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The space of ω-limit sets of piecewise continuous maps of the interval

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Abstract. According to a well-known result the collection of all ω-limit sets of a continuous map of the interval equipped with the Hausdorff metric is a compact metric space. In this paper a similar result is proved for piecewise continuous maps with finitely many points of discontinuity, if the points of discontinuity are not periodic for any variant of the map. A variant of f is a map g coinciding with f at any point of continuity and being continuous from one side at any point of discontinuity. It is also shown that ω-limit sets of these maps are locally saturating, another property known for continuous maps. However, contrary to the situation for continuous maps, there are piecewise continuous maps having locally saturating sets which are not ω-limit sets. A condition implying that a locally saturating set is an ω-limit set is presented.

1. Introduction

In [3] it is proved that, for an f in the class $\mathcal{C}$ of continuous maps of the interval $I = [0,1]$, the space $H(f)$ of ω-limit sets of f equipped with the Hausdorff metric ρ_H is a compact metric space. An analogous result is not true for continuous maps of the square, even for the special class of triangular (or, skew-product) maps $(x,y) \mapsto \big(f(x), g(x,y)\big)$ (see [6]). However, as we show in this paper, a similar theorem holds in some classes of piecewise continuous maps of the interval.

The topological structure of the ω-limit set and of the space of ω-limit sets of continuous interval maps has been investigated in several papers, e.g.

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in [2] and [14]. It is also treated in some textbooks, e.g. in [1], [4] and [7], where one can find further literature on this subject.

We say that a function $f : I \to I$ is *piecewise continuous* if there are points $0 < c_1 < c_2 < \cdots < c_k < 1$ such that f is discontinuous at any c_i, continuous at any $x \neq c_i$, and has both one-sided limits at any c_i. Moreover we assume that f is continuous from one side at any *discontinuity point* c_i. Let $C(f) = \bigcup_{i=1}^{k} \bigcup_{j=0}^{\infty} f^{-j}(c_i) \setminus \{0,1\}$ be the union of all f-preimages of the discontinuity points (except 0 and 1). We denote by $\mathcal{P}$ the class of all maps $f : I \to I$ which are piecewise continuous and such that $f^{-i}(c_j)$ is finite, for any i and j.

Special cases of these piecewise continuous maps are the well known piecewise monotone maps. Piecewise monotone maps are investigated e.g. in [8], [9], [10], [11] and [12].

For any $f \in \mathcal{P}$ with discontinuity points $c_1, c_2, \ldots, c_k$ let $\mathcal{E}(f)$ be the collection of the 2^k functions obtained from f by changing its values at discontinuity points but keeping the condition that the map must be continuous at any discontinuity point from one side. Denote by $\mathcal{P}_0$ the class of all $f \in \mathcal{P}$ such that no discontinuity point of f is a periodic point of a map in $\mathcal{E}(f)$. Finally, let $\mathcal{P}_1$ be the class of all maps $f \in \mathcal{P}$ such that if a discontinuity point c of f is periodic with respect to a $g \in \mathcal{E}(f)$, with period $m \geq 1$ then, for a sufficiently small $\varepsilon > 0$, and for $G_+ = (c, c+\varepsilon)$, $G_- = (c-\varepsilon, c)$, the condition $f^m(G_+) \cap G_- = \emptyset$ and $f^m(G_-) \cap G_+ = \emptyset$ holds.

Given $x \in I$, the ω-limit set $\omega_f(x)$ of x with respect to the map f is the set of all limit points of the sequence $\{f^n(x)\}_{n=0}^{\infty}$. We call a set $A \subseteq I$ an ω-limit set of f, if there is an $x \in I$ with $A = \omega_f(x)$.

The following results are the main results of this paper.

Theorem 1. *If $f \in \mathcal{P}_0$ then $(H(f), \rho_H)$ is a compact metric space.*

Theorem 2. *Let $f \in \mathcal{P}_1$, and let $\omega_n \in H(f)$, for $n \geq 1$. If $\lim_{n\to\infty} \omega_n = \omega$ in the Hausdorff metric then $\omega \in H(g)$, for some $g \in \mathcal{E}(f)$.*

Remark 1. Theorem 1 is not true for arbitrary $f \in \mathcal{P}$. To see this let $f \in \mathcal{P}$, $f(x) := x + \frac{1}{2}$ if $0 \leq x < \frac{1}{2}$, and $f(x) := x - \frac{1}{2}$ otherwise. This map is shown in Figure 1. Then $f \notin \mathcal{P}_0$ and, for any $n > 2$, $\omega_n := \{\frac{1}{2} - \frac{1}{n}, 1 - \frac{1}{n}\}$ is an ω-limit set of f consisting of a periodic orbit of period 2, while $\lim_{n\to\infty} \omega_n = \{\frac{1}{2}, 1\}$ is not invariant, and $\omega_f(\frac{1}{2}) = \omega_f(1) = \{0, \frac{1}{2}\}$. On the other hand, $f \in \mathcal{P}_1$, hence Theorem 2 is true for this f. Since $f \notin \mathcal{P}_0$ we get that $\mathcal{P}_0 \subset \mathcal{P}_1$ are different sets. It is easy to see that $\mathcal{P}_1$ is a proper subset of $\mathcal{P}$, cf. also Remarks 3 and 4.

In order to describe ω-limit sets a technical condition has been introduced in [3]. We call this condition "locally saturating". Note that the property

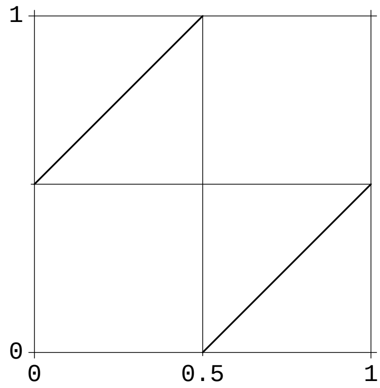

Figure 1: The map f defined in Remark 1.

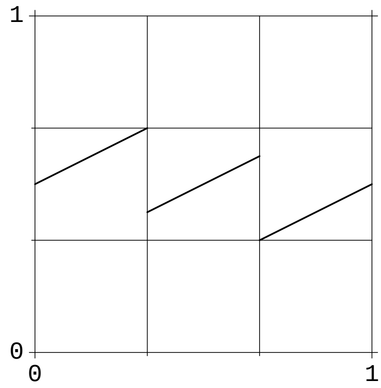

Figure 2: The map f defined in Remark 2.

called "locally saturating" in this paper is called "locally expanding" in [3]. Roughly spoken, a compact set A is called locally saturating, if for any neighbourhood U of A and for any $x \in A$ there is a side T (where T stands for "left" or "right") such that for any T-neighbourhood V of x the union $\bigcup_{n=1}^{\infty} (f_U)^n (V)$ contains a finite union J of intervals with $A \in \overline{J}$, where $(f_U)^n (V)$ denotes the set of all $f^n(x)$ with $x \in V$ and $f^j(x) \in U$ for all $j \in \{1, 2, \dots, n\}$. The exact definition of this notion will be given in Section 4. Because of the discontinuities of the map it is necessary to modify the original definition from [3]. For continuous maps it has been shown in [3] that a compact set is locally saturating if and only if it is an ω-limit set. In our case one implication, namely every ω-limit set is locally saturating, still holds, while the reverse implication is not true in general. For piecewise monotone maps we present a condition on compact sets ensuring that locally saturating sets are ω-limit sets.

The paper is organized as follows. In Section 2 we introduce the main tools for our investigation, and present the proofs of Theorem 1 and Theorem 2. A certain condition equivalent to the compactness of the space of ω-limit sets is investigated in Section 3. The counterexamples presented in Section 2 indicate that it is not easy to find nicer equivalent conditions. Finally, in Section 4 we consider connections between local saturation and ω-limit sets.

2. Compactness of the space of ω-limit sets

This section is devoted to the proof of Theorems 1 and 2. In the proof we will use several lemmas. Before stating them we introduce further terminology and notation.

Assume that $f \in \mathcal{P}$, with discontinuity points $c_1 < c_2 < \cdots < c_k$. Assign to any $x \in C(f)$ a compact interval $I_x = [x_-, x_+] \subseteq (0,1)$ of positive length such that $\sum_{x \in C(f)} |I_x| < 1$, where $|J|$ denotes the length of the interval J. Moreover, we choose these intervals I_x such that they are pairwise disjoint and $I_x < I_y$ for $x < y$, where $A < B$ for sets $A, B \subseteq \mathbb{R}$ means that $a < b$ for all $a \in A$ and all $b \in B$. Let τ be an order-preserving set-valued map from I onto I (this means τ is a map from I to the family of all subsets of I satisfying that $x < y$ implies $\tau(x) < \tau(y)$ and $\bigcup_{x \in I} \tau(x) = I$) such that $\tau(x) = I_x$ if $x \in C(f)$, and $\tau(x)$ is a singleton otherwise. Then τ^{-1} can be considered as an element of $\mathcal{C}$. More exactly, there exists an increasing function $\varphi \in \mathcal{C}$ satisfying $\varphi^{-1}\bigl(\{x\}\bigr) = \tau(x)$ for all $x \in I$. Denote

$$X := I \setminus \bigcup_{x \in C(f)} (x_-, x_+) \quad \text{and} \quad X_0 := I \setminus \bigcup_{x \in C(f)} I_x \,. \tag{1}$$

Observe that $\varphi\big|_{X_0} : X_0 \to \bigl(I \setminus C(f)\bigr)$ is bijective.

Now we describe two "canonical" extensions f^* and $\widetilde{f}$ of f which will be useful in the sequel. The map $\widetilde{f}$ will be in $\mathcal{C}$, i.e. a continuous map $I \to I$, whereas $f^* = \widetilde{f}\big|_X$ for a certain subset X of I (X may be a Cantor set). Using a similar construction Arnaud Denjoy gave examples of diffeomorphisms with wandering intervals in [5].

Since $\varphi\big|_{X_0}$ is a continuous bijection onto $I \setminus C(f)$ we can define

$$f^*(x) := \varphi^{-1} \circ f \circ \varphi(x) \quad \text{if } x \in X_0. \tag{2}$$

Therefore $f^*\big|_{X_0}$ is continuous, and $f^*\big|_{X_0}$ is conjugate to $f\big|_{I \setminus C(f)}$ since both $\varphi\big|_{X_0}$ and $\varphi^{-1}\big|_{I \setminus C(f)}$ are continuous. For $x \in C(f)$ (note that x_- and x_+ are in $X \setminus X_0$) set

$$\begin{aligned} f^*(x_-) &:= \lim_{y \in X_0,\, y \to x_-} f^*(y) \quad \text{and} \\ f^*(x_+) &:= \lim_{y \in X_0,\, y \to x_+} f^*(y) \,. \end{aligned} \tag{3}$$

By (2) and (3), $f^* : X \to X$ is a continuous map on the compact metric space $X \subseteq I$, where X is equipped with the relative topology.

Observe that the map f^* is conjugate to the map obtained from f by a standard doubling points construction (see e.g. [15] or [13] for details). Therefore one may consider f^* as the map $\widehat{f}$ considered e.g. in [13].

Next we extend f^* to a map $\widetilde{f} \in \mathcal{C}$ in such a way that, for any interval (a_-, a_+) complementary to X, we let $\widetilde{f}(x) := f^*(a_-)$ if $a_- < x \leq a_- + \frac{1}{3}(a_+ - a_-)$, $\widetilde{f}(x) := f^*(a_+)$ if $a_+ - \frac{1}{3}(a_+ - a_-) \leq x < a_+$, and we let $\widetilde{f}$ be linear otherwise. Then

$$\varphi \circ \widetilde{f}(x) = f \circ \varphi(x) \quad \text{if } \varphi(x) \notin \{c_1, c_2, \dots, c_k\}, \text{ and} \tag{4}$$

$$f \circ \varphi(\{(c_i)_-, (c_i)_+\}) \in \varphi \circ \widetilde{f}(\{(c_i)_-, (c_i)_+\}) \ , \quad \text{for } 1 \leq i \leq k. \tag{5}$$

Thus, in this sense, the map $\widetilde{f}$ is semiconjugate to f via φ.

An example for this construction is shown in Figures 3 and 4. Figure 3 shows the original map f while Figure 4 shows the map $\widetilde{f}$ constructed from f.

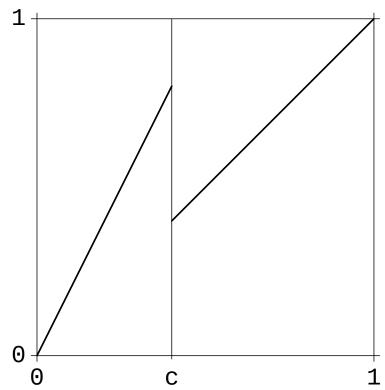

Figure 3: A map $f \in \mathcal{P}$. In Figure 4 the map $\widetilde{f}$ is constructed from f.

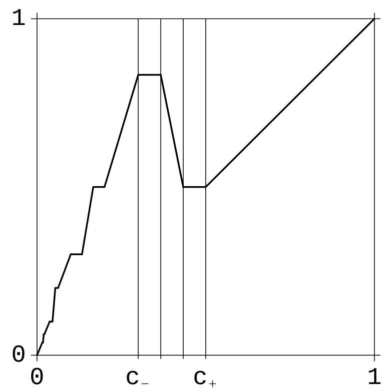

Figure 4: The map $\widetilde{f}$ constructed from the map f shown in Figure 3.

Lemma 1. *Suppose that $f \in \mathcal{P}$. Then the following properties hold.*

(1) *Assume that $x \in I$ is such that that there is an $N \in \mathbb{N}$ such that $f^n(x) \notin \{c_1, c_2, \dots, c_k\}$ for all $n \geq N$. Then there exists an $\omega^* \in H(f^*)$ such that $\omega_f(x) = \varphi(\omega^*)$.*

(2) *If $x \in X$ is such that that there is an $N \in \mathbb{N}$ such that $f^{*n}(x) \in X_0$ for all $n \geq N$, then $\varphi\big(\omega_{f^*}(x)\big) \in H(f)$.*

Proof. First we prove (1). As $\omega_f\big(f^n(x)\big) = \omega_f(x)$ for any n we may assume that $f^n(x) \notin \{c_1, c_2, \dots, c_k\}$ for all $n \geq 0$. Then the trajectory $\{x_j\}_{j=0}^{\infty}$ ($x_j := f^j(x)$) of x is disjoint from $C(f)$. Since $\varphi\big|_{X_0} : X_0 \to \big(I \setminus C(f)\big)$ is bijective, $\{\varphi^{-1}(x_j)\}_{j=0}^{\infty}$ is an f^*-trajectory contained in X_0 by (2). Set $\omega^* := \omega_{f^*}\big(\varphi^{-1}(x)\big)$. Then by the continuity of φ and by (2) we obtain $\varphi(\omega^*) = \omega_f(x)$.

In order to prove (2) we may assume that $f^{*n}(x) \in X_0$ for all $n \geq 0$, as $\omega_{f^*}\big(f^{*n}(x)\big) = \omega_{f^*}(x)$. Again by the continuity of φ and by (2) we get $\varphi\big(\omega_{f^*}(x)\big) = \omega_f\big(\varphi(x)\big)$. This shows that $\varphi\big(\omega_{f^*}(x)\big) \in H(f)$. □

Lemma 2. *If $f \in \mathcal{P}_0$ then $\varphi(H(f^*)) = H(f)$.*

Proof. Assume that $\omega = \omega_f(x)$ for some $x \in I$. Since $f \in \mathcal{P}_0$ every trajectory contains at most finitely many discontinuity points. Hence $f^n(x) \notin \{c_1, c_2, \dots, c_k\}$ for all sufficiently large n. Then $\omega \in \varphi(H(f^*))$ by (1) of Lemma 1.

Now suppose that $\omega^* \in H(f^*)$. Then $\omega^* = \omega_{f^*}(y)$ for some $y \in X$. If $f^{*n}(y) \notin X_0$ for infinitely many n then there is a $c \in \{c_{1-}, c_{1+}, \dots, c_{k-}, c_{k+}\}$ such that c is f^*-periodic and $\omega^* = \{c, f^*(c), \dots, f^{*m-1}(c)\}$, where m denotes the period of c. Suppose that ω^* contains both d_- and d_+ for some discontinuity point d. By the finiteness of ω^* we may assume that $c \in \{d_-, d_+\}$, $0 < l < m$ is such that $f^{*l}(c) \in \{d_-, d_+\}$, and there are no discontinuity points c_i such that $\{f^*(c), f^{*2}(c), \dots, f^{*l}(c)\}$ contains both c_{i-} and c_{i+}. Defining $g\big(\varphi(f^{*j}(c))\big) := \varphi(f^{*j+1}(c))$ for $0 \leq j \leq l$, and $g(x) := f(x)$ otherwise, we obtain a map $g \in \mathcal{E}(f)$ satisfying $g^l(\varphi(c)) = \varphi(f^{*l}(c))$. Since $\varphi(c) = \varphi(f^{*l}(c)) = d$ this gives $g^l(d) = d$, which contradicts $f \in \mathcal{P}_0$. Hence there are no discontinuity points d such that ω^* contains both d_- and d_+. Define $g : I \to I$ by $g(\varphi(x)) := \varphi(f^*(x))$ for $x \in \omega^*$, and $g(x) := f(x)$ otherwise. Then $g \in \mathcal{E}(f)$ and $g^m(\varphi(c)) = \varphi(f^{*m}(c)) = \varphi(c)$, again contradicting the fact $f \in \mathcal{P}_0$.

Therefore $f^{*n}(y) \in X_0$ for all sufficiently large n. By (2) of Lemma 1 we obtain $\varphi(\omega^*) \in H(f)$. □

The next result shows that for the map obtained from f using a doubling points construction the collection of all ω-limit sets equipped with the Hausdorff metric is compact.

Proposition 1. *If $f \in \mathcal{P}$ then $(H(f^*), \rho_H)$ is a compact metric space.*

Proof. Let $\omega_n^* := \omega_{f^*}(z_n)$ be an ω-limit set of f^*, generated by a point $z_n \in X$, $n \geq 1$, and assume that the sequence $\{\omega_n^*\}_{n=1}^{\infty}$ converges in the Hausdorff metric to a set ω^*. Since $\widetilde{f}\big|_X = f^*$, any ω_n^* is an ω-limit set of the map $\widetilde{f} \in \mathcal{C}$.

Hence by [3] we obtain $\omega^* = \omega_{\widetilde{f}}(z)$ for some $z \in I$. But then the trajectory $\{\widetilde{f}^n(z)\}_{n=1}^\infty$ of z is eventually in X. Indeed, if $\widetilde{f}^n(z) \notin X$ for infinitely many n then, for some $c := c_i$, the trajectory of z enters the interior (c_-, c_+) of the critical interval I_c infinitely many times. As $\widetilde{f}\big|_{I_c}$ has values in X, except for the middle third, and X is $\widetilde{f}$-invariant, the trajectory enters the middle third J_c of I_c infinitely many times. Consequently, ω^* must have a point in J_c, contrary to the fact that $\operatorname{dist}(\omega_n^*, J_c) \geq \frac{1}{3}(c_+ - c_-)$. Hence $\widetilde{f}^n(z) \in X$ for some n. Since $\omega_{\widetilde{f}}(z) = \omega_{\widetilde{f}}\big(\widetilde{f}^n(z)\big)$, X is closed and $\widetilde{f}$-invariant, and $\widetilde{f}\big|_X = f^*$, we obtain $\omega^* = \omega_{f^*}\big(\widetilde{f}^n(z)\big)$. $\square$

Now we are able to prove Theorem 1.

Proof of Theorem 1. Let $\omega_n := \omega_f(x_n)$, $n \geq 1$. By Lemma 2 there are sets $\omega_n^* := \omega_{f^*}(y_n)$, $n \geq 1$, such that $\varphi(\omega_n^*) = \omega_n$, for any n. Now Proposition 1 implies that there is an $\omega^* \in H(f^*)$ and a subsequence $\{\omega_{n_k}^*\}_{k=1}^\infty$ with $\lim_{k\to\infty} \omega_{n_k}^* = \omega^*$ with respect to ρ_H. Hence $\omega := \varphi(\omega^*) \in H(f)$ by Lemma 2. By the continuity of φ, $\lim_{k\to\infty} \rho_H(\omega_{n_k}, \omega) = 0$. $\square$

Lemma 3. *If $f \in \mathcal{P}_1$ then $\varphi(H(f^*)) \subseteq \bigcup_{g\in\mathcal{E}(f)} H(g)$.*

Proof. Let $\omega^* = \omega_{f^*}(x)$, for some $x \in X$. If the trajectory of x eventually is disjoint from the critical set $\{c_{1-}, c_{1+}, \ldots, c_{k-}, c_{k+}\}$ then by (2) of Lemma 1 we obtain that $\varphi(\omega^*) \in H(f)$. Otherwise the trajectory contains a discontinuity point, say, $c := c_{i+}$ at least twice. Then c is f^*-periodic with some period $p > 0$ and the trajectory consists of finitely many elements. So assume that the trajectory passes periodically through the discontinuity points

(6) $\quad c_{i(1)s(1)}, c_{i(2)s(2)}, \ldots, c_{i(m)s(m)}\ ,\quad$ where $1 \leq m \leq 2k$, and $s(j) \in \{+, -\}$,

and that it contains no other discontinuity points. Since $f \in \mathcal{P}_1$, the sequence $\{c_{i(j)s(j)}\}$ in (6) contains at most one element from any pair c_{q-}, c_{q+}. Define $g \in \mathcal{E}(f)$ by $g(c_{i(j)}) := \varphi(f^*(c_{i(j)s(j)})$ for $1 \leq j \leq m$, and $g(x) := f(x)$ otherwise. Then obviously $\omega_g(c_i) = \varphi(\omega^*)$. $\square$

We like to give some remarks concerning Lemma 3. At first we will give an example of an $f \in \mathcal{P}_1$ showing that the reverse inclusion in the conclusion of Lemma 3 is not true in general. The second example will be an $f \in \mathcal{P} \setminus \mathcal{P}_1$ such that the conclusion of Lemma 3 and Theorem 2 holds. In our third example we will consider again a map $f \in \mathcal{P} \setminus \mathcal{P}_1$, but in this example the conclusion of Lemma 3 and Theorem 2 does not hold.

Remark 2. In general $\bigcup_{g\in\mathcal{E}(f)} H(g) \subseteq \varphi(H(f^*))$ does not hold for maps f in $\mathcal{P}_1$ (i.e. the reverse of Lemma 3 does not hold). Now we give an example of an $f \in \mathcal{P}_1$ with an $\omega \in H(f)$ such that $\omega \notin \varphi(H(f^*))$. Set

$$f(x) := \begin{cases} \frac{x}{2} + \frac{1}{2}\,, & \text{for } x \in \left[0, \frac{1}{3}\right], \\ \frac{x}{2} + \frac{1}{4}\,, & \text{for } x \in \left(\frac{1}{3}, \frac{2}{3}\right), \\ \frac{x}{2}\,, & \text{for } x \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

The map f is shown in Figure 2. One easily checks that $f \in \mathcal{P}_1$. For the map f^* we have $f^*(X) \subseteq \left[\frac{1}{3}_+, \frac{2}{3}_-\right]$ (we also get $\widetilde{f}(I) \subseteq \left[\frac{1}{3}_+, \frac{2}{3}_-\right]$). As $\lim_{n\to\infty} f^n(x) = \frac{1}{2}$ for all $x \in \left(\frac{1}{3}, \frac{2}{3}\right)$ and $\left\{\varphi\left(f^*\left(\frac{1}{3}_+\right)\right), \varphi\left(f^*\left(\frac{2}{3}_-\right)\right)\right\} \subseteq \left(\frac{1}{3}, \frac{2}{3}\right)$, this implies $\varphi(H(f^*)) = \left\{\frac{1}{2}\right\}$. On the other hand $B := \left\{\frac{1}{3}, \frac{2}{3}\right\}$ is a periodic orbit of f. Therefore $B \in H(f)$, but $B \notin \varphi(H(f^*))$.

Remark 3. We have $\mathcal{P} \setminus \mathcal{P}_1 \neq \emptyset$. To see this consider the map f defined by

$$f(x) := \begin{cases} 1 - x\,, & \text{for } x \in \left[0, \frac{1}{2}\right), \\ x - \frac{1}{2}\,, & \text{for } x \in \left[\frac{1}{2}, 1\right]. \end{cases} \tag{7}$$

This map is shown in Figure 5. Then $f \in \mathcal{P}$, and f has a unique discontinuity point $c = \frac{1}{2}$. For any $\delta \in \left(0, \frac{1}{2}\right)$ we obtain

$$(c - \delta, c) \mapsto (c, c + \delta) \mapsto (0, \delta) \mapsto (1 - \delta, 1) \mapsto (c - \delta, c)\ . \tag{8}$$

In fact, every $x \in [0,1] \setminus \left\{0, c, 1, \frac{1}{4}, \frac{3}{4}\right\}$ is a point of period 4 for f, and for the map f^* every $x \in X \setminus \{x_1, x_2\}$ is a point of period 4, where x_1 and x_2 are the points of period 2 of f^* corresponding to $\frac{1}{4}$ and $\frac{3}{4}$. In this example we have $\varphi(H(f^*)) \subseteq \bigcup_{g\in\mathcal{E}(f)} H(g)$, hence the conclusion of Lemma 3 holds also for this example (therefore by Lemma 4 also the conclusion of Theorem 2 holds).

Remark 4. Next we like to give an example showing that Lemma 3 does not hold in general for maps in $\mathcal{P}\setminus\mathcal{P}_1$. To this end define the map $f : [0,1] \to [0,1]$ by

$$f(x) := \begin{cases} 1 - x\,, & \text{for } x \in \left[0, \frac{1}{3}\right), \\ x - \frac{1}{3}\,, & \text{for } x \in \left[\frac{1}{3}, \frac{3}{4}\right], \\ \frac{7}{6} - x\,, & \text{for } x \in \left[\frac{3}{4}, \frac{11}{12}\right], \\ x - \frac{2}{3}\,, & \text{for } x \in \left[\frac{11}{12}, 1\right]. \end{cases} \tag{9}$$

This map f is shown in Figure 6. Observe that the set $A := \left[0, \frac{1}{12}\right] \cup \left[\frac{1}{4}, \frac{5}{12}\right] \cup \left[\frac{2}{3}, \frac{3}{4}\right] \cup \left[\frac{11}{12}, 1\right]$ is invariant, and $f^3\big([0,1]\big) \subseteq A$. Moreover, note that

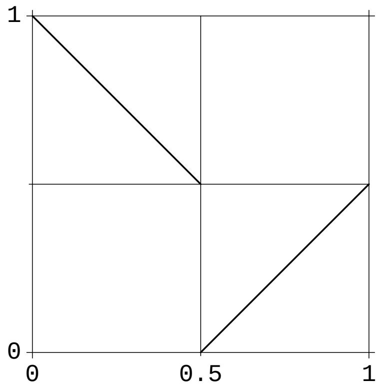

Figure 5: The map f defined in Remark 3.

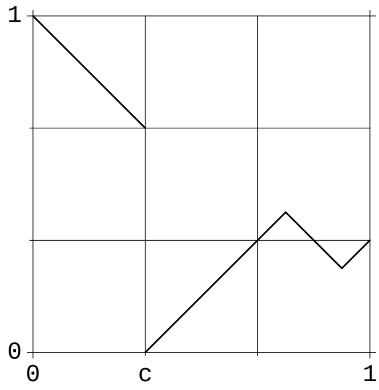

Figure 6: The map f defined in Remark 4.

every $x \in A \setminus \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$ is a point of period 5 for f. In fact, $f^3(x)$ is a point of period 5 for every $x \in [0,1] \setminus F$, where F is finite (F consists only of points of the form $\frac{k}{12}$ for some $k \in \{0, 1, 2, \ldots, 12\}$). Obviously $f \in \mathcal{P}$ and f has a unique discontinuity point $c = \frac{1}{3}$. Then we obtain that for any $x \in X$ the point $f^{*3}(x)$ is a point of period 5 for f^*. In particular we have

$$c_- \mapsto \frac{2}{3}_+ \mapsto c_+ \mapsto 0 \mapsto 1 \mapsto c_- \,. \tag{10}$$

We have $\mathcal{E}(f) = \{f, g\}$, where g is the map coinciding with f on $[0,1] \setminus \{c\}$ and satisfying $g(c) = \frac{2}{3}$. Then we obtain the orbits

$$\begin{aligned} c \mapsto 0 \mapsto 1 \mapsto c \quad &\text{for } f, \text{ and} \\ c \mapsto \frac{2}{3} \mapsto c \quad &\text{for } g. \end{aligned} \tag{11}$$

Hence $\{0, c, 1\}$ is an ω-limit set for f. On the other hand, every ω-limit set ω^* of f^* satisfying $c \in \varphi(\omega^*)$ has to contain 0, c, $\frac{2}{3}$ and 1 by (10). This shows that $\{0, c, 1\} \in H(f)$, but $\{0, c, 1\} \notin \varphi(H(f^*))$. Furthermore, by (10) we obtain $\left\{0, c_-, c_+, \frac{2}{3}_+, 1\right\}$ is an ω-limit set for f^*, hence $B := \left\{0, c, \frac{2}{3}, 1\right\} \in \varphi(H(f^*))$. By (11) B is neither an ω-limit set for f nor an ω-limit set for g. Therefore $B \in \varphi(H(f^*))$, but $B \notin \bigcup_{h \in \mathcal{E}(f)} H(h)$. For $n \geq 12$ one easily checks that $\omega_n := \left\{\frac{1}{n}, c - \frac{1}{n}, c + \frac{1}{n}, \frac{2}{3} + \frac{1}{n}, 1 - \frac{1}{n}\right\}$ is an ω-limit set for f (as it is an orbit of period 5). Obviously $\lim_{n \to \infty} \omega_n = B$ in the Hausdorff metric. Since B is neither an ω-limit set for f nor an ω-limit set for g we get that the conclusion of Theorem 2 does not hold for f.

Lemma 4. *Let $f \in \mathcal{P}$ and suppose that $\varphi(H(f^*)) \subseteq \bigcup_{g \in \mathcal{E}(f)} H(g)$. Moreover, assume that $\{\omega_n\}_{n=1}^{\infty}$ is a sequence in $H(f)$, and that $\lim_{n\to\infty} \omega_n = \omega$ in the Hausdorff metric. Then $\omega \in H(g)$ for some $g \in \mathcal{E}(f)$.*

Proof. As $\omega_n \in H(f)$ there is an $x_n \in I$ with $\omega_n = \omega_f(x_n)$. At first we assume that there is an N such that for every $n \geq N$ there exists a J_n with $f^j(x_n) \notin \{c_1, c_2, \dots, c_k\}$ for every $j \geq J_n$. Then by (1) of Lemma 1 for every $n \geq N$ there is an $\omega_n^* \in H(f^*)$ with $\omega_n = \varphi(\omega_n^*)$. Now Proposition 1 implies the existence of a subsequence $\{\omega_{n_j}^*\}_{j=1}^{\infty}$ and an $\omega^* \in H(f^*)$ with $\omega_{n_j}^* \to \omega^*$ in the Hausdorff metric. Since φ is continuous we get $\omega_{n_j} = \varphi(\omega_{n_j}^*) \to \varphi(\omega^*)$, and therefore $\varphi(\omega^*) = \omega$. By our assumption this implies $\omega \in H(g)$ for some $g \in \mathcal{E}(f)$.

It remains to consider the case that for infinitely many n there are infinitely many j with $f^j(x_n) \in \{c_1, c_2, \dots, c_k\}$. In this case there is a discontinuity point c and an infinite set $K \subseteq \mathbb{N}$, such that for all $n \in K$ there are infinitely many j with $f^j(x_n) = c$. Then c is a periodic point of f, $\omega_f(c)$ is the periodic orbit containing c, and $\omega_n = \omega_f(x_n) = \omega_f(c)$ for all $n \in K$. As $\omega_n \to \omega$ in the Hausdorff metric, and there is subsequence equal to $\omega_f(c)$ we obtain $\omega = \omega_f(c)$. Therefore $\omega \in H(f)$ (obviously $f \in \mathcal{E}(f)$). □

Observe that in order to check that the assumption of Lemma 4 is satisfied, by (2) of Lemma 1 it suffices to prove that for every periodic orbit ω of f^* with $\varphi(\omega)$ containing a discontinuity point the set $\varphi(\omega)$ is contained in $H(g)$ for some $g \in \mathcal{E}(f)$.

Now we are able to prove Theorem 2.

Proof of Theorem 2. As $f \in \mathcal{P}_1$ Lemma 3 gives $\varphi(H(f^*)) \subseteq \bigcup_{g \in \mathcal{E}(f)} H(g)$. By Lemma 4 this implies the desired result. □

3. A condition equivalent to "compactness"

By Theorem 2 the space $H(f)$ has a certain "compactness property", if $f \in \mathcal{P}_1$. However, in Remark 3 we have seen an example of an $f \in \mathcal{P} \backslash \mathcal{P}_1$ such that the conclusion of Theorem 2 holds also for this example. In Lemma 4 a condition implying the conclusion of Theorem 2 is given. Modifying the example given in Remark 4 we can find an example such that the assumption of Lemma 4 does not hold, but the conclusion of Theorem 2 holds (another example with these properties is the example given in Remark 5 in Section 4, in order to prove that the conclusion of Theorem 2 holds one can use Proposition 2). The problem addressed in this section is to find a condition equivalent to the conclusion of Theorem 2. As mentioned above this condition cannot be that easy. We will state it in terms of the "doubling points construction" of f, i.e. in terms of the map f^*.

Denote by $\mathcal{P}_2$ the set of all $f \in \mathcal{P}$ such that for every $\omega^* \in H(f^*)$ satisfying $c_j \in \varphi(\omega^*)$ for some discontinuity point c_j of f and $\{\omega^*\}$ is not isolated in $H(f^*)$ there exists a $g \in \mathcal{E}(f)$ with $\varphi(\omega^*) \in H(g)$. Finally, let $\mathcal{P}_3$ be the collection of all $f \in \mathcal{P}$ such that $\omega = \lim_{n\to\infty} \omega_n$ in the Hausdorff metric for a sequence $\omega_n \in H(f)$ implies the existence of a $g \in \mathcal{E}(f)$ with $\omega \in H(g)$. This means $\mathcal{P}_3$ consists of exactly those $f \in \mathcal{P}$ satisfying the conclusion of Theorem 2.

Proposition 2. *Let $f \in \mathcal{P}$. Then $f \in \mathcal{P}_2$ if and only if $f \in \mathcal{P}_3$.*

Proof. Assume that $f \in \mathcal{P}_2$. Suppose that $\omega = \lim_{n\to\infty} \omega_n$ in the Hausdorff metric for a sequence $\omega_n \in H(f)$. Then there are $x_n \in [0,1]$ with $\omega_n = \omega_f(x_n)$. If $f^j(x_n) \in \{c_1, c_2, \ldots, c_k\}$ for infinitely many $j \geq 0$ then there is a $d_n \in \{c_1, c_2, \ldots, c_k\}$ which is f-periodic and $\omega_n = \omega_f(d_n)$. First we assume that there are infinitely many n with $\omega_n = \omega_f(d_n)$ for some $d_n \in \{c_1, c_2, \ldots, c_k\}$. In this case there is a $c \in \{c_1, c_2, \ldots, c_k\}$ with $d_n = c$ for infinitely many n. Hence $\omega = \omega_f(c) \in H(f)$ (obviously $f \in \mathcal{E}(f)$).

Otherwise there is an N such that for all $n \geq N$ there is a K_n with $f^j(x_n) \notin \{c_1, c_2, \ldots, c_k\}$ for all $j \geq K_n$. For $n \geq N$ there is an $\omega_n^* \in H(f^*)$ with $\omega_n = \varphi(\omega_n^*)$ by (1) of Lemma 1. By Proposition 1 there exists an $\omega^* \in H(f^*)$ and a subsequence $\omega_{n_k}^*$ converging to ω^* in the Hausdorff metric. There is an $x \in [0,1]$ with $\omega^* = \omega_{f^*}(x)$. Moreover, the continuity of φ implies that $\omega = \varphi(\omega^*)$. If $f^{*j}(x) \in X_0$ for all sufficiently large j, then (2) of Lemma 1 gives that $\omega \in H(f)$. Otherwise there is a $c \in \{c_1, c_2, \ldots, c_k\}$ such that c_- is f^*-periodic and $\omega^* = \omega_{f^*}(c_-)$, or c_+ is f^*-periodic and $\omega^* = \omega_{f^*}(c_+)$. Then $c \in \varphi(\omega^*)$. If $\{\omega^*\}$ is isolated in $H(f^*)$, then $\omega_{n_k}^* = \omega^*$ for all sufficiently large k, and therefore $\omega = \omega_{n_k} \in H(f)$. It remains to consider the case $\{\omega^*\}$ is not isolated in $H(f^*)$. In this case the definition of $\mathcal{P}_2$ implies that there is a $g \in \mathcal{E}(f)$ with $\omega = \varphi(\omega^*) \in H(g)$. This completes the proof of $\mathcal{P}_2 \subseteq \mathcal{P}_3$.

In order to show $\mathcal{P}_3 \subseteq \mathcal{P}_2$ let $f \in \mathcal{P}_3$. Assume that $\omega^* \in H(f^*)$ is such that $c \in \varphi(\omega^*)$ for some discontinuity point c and $\{\omega^*\}$ is not isolated in $H(f^*)$. Since $\{\omega^*\}$ is not isolated in $H(f^*)$ there exists a sequence $\omega_n^* \in H(f^*)$ with $\omega_n^* \neq \omega^*$ for all n and $\omega^* = \lim_{n\to\infty} \omega_n^*$ in the Hausdorff metric. For every n there is an $x_n \in [0,1]$ with $\omega_n^* = \omega_{f^*}(x_n)$. If $f^{*j}(x_n) \notin X_0$ for infinitely many j, then there is a $d_n \in \{c_{1-}, c_{1+}, \ldots, c_{k-}, c_{k+}\}$ which is f^*-periodic and $\omega_n^* = \omega_{f^*}(d_n)$. Suppose that there are infinitely many n such that $f^{*j}(x_n) \notin X_0$ for infinitely many j. Then there is a $d \in \{c_{1-}, c_{1+}, \ldots, c_{k-}, c_{k+}\}$ with $d_n = d$ for infinitely many n. This implies $\omega_n^* = \omega_{f^*}(d)$ for infinitely many n, and therefore $\omega^* = \omega_{f^*}(d)$ contradicting the fact $\omega_n^* \neq \omega^*$ for all n.

Hence there is an N such that for all $n \geq N$ there exists a K_n with $f^{*j}(x_n) \in X_0$ for all $j \geq K_n$. By (2) of Lemma 1 we obtain for $n \geq N$ that $\varphi(\omega_n^*) \in H(f)$. The continuity of φ gives $\varphi(\omega^*) = \lim_{n\to\infty} \varphi(\omega_n^*)$ in the Hausdorff metric. As $f \in \mathcal{P}_3$ there exists a $g \in \mathcal{E}(f)$ with $\varphi(\omega^*) \in H(g)$, completing the proof. □

4. Characterization of ω-limit sets

In this section we will show that an ω-limit set of an $f \in \mathcal{P}$ is locally saturating. The converse implication is not true in general. However, we can give a sufficient condition for locally saturating compact sets to be an ω-limit.

The notion "locally saturating" has been defined in [3] (it is called "locally expanding" there). We recall this definition. For $x \in I$ a set V is called a *one-sided neighbourhood* of x, if there is a $\delta > 0$ such that $V = [x-\delta, x]$ (in this case it is called a *left neighbourhood*) or $V = [x, x+\delta]$ (in this case it is called a *right neighbourhood*). Given a map $f : I \to I$ and $U \subseteq I$ with $U \neq \emptyset$, define $f_U(B) := \overline{f(\mathrm{int}(B)) \cap U} \cup \big(f(B) \cap C(f) \cap \overline{U}\big)$, for any $B \subseteq I$. Note that this definition is slightly different from the definition of f_U in [3], but for continuous f and closed B both definitions coincide. For $n \in \mathbb{N}$ we define inductively $(f_U)^n(B)$ by $(f_U)^n(B) := f_U\big((f_U)^{n-1}(B)\big)$. A side T (where T stands for "left" or "right") of a point x of a compact set $A \subseteq I$ is called A*-covering*, if for any compact neighbourhood U of A and for any T-neighbourhood V of x there are intervals $J_1, J_2, \dots, J_m$ such that

$$A \subseteq \overline{J_1 \cup J_2 \cup \cdots \cup J_m} \quad \text{and} \quad J_1 \cup J_2 \cup \cdots \cup J_m \subseteq \bigcup_{n=1}^{\infty} (f_U)^n(V)\,. \tag{12}$$

We call a compact set A *locally saturating* if any point of A has an A-covering side.

Lemma 5. *Suppose that $f \in \mathcal{P}$ and assume that ω is an infinite ω-limit set for f. Then there exists an $\omega^* \in H(f^*)$ such that $\varphi(\omega^*) = \omega$.*

Proof. We have $\omega = \omega_f(x)$ for some $x \in I$. If the trajectory of x contains a discontinuity point twice, then this discontinuity point has to be periodic and $\omega_f(x)$ has to be this periodic orbit, contradicting the fact that ω is infinite. As there are only finitely many discontinuity points this implies that $f^n(x) \notin \{c_1, c_2, \dots, c_k\}$ for all sufficiently large n. Hence (1) of Lemma 1 shows that there is an $\omega^* \in H(f^*)$ with $\varphi(\omega^*) = \omega$. □

Theorem 3. *Let $f \in \mathcal{P}$. Then any ω-limit set for f is locally saturating for f.*

Proof. Suppose that $A = \omega_f(y)$ is an ω-limit set for f, this means $A \in H(f)$. Assume first that $A = \varphi(A^*)$ for some $A^* \in H(f^*)$. Let $U = U_1 \cup U_2 \cup \cdots \cup U_r$ be a compact neighbourhood of A which is the union of finitely many disjoint compact intervals U_i. Without loss of generality we may assume that the set of the endpoints of these intervals U_i is disjoint from $C(f)$. For a discontinuity point c set $J_c := \left(c_- + \frac{1}{3}(c_+ - c_-), c_+ - \frac{1}{3}(c_+ - c_-)\right)$. Define $U^* := \varphi^{-1}(U) \setminus \left(\bigcup_{j=1}^k J_{c_j}\right)$. By the continuity of φ we get that U^* is a compact neighbourhood of A^* and $\varphi(U^*) \subseteq U$. Let $V \subseteq U^*$ be a closed interval such that $\operatorname{card}(\varphi(V)) > 1$, if $\varphi(V) \cap C(f) \neq \emptyset$. Moreover, we assume that if there is an $x \in C(f) \setminus \{c_1, c_2, \ldots, c_k\}$ with $V \cap (x_-, x_+) \neq \emptyset$ then $[x_-, x_+] \subseteq V$. We get

$$\varphi(U^* \cap \widetilde{f}(V)) \subseteq U \cap \overline{f(\varphi(V))}\,, \tag{13}$$

since for $x \in \varphi(U^* \cap \widetilde{f}(V))$ the inclusion is satisfied by (4) in the case $x \notin \{c_1, c_2, \ldots, c_k\}$, and it follows from our assumptions on V, if x is a discontinuity point. If $\operatorname{card}(\varphi(V)) = 1$, then $V = \{x\}$ for some x, and $\varphi(\widetilde{f}^n(x)) \notin C(f)$ for all $n \in \mathbb{N}$. The set $U^* \cap \widetilde{f}(V)$ is always a union of finitely many pairwise disjoint compact intervals. Let W be one of these intervals. If $\operatorname{card}(\varphi(V)) > 1$ and $\operatorname{card}(\varphi(W)) = 1$ then $\varphi(W) = \{x\}$, where x is an endpoint of U_i for some $i \in \{1, 2, \ldots, r\}$. Hence $\varphi(W) \cap C(f) = \emptyset$ by our choice of U. Therefore $\varphi(W) \cap C(f) \neq \emptyset$ implies $\operatorname{card}(\varphi(W)) > 1$. Furthermore, if there is an $x \in C(f) \setminus \{c_1, c_2, \ldots, c_k\}$ with $W \cap (x_-, x_+) \neq \emptyset$ then $[x_-, x_+] \subseteq W$. Therefore we can iterate (13), and we obtain

$$\varphi\left(\bigcup_{n=1}^{\infty} (\widetilde{f}_{U^*})^n(V)\right) \subseteq \bigcup_{n=1}^{\infty} (f_U)^n(\varphi(V))\,. \tag{14}$$

Now let $x \in A$. Then there is a $z \in A^*$ with $\varphi(z) = x$. Since $A^* \in H(\widetilde{f})$, by Theorem 2.12 of [3] we get that A^* is locally saturating. Hence z has an A^*-covering side T, say, the left side. Suppose that $x \in C(f)$ and $z = x_+$. Choose a left neighbourhood V of z such that $V \subseteq \left[x_+ - \frac{1}{3}(x_+ - x_-), x_+\right]$. Then $\widetilde{f}^n(V) = \left\{\widetilde{f}^n(z)\right\}$ for all $n \in \mathbb{N}$, and therefore also the right side is A^*-covering. Hence we may assume that $z = x_-$, if $x \in C(f)$. Let $V = [u, z]$ be a left neigbourhood of z. We may assume that $u \notin (v_-, v_+)$ for some $v \in C(f)$, because otherwise we can replace $[u, z]$ by $[v_+, z]$. As $z = x_-$, if $x \in C(f)$ we get that $\operatorname{card}(\varphi(V)) > 1$, and $\varphi(V)$ is a left neighbourhood of x. Moreover, by the choice of V we obtain that $v \in C(f)$ with $V \cap (v_-, v_+) \neq \emptyset$ implies $[v_-, v_+] \subseteq V$. Therefore (14) shows that the left side is A-covering. Hence A is locally saturating for f.

Consider the case that A is infinite. Then Lemma 5 shows that $A = \varphi(A^*)$ for some $A^* \in H(f^*)$, and therefore A is locally saturating for f. Now assume that A is finite. If A is an f-cycle we are done since the map f is continuous at any discontinuity point from one side and hence the A-covering sides are the sides from which f is continuous. In the other case we may assume that the trajectory of y is disjoint from any point in A. This implies that this trajectory is infinite, and therefore $f^n(x) \notin \{c_1, c_2, \dots, c_k\}$ for all sufficiently large n. By (1) of Lemma 1 we get that $A = \varphi(A^*)$ for some $A^* \in H(f^*)$, and hence A is locally saturating for f. $\square$

Remark 5. The converse statement is far not true. To show the difficulties consider the following example of a map $f \in \mathcal{P}$ with unique discontinuity point $c = \frac{1}{2}$. We let f connect linearly the points $(0, \frac{1}{2})$, $(\frac{1}{4}, 0)$, $((\frac{1}{2})_-, 1)$, $((\frac{1}{2})_+, 0)$, $(\frac{3}{4}, 1)$ and $(1, \frac{1}{2})$, and we let $f(\frac{1}{2})$ be 0 or 1. This map f is shown in Figure 7. Then $A = \{0, \frac{1}{2}, 1\}$ is locally saturating but it is an ω-limit set for no $g \in \mathcal{E}(f)$, since the set A is locally repelling and either $\omega_g(0) = \{\frac{1}{2}, 1\}$ or $\omega_g(1) = \{0, \frac{1}{2}\}$. Note that $f \notin \mathcal{P}_1$, but $f \in \mathcal{P}_3$.

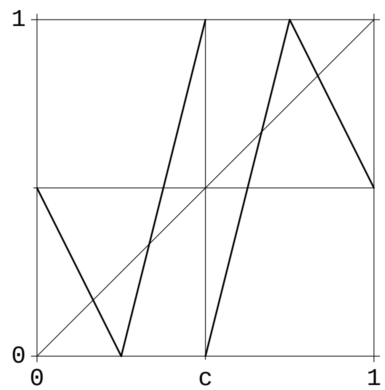

Figure 7: The map f defined in Remark 5.

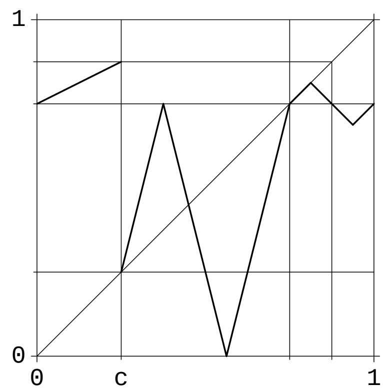

Figure 8: The map f defined in Remark 6.

However, we are able to prove the following weaker form of the converse statement. We call $f \in \mathcal{P}$ a *piecewise monotone map*, if there exist $d_0 = 0 < d_1 < d_2 < \dots < d_{q-1} < d_q = 1$ such that $f\big|_{(c_{j-1}, c_j)}$ is continuous and strictly monotone for all $j \in \{1, 2, \dots, q\}$. Then $\{c_1, c_2, \dots, c_k\} \subseteq \{d_1, d_2, \dots, d_{q-1}\}$.

An element of $\{d_1, d_2, \dots, d_{q-1}\} \setminus \{c_1, c_2, \dots, c_k\}$ is called a *turning point* of f. Denote the class of all piecewise monotone maps by $\mathcal{M}$.

Theorem 4. *Let $f \in \mathcal{M}$, and let A be a compact locally saturating set for f such that every point in $C(f) \cap A$ is accumulated from the left and from the right by points of A. Then A is an ω-limit set for f.*

Proof. Define $B := \varphi^{-1}(A) \cap X$. Then B is compact, and $\varphi(B) = A$. Moreover, either B is infinite, or $A \cap C(f) = \emptyset$ and $B = \varphi^{-1}(A)$. At first we show that B is locally saturating for $\widetilde{f}$.

Let $\widetilde{U}$ be a compact neighbourhood of B. We may assume that $\widetilde{U}$ is a finite union of pairwise disjoint closed intervals such that each of these intervals has nonempty intersection with X_0. Making $\widetilde{U}$ a bit smaller, if necessary, we may assume that $\varphi(\widetilde{U})$ does not contain discontinuity points c which are not in A. Consider an endpoint x of one of the pairwise disjoint closed intervals forming $\widetilde{U}$. As $B = \varphi^{-1}(A)$ another of these intervals has an endpoint y with $\varphi(y) = \varphi(x)$, if $\varphi(x) \in C(f) \cap A$. Making $\widetilde{U}$ a bit smaller, if necessary, we may assume that $\widetilde{U}$ is disjoint from $\big(u_- + \frac{1}{3}(u_+ - u_-), u_+ - \frac{1}{3}(u_+ - u_-)\big)$, where $u = \varphi(x)$. If $\varphi(x) \in C(f) \setminus A$, then we can make $\widetilde{U}$ a bit smaller in order to avoid $\varphi(x) \in C(f) \setminus A$. Therefore we may assume without loss of generality $\varphi(x) \notin C(f) \setminus A$ for every endpoint of one of the pairwise disjoint intervals which form $\widetilde{U}$. Moreover, we may assume that $\widetilde{U}$ is disjoint from $\big(c_- + \frac{1}{3}(c_+ - c_-), c_+ - \frac{1}{3}(c_+ - c_-)\big)$ for every discontinuity point c.

Denote by C_0 the set of all $u \in C(f) \cap A$ such that there is an endpoint x of one of the intervals forming $\widetilde{U}$ with $\varphi(x) = u$. Set $U := \varphi(\widetilde{U})$. Obviously U is the union of finitely many compact intervals. By the choice of $\widetilde{U}$, we obtain that U is a neighbourhood of any point of $A = \varphi(B)$. Moreover, again by the choice of $\widetilde{U}$, the endpoints of the pairwise disjoint intervals forming U are not in $C(f)$. The set $\varphi^{-1}(U) \setminus \widetilde{U}$ is the union of finitely many intervals each of which is contained in $\varphi^{-1}(\{u\})$ for some $u \in C_0$.

Assume that $\widetilde{V} \subseteq \widetilde{U}$ is a closed interval such that $\operatorname{card}\Big(\varphi(\widetilde{V})\Big) > 1$, if $\varphi(\widetilde{V}) \cap C(f) \neq \emptyset$. Moreover, we assume that if an endpoint x of $\widetilde{V}$ satisfies $u = \varphi(x) \in C(f)$ then $\widetilde{V} \cap (u_-, u_+) = \emptyset$ or $[u_-, u_+] \subseteq \widetilde{V}$ or $u \in C_0$. In the case $\operatorname{card}\Big(\varphi(\widetilde{V})\Big) = 1$ we get $\widetilde{V} = \{x\}$ for some x and $\varphi\left(\widetilde{f}^n(x)\right) \notin C(f)$ for all $n \in \mathbb{N}$. We have in any case that $\widetilde{U} \cap \widetilde{f}(\widetilde{V})$ is a finite union of pairwise disjoint closed intervals. Let $\widetilde{W}$ be one of these intervals. If $\operatorname{card}\Big(\varphi(\widetilde{V})\Big) > 1$ and $\operatorname{card}\Big(\varphi(\widetilde{W})\Big) = 1$ then $\varphi(\widetilde{W}) = \{u\}$, where u is an endpoint of one of the pairwise disjoint intervals forming U. This implies $u \notin C(f)$, and hence card $\left(\varphi(\widetilde{W})\right) > 1$, if $\varphi(\widetilde{W}) \cap C(f) \neq \emptyset$. Suppose that $u \in C(f)$ is such that $u \in \varphi(\widetilde{W}) \cap C(f)$, $u \notin C_0$ and $[u_-, u_+]$ is not contained in $\widetilde{W}$. Then there is an endpoint x of $\widetilde{W}$ such that $\varphi(x) = u$. Since f is a piecewise monotone map we get that $x = \widetilde{f}(y)$, where y is an endpoint of $\widetilde{V}$ or $\varphi(y)$ is a turning point of f. In both cases we have that $\widetilde{W} \cap (u_-, u_+) = \emptyset$. For $B \subseteq I$ define $\widehat{f}_U(B) := \overline{f\left(\operatorname{int}(B)\right) \cap U}$. By the choice of $\widetilde{U}$ and $\widetilde{V}$ we get

$$\widehat{f}_U\left(\varphi(\widetilde{V})\right) \subseteq \varphi\left(\widetilde{f}_{\widetilde{U}}(\widetilde{V})\right) . \tag{15}$$

Denote by $E_{\widetilde{V}}$ the set of all y such that y is an endpoint of $\widetilde{V}$ or $\varphi(y)$ is a turning point of f or $\varphi(y) \in C_0$. Set $C_{\widetilde{V}} := \left\{\varphi\left(\widetilde{f}^n(y)\right) : y \in E_{\widetilde{V}}, n \geq 0\right\} \cap C(f)$.

If $\varphi\left(\widetilde{f}(y)\right) \in C(f)$ then there is an $n \geq 0$ such that $\varphi\left(\widetilde{f}^n(y)\right)$ is a discontinuity point. Then y is eventually periodic, if $\varphi\left(\widetilde{f}^n(y)\right) \in C(f)$ for infinitely many n. Therefore $C_{\widetilde{V}}$ is finite. Let $n \in \mathbb{N}$. The above shows that $\left(\widetilde{f}_{\widetilde{U}}\right)^n\left(\widetilde{V}\right)$ is a finite union of pairwise disjoint compact intervals. Suppose that $\widetilde{W}$ is one of these intervals. Then card $\left(\varphi(\widetilde{W})\right) > 1$, if $\varphi(\widetilde{W}) \cap C(f) \neq \emptyset$. Furthermore $u \in C(f)$, $u \in \varphi(\widetilde{W}) \cap C(f)$, $u \notin C_0$ and $[u_-, u_+]$ is not contained in $\widetilde{W}$ implies that $u \in C_{\widetilde{V}}$ and $\widetilde{W} \cap (u_-, u_+) = \emptyset$. From (15) we obtain by induction that

$$\left(\widehat{f}_U\right)^n\left(\varphi(\widetilde{V})\right) \subseteq \varphi\left(\left(\widetilde{f}_{\widetilde{U}}\right)^n(\widetilde{V})\right) . \tag{16}$$

Observe that the definition of f_U and $\widehat{f}_U$ imply that $\bigcup_{n=1}^{\infty} (f_U)^n\left(\varphi(\widetilde{V})\right) = F \cup \bigcup_{n=1}^{\infty}\left(\widehat{f}_U\right)^n\left(\varphi(\widetilde{V})\right)$, where F is finite. Set $U_0 := \overline{\widetilde{U} \setminus \left(\bigcup_{u \in C_{\widetilde{V}}} \varphi^{-1}\{u\}\right)}$. Then U_0 is a finite union of intervals, since $C_{\widetilde{V}}$ is finite.

Let $b \in B$, set $a := \varphi(b)$, and let T be an A-covering side of a. Set $\widetilde{b} := b$, if $a \notin C(f)$, $\widetilde{b} := a_+$, if $a \in C(f)$ and T is the right side, and $\widetilde{b} := a_-$, if $a \in C(f)$ and T is the left side. We show that T is a B-covering side of $\widetilde{b}$. Let $\widetilde{V} \subseteq \widetilde{U}$ be a T-neighbourhood of $\widetilde{b}$. Then $\widetilde{V} = [x, \widetilde{b}]$ or $\widetilde{V} = [\widetilde{b}, x]$. Making $\widetilde{V}$ a bit smaller, if necessary, we may assume that $\varphi(x) \notin C(f)$. The set $V := \varphi(\widetilde{V})$ is a T-neighbourhood of a. As T is an A-covering side of a and as $\bigcup_{n=1}^{\infty} (f_U)^n\left(\varphi(\widetilde{V})\right) \setminus \bigcup_{n=1}^{\infty}\left(\widehat{f}_U\right)^n\left(\varphi(\widetilde{V})\right)$ is finite, there are intervals $J_1, J_2, \ldots, J_m$ such that $A \subseteq \overline{J_1 \cup J_2 \cup \cdots \cup J_m}$ and

$J_1 \cup J_2 \cup \cdots \cup J_m \subseteq \bigcup_{n=1}^{\infty} (\widehat{f}_U)^n(V)$. Using also (16) we get

$$J_1 \cup J_2 \cup \cdots \cup J_m \subseteq \bigcup_{n=1}^{\infty} (\widehat{f}_U)^n(V) \subseteq$$
$$\bigcup_{n=1}^{\infty} \varphi\left(\left(\widetilde{f}_{\widetilde{U}}\right)^n (\widetilde{V})\right) = \varphi\left(\bigcup_{n=1}^{\infty} \left(\widetilde{f}_{\widetilde{U}}\right)^n (\widetilde{V})\right).$$

For $j \in \{1, 2, \ldots, m\}$ define $\widetilde{J}_j := \varphi^{-1}(J_j) \cap U_0$. Since U_0 is a finite union of intervals we get that $\widetilde{J}_j$ is a finite union of intervals. Therefore $\widetilde{J} := \widetilde{J}_1 \cup \widetilde{J}_2 \cup \cdots \cup \widetilde{J}_m$ is a finite union of intervals. Since every point in $C(f) \cap A$ is accumulated from the left and from the right by points of A we get that $B \subseteq \overline{\widetilde{J}}$. Let $x \in \widetilde{J}$. Then $\varphi(x) \in \bigcup_{n=1}^{\infty} (\widehat{f}_U)^n(V)$. If $\varphi(x) \notin C(f)$, then $x \in \bigcup_{n=1}^{\infty} \left(\widetilde{f}_{\widetilde{U}}\right)^n (\widetilde{V})$. Otherwise $u := \varphi(x) \in C(f) \setminus C_{\widetilde{V}}$, and by our choice of $\widetilde{U}$ and $\widetilde{V}$ we obtain $[c_-, c_+] \subseteq \bigcup_{n=1}^{\infty} \left(\widetilde{f}_{\widetilde{U}}\right)^n (\widetilde{V})$. Therefore $\widetilde{J} \subseteq \bigcup_{n=1}^{\infty} \left(\widetilde{f}_{\widetilde{U}}\right)^n (\widetilde{V})$, and hence the side T is a B-covering side of $\widetilde{b}$.

In particular this shows that T is a B-covering side of b, if $b \in B \setminus \varphi^{-1}(C(f))$. Otherwise $b = a_+$ or $b = a_-$. We consider at first the case $b = a_+$. Let $\widetilde{V} \subseteq \widetilde{U}$ be a right neighbourhood of b. Then $V := \varphi(\widetilde{V})$ is a right neighbourhood of a. Since a is accumulated from the right by points of A there is an $x \in A \cap \operatorname{int}(V)$. As x has an A-covering side T, the side T is B-covering for $\widetilde{x}$, where $\varphi(\widetilde{x}) = x$, and $\widetilde{x} = x_-$, if $x \in C(f)$ and T is the left side, and $\widetilde{x} = x_+$, if $x \in C(f)$ and T is the right side. Taking a T-neighbourhood $\widetilde{V}_1 \subseteq \widetilde{V}$ of $\widetilde{x}$, there exist intervals $\widetilde{J}_1, \widetilde{J}_2, \ldots, \widetilde{J}_m$ such that (12) holds. Hence $B \subseteq \overline{\widetilde{J}_1 \cup \widetilde{J}_2 \cup \cdots \cup \widetilde{J}_m}$ and $\widetilde{J}_1 \cup \widetilde{J}_2 \cup \cdots \cup \widetilde{J}_m \subseteq \bigcup_{n=1}^{\infty} \left(\widetilde{f}_{\widetilde{U}}\right)^n (\widetilde{V}_1) \subseteq \bigcup_{n=1}^{\infty} \left(\widetilde{f}_{\widetilde{U}}\right)^n (\widetilde{V})$, showing that for $b = a_+$ the right side is B-covering. An analogous argument shows that for $b = a_-$ the left side is B-covering. Hence B is locally saturating.

Since $\widetilde{f}$ is continuous and B is locally saturating, Theorem 2.12 of [3] implies that B is an ω-limit set for $\widetilde{f}$. Hence there is an x_0 with $B = \omega_{\widetilde{f}}(x_0)$. Assume that $\widetilde{f}^n(x_0) \notin X_0$ for infinitely many n. Then there is a discontinuity point c such that $\varphi\left(\widetilde{f}^n(x_0)\right) = c$ for infinitely many n. As $B \subseteq X$, $\widetilde{f}^n(x_0)$ cannot be in the middle third of I_c for infinitely many n. Therefore there is an n_0 with $\widetilde{f}^{n_0}(x_0) \in X$, and hence $\widetilde{f}^n(x_0) \in X$ for all $n \geq n_0$, because X is $\widetilde{f}$-invariant. This implies that $\widetilde{f}^n(x_0) = c_-$ or $\widetilde{f}^n(x_0) = c_+$ for infinitely many n. Hence B is a periodic orbit, and $A = \varphi(B)$ is finite and contains c. Then $c \in C(f) \cap A$ is isolated in A both from the left and the right, which contradicts our assumption on A. Therefore there is an N such that $\widetilde{f}^n(x_0) \in X_0$ for all $n \geq N$. By (2) of Lemma 1, $A = \varphi(B)$ is an ω-limit set for f. $\square$

Remark 6. In Theorem 4, the condition that any discontinuity point in A is isolated in A neither from the left, nor from the right, cannot be omitted as the following example shows. Define $f : [0,1] \to [0,1]$ by $f(x) := \frac{x}{2} + \frac{3}{4}$ for $x \in \left[0, \frac{1}{4}\right]$, $f(x) := 4x - \frac{3}{4}$ for $x \in \left(\frac{1}{4}, \frac{3}{8}\right]$, $f(x) := \frac{9}{4} - 4x$ for $x \in \left[\frac{3}{8}, \frac{9}{16}\right]$, $f(x) := 4x - \frac{9}{4}$ for $x \in \left[\frac{9}{16}, \frac{3}{4}\right]$, $f(x) := x$ for $x \in \left[\frac{3}{4}, \frac{13}{16}\right]$, $f(x) := \frac{13}{8} - x$ for $x \in \left[\frac{13}{16}, \frac{15}{16}\right]$, and $f(x) := x - \frac{1}{4}$ for $x \in \left[\frac{15}{16}, 1\right]$. This map is shown in Figure 8. Obviously $f \in \mathcal{M}$, and $c := \frac{1}{4}$ is the unique discontinuity point of f. Set

$$A := \left(\left(\frac{1}{4}, \frac{3}{4} \right] \setminus \bigcup_{n=0}^{\infty} f^{-n} \left[0, \frac{1}{4} \right) \right) \cup \left\{ \frac{1}{4} \right\} \cup \left\{ \frac{7}{8} \right\} .$$

Note that c is not isolated from the right in A. This set A cannot be an ω-limit set for f, because every orbit coming close to $\frac{7}{8}$ is eventually constant (since $f(x) \leq \frac{7}{8}$ for all x). On the other hand, A is locally saturating. For the point c the right side is A-covering, because any right neighbourhood of c eventually covers $\left(\frac{1}{4}, \frac{3}{4}\right]$, the image of this interval contains $\left[0, \frac{1}{4}\right]$, and the latter interval is mapped to $\left[\frac{3}{4}, \frac{7}{8}\right]$. In the case of $\frac{7}{8}$ the right side is A-covering (using similar arguments as above). For any other point $x \in A$ this point is not isolated in A from at least one side, and this side is A-covering (again by similar reasoning as above).

Open problems. The results given in our paper are not optimal. Actually, it seems that more detailed analysis of the results obtained in the paper would lead to a characterization of maps $f \in \mathcal{P}$ for which the space $H(f)$ of ω-limit sets equipped with the Hausdorff metric is compact. The result in Theorem 4 is the first approximation of the condition sufficient for a set to be an ω-limit set. We conjecture that the condition "no point in $C(f) \cap A$ is isolated in A from the left or from the right" can be replaced by the weaker condition "no discontinuity point in A is isolated in A from the left or from the right". To obtain this, or even a stronger result, it would be good to follow the original argument for continuous maps given in [3].

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References

[1] L. Block, W. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics 1513, Springer-Verlag, Berlin, 1992.

[2] L. Block, E. Coven, *ω-limit sets for maps of the interval*, Ergodic Theory Dynam. Systems **6** (1986), 335–344.

[3] A. Blokh, A. M. Bruckner, P. Humke, J. Smítal, *The space of ω-limit sets of a continuous map of the interval*, Trans. Amer. Math. Soc. **348** (1996), 1357–1372.

[4] K. Brucks, H. Bruin, *Topics from one-dimensional dynamics*, Cambridge University Press, Cambridge, 2004.

[5] A. Denjoy, *Sur les courbes définies par les équations différentielles à la surface du tore*, J. Math. Pures Appl. **11** (1932), 333–375.

[6] J. L. García Guirao, *Hausdorff compactness on a space of ω-limit sets*, Topology Appl. **153** (2005), 833–843.

[7] W. Gottschalk, G. Hedlund, *Topological dynamics*, Amer. Math. Soc. Coll. Publ. 36, American Mathematical Society, Providence, 1955.

[8] F. Hofbauer, *On intrinsic ergodicity of piecewise monotonic transformations with positive entropy*, Israel J. Math. **34** (1979), 213–237; *Part 2*, Israel J. Math. **38** (1981), 107–115.

[9] F. Hofbauer, *Piecewise invertible dynamical systems*, Probab. Theory Related Fields **72** (1986), 359–386.

[10] F. Hofbauer, G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, Math. Z. **180** (1982), 119–140.

[11] M. Misiurewicz, P. Raith, *Strict inequalities for the entropy of transitive piecewise monotone maps*, Discrete Contin. Dyn. Syst. **13** (2005), 451–468.

[12] M. Misiurewicz, W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. **67** (1980), 45–63.

[13] P. Raith, *Continuity of the Hausdorff dimension for piecewise monotonic maps*, Israel J. Math. **80** (1992), 97–133.

[14] T. H. Steele, *The persistence of ω-limit sets under perturbation of the generating function*, Real Anal. Exchange **26** (2000/2001), 963–974.

[15] P. Walters, *Equilibrium states for β-transformations and related transformations*, Math. Z. **159** (1978), 65–88.

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