, \ldots, i_{l+j})$ define $[i_l, i_{l+1}, \ldots, i_{l+j}] := \{\mathbf{j} = (j_u)_{u \in \mathbb{Z}} : j_u = i_u \text{ for } u \in \{l, l+1, \ldots, l+j\}\}$.

Since $T|_{D_{k_0}}$ is topologically transitive by Proposition 1 we have that A is irreducible. Then we know that there exist $\mathbf{u} = (u_1, \ldots, u_h)$ and $\mathbf{v} = (v_1, \ldots, v_h) \in \mathbb{R}^h$ (note that each of them depend on k_0) such that $u_i, v_i > 0$ for all i, $\mathbf{u} \cdot A = \lambda^{-s}\mathbf{u}$, $A \cdot \mathbf{v} = \lambda^{-s}\mathbf{v}$, and $\sum_{i=1}^{h} u_i v_i = 1$. We define $\mathbf{w} = (w_1, \ldots, w_h)$ by $w_i := u_i \cdot v_i$, and the $h \times h$-matrix $P = (p_{i,j})_{1 \leq i,j \leq h}$ by

$$p_{i,j} := \lambda^s a_{i,j} \frac{v_j}{v_i} .$$

Then $\sum_{i=1}^{h} w_i = 1$, P is a stochastic matrix and

$$\mathbf{w} \cdot P = \mathbf{w} \ .$$

Let μ be the corresponding Markov measure on Σ_A. Then there exist positive constants d_2, d_3 (depending only on k_0) such that for all $l \in \mathbb{Z}$, all $j \geq 1$, and for all admissible words $(i_l, i_{l+1}, \dots, i_{l+j-1})$ we have

$$d_2 \cdot \lambda^{js} < \mu([i_l, i_{l+1}, \dots, i_{l+j-1}]) < d_3 \cdot \lambda^{js} \ . \tag{36}$$

Now define

$$Q_{k_0} := D_{k_0} \times [0,1] \quad \text{and} \quad F_{k_0} := F\big|_{Q_{k_0}} \ .$$

Since $T(D_{k_0}) \subset D_{k_0}$ we get that $F_{k_0}(Q_{k_0}) \subset Q_{k_0}$. Define

$$\Lambda_{k_0} := \bigcap_{n=0}^{\infty} F_{k_0}^n(Q_{k_0}) \ .$$

Then it is obvious that

$$\Lambda_{k_0} \subset \Lambda \tag{37}$$

holds for all k_0. For every $l \in \{1, \dots, h\}$ we define

$$\Delta_l := (J_l \cap D_{k_0}) \times [0,1] \ .$$

These are the first level vertical cylinders of Λ_{k_0}. We define for an admissible word $(i_1, \dots, i_q) \in \{1, \dots, h\}^q$ (according to the matrix A) the corresponding q-level vertical cylinder as

$$\Delta_{i_1,\dots,i_q} := \Delta_{i_1} \cap F_{k_0}^{-1}\Delta_{i_2} \cap \dots \cap F_{k_0}^{q-1}\Delta_{i_q} \ .$$

Let $p \geq 1$. For an admissible word $(i_{-(p-1)}, \dots, i_0)$ the corresponding "horizontal" (actually almost horizontal) cylinder of Λ_{k_0} is

$$\widetilde{S}_{i_{-(p-1)},\dots,i_0} := F_{k_0}^p \underbrace{\left(\Delta_{i_{-(p-1)}} \cap F_{k_0}^{-1}\Delta_{i_{-(p-2)}} \cap \dots \cap F_{k_0}^{-(p-1)}\Delta_{i_0}\right)}_{\Delta_{i_{-(p-1)},\dots,i_0}} \ .$$

For an admissible word $\left(i_{-(p-1)}, \dots, i_0, i_1, \dots, i_q\right)$ we call the set

$$\widetilde{S}_{i_{-(p-1)},\dots,i_0} \cap \Delta_{i_1,\dots,i_q}$$

a (p,q)-cylinder of Λ_{k_0}.

Given $\mathbf{i} \in \Sigma_A$, the (p,q)-cylinders obtained from $\mathbf{i}$ converge to a point in Λ_{k_0}, if we let $p \to \infty$ and $q \to \infty$. This point is called the natural projection of $\mathbf{i} \in \Sigma_A$ to Λ_{k_0}. To be more formal, we define this natural projection Π by

$$\Pi(\mathbf{i}) := \bigcap_{p=1}^{\infty} \bigcap_{q=0}^{\infty} \left(\widetilde{S}_{i_{-(p-1)},\dots,i_0} \cap \Delta_{i_1,\dots,i_q} \right) .$$

Moreover, for an admissible word $(i_{-(p-1)}, \dots, i_0)$ we write

$$\left[i_{-(p-1)}, \dots, i_0\right]_0 := \{\mathbf{j} = (j_u)_{u\in\mathbb{Z}} : j_{-k} = i_{-k} \text{ for all } 0 \le k \le p-1\} \tag{38}$$

and we say that $\left[i_{-(p-1)}, \dots, i_0\right]_0$ is a *non-positive p-cylinder* of Σ_A.

It is easy to see that the following diagram is commutative.

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ \Pi \downarrow & & \downarrow \Pi \\ \Lambda_{k_0} & \xrightarrow{F_{k_0}} & \Lambda_{k_0} \end{array} \tag{39}$$

The measure μ gives rise to a measure $\Pi_*\mu$ on $[a,b] \times [0,1]$ defined by

$$\Pi_*\mu(B) := \mu\left(\Pi^{-1}(B \cap \Lambda_{k_0})\right) . \tag{40}$$

Note that $\Pi_*\mu$ depends on k_0. However, for every k_0, $\Pi_*\mu$ is a probability measure concentrated on Λ_{k_0}.

6. Dimension of Λ_{k_0}

Now we use the Markov subsets of Proposition 1 to give a lower estimate for the Hausdorff dimension of the attractor. Note that the condition $\lambda < \inf \frac{1}{(T')^2}$ implies that $\frac{-\log \lambda}{\log \sup |T'|} > 2$.

Proposition 4. *Let $T : [a,b] \to [a,b]$ be a piecewise monotonic map with critical points $c_0 := a < c_1 < c_2 < \cdots < c_{m_0} := b$, which is expanding and topologically transitive. Suppose that $\lambda \in \left(0, \min\left\{\frac{1}{m_0}, \inf \frac{1}{(T')^2}\right\}\right)$, and suppose that $\varphi : [a,b] \to \left[\frac{\lambda}{2}, 1 - \frac{\lambda}{2}\right]$ is a function, which is continuously differentiable on $(a,b) \setminus \{c_0, c_1, \dots, c_{m_0}\}$ and satisfies $\inf |\varphi'| > 0$ and $\sup |\varphi'| < \infty$. Define $Q := [a,b] \times [0,1]$ and define $F : Q \to Q$ by*

$F(x,y) := \big(T(x), \varphi(x) + \lambda(y - \frac{1}{2})\big)$. *Assume that the transversality condition holds. Suppose that* k_0 *is so large that the set* D_{k_0} *in Proposition* 1 *satisfies*

$$\dim_{\mathrm{H}}(D_{k_0}) > \dim_{\mathrm{H}}\big(\pi_1(\Phi(B))\big) \quad \textit{and} \tag{41}$$

$$2\, h_{\mathrm{top}}(T) < \frac{-\log\lambda}{\log \sup |T'|} \cdot h_{\mathrm{top}}\left(T\big|_{D_{k_0}}\right) , \tag{42}$$

where B *is the set defined in* (31). *Then we have that*

$$\dim_{\mathrm{H}}(\Lambda_{k_0}) \geq \dim_{\mathrm{H}}(D_{k_0}) + \frac{h_{\mathrm{top}}\left(T\big|_{D_{k_0}}\right)}{-\log\lambda} . \tag{43}$$

The proof of this proposition is divided into several lemmas. For the rest of this section we fix k_0 satisfying (41) and (42) and we write

$$s := s_{k_0} = \frac{h_{\mathrm{top}}\left(T|_{D_{k_0}}\right)}{-\log\lambda} .$$

Moreover, choose a λ_2 with $\sqrt{\lambda} < \lambda_2 < \frac{1}{\sup|T'|}$, such that

$$2\, h_{\mathrm{top}}(T) < \frac{\log\lambda}{\log\lambda_2} \cdot h_{\mathrm{top}}\left(T\big|_{D_{k_0}}\right) \tag{44}$$

holds, which is possible because of (42).

Lemma 6. *Under the assumptions of Proposition* 4 *we have*

$$\Pi_*\mu\big(\pi_1(\Phi(B)) \times [0,1]\big) = 0 .$$

Proof. Using the notations of the proof of Lemma 5 it is enough to prove that for every $u \geq 0$ we have

$$\Pi_*\mu\left(\big(T^u(\pi_1(B_1))\big) \times [0,1]\right) = 0 . \tag{45}$$

Let $N > u$. Then by (32) and (33) the set $T^u(\pi_1(B_1))$ can be covered by at most $(\#\mathcal{V}_{N-1})^2$ intervals, whose lengths are at most $d_1\,(\sup|T'|)^u\,\lambda^N$. We choose $q = q(N)$ such that $\lambda_2^q = \lambda^N$. Since by Proposition 1 every $[q]$-level cylinder of D_{k_0} has length at least $d\lambda_2^{[q]}$ we obtain that the number of $[q]$-level cylinders of D_{k_0} needed to cover $T^u(\pi_1(B_1)) \cap D_{k_0}$ is at most $\frac{d_1}{d}\,(\#\mathcal{V}_{N-1})^2\,(\sup|T'|)^u$. It follows from (36) that the μ-measure of such a $[q]$-level cylinder is at most $d_3\lambda^{[q]s}$. Hence

$$\begin{aligned} &\Pi_*\mu\left(\big(T^u(\pi_1(B_1))\big) \times [0,1]\right) \leq \\ &\quad \leq \frac{d_1 d_3}{d}\lambda^{-s}\,(\sup|T'|)^u \cdot (\#\mathcal{V}_{N-1})^2\,\lambda^{sN\log\lambda/\log\lambda_2} . \end{aligned} \tag{46}$$

Using that $\lambda^s = \exp\left(-h_{\mathrm{top}}(T|_{D_{k_0}})\right)$ by the choice of s, and also using the fact that

$$\lim_{n\to\infty} \frac{1}{n} \log \#\mathcal{V}_{n-1} = h_{\mathrm{top}}(T)$$

we obtain by (44) that the right hand side of (46) tends to zero. □

Corollary 1. *Under the assumptions of Proposition* 4 *we have*

$$\Pi_*\mu(\Phi(B)) = 0.$$

Now we are ready to prove a finer result. We have seen that the set of those point which have not unique coding is a set of $\Pi_*\mu$-measure zero. Now we will point out that most of those points which have only one code do have only the same code of the first n non-positive coordinates in a neighborhood of size close to the size of a horizontal n-cylinder.

Let $x \in \Lambda_{k_0} \setminus \Phi(B)$. Then there exists a unique $\mathbf{i} \in \Sigma_A$ such that $x = \Pi(\mathbf{i})$. We define

$$L(x) := \min\left\{\mathrm{dist}\,(x, \Pi(\mathbf{j})) : i_k = j_k\, \forall k > 0 \text{ and } i_0 \neq j_0\right\} .$$

The meaning of $L(x)$ is as follows: Let l be the vertical line which contains x. Then the symbolic representation of all points in the open interval on l of radius $L(x)$ centered at x has the same zero coordinate as the zero coordinate of $\Pi^{-1}(x)$.

It follows from Corollary 1 that the function L is defined for $\Pi_*\mu$-almost all points of Λ_{k_0}.

For $\varepsilon > 0$ and $n \in \mathbb{N}$ we define

$$O_n(\varepsilon) := \left\{\mathbf{i} \in \Sigma_A \setminus \Pi^{-1}\Phi(B) : L\left(\Pi(\sigma^{-n}\mathbf{i})\right) < e^{-\varepsilon n}\right\} . \tag{47}$$

Note that $\Sigma_A \setminus \Pi^{-1}\Phi(B)$ is the set of those $\mathbf{i} \in \Sigma_A$ for which there is a unique $I \in \mathcal{I}^F_\infty$ such that $\Pi(\mathbf{i}) \in I$. So, there is no overlap in Λ at $\Pi(\mathbf{i})$.

Lemma 7. *Under the assumptions of Proposition* 4 *we obtain that for every $\varepsilon \in \left(0, \log\sup|T'|\right)$ there exists an $\eta \in (0,1)$, a $d_* > 0$, and a sequence $\left(R_n(\varepsilon)\right)_{n\in\mathbb{N}}$ of subsets of Σ_A such that for every n we have*

(a) $O_n(\varepsilon) \subset R_n(\varepsilon)$,

(b) *$R_n(\varepsilon)$ is a union of non-positive $\left[\left(1 + \frac{\varepsilon}{-\log\lambda}\right) n\right]$-cylinders, and*

(c) $\mu(R_n(\varepsilon)) < d_*\eta^n$.

Proof. We assume that $n > (k_0+1)\frac{-\log\lambda}{\varepsilon}$, because it suffices to construct the sets $R_n(\varepsilon)$ in this case. Set $Q_n(\varepsilon) := \{\mathbf{i} \in \Sigma_A \setminus \Pi^{-1}\Phi(B) : L(\Pi\mathbf{i}) < e^{-\varepsilon n}\}$. Clearly,

$$O_n(\varepsilon) \subset \sigma^n(Q_n(\varepsilon)) \; . \tag{48}$$

We define m by

$$m := \left[\frac{\varepsilon}{-\log\lambda} n\right] \; .$$

Let $\mathcal{I}_m$ be the pairs of non-positive m-cylinders of Σ_A with different zero coordinates, this means

$$\mathcal{I}_m := \big\{(\tau,\omega) : \tau = [\tau_{-(m-1)},\dots,\tau_0]_0, \omega = [\omega_{-(m-1)},\dots,\omega_0]_0 \text{ admissible words, } \tau_0 \neq \omega_0\big\} \; .$$

The corresponding horizontal m-cylinders of Λ_{k_0} are

$$S_\tau := \Pi(\tau) \text{ and } S_\omega := \Pi(\omega).$$

Put

$$X_n(\omega) := \left\{y \in \Lambda_{k_0} : \exists\, x \in S_\omega \text{ such that } \pi_1(y) = \pi_1(x), \operatorname{dist}(x,y) < e^{-\varepsilon n}\right\} .$$

Then

$$Q_n(\varepsilon) \subset \bigcup_{(\omega,\tau)\in\mathcal{I}_m} \Pi^{-1}\left(X_n(\omega) \cap S_\tau\right) . \tag{49}$$

It follows from Lemma 4 that

$$\left|\pi_1\left(X_n(\omega)\cap S_\tau\right)\right| < 3d(k_0)\cdot\lambda^m \; . \tag{50}$$

Now we define k by

$$k := \left[\frac{\log\lambda}{\log\lambda_2} m\right] \; . \tag{51}$$

Then

$$\begin{gathered} k \leq \frac{\log\lambda}{\log\lambda_2}\cdot m < k+1 \; , \\ m \leq \frac{\varepsilon}{-\log\lambda}\cdot n < m+1 \; , \text{ and} \\ k \leq \frac{\varepsilon}{-\log\lambda_2}\cdot n < k+1+\frac{\log\lambda}{\log\lambda_2} \; . \end{gathered} \tag{52}$$

This implies that $k > m$, and by the choice of n we have $k > m > k_0$. Using also (50) we obtain that

$$|\pi_1 (X_n(\omega) \cap S_\tau)| < 3d(k_0) \cdot \lambda_2^k. \tag{53}$$

Set $\mathcal{V}_{k-1}(\omega,\tau) := \{V \in \mathcal{V}_{k-1} : V \cap \pi_1 (X_n(\omega) \cap S_\tau) \neq \emptyset\}$. Using that the length of every k-cylinder of D_{k_0} is at least $d\lambda_2^k$ by Proposition 1 and that $\pi_1 (X_n(\omega) \cap S_\tau) \subset D_{k_0}$ we obtain from (50) that

$$\#\mathcal{V}_{k-1}(\omega,\tau) \leq \frac{3d(k_0)}{d} \,. \tag{54}$$

For a $V \in \mathcal{V}_{k-1}$ we can find $v_1, \ldots, v_{k-k_0} \in \{1, \ldots, h\}$ such that $V \cap D_{k_0} = \pi_1\Pi\left([v_1, \ldots, v_{k-k_0}]_{k-k_0}\right)$. Recall that $S_\tau = \Pi\left([\tau_{-(m-1)}, \ldots, \tau_0]_0\right)$. It follows from (36) that

$$\mu[\tau_{-(m-1)}, \ldots, \tau_0, v_1, \ldots, v_{k-k_0}]_{k-k_0} \leq d_3\lambda^{-k_0 s} \cdot \lambda^{(k+m)s} \,. \tag{55}$$

We define the set $P_n(\varepsilon) \subset \Sigma_A$ by

$$P_n(\varepsilon) := \bigcup_{(\omega,\tau)\in\mathcal{I}_m} \quad \bigcup_{\substack{V\in\mathcal{V}_{k-1}(\omega,\tau) \\ V\cap D_{k_0}=\pi_1\Pi\left([v_1,\ldots,v_{k-k_0}]_{k-k_0}\right)}} \left[\tau_{-(m-1)}, \ldots, \tau_0, v_1, \ldots, v_{k-k_0}\right]_{k-k_0} \,.$$

Then clearly (use (49) and the definitions)

$$Q_n(\varepsilon) \subset P_n(\varepsilon) \,. \tag{56}$$

As $\varepsilon < \log\sup|T'| < -\log\lambda_2$ by the choice of ε and λ_2 we get by (52) that $k < n$, and therefore $k - k_0 - n < 0$. Hence it follows that the set

$$R_n(\varepsilon) := \sigma^n P_n(\varepsilon) =$$
$$= \bigcup_{(\omega,\tau)\in\mathcal{I}_m} \quad \bigcup_{\substack{V\in\mathcal{V}_{k-1}(\omega,\tau) \\ V\cap D_{k_0}=\pi_1\Pi\left([v_1,\ldots,v_{k-k_0}]_{k-k_0}\right)}} \left[\tau_{-(m-1)}, \ldots, \tau_0, v_1, \ldots, v_{k-k_0}\right]_{k-k_0-n}$$

can be represented as a union of horizontal (non-positive coordinates) $n+m$-cylinders. Using (52) we see that $n + m = \left[\left(1 + \frac{\varepsilon}{-\log\lambda}\right) n\right]$, showing (b). Furthermore, (48) and (56) imply that $O_n(\varepsilon) \subset R_n(\varepsilon)$, hence (a) is shown.

In order to prove (c) we have to verify that

$$\mu(R_n(\varepsilon)) = \mu(P_n(\varepsilon)) \to 0 \quad \text{exponentially fast.}$$

To see this note that it follows from (52), (54) and (55) that

$$
\begin{aligned}
\mu(P_n(\varepsilon)) &\le \sum_{(\omega,\tau)\in\mathcal{I}_m} \sum_{V\in\mathcal{V}_{k-1}(\omega,\tau)} \mu\left([\tau_{-(m-1)},\dots,\tau_0,v_1,\dots,v_{k-k_0}]_{k-k_0}\right) \\
&\le \frac{3d(k_0)}{d} d_3 \lambda^{-k_0 s}\cdot \left(\#\mathcal{V}_{m-1}\right)^2 \lambda^{(k+m)s} \\
&\le \frac{3d(k_0)}{d} d_3 \lambda^{-(k_0+1)s}\cdot \left(\#\mathcal{V}_{m-1}\right)^2 \lambda^{sm(1+\log\lambda/\log\lambda_2)} \,.
\end{aligned} \tag{57}
$$

Since $\lim_{r\to\infty}\frac{1}{r}\log\#\mathcal{V}_{r-1} = h_{\text{top}}(T)$, and since $n\to\infty$ implies $m\to\infty$ by (52), we get by (44) that $\frac{2}{m}\log\#\mathcal{V}_{m-1} < \frac{\log\lambda}{\log\lambda_2} h_{\text{top}}\left(T|_{D_{k_0}}\right)$ for all sufficiently large n. Therefore, using also the definition of s we obtain

$$
\begin{aligned}
&\frac{2}{m}\log\#\mathcal{V}_{m-1} + s\left(1+\frac{\log\lambda}{\log\lambda_2}\right)\log\lambda = \\
&= \frac{2}{m}\log\#\mathcal{V}_{m-1} - \frac{\log\lambda}{\log\lambda_2} h_{\text{top}}\left(T|_{D_{k_0}}\right) - h_{\text{top}}\left(T|_{D_{k_0}}\right) < \\
&< -h_{\text{top}}\left(T|_{D_{k_0}}\right) < 0
\end{aligned} \tag{58}
$$

for all sufficiently large n. If n is sufficiently large, then (52), (57) and (58) give

$$
\begin{aligned}
&\frac{1}{n}\log\mu(P_n(\varepsilon)) - \frac{1}{n}\log\left(\frac{3d(k_0)}{d} d_3\lambda^{-(k_0+1)s}\right) \le \\
&\le \frac{1}{n}\left(2\log\#\mathcal{V}_{m-1} + sm\left(1+\frac{\log\lambda}{\log\lambda_2}\right)\log\lambda\right) = \\
&= \frac{m}{n}\left(\frac{2}{m}\log\#\mathcal{V}_{m-1} + s\left(1+\frac{\log\lambda}{\log\lambda_2}\right)\log\lambda\right) \le \\
&\le \frac{m}{m+1}\cdot\frac{\varepsilon}{-\log\lambda}\cdot\left(\frac{2}{m}\log\#\mathcal{V}_{m-1} + s\left(1+\frac{\log\lambda}{\log\lambda_2}\right)\log\lambda\right) \le \\
&\le \frac{m}{m+1}\cdot\frac{\varepsilon}{-\log\lambda}\cdot\left(-h_{\text{top}}\left(T|_{D_{k_0}}\right)\right) \,.
\end{aligned}
$$

As $n\to\infty$ implies $m\to\infty$ by (52), we obtain that

$$
\limsup_{n\to\infty}\frac{1}{n}\log\mu(P_n(\varepsilon)) \le \frac{\varepsilon}{-\log\lambda}\left(-h_{\text{top}}\left(T|_{D_{k_0}}\right)\right) < 0 \,,
$$

which completes the proof of this lemma. □

Proof of Proposition 4. For the rest of the proof fix an $\varepsilon \in \big(0, \log\sup|T'|\big)$. We need some further notation. For an $\mathbf{i} \in \Sigma_A$ we write

$$\mathbf{i}^- := \{\mathbf{j} \in \Sigma_A : j_k = i_k \text{ for every } k \leq 0\} \ .$$

This corresponds to the unstable fibre which contains $\Pi(\mathbf{i})$. Geometrically the projection $\Pi(\mathbf{i}^-)$ is the almost horizontal line above D_{k_0} of the attractor which contains $\Pi(\mathbf{i})$.

If $p \in D_{k_0}$ define the segment $l_p := \{p\} \times [0,1]$. For every $p \in D_{k_0}$ we have

$$\mu\left\{\mathbf{i} \in \Sigma_A : p \in \pi_1\Pi(\mathbf{i}^-)\right\} > 0 \ , \tag{59}$$

because choosing $\mathbf{j} = (\dots, j_0, j_1, j_2, \dots)$ such that $p = \pi_1\Pi(\mathbf{j})$, the set

$$\left\{\mathbf{i} \in \Sigma_A : p \in \pi_1\Pi(\mathbf{i}^-)\right\} = \{\mathbf{i} \in \Sigma_A : a_{i_0,j_1} = 1\}$$

has positive μ-measure.

Set

$$G := \left\{\mathbf{i} \in \Sigma_A \setminus \Pi^{-1}\Phi(B) : \exists\, N(\mathbf{i}) \text{ such that } \mathbf{i} \notin R_n(\varepsilon) \text{ for all } n \geq N(\mathbf{i})\right\} \ ,$$

where $R_n(\varepsilon)$ is as in Lemma 7. It follows from Corollary 1, from Lemma 7 and from the Borel Cantelli lemma that

$$\mu(G) = 1 \ . \tag{60}$$

Fix an arbitrary $p \in D_{k_0} \setminus \pi_1(\Phi(B))$. Set

$$G_p := \left\{\mathbf{i} \in G : p \in \pi_1\Pi(\mathbf{i}^-)\right\} \ .$$

Using (b) of Lemma 7 we obtain that for any $\mathbf{i}, \mathbf{j} \in \Sigma_A$ with $\mathbf{i}^- = \mathbf{j}^-$ we have that either both $\mathbf{i}$ and $\mathbf{j}$ are contained in G_p or neither of them is contained in G_p. Now we obtain from (59) that

$$\mu(G_p) > 0 \ . \tag{61}$$

We define the measure ν_p on the segment l_p as follows: for a Borel set $H \subset [0,1]$ define

$$\nu_p\big(\{p\} \times H\big) := \mu\left\{\mathbf{i} \in G_p : \pi_2(\Pi(\mathbf{i}^-) \cap l_p) \in H\right\} \ .$$

From (61) we see that $\nu_p\big(\{p\} \times [0,1]\big) > 0$. By the considerations above, the set of all $x \in l_p$ such that there exists an $\mathbf{i} \in G_p$ with $\Pi(\mathbf{i}) = x$ forms a set of full ν_p-measure.

Now we fix an arbitrary $\mathbf{i} \in G_p$ such that $p = \pi_1 \Pi(\mathbf{i})$. Put $x := \Pi(\mathbf{i})$. Then by the definition of G we have that there exists an $N(\mathbf{i})$ such that for all $n \geq N(\mathbf{i})$ we have $\mathbf{i} \notin R_n(\varepsilon)$. For $n \in \mathbb{N}$ set

$$\rho_n := \max\left\{ r : B(x,r) \cap l_p \cap \Lambda_{k_0} \subset \Pi[i_{-(n-1)}, \ldots, i_0]_0 \right\} .$$

Then for every n we have $\rho_n > 0$ by the choice of p. In this way

$$\nu_p\big(B(x,\rho_n) \cap l_p\big) \leq \mu\big([i_{-(n-1)}, \ldots, i_0]_0\big) . \tag{62}$$

Note that obviously $\lim_{n\to\infty} \rho_n = 0$.

Our next step is to estimate the magnitude of ρ_n. As $\mathbf{i} \notin \Pi^{-1}\Phi(B)$ we have that $\min_{0\leq k \leq N(\mathbf{i})} L(F^{-k}(x)) > 0$. Therefore, using also (a) of Lemma 7 and (47), we can find $N_1 \geq N(\mathbf{i})$ such that $\min_{0\leq k\leq n} L(F^{-k}(x)) \geq e^{-\varepsilon n}$ for all $n > N_1$. Hence by definition we get

$$\rho_n \geq e^{-\varepsilon n} \lambda^n \tag{63}$$

for all $n \geq N_1$.

By Frostman's Lemma we get that ν_p- ess sup $\liminf_{r\to 0} \frac{\log \nu_p(B(x,r)\cap l_p)}{\log r}$ is a lower bound for the Hausdorff dimension of $\Lambda_{k_0} \cap l_p$. To use this we recall that for a ν_p-typical $x \in l_p$ there exists an $\mathbf{i} \in G_p$ with $\Pi(\mathbf{i}) = x$. Hence using (36), (62) and (63) we obtain

$$\lim_{n\to\infty} \frac{\log \nu_p(B(x,\rho_n) \cap l_p)}{\log \rho_n} \geq \lim_{n\to\infty} \frac{-\log \mu[i_{-(n-1)}, \ldots, i_0]_0}{\varepsilon n - n \log \lambda} =$$
$$= \lim_{n\to\infty} \frac{\frac{-\log \mu[i_{-(n-1)},\ldots,i_0]_0}{-n\log\lambda}}{1 - \frac{\varepsilon}{\log \lambda}} \geq \frac{s}{1 - \frac{\varepsilon}{\log\lambda}} .$$

As $\varepsilon \in \big(0, \log \sup |T'|\big)$ was arbitrary, the inequality above implies that $\dim_{\mathrm{H}}(\Lambda_{k_0} \cap l_p) \geq s$ holds for all $p \in D_{k_0} \setminus \pi_1(\Phi(B))$. Using (41) it follows from Theorem 5.8 in [2] that $\dim_{\mathrm{H}}(\Lambda_{k_0}) \geq \dim_{\mathrm{H}}(D_{k_0}) + s$, completing the proof of the proposition. □

7. The proof of Theorem 2

Before we give the proof of Theorem 2 we show that under the assumption of the transversality condition a more general result holds.

Proposition 5. *Let $T : [a,b] \to [a,b]$ be a piecewise monotonic map with critical points $c_0 := a < c_1 < c_2 < \cdots < c_{m_0} := b$, which is expanding and topologically transitive. Suppose that $\lambda \in \left(0, \min\left\{\frac{1}{m_0}, \inf \frac{1}{(T')^2}\right\}\right)$,* *and suppose that* $\varphi : [a,b] \to \left[\frac{\lambda}{2}, 1-\frac{\lambda}{2}\right]$ *is a function, which is continuously differentiable on* $(a,b) \setminus \{c_0, c_1, \ldots, c_{m_0}\}$ *and satisfies* $\inf|\varphi'| > 0$ *and* $\sup|\varphi'| < \infty$*. Define* $Q := [a,b] \times [0,1]$ *and define* $F : Q \to Q$ *by* $F(x,y) := \big(T(x), \varphi(x) + \lambda(y - \frac{1}{2})\big)$*. Assume that the transversality condition holds. Then the attractor* $\Lambda := \bigcap_{n=0}^{\infty} F^n(Q)$ *satisfies*

$$\dim_{\mathrm{H}}(\Lambda) = 1 + \frac{h_{\mathrm{top}}(T)}{-\log\lambda} .$$

Proof. The upper estimate follows immediately from Proposition 3.

Let $\varepsilon > 0$ be arbitrary. As the transversality condition holds Lemma 5 gives $\dim_{\mathrm{H}}\big(\pi_1(\Phi(B))\big) < 1$, where B is the set defined in (31). Observing that $\frac{-\log\lambda}{\log\sup|T'|} > 2$ as $\lambda < \inf\frac{1}{(T')^2}$ we obtain by Proposition 1 that there exists a $k_0 \geq 2$ such that

$$\dim_{\mathrm{H}}(D_{k_0}) > \max\left\{1-\varepsilon, \dim_{\mathrm{H}}\big(\pi_1(\Phi(B))\big)\right\} \quad \text{and}$$
$$h_{\mathrm{top}}\left(T\big|_{D_{k_0}}\right) > \max\left\{h_{\mathrm{top}}(T) - \varepsilon, 2\frac{\log\sup|T'|}{-\log\lambda}h_{\mathrm{top}}(T)\right\} .$$

Hence (41) and (42) are satisfied. Now (37) and Proposition 4 give

$$\dim_{\mathrm{H}}(\Lambda) \geq \dim_{\mathrm{H}}(\Lambda_{k_0}) \geq \dim_{\mathrm{H}}(D_{k_0}) + \frac{h_{\mathrm{top}}\left(T\big|_{D_{k_0}}\right)}{-\log\lambda} >$$
$$> 1 - \varepsilon + \frac{h_{\mathrm{top}}(T) - \varepsilon}{-\log\lambda} .$$

Since $\varepsilon > 0$ was arbitrary this gives $\dim_{\mathrm{H}}(\Lambda) \geq 1 + \frac{h_{\mathrm{top}}(T)}{-\log\lambda}$, which completes the proof. □

Finally we prove Theorem 2.

Proof of Theorem 2. By Proposition 2 we get that the transversality condition holds. As $m_0 = 2$ all assumptions of Proposition 5 are satisfied. Therefore $\dim_{\mathrm{H}}(\Lambda) = 1 + \frac{h_{\mathrm{top}}(T)}{-\log\lambda}$ by Proposition 5. □

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