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Stability of the omega limit set for unimodal transformations

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Abstract. Consider the map $T \mapsto \omega(T)$ on the collection of unimodal transformations, where $\omega(T)$ is the ω-limit set of T. The collection of unimodal transformations is endowed with the topology of uniform convergence, and the collection of subsets of $[0,1]$ is endowed with the Hausdorff metric. Conditions on a unimodal transformation T implying the continuity of the map $S \mapsto \omega(S)$ at T are investigated.

Introduction

Let $T : [0,1] \to [0,1]$ be a continuous transformation. For a point $x \in [0,1]$ let $\omega(x)$ be the set of all limit points of the sequence $\big(T^n(x)\big)_{n\in\mathbb{N}}$. We call $\omega(T) := \bigcup_{x\in[0,1]} \omega(x)$ the *ω-limit set* of the transformation T. It is proved in [10] that $\omega(T)$ is closed. Let $\mathcal{F}$ be the collection of all continuous transformations on $[0,1]$, and endow this space with the uniform metric ϱ. Moreover, let $\mathcal{C}$ be the collection of all closed subsets of $[0,1]$ endowed with the Hausdorff metric d. We consider the map $\omega : \mathcal{F} \to \mathcal{C}$ which assigns to each $S \in \mathcal{F}$ its ω-limit set $\omega(S)$. In this paper we investigate the continuity of the map ω on certain subsets of $\mathcal{F}$.

A transformation $T : [0,1] \to [0,1]$ is called *piecewise monotonic*, if there is a finite partition $\mathcal{Z}$ of $[0,1]$ into intervals, such that $T|_Z$ is strictly monotonic and continuous for all $Z \in \mathcal{Z}$. If c is an endpoint of an interval of monotonicity and $c \notin \{0,1\}$, then c is called a *critical point* of T. Denote by $\mathcal{M}$ the collection of all continuous piecewise monotonic transformations. The

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finite partition $\mathcal{Z}$ is called a *generator*, if for every sequence $(Z_j)_{j\in\mathbb{N}_0}$ in $\mathcal{Z}$ the set $\bigcap_{j=0}^{\infty} \overline{T^{-j}Z_j}$ contains at most one element. If a map $T : [0,1] \to [0,1]$ is piecewise monotonic with respect to a finite partition $\mathcal{Z}$, and if $\mathcal{Z}$ is a generator, then we call T piecewise monotonic with *generating partition*.

We call a continuous transformation $T : [0,1] \to [0,1]$ *unimodal*, if there exists a point $c \in (0,1)$ such that T is strictly increasing on $[0,c]$ and strictly decreasing on $[c,1]$. The point c is the unique critical point of T. Then $\mathcal{Z} := \big\{[0,c],(c,1]\big\}$ is a finite partition of $[0,1]$ and T is piecewise monotonic with respect to $\mathcal{Z}$. We denote the collection of all unimodal transformations by $\mathcal{U}$. Obviously we have $\mathcal{U} \subseteq \mathcal{M} \subseteq \mathcal{F}$.

For a transformation $T : [0,1] \to [0,1]$ a point $p \in [0,1]$ is called *periodic* (sometimes T-*periodic*), if $T^n(p) = p$ for some $n \geq 1$. The smallest $n \geq 1$ satisfying $T^n(p) = p$ is called the *period* of p, and it is denoted by $\mathrm{per}(p)$. Let p be a periodic point of a transformation T with period n. We say that p is *transversal*, if the graph of the map T^n crosses the diagonal at the point (p,p). Otherwise, we call p *nontransversal*.

If there are two different $n_1, n_2 \in \mathbb{N}$ with $T^{n_1}(p) = T^{n_2}(p)$, then p is called *eventually periodic*. This is equivalent to the property that $\{T^n(p) : n \geq 0\}$ (the orbit of p) is finite. Moreover, in this case we get that $T^k(p)$ is periodic for some $k \geq 0$.

In general the map ω is neither lower semicontinuous nor upper semicontinuous. We consider the lower semicontinuity of the map $\omega : \mathcal{F} \to \mathcal{C}$ in Section 1. We show the following result.

Theorem. *Suppose that $T \in \mathcal{M}$ and that all periodic points of T, which are separated from periodic points of different period, are transversal. Then the map $\omega : \mathcal{F} \to \mathcal{C}$ is lower semicontinuous at T.*

If the transformation $T \in \mathcal{M}$ has a generating partition and the critical points of T are not periodic, then the condition in the theorem above is satisfied. Therefore we obtain the lower semicontinuity of the map ω in this case.

The rest of the paper deals with unimodal transformations. We fix a transformation $T \in \mathcal{U}$, which has a generating partition. Let c be the critical point of T. We consider the restriction of T to the interval $[T^2(c), T(c)]$, which is T-invariant, and denote now the ω-limit set of this restricted transformation by $\omega(T)$. Doing this we loose at most one ω-limit point, which is the point 0, in case it is a fixed point. Our aim is then to characterize the continuity of the map $\omega : \mathcal{U} \to \mathcal{C}$ at T by a condition on the orbit of the critical point under T. Writing $x \sim y$, if the critical point c is not contained in the open interval with endpoints x and y, we have the following result.

Theorem. *Let $T : [0,1] \to [0,1]$ be a unimodal transformation with critical point c. Suppose that T has a generating partition. Then the map $\omega : \mathcal{U} \to \mathcal{C}$ is not continuous at T if and only if there is n with $T^j(c) \sim T^{n+j}(c)$ for $1 \le j \le n-1$ and $T^n(c) \ne T^{2n}(c)$, but $T^{n+1}(c)$ is a periodic point of period n.*

We can apply this theorem to a unimodal transformations T with negative Schwarzian derivative. It is well known, that T has either a generating partition or the orbit of the critical point is attracted by a periodic orbit. In the first case we can apply the above theorem. It is explained at the end of Section 5 how one can show that the map $\omega : \mathcal{U} \to \mathcal{C}$ is continuous at T, if the periodic orbit of T, which attracts the orbit of the critical point, is attracting on both sides, and that the map ω is not continuous at T, if this periodic orbit is attracting on one side and repelling on the other side.

The stability of the ω-limit set has also been investigated in [11] and [12] for the collection $\mathcal{F}$ of all continuous transformations on the interval (also endowed with the uniform metric). In these papers it is proved that the function $\omega : \mathcal{F} \to \mathcal{C}$ is continuous at f, if and only if the closure of the periodic points of f equals the chain recurrent points of f. This is a characterization of rather different kind.

A similar question has been dealt with in [1]. The continuity of the map $x \mapsto \omega(x)$ as a map $[0,1] \to \mathcal{C}$ has been investigated in this paper. Properties of the map $(T,x) \mapsto \omega_T(x)$ as a map $\mathcal{F} \times [0,1] \to \mathcal{C}$ have been investigated in [13], where $\omega_T(x)$ denotes the ω-limit of x with respect to the transformation T (i.e. the set of all limit points of $\big(T^n(x)\big)_{n\in\mathbb{N}}$). However, the maps considered in these papers are slightly different from the map ω considered here.

In [8] the stability of maximal topologically transitive subsets of a piecewise monotonic transformation T under small perturbations of T is investigated. However, this question is different from the problem considered in this paper. Only for topologically transitive transformations T we obtain directly from Theorem 2 in [8] (see also Theorem 3 in [9]) that the map $\omega : \mathcal{M} \to \mathcal{C}$ is continuous at T.

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1. Lower semicontinuity for continuous transformations

Assume that T is a continuous piecewise monotonic transformation, which means $T \in \mathcal{M}$. In this section we investigate the lower semicontinuity of the function $\omega : \mathcal{F} \to \mathcal{C}$ at T. Here lower semicontinuity at T means that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $S \in \mathcal{F}$ with $\varrho(S,T) < \delta$ there is a set $G \subseteq \omega(S)$ with $d\big(G, \omega(T)\big) < \varepsilon$. The following simple examples show that lower semicontinuity cannot be true in general.

First, for $s \in [0, \frac{1}{2}]$, we consider the map $T_s : [0,1] \to [0,1]$ defined by $T_s x := (1-s)x^2$. Obviously $\lim_{s\to 0+} \|T_s - T_0\|_\infty = 0$, $\omega(T_0) = \{0,1\}$ and $\omega(T_s) = \{0\}$ for $s > 0$. This shows that the map $S \mapsto \omega(S)$ is not lower semicontinuous at T_0. Here 1 is an isolated fixed point of T_0 which is not attracting.

For our second example we define for $s \in [0, \frac{1}{2}]$ the map $T_s : [0,1] \to [0,1]$ by $T_s x := (1-s)x$. Then $\lim_{s\to 0+} \|T_s - T_0\|_\infty = 0$. As $\omega(T_0) = [0,1]$ and $\omega(T_s) = \{0\}$ for $s > 0$ also in this case the map $S \mapsto \omega(S)$ is not lower semicontinuous at T_0. In this example the non-attracting fixed point 1 is not isolated in the set of periodic points of T_0, but it is separated from those periodic points having a period different from 1.

Assume that $\varepsilon > 0$. A T-periodic point p is called *ε-persistent*, if there is a $\delta > 0$, such that every $S \in \mathcal{F}$ with $\varrho(T,S) < \delta$ has an S-periodic point q with $|p - q| < \varepsilon$.

It is obvious from these definitions that every transversal periodic point p of a transformation T is ε-persistent for every $\varepsilon > 0$.

Lemma 1. *Assume that $T \in \mathcal{M}$ and suppose that T has no nontransversal periodic point p, which is separated from the set of all periodic points q with* $\operatorname{per}(q) \neq \operatorname{per}(p)$. *Then for every $\varepsilon > 0$ the transformation T has at most finitely many periodic points, which are not ε-persistent.*

Proof. Let K be the finite set of endpoints of intervals on which T is monotonic and set $L := \bigcup_{j=0}^{\infty} T^{-j}(K)$. Then L is the set of endpoints of maximal intervals, on which some iterate of T is monotonic. As for every $x \in L$ there is a $j \geq 0$ with $T^j(x) \in K$, and as K is finite, there are at most finitely many T-periodic points in L.

Fix $\varepsilon > 0$. Suppose that $p \in [0,1] \setminus L$ is a periodic point of T. Set $k := \operatorname{per}(p)$. Since $p \notin L$ there exists a maximal open interval I containing p, on which T^k is monotonic. Now set $U_p := (p - \varepsilon, p + \varepsilon) \cap I$.

Suppose that U_p contains a point u with $T^k(u) > u$ and a point v with $T^k(v) < v$. Then there is $\delta > 0$ such that $S^k(u) > u$ and $S^k(v) < v$ hold for every $S \in \mathcal{M}$ satisfying $\varrho(T,S) < \delta$. If $S \in \mathcal{M}$ and $\varrho(T,S) < \delta$ then $S^k(u) > u$ and $S^k(v) < v$ imply the existence of a point $q \in U_p$ satisfying $S^k(q) = q$. Therefore p is ε-persistent.

Now we assume that $T^k(x) \leq x$ holds for all $x \in U_p$. Set $U_p^+ := (p, p + \varepsilon) \cap I$. If the open interval U_p^+ contains a point r with $T^k(r) = r$, then r is a T-periodic point and $\operatorname{per}(r)$ divides k. In this case choose $r_0 \in U_p^+$ with

$T^k(r_0) = r_0$ such that $\operatorname{per}(r_0)$ equals $\max\{\operatorname{per}(r) : r \in U_p^+, T^k(r) = r\}$. If r_0 is transversal, then r_0 is $(\varepsilon - (r_0 - p))$-persistent. Therefore there is a $\delta > 0$ such that for every $S \in \mathcal{M}$ with $\varrho(T, S) < \delta$ there is a S-periodic point q with $|q - r_0| < \varepsilon - (r_0 - p)$, which implies $|q - p| < \varepsilon$. Hence p is ε-persistent in this case. Otherwise r_0 is nontransversal. By our assumption r_0 can be approximated by periodic points r with $\operatorname{per}(r) \neq \operatorname{per}(r_0)$. As U_p^+ is open and the period of periodic points in U_p^+ is at most $\operatorname{per}(r_0)$ there is an $m < \operatorname{per}(r_0)$ such that r_0 can be approximated by points r with $\operatorname{per}(r) = m$. Because of $T^m(r) = r$ the continuity of T^m implies $T^m(r_0) = r_0$. This contradicts $m < \operatorname{per}(r_0)$. Therefore U_p^+ cannot contain nontransversal periodic points. It remains to consider the case $T^k(x) < x$ for all $x \in U_p^+$. In this case $\lim_{n\to\infty} T^{nk}(x) = p$ for all $x \in U_p^+$. Hence p attracts an interval of length ε or an endpoint of I and therefore a point in K.

A similar argument works, if $T^k(x) \geq x$ holds for all $x \in U_p$. Therefore we obtain that p is ε-persistent or p attracts an interval of length ε or p attracts a point in K. Obviously there are at most finitely many T-periodic points attracting an interval of length ε and there are at most finitely many T-periodic points attracting a point in K. Since there are also at most finitely many T-periodic points in L, we have shown that there are at most finitely many T-periodic points, which are not ε-persistent. □

Theorem 1. *Suppose that $T \in \mathcal{M}$ and that T has no nontransversal periodic point, which is separated from periodic points of different period. Then the map $\omega : \mathcal{F} \to \mathcal{C}$ is lower semicontinuous at T.*

Proof. Fix $\varepsilon > 0$. By [7] the periodic points are dense in $\omega(T)$. Therefore Lemma 1 implies that there is a finite set F of $\frac{\varepsilon}{2}$-persistent T-periodic points with $d\big(F, \omega(T)\big) < \frac{\varepsilon}{2}$. Hence there is a $\delta > 0$ such that $\varrho(T, S) < \delta$ implies the existence of a finite set G of S-periodic points with $d(G, F) < \frac{\varepsilon}{2}$. This implies $d\big(G, \omega(T)\big) < \varepsilon$. As $G \subseteq \omega(S)$ this shows the lower semicontinuity of $\omega : \mathcal{F} \to \mathcal{C}$ at T. □

2. Unimodal transformations

In this section we collect some results about unimodal transformations we shall need later. Let c be the critical point of the unimodal transformation T. We assume that the partition $\big\{[0, c], (c, 1]\big\}$ is a generator for T. This implies that $T(c) > c$. If 0 is not a fixed point, then $\omega(T)$ is contained in $[T^2(c), T(c)]$. In the case that 0 is a fixed point and $T^2(c) \neq 0$, then 0 is the only point in $\omega(T)$ which is not contained in $[T^2(c), T(c)]$ (this fixed point can be destroyed by arbitrary small perturbations). Therefore we restrict

T to the interval $[T^2(c), T(c)]$. Then the partition $\{[T^2(c), c], (c, T(c)]\}$ is a generator for this restriction. For unimodal transformations $\omega(T)$ denotes now the set of ω-limit points contained in $[T^2(c), T(c)]$.

Next we describe the structure of the set $\omega(T)$. We write $x \sim y$, if x and y are either both $\leq c$ or both $\geq c$. The closed interval with the endpoints a and b is denoted by $[a, b]$, even if $b < a$. A subset K of $[T^2(c), T(c)]$ is called a *fundamental set*, if there is $n \in \mathbb{N}$ such that

(a) $K = \bigcup_{j=1}^{n} [T^j(c), T^{n+j}(c)]$, where the interiors of these intervals are pairwise disjoint, and

(b) $T^j(c) \sim T^{n+j}(c)$ for $1 \leq j \leq n-1$, $T^n(c) \not\sim T^{2n}(c)$ and $T^{n+1}(c) \leq T^{2n+1}(c) \leq T(c)$.

In this case, for $1 \leq j \leq n-1$ the interval $[T^j(c), T^{n+j}(c)]$ is mapped monotonically onto the interval $[T^{j+1}(c), T^{n+j+1}(c)]$. Moreover, we have that the two intervals $[T^n(c), c]$ and $[T^{2n}(c), c]$ are mapped monotonically to the interval $[T^{n+1}(c), T(c)]$ (the second one may be not onto). In particular we have $T(K) \subseteq K$.

If G and H are T-invariant sets which are finite unions of intervals, and if $H \subsetneqq G$, then define

$$B_T(G, H) := \bigcap_{j=0}^{\infty} \overline{G \setminus T^{-j} H} \,. \tag{1}$$

These sets are used to desribe the ω-limit set of a unimodal transformation. We have the following well known result.

Proposition 1. *Assume that T is a unimodal transformation with generating partition. Then there is a sequence*

$$[T^2(c), T(c)] = K_1 \supsetneqq K_2 \supsetneqq K_3 \supsetneqq \cdots \tag{2}$$

of fundamental sets, which ends either with K_q for some integer $q \geq 1$, or is infinite. In the first case $\omega(T) = K_q \cup \bigcup_{j=1}^{q-1} B_T(K_j, K_{j+1})$, and in the second case $\omega(T) = B_\infty \cup \bigcup_{j=1}^{\infty} B_T(K_j, K_{j+1})$ where $B_\infty := \bigcap_{j=1}^{\infty} K_j$ is a Cantor set.

The proof of this proposition can be found in several papers (see for example [6]). We give some remarks on the proof, following [3] and [4]. The tool used there is an oriented graph called Markov diagram, whose paths correspond to the orbits of the unimodal transformation T. Let c be its critical point. We define integers $r_1, r_2, \ldots$ and $S_0, S_1, S_2, \ldots$ in the following way. Set $r_1 := 1$ and $S_0 := 1$. If r_j and S_{j-1} are defined, we set $S_j := S_{j-1} + r_j$

and define r_{j+1} such that $T^{S_j+k}(c) \sim T^k(c)$ for $1 \le k < r_{j+1}$ and $T^{S_j+k}(c) \not\sim T^k(c)$ for $k = r_{j+1}$. In the case of $T^{S_j+k}(c) \sim T^k(c)$ for all $k \ge 1$ we set $r_{j+1} := \infty$, and do not define r_u for $u > j+1$ (hence $S_{j+1} = \infty$ and S_u is not defined for $u > j+1$). For $n \ge 1$ define $J_n := [T^{n+1}(c), T^{n+1-S_l}(c)]$, where l is such that $S_l \le n < S_{l+1}$. The oriented graph $\mathcal{D} = \{J_n : n \ge 1\}$ with arrows $J_n \to J_{n+1}$ for $n \ge 1$ and $J_{S_m-1} \to J_{r_m}$ for $m \ge 1$ is called the Markov diagram of the unimodal transformation T. We call $I_0 I_1 I_2 \dots$ a *path* in the Markov diagram, if $I_n \in \mathcal{D}$ and $I_n \to I_{n+1}$ for all $n \in \mathbb{N}_0$.

In [3] and [4] the numbers $R_j = S_j - 1$ are used instead of S_j and the proofs are given for a shift space which is conjugate to T. This shift space is characterized by the expansion $e_1 e_2 e_3 \dots$ of $T(c)$, which is defined by $e_j = 0$, if $T^j(c) < c$ and $e_j = 1$, if $T^j(c) > c$ (for simplicity we exclude the case that c is periodic). Set $0' = 1$ and $1' = 0$. Then the numbers $r_1, r_2, \dots$ and $S_0, S_1, S_2, \dots$ are uniquely determined by $r_1 = S_0 = 1$ and

$$e_{S_j+1} e_{S_j+2} \dots e_{S_{j+1}-1} e_{S_{j+1}} = e_1 e_2 \dots e_{r_{j+1}-1} e'_{r_{j+1}} \quad \text{for } j \ge 0.$$

A subset of the oriented graph $\mathcal{D}$ is called closed, if there is no arrow from a vertex in this subset to a vertex which is outside of this subset. By Lemma 1 in [4] for every $j \ge 1$ there is $m \ge 0$ with $r_j = S_m$. Therefore, a closed subset of $\mathcal{D}$ is of the form $\{J_k, J_{k+1}, \dots\}$ with $k = S_m$ for some $m \ge 0$. Let $\mathcal{D}_1 = \mathcal{D} \supsetneqq \mathcal{D}_2 \supsetneqq \mathcal{D}_3 \supsetneqq \cdots$ be the closed subsets of $\mathcal{D}$. It may happen that $\mathcal{D}_1$ is the only one, but there may be also infinitely many. For $j \ge 1$ set $K_j = \bigcup_{I \in \mathcal{D}_j} I$. If m is such that $\mathcal{D}_j = \{J_{S_m}, J_{S_m+1}, \dots\}$, then $K_j = J_{S_m} \cup J_{S_m+1} \cup \dots \cup J_{S_{m+1}-1}$ and $J_{S_{m+1}} \subseteq J_{S_m}$ by Lemma 5 in [5]. Furthermore, $r_{m+1} = S_m$ and S_m divides r_l for $l \ge m+1$. Therefore, the graph restricted to $\mathcal{D}_j$ has period S_m (see Lemma 4 in [5]). This implies that the intervals J_i with $S_m \le i < S_{m+1}$ have pairwise disjoint interiors. If there was a point x in the interior of two of these intervals, the orbit of x could be represented by two different paths in the graph starting at vertices J_u and J_v with $S_m \le u < v < S_{m+1}$ (we say that the orbit of x is represented by a path $I_0 I_1 I_2 \dots$ in the Markov diagram, if $T^j(x) \in I_j$ for $j \ge 0$). By Theorem 1 in [5] these two paths must be the same except for a finite initial segment. But this is not possible, since the graph restricted to $\mathcal{D}_j$ has period S_m. Hence such a point x cannot exist. Therefore we have shown that K_j satisfies the definition of a fundamental set, since K_j equals $\bigcup_{j=1}^n [T^j(c), T^{n+j}(c)]$ with $n = S_m = r_{m+1}$.

Because of $\mathcal{D}_1 \supsetneqq \mathcal{D}_2 \supsetneqq \mathcal{D}_3 \supsetneqq \cdots$ we have $K_1 \supsetneqq K_2 \supsetneqq K_3 \supsetneqq \cdots$, and because of $\mathcal{D}_1 = \mathcal{D}$ and $S_0 = r_1 = 1$ we get $K_1 = [T^2(c), T(c)]$. The results about $\omega(T)$ in Proposition 1 follow now easily from the results in [5].

The following lemma shows a stronger result about the disjointness of the intervals, of which fundamental sets consists.

Lemma 2. *Let K be a fundamental set. Then there is $n \geq 1$, such that the boundary ∂K of K consists of the points $T(c), T^2(c), \dots, T^{2n}(c)$, which are different, and $T^{2n+1}(c)$ is in the interior of K, if and only if ∂K is not invariant under T.*

Proof. We have $K = \bigcup_{j=1}^{n} [T^j(c), T^{n+j}(c)]$ for some n by definition and the intervals $[T^j(c), T^{n+j}(c)]$ for $1 \leq j \leq n$ have disjoint interiors. Notice that $T^{2n+1}(c) \neq T(c)$, because otherwise we would have $T^{2n}(c) = c$ contradicting $T^n(c) \not\sim T^{2n}(c)$.

Suppose first that $T^{n+1}(c) < T^{2n+1}(c)$. If now $T^k(c) = T^l(c)$ holds with $1 \leq k < l \leq 2n+1$ then $T^r(c) = T^{2n+1}(c)$ with $r = 2n+1+k-l \leq 2n$. We have also $r \geq 2$, since $r = 1$ implies $k = 1$ and $l = 2n+1$ contradicting $T^{2n+1}(c) \neq T(c)$, and $r \neq n+1$ because of $T^{n+1}(c) < T^{2n+1}(c)$. It follows that the intervals $[T^j(c), T^{n+j}(c)]$ and $[T^{n+1}(c), T(c)]$ have nondisjoint interiors, where $j = r$, if $2 \leq r \leq n$, and $j = r - n$, if $n+2 \leq r \leq 2n$. This is again a contradiction. We have shown that the points $T(c), T^2(c), \dots, T^{2n}(c)$ are different and therefore form the boundary of K, and that $T^{2n+1}(c)$ is in the interior of K because of $T^{n+1}(c) < T^{2n+1}(c) < T(c)$. Hence the lemma is proved, if $T^{n+1}(c) < T^{2n+1}(c)$.

Now suppose that $T^{n+1}(c) = T^{2n+1}(c)$. If we assume that the points $T(c), T^2(c), \dots, T^{2n}(c)$ are different, then they form the boundary of K. Since ∂K is invariant under T the lemma is proved in this case. Hence suppose that $T^k(c) = T^l(c)$ holds with $1 \leq k < l \leq 2n$. This implies $T^s(c) = T^{2n}(c)$ with $s = 2n + k - l \leq 2n - 1$ and $s \geq 1$. The definition of a fundamental set implies $T^n(c) \neq T^{2n}(c)$ and hence $s \neq n$. Let p be the period of the periodic point $T^{n+1}(c)$. Then p divides n and $n - s$ which implies $n = dp$ with $d \in \{2, 3, \dots\}$. For $d \geq 3$ at least three of the intervals $[T^j(c), T^{n+j}(c)]$ with $1 \leq j \leq n$ would have a common endpoint, which is a contradiction to the disjointness of the interiors of these intervals. Therefore we have $d = 2$. For $1 \leq j \leq p$ we get $T^{n+j}(c) = T^{n+p+j}(c)$ and $[T^j(c), T^{n+j}(c)] \cup [T^{n+p+j}(c), T^{p+j}(c)] = [T^j(c), T^{p+j}(c)]$, since the interiors of the intervals $[T^j(c), T^{n+j}(c)]$ and $[T^{n+p+j}(c), T^{p+j}(c)]$ are disjoint. We get $K = \bigcup_{j=1}^{p} [T^j(c), T^{p+j}(c)]$. Since p is the period of the periodic point $T^{2p+1}(c)$ and $T^j(c) \neq T^{p+j}(c)$ for $1 \leq j \leq p$ the points $T(c), T^2(c), \dots, T^{2p}(c)$ are different and therefore form the boundary of K. Furthermore, $T^{2p+1}(c)$ is in the interior of the interval $[T(c), T^{p+1}(c)]$ and hence in the interior of K. Hence the lemma is proved with p in the place of n. □

3. Stable behaviour of unimodal transformations

Consider a unimodal transformation with generating partition. In this section we find conditions under which the sets occurring in Proposition 1 are stable. We need information about the stability of fundamental sets and their inverse images. The boundary of a fundamental set is contained in the orbit of the critical point. Hence we have to investigate the stability of points in this orbit and of inverse images of these points. This is done in the next two lemmas.

Lemma 3. *Fix $m \in \mathbb{N}$ and let T be unimodal with critical point c. Then for $\varepsilon > 0$ there is $\delta > 0$ such that for every unimodal transformation S with $\varrho(S,T) < \delta$ we have $|S^j(\widetilde{c}) - T^j(c)| < \varepsilon$ for $0 \le j \le m$, where $\widetilde{c}$ is the critical point of S.*

Proof. Set $\eta_0 := \varepsilon$. If η_j is defined, choose $\eta_{j+1} \le \eta_j$ such that $|x-y| < \eta_{j+1}$ implies $|T(x) - T(y)| < \frac{\eta_j}{2}$. Furthermore, set $\beta := \min\{T(c) - T(c+\eta_m), T(c) - T(c-\eta_m)\}$. Finally choose $\delta \in (0, \frac{1}{2}\min(\eta_m, \beta))$.

Now let S be a unimodal transformation which satisfies $\varrho(S,T) < \delta$ and has critical point $\widetilde{c}$. In particular we have $S(c) > S(c+\eta_m)$ and $S(c) > S(c-\eta_m)$, which implies $|\widetilde{c} - c| < \eta_m$. Suppose that $0 \le j \le m-1$ and that $|S^j(\widetilde{c}) - T^j(c)| < \eta_{m-j}$ is already shown. Then we get $|S^{j+1}(\widetilde{c}) - T^{j+1}(c)| \le |S^{j+1}(\widetilde{c}) - T(S^j(\widetilde{c}))| + |T(S^j(\widetilde{c})) - T^{j+1}(c)| < \delta + \frac{\eta_{m-j-1}}{2} \le \eta_{m-j-1}$. We have shown by induction that $|S^j(\widetilde{c}) - T^j(c)| < \eta_{m-j} \le \varepsilon$ holds for $0 \le j \le m$. $\square$

Lemma 4. *Fix $k \in \mathbb{N}$ and let T be unimodal with critical point c. Let u be a point with $T^k(u) = T^l(c)$ for some $l \ge 0$ and suppose that $T^j(u)$ is not contained in the orbit of c for $0 \le j \le k-1$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that for every unimodal transformation S with $\varrho(S,T) < \delta$ we find a point v with $S^k(v) = S^l(\widetilde{c})$ and $|S^j(v) - T^j(u)| < \varepsilon$ for $0 \le j \le k$, where $\widetilde{c}$ is the critical point of S.*

Proof. Since $\{u, T(u), \dots, T^{k-1}(u)\}$ is disjoint from the orbit of c, we find $\kappa > 0$ such that the intervals $J_1 = [T^2(c), T^2(c)+\kappa]$, $J_2 = [c-\kappa, c]$, $J_3 = [c, c+\kappa]$ and $J_4 = [T(c)-\kappa, T(c)]$ are disjoint from $\{u, T(u), \dots, T^{k-1}(u)\}$. Let ν be the minimum of the lengths of the intervals $T(J_1)$, $T(J_2)$, $T(J_3)$ and $T(J_4)$.

Denote the two monotonic pieces of T by T_1 and T_2. Set $\eta_0 = \min(\varepsilon, \frac{\nu}{2})$. If η_j is defined, choose $\eta_{j+1} \le \eta_j$ such that $|x-y| < \eta_{j+1}$ implies $|T_1^{-1}(x) - T_1^{-1}(y)| < \frac{\eta_j}{2}$ and $|T_2^{-1}(x) - T_2^{-1}(y)| < \frac{\eta_j}{2}$, provided these inverse images exist.

For $\alpha > 0$ set $\gamma_\alpha(T) = \min\{|T(x) - T(y)| : x \sim y, |x-y| = \alpha\}$. Then we have $\gamma_\alpha(T) > 0$ and if S is a unimodal transformation with $\varrho(S,T) < \gamma_\alpha(T)$

then $|S_1^{-1}(x) - T_1^{-1}(x)| < \alpha$ and $|S_2^{-1}(x) - T_2^{-1}(x)| < \alpha$ hold for all x for which the inverse images exist. Let $\widehat{\delta}$ be the δ in Lemma 3 for $\varepsilon = \eta_k$ and $m = l$. Then choose $\delta \in (0, \min(\gamma_{\eta_k/2}(T), \frac{\nu}{2}, \widehat{\delta}))$.

Now let S be a unimodal transformation which satisfies $\varrho(S,T) < \delta$ and has critical point $\widetilde{c}$. Set $v_k = S^l(\widetilde{c})$. Because of $\delta < \widehat{\delta}$ and $T^k(u) = T^l(c)$ we have $|T^k(u) - v_k| < \eta_k$. Suppose that $1 \le j \le k$ and that $|T^j(u) - v_j| < \eta_j$ is already shown. Choose $i \in \{1,2\}$ such that $T_i^{-1}(T^j(u)) = T^{j-1}(u)$ and set $v_{j-1} = S_i^{-1}(v_j)$. This inverse image exists, since $|T^j(u) - v_j| < \eta_j \le \frac{\nu}{2}$ and $\delta \le \frac{\nu}{2}$. Also $T_i^{-1}(v_j)$ exists because of $|T^j(u) - v_j| < \nu$. Then we get $|T^{j-1}(u) - v_{j-1}| \le |T_i^{-1}(T^j(u)) - T_i^{-1}(v_j)| + |T_i^{-1}(v_j) - S_i^{-1}(v_j)| < \frac{\eta_{j-1}}{2} + \frac{\eta_k}{2} \le \eta_{j-1}$. We have found $v_k, \dots, v_1, v_0$ with $S(v_{j-1}) = v_j$ and shown by induction that $|T^j(u) - v_j| < \eta_j$ holds for $0 \le j \le k$.

Set $v = v_0$. Then we have $S^j(v) = v_j$ for $1 \le j \le k$. In particular we have $S^k(v) = S^l(\widetilde{c})$. For $0 \le j \le k$ we have $\eta_j \le \varepsilon$ and therefore $|T^j(u) - v_j| < \varepsilon$. □

For a subset A of $[0,1]$ and $\varepsilon > 0$ define $U_\varepsilon(A) := \{x \in [0,1] : \inf_{y \in A} |x - y| < \varepsilon\}$.

Proposition 2. *Let $T : [0,1] \to [0,1]$ be a unimodal transformation with critical point c and let $[T^2(c), T(c)] = K_1 \supsetneqq K_2 \supsetneqq K_3 \supsetneqq \cdots \supsetneqq K_q$ be fundamental sets such that ∂K_q is not invariant under T. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that every unimodal transformation S with $\varrho(S,T) < \delta$ has fundamental sets $[S^2(\widetilde{c}), S(\widetilde{c})] = L_1 \supsetneqq L_2 \supsetneqq L_3 \supsetneqq \cdots \supsetneqq L_q$ satisfying $B_S(L_j, L_{j+1}) \subseteq U_\varepsilon(B_T(K_j, K_{j+1}))$ for $1 \le j \le q-1$ and $L_q \subseteq U_\varepsilon(K_q)$, where $\widetilde{c}$ is the critical point of S.*

Proof. Fix $\varepsilon > 0$. Set $C_{j,k} = \overline{K_j \setminus T^{-k}(K_{j+1})}$ for $1 \le j \le q-1$ and $k \ge 1$. Since K_{j+1} is T-invariant, we have $C_{j,k+1} \subseteq C_{j,k}$ for $k \ge 1$. Using compactness, this implies $B_T(K_j, K_{j+1}) = \bigcap_{k=0}^{\infty} C_{j,k}$. Therefore we find r with

$$C_{j,r} \subseteq U_{\varepsilon/2}\big(B_T(K_j, K_{j+1})\big) \quad \text{for } 1 \le j \le q-1. \tag{3}$$

Using Lemma 2 we obtain that there exists a natural number m such that ∂K_q consists of the points $T(c), T^2(c), \dots, T^{m-1}(c)$ which are all different and $T^m(c)$ is in the interior of K_q. By definition of a fundamental set, c is in the interior of K_q and hence $T^i(c) \ne c$ for $1 \le i \le m-1$. Again by Lemma 2 we get $\partial K_j \subseteq \{T(c), T^2(c), \dots, T^{m-1}(c)\}$ for $1 \le j \le q$.

Set $W = T^{-r}(\{T(c), T^2(c), \dots, T^{m-1}(c)\})$. Let $\eta > 0$ be the minimal distance of two different points in $\{c, T^m(c)\} \cup \bigcup_{j=0}^{r} T^{-j}(W)$ and set $\alpha := \frac{1}{2}\min(\varepsilon, \eta)$. Choose $\widehat{\delta}$ as the δ in Lemma 3 for α instead of ε. For every

$u \in W$ let δ_u be the δ in Lemma 4 for r instead of k and for α instead of ε. Finally choose $\delta > 0$ satisfying $\delta < \widehat{\delta}$ and $\delta < \delta_u$ for all $u \in W$.

Let S be a unimodal transformation with critical point $\widetilde{c}$ which satisfies $\varrho(S, T) < \delta$. In particular, we have $|S^j(\widetilde{c}) - T^j(c)| < \alpha$ for $0 \leq j \leq m$ by Lemma 3.

For $1 \leq j \leq q-1$ we have $K_j = \bigcup_{j=1}^n [T^j(c), T^{n+j}(c)]$ with $2n+1 \leq m$, since $\partial K_j \subseteq \{T(c), T^2(c), \dots, T^{m-1}(c)\}$ (see Lemma 2), and we set $L_j := \bigcup_{j=1}^n [S^j(\widetilde{c}), S^{n+j}(\widetilde{c})]$. Since the ordering of the orbit segment $\{c, T(c), T^2(c), \dots, T^m(c)\}$ is the same as that of $\{\widetilde{c}, S(\widetilde{c}), S^2(\widetilde{c}), \dots, S^m(\widetilde{c})\}$ by the choice of α and η, the sets L_j are fundamental sets for S satisfying $[S^2(\widetilde{c}), S(\widetilde{c})] = L_1 \supsetneqq L_2 \supsetneqq L_3 \supsetneqq \cdots \supsetneqq L_q$. Because of $|S^j(\widetilde{c}) - T^j(c)| < \alpha$ for $0 \leq j \leq m$ we have $d(L_j, K_j) < \alpha$ for $1 \leq j \leq q$. In particular, $L_q \subseteq U_\varepsilon(K_q)$ is shown.

Now fix $j \in \{1, 2, \dots, q-1\}$ and set $D_{j,r} := \overline{L_j \setminus S^{-r}(L_{j+1})}$. The set $T^{-r}(K_{j+1})$ is a finite union of pairwise disjoint closed intervals. We have that the endpoints of these intervals are contained in orbit segments $\{u, T(u), \dots, T^r(u)\}$ with $T^r(u) = T^l(c)$ for some $l \in \{1, 2, \dots, m-1\}$ and these orbit segments have no other intersection with the orbit of c . By Lemma 4, for each such orbit segment $\{u, T(u), \dots, T^r(u)\}$ we find an orbit segment $\{v, S(v), \dots, S^r(v)\}$ with $S^r(v) = S^l(\widetilde{c})$, such that $|S^j(v) - T^j(u)| < \alpha$ for $0 \leq j \leq r$. These orbit segments $\{v, S(v), \dots, S^r(v)\}$ form the endpoints of the intervals of which $S^{-r}(L_{j+1})$ consists. By the choice of η and α, there is a one-to-one correspondence between the intervals of which $S^{-r}(L_{j+1})$ consists and the intervals of which $T^{-r}(K_{j+1})$ consists. The distance between left endpoints and between right endpoints of corresponding intervals is at most $\frac{\eta}{2}$. This means that there are still gaps between the intervals of which $S^{-r}(L_{j+1})$ consists. We get $d(T^{-r}(K_{j+1}), S^{-r}(L_{j+1})) < \alpha$ and because of $d(L_j, K_j) < \alpha$ we have also $d(D_{j,r}, C_{j,r}) < \alpha$. By (3) and since $\alpha < \frac{\varepsilon}{2}$ we get $D_{j,r} \subseteq U_\varepsilon(B_T(K_j, K_{j+1}))$. Because of $B_S(L_j, L_{j+1}) \subseteq D_{j,r}$ this implies $B_S(L_j, L_{j+1}) \subseteq U_\varepsilon(B_T(K_j, K_{j+1}))$. ☐

4. Unstable behaviour of unimodal transformations

In this section we consider a unimodal transformation with generating partition and find conditions under which its ω-limit set is unstable.

Lemma 5. *Let T be a unimodal transformation. If there is n such that $T^n(c) \neq T^{2n}(c)$ and $T^{2n+1}(c) = T^{n+1}(c)$, then $T^k(c) \neq c$ for all $k \geq 1$.*

Proof. Suppose that there is $k \geq 1$ with $T^k(c) = c$. We may assume that k is minimal with this property. Hence c is a periodic point of period k, and therefore also $T^{n+1}(c)$ is a periodic point of period k. As $T^n(T^{n+1}(c)) = T^{n+1}(c)$ we obtain $n = dk$ for some $d \geq 1$. This implies $T^{2n}(c) = T^n(T^n(c)) = T^n(T^{dk}(c)) = T^n(c)$ contradicting the assumption $T^n(c) \neq T^{2n}(c)$. □

Proposition 3. *Assume that T is a unimodal transformation with generating partition and that T has $q \geq 2$ fundamental sets $K_1 \supsetneqq K_2 \supsetneqq \cdots \supsetneqq K_q$, such that $T(\partial K_q) \subseteq \partial K_q$ holds. Then the map $\omega : \mathcal{U} \to \mathcal{C}$ is not continuous at T.*

Proof. We have $K_q = \bigcup_{j=1}^n I_j$ for some n, where $I_j = [T^j(c), T^{n+j}(c)]$. By Lemma 2 and its proof the intervals I_j are pairwise disjoint and $T^{n+1}(c) = T^{2n+1}(c) < T(c)$. Set $a := T^{n+1}(c)$ and $b := T(c)$. For $1 \leq j \leq n-1$ the interval I_j is mapped monotonically to I_{j+1} under T. The interval I_n contains the critical point c in its interior and each of the intervals $I_n^- := [T^n(c), c]$ and $I_n^+ := [c, T^{2n}(c)]$ is mapped monotonically to I_1. Therefore the interval $I_1 = [a, b]$ is invariant under T^n with $T^n(a) = T^n(b) = a$ and there is $d \in (a, b)$ with $T^n(d) = b$, such that T^n is strictly increasing on $[a, d]$ and strictly decreasing on $[d, b]$. There is a unique $p \in (d, b)$ which is fixed under T^n.

By Lemma 2 and Lemma 5 the critical point c is not periodic, hence a cannot be an inverse image of c. This implies that there is an open interval $U = (a - \gamma, a)$ with $\gamma > 0$ such that T^n is strictly increasing on U and $U \cap K_q$ is empty. Since $\mathcal{Z}$ is a generator we have $|T^n(x) - a| > |x - a|$ for all $x \in U$. As $\omega(T) \setminus K_q$ is finite or a Cantor set there is an open interval $J \subseteq U$ with $J \cap \omega(T) = \emptyset$. Because $\mathcal{Z}$ is a generator the inverse images of p are dense. Therefore there is $y \in J$ with $T^m(y) = p$ for some $m \geq 1$. In particular $\delta := d\big(\omega(T), \{y\}\big)$ satisfies $\delta > 0$.

Let $\varepsilon > 0$ be arbitrary. Define $A := \{T^j(c) : 0 \leq j < 2n\} \cup \{T^j(y) : j \geq 0\}$. This is a finite set, which does not contain $T^{2n}(c)$. Hence there exists an open interval V with $T^{2n}(c) \in V$ and $\overline{V} \cap A = \emptyset$. Let S be a unimodal transformation with $S(x) = T(x)$ for all $x \in [0,1] \setminus V$, $\varrho(S, T) < \varepsilon$, and $|S^n(x) - a| > |x - a|$ for all $x \in U \cup \{a\}$. Choose $v \geq 1$ minimal with $|S^{vn}(a) - a| > |y - a|$. For $\alpha \in [0,1]$ set $T_\alpha := \alpha S + (1 - \alpha)T$. Then also T_α is a unimodal transformation with $\varrho(T_\alpha, T) < \varepsilon$. As the map $\alpha \mapsto {T_\alpha}^{vn}(a)$ is continuous there exists $\alpha \in [0,1]$ with ${T_\alpha}^{vn}(a) = y$.

It is obvious that c is also the critical point of T_α. As $c, T(c), \ldots, T^n(c) \in A$ we get ${T_\alpha}^{n+1}(c) = T^{n+1}(c) = a$. Therefore $y = {T_\alpha}^{vn}(a) = {T_\alpha}^{vn+n+1}(c)$ is in the T_α-orbit of c. Since also the T-orbit of y is in A, we get ${T_\alpha}^{m+vn+n+1}(c) = {T_\alpha}^m(y) = T^m(y) = p$. This means that the orbit of the critical point c of T_α is finite and therefore T_α has finitely many fundamental sets. Let L be the last one, which contains $\omega(c)$. We do not know whether T_α has a generating partition. There may be atomic intervals, which are open intervals I such that $c \notin T^k(I)$ for all $k \geq 0$. The structure theorem can be applied to T_α considering atomic intervals as single points (see Theorem 11 in [5]). Hence all points in L which do not lie in the closure of an atomic interval, are contained in $\omega(T_\alpha)$. In particular, if $y = {T_\alpha}^{vr+n+1}(c)$, which is in L, is not in the closure of an atomic interval, we have $y \in \omega(T_\alpha)$ and $d(\omega(T_\alpha), \omega(T)) \geq \delta$ is shown.

Now it remains to show that y is not in the closure of an atomic interval. To this end suppose that I is an atomic interval with $y \in \overline{I}$. Then $J := {T_\alpha}^m(I)$ is also an atomic interval with $p \in \overline{J}$. We have $T^{j-1}(p) \in I_j$ for $1 \leq j \leq n-1$ and $T^{n-1}(p) \in I_n^-$, since b and p are on the same side of d. By the choice of the set A and the definition of S, the transformations T and T_α coincide on the intervals $I_1, I_2, \ldots, I_{n-1}, I_n^-$. Since ${T_\alpha}^j(J)$ does not contain the critical point c and has $T^j(p)$ in its closure for $j \geq 1$, we get $T^{nk-1}(J) = {T_\alpha}^{nk-1}(J) \subseteq I_n$ for $k \geq 1$ and $T^{nk+i-1}(J) = {T_\alpha}^{nk+i}(J) \subseteq I_i$ for $k \geq 0$ and $1 \leq i < n$. This shows that I is also an atomic interval for the transformation T contradicting the assumption that T has a generating partition. $\square$

5. Conditions for the continuity

Conditions for stable and unstable behaviour of the ω-limit set of a unimodal transformation have been found in the previous two sections. We have seen that stability depends on properties of the boundary of fundamental sets. In this section we give conditions for stability in terms of the orbit of the critical point.

Proposition 4. *Let $T : [0,1] \to [0,1]$ be a unimodal transformation with generating partition. Then the following properties are equivalent.*

(a) *The map T has $q \geq 2$ fundamental sets $K_1 \supsetneqq K_2 \supsetneqq \cdots \supsetneqq K_q$ and $T(\partial K_q) \subseteq \partial K_q$.*

(b) *There is n, such that $T^j(c) \sim T^{n+j}(c)$ holds for $1 \leq j \leq n-1$ and $T^{n+1}(c)$ is a periodic point of period n, but $T^n(c) \neq T^{2n}(c)$.*

(c) *There exists $m \geq 1$ such that $r_j = S_m$ for all $j > m$ and $r_m < S_{m-1}$.*

Proof. Suppose that (a) holds. By the definition of the fundamental set K_q there is n with $T^{n+j}(c) \sim T^j(c)$ for $1 \leq j \leq n-1$ and $T^n(c) \not\sim T^{2n}(c)$. Furthermore $T^{2n+1}(c) = T^{n+1}(c)$ or $T^{2n+1}(c) = T(c)$, since we assume $T(\partial K_q) \subseteq \partial K_q$. As the unique pre-image of $T(c)$ is c, the latter case implies $T^{2n}(c) = c$ contradicting $T^n(c) \not\sim T^{2n}(c)$. Hence $T^{2n+1}(c) = T^{n+1}(c)$, which says that $T^{n+1}(c)$ is a periodic point. Let k be its period. Then $n = dk$ for some $d \in \mathbb{N}$ and $T^{n+1}(c) = T^{n+k+1}(c)$. Suppose that $d \geq 2$. We have then

$[T^{n+1}(c), T(c)] \subseteq K_q$ and $[T^{n+k+1}(c), T^{k+1}(c)] \subseteq K_q$. Since $T(\partial K_q) \subseteq \partial K_q$ we get that $T^j(c)$ is on the boundary of K_q for $j \geq 1$. This is only possible, if $T^{k+1}(c) = T(c)$. Since c is the only inverse image of $T(c)$, this gives $T^k(c) = c$, contradicting Lemma 5. Hence $d = 1$ and (b) is shown.

Next we assume that (b) holds. Since $T^{2n+1}(c) = T^{n+1}(c)$, we get that the points $T^{2n}(c)$ and $T^n(c)$ are inverse images of the same point. As they are not equal, they have to be on different sides of c, which gives $e'_n = e_{2n}$. Furthermore, we assume $T^{n+j}(c) \sim T^j(c)$ for $1 \leq j \leq n-1$. Lemma 5 implies that $T^k(c) \neq c$ for all $k \geq 1$. It follows from the definitions in Section 2 that $e_{n+j} = e_j$ holds for $1 \leq j \leq n-1$. Together this gives

$$e_{n+1}e_{n+2}\dots e_{2n-1}e_{2n} = e_1e_2\dots e_{n-1}e'_n \,.$$

Hence $e_1e_2\dots e_{n-1}e'_n$ occurs in $e_1e_2e_3\dots$ and is therefore an admissible word in the shift space conjugate to T. By Lemma 1 in [3] we get $n = S_m$ for some m (we have $S_j < \infty$ for all j, since $T^k(c) \neq c$ for all $k \geq 1$). Now we have

$$e_{S_m+1}e_{S_m+2}\dots e_{2S_m-1}e_{2S_m} = e_1e_2\dots e_{S_m-1}e'_{S_m} \,,$$

which, by the definition of the numbers r_i, gives $r_{m+1} = S_m$. Since $T^{S_m+1}(c)$ has period S_m, we get also

$$e_{kS_m+1}e_{kS_m+2}\dots e_{(k+1)S_m-1}e_{(k+1)S_m} = e_1e_2\dots e_{S_m-1}e'_{S_m} \quad \text{for } k \geq 2,$$

implying $r_{m+k} = S_m$ for $k \geq 2$. It is not possible, that $r_m > S_{m-1}$, since $r_m < S_m$ by definition and $r_m = S_j$ for some j by Lemma 1 in [3]. In order to show $r_m < S_{m-1}$, suppose that $r_m = S_{m-1}$ holds. By definition of r_m we get

$$e_{S_{m-1}+1}e_{S_{m-1}+2}\dots e_{S_m-1}e_{S_m} = e_1e_2\dots e_{S_{m-1}-1}e'_{S_{m-1}} \,.$$

This implies that the word $e_1e_2\dots e_{S_m-1}e'_{S_m}$ has identical first and second halves. It follows that the sequence $e_{S_m+1}e_{S_m+2}\dots$ has period $S_{m-1} = \frac{n}{2}$. As $\mathcal{Z}$ is a generator, this means that $T^{S_m+1}(c) = T^{n+1}(c)$ has also period $\frac{n}{2}$, contradicting (b). Therefore $r_m < S_{m-1}$ is shown and the proof of (c) is finished.

Suppose that (c) holds. It follows, that the subset $\mathcal{E} = \{J_{S_m}, J_{S_{m+1}}, \dots\}$ of the Markov diagram $\mathcal{D}$ is a closed set, and that $\mathcal{E}$ has no strict subset, which is closed. By the results described in Section 2 this implies that there are finitely many fundamental sets, and the last one, which corresponds to $\mathcal{E}$, is $K_q = \bigcup_{j=1}^{n}[T^{n+j}(c), T^j(c)]$ with $n = S_m$.

By (c) and the definition of the numbers r_i, we get $T^{n+l}(c) \sim T^{2n+l}(c)$ for all $l \geq 1$, which implies $T^{n+1}(c) = T^{2n+1}(c)$, since $\mathcal{Z}$ is a generator. Again by (c) and the definition of r_{m+1}, we get $T^n(c) \not\sim T^{2n}(c)$, which implies

$T^n(c) \neq T^{2n}(c)$. Suppose that $T^{n+1}(c)$ has period k, with $n = dk$. If $d \geq 3$, then $T^{n+1}(c) = T^{n+k+1}(c) = T^{n+2k+1}(c)$ are endpoints of three different intervals in the union defining K_q. One of these intervals must then be contained in another one, which contradicts the disjointness of the interiors of these intervals. Hence $d = 2$ and $n = 2k$. Then the sequence $e_{S_m+1}e_{S_m+2}\dots$ has period k, implying that the first and second halves of $e_{S_m+1}e_{S_m+2}\dots e_{2S_m}$ are the same. Because of $e_1e_2\dots e_{S_m-1}e'_{S_m} = e_{S_m+1}e_{S_m+2}\dots e_{2S_m-1}e_{2S_m}$ this gives $e_1e_2\dots e_{k-1}e'_k = e_{k+1}e_{k+2}\dots e_{S_m-1}e_{S_m}$. Now it follows that $k = S_{m-1}$ and that $r_m = S_{m-1}$ using Lemma 1 in [3]. This contradicts (c). Hence it is shown that $T^{n+1}(c)$ is a periodic point of period n.

Since $T^n(c)$ is not periodic, also $T^j(c)$ is not periodic for $j < n$. This implies that $T^l(c) \neq T^j(c)$ holds for $1 \leq l < j \leq n$. Therefore we get that the points $T(c), T^2(c), \dots, T^{2n}(c)$ are all different and form the boundary of K_q. Because of $T^{2n+1}(c) = T^{n+1}(c)$, we get $T(\partial K_q) \subseteq \partial K_q$. This completes the proof of (a). □

Now we are able to prove the main result of this paper.

Theorem 2. *Let $T : [0,1] \to [0,1]$ be a unimodal transformation with critical point c. Suppose that T has a generating partition. Then the map $\omega : \mathcal{U} \to \mathcal{C}$ is not continuous at T if and only if there is n, such that $T^j(c) \sim T^{n+j}(c)$ holds for $1 \leq j \leq n-1$ and $T^{n+1}(c)$ is a periodic point of period n, but $T^n(c) \neq T^{2n}(c)$.*

Proof. As T has a generating partition, every periodic point $x \notin \{T^{-j}(c) : j \geq 0\}$ is transversal. Hence the condition in Theorem 1 is satisfied, if c is not periodic. If c is periodic, then Proposition 1 implies that there are only finitely many fundamental sets $K_1 \supsetneqq K_2 \supsetneqq \cdots \supsetneqq K_q$, the orbit of c is contained in K_q, and K_q is a finite union of intervals. By [7] the periodic points are dense in $\omega(T)$. Hence c is not separated from periodic points of different period, and also in this case the condition in Theorem 1 is satisfied. Therefore lower semicontinuity follows from Theorem 1 and it remains to investigate the upper semicontinuity.

By Proposition 4 the condition in the theorem implies that T has $q \geq 2$ fundamental sets $K_1 \supsetneqq K_2 \supsetneqq \cdots \supsetneqq K_q$ and $T(\partial K_q) \subseteq \partial K_q$. Proposition 3 implies then that the map ω is not continuous at T.

Using again Proposition 4, if the condition in the theorem is not satisfied, then T has either infinitely many fundamental sets or $q \geq 1$ fundamental sets $[T^2(c), T(c)] = K_1 \supsetneqq K_2 \supsetneqq \cdots \supsetneqq K_q$ such that $q = 1$ or ∂K_q is not invariant under T. In the second case, by Proposition 2 there is $\delta > 0$ such that every unimodal transformation S with $\varrho(S,T) < \delta$ has fundamental sets $[S^2(\tilde{c}), S(\tilde{c})] = L_1 \supsetneqq L_2 \supsetneqq L_3 \supsetneqq \cdots \supsetneqq L_r$ satisfying $B_S(L_j, L_{j+1}) \subseteq$

$U_\varepsilon(B_T(K_j, K_{j+1}))$ for $1 \leq j \leq q-1$ and $L_q \subseteq U_\varepsilon(K_q)$. By Proposition 1 we have $\omega(S) \subseteq L_q \cup \bigcup_{j=1}^{q-1} B_S(L_j, L_{j+1})$, which gives $\omega(S) \subseteq U_\varepsilon(\omega(T))$. Therefore, upper semicontinuity of ω at T is shown.

It remains to show upper semicontinuity of ω at T, if T has infinitely many fundamental sets K_j. Fix $\varepsilon > 0$. There is r such that every interval of which K_r consists has length at most $\frac{\varepsilon}{2}$. Since every such interval has nonempty intersection with $\omega(T)$ we have $K_r \subseteq U_{\varepsilon/2}\big(\omega(T)\big)$. Furthermore, ∂K_r cannot be invariant under T, because otherwise the critical point would be eventually periodic and its orbit could not form the boundaries of infinitely many fundamental sets. Again by Proposition 2 we find $\delta > 0$ such that every unimodal transformation S with $\varrho(S,T) < \delta$ has fundamental sets $[S^2(\widetilde{c}), S(\widetilde{c})] = L_1 \supsetneqq L_2 \supsetneqq L_3 \supsetneqq \cdots \supsetneqq L_r$ satisfying $B_S(L_j, L_{j+1}) \subseteq U_\varepsilon(B_T(K_j, K_{j+1}))$ for $1 \leq j \leq r-1$ and $L_r \subseteq U_{\varepsilon/2}(K_r)$. By Proposition 1 we have $\omega(S) \subseteq L_r \cup \bigcup_{j=1}^{r-1} B_S(L_j, L_{j+1})$ and we get again $\omega(S) \subseteq U_\varepsilon(\omega(T))$. This means that upper semicontinuity of ω at T is shown. □

We can apply Theorem 2 to unimodal transformations with negative Schwarzian derivative. It is well known (see [2]) that a unimodal transformations T with negative Schwarzian derivative has either a generating partition or the orbit of the critical point is attracted by a periodic orbit. In the first case we can apply Theorem 2.

Therefore, suppose that the orbit of the critical point is attracted by the periodic orbit of a point x, which has period k. There are disjoint closed intervals $I_0, I_1, \ldots, I_{k-1}$ with $T^j(x) \in I_j$ for $0 \leq j \leq k-1$ such that $T(I_j) \subseteq I_{j+1}$ for $0 \leq j \leq k-1$ (setting $I_k = I_0$) and these intervals are attracted by the periodic orbit of x. There are two cases. Either the periodic point x is attracting on both sides or it is attracting on one side and repelling on the other side.

If x is attracting on both sides, then we have the same situation as in Proposition 2, replacing the last fundamental set K_q by $\bigcup_{j=0}^{k-1} I_j$. Since the periodic orbit of x and the intervals $I_0, I_1, \ldots, I_{k-1}$ are stable under small perturbations, similar methods as in Section 3 show that the map ω is continuous at T.

Finally consider the case that x is attracting on one side and repelling on the other side. Then the periodic orbit of x is on the boundary of the intervals $I_0, I_1, \ldots, I_{k-1}$ with the attracting side inside these intervals. The same method of proof as used for Proposition 3, where x plays the role of the periodic point p in that proof, shows then that the map ω is not continuous at T.

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