

This paper has been published after editorial input in
Discrete Contin. Dyn. Syst. **38** (2018), 2527–2539,
© American Institute of Mathematical Sciences, Springfield.

Stability of the distribution function for piecewise monotonic maps on the interval

MICHAL MÁLEK AND PETER RAITH

Abstract. For piecewise monotonic maps the notion of approximating distribution function is introduced. It is shown that for a mixing basic set it coincides with the usual distribution function. Moreover, it is proved that the approximating distribution function is upper semi-continuous under small perturbations of the map.

Introduction

The notion of distributional chaos has been introduced by Schweizer and Smítal in [12]. To this end distribution functions were introduced. Given $t \in \mathbb{R}$ and two points x, y in a dynamical system one counts the relative number of times when the distance of the orbits of x and y is smaller than t . Denote the limit superior of this sequence by $U_{x,y}(t)$ and its limit inferior by $F_{x,y}(t)$. In this way one obtains two functions $U, F : \mathbb{R} \rightarrow [0, 1]$, called the upper and lower distribution function. We will call the lower distribution function simply the distribution function. A dynamical system is called distributionally chaotic, if there are x, y and $t > 0$ with $F_{x,y}(t) < 1$ and $U_{x,y}(s) = 1$ for all $s > 0$. More details on distributional chaos can be found in the nice survey paper [11] and also in [3].

We like to consider the distribution function for piecewise monotonic maps $T : [0, 1] \rightarrow [0, 1]$. This means that there exists a finite partition

2010 *Mathematics Subject Classification.* 37E05, 37D45, 37C75.

Key words and phrases. Distributional chaos, piecewise monotonic map, distribution function, interval map, perturbation, basic set.

The research was partially supported by the projects 42p11 and 38p10 of AKTION Česká republika – Österreich, and by project MSM4781305904 of the Czech Ministry of Education.

of $[0, 1]$ into intervals such that the restriction of T to such an interval is continuous and strictly monotonic. However, T needs not be continuous at the endpoints of intervals of monotonicity. Because of these discontinuities it is much harder to work with the distribution function. Therefore we introduce the notion of approximating distribution functions. It is defined similar as the usual distribution function but taking into account only points x, y in Markov subsets avoiding the points of discontinuity and turning points.

For a piecewise monotonic map a set B is called basic set, if it is a maximal topologically transitive subset with positive topological entropy. It will be proved in Theorem 1 that the approximating distribution function of a mixing basic set coincides with usual one in all points where the usual distribution function is right continuous. Hence they differ in at most countably many points. In particular defining distributional chaos one can use the approximating distribution function instead of the usual one.

The main question addressed in this paper is the behaviour of the approximating distribution function under small perturbations of the piecewise monotonic map. For continuous interval maps (but not necessarily piecewise monotonic) the upper semi-continuity of distribution functions has been proved by Francisco Balibrea, Bert Schweizer, Abe Sklar and Jaroslav Smítal in [2] (Theorem 4.5 and Corollary 4.6 of [2]). We will prove in Theorem 2 that for mixing basic sets of (not necessarily continuous) piecewise monotonic maps the approximating distribution function is upper semi-continuous. This implies that distributional chaos (defined using the approximating distribution functions) is stable under small perturbations of the map.

Finally we consider again a mixing basic set B (as defined above) of a piecewise monotonic map and its approximating distribution function G . Consider finitely many $x_1, x_2, \dots, x_u \in [0, 1]$ and a decreasing sequence (J_n) of unions of u intervals forming a neighbourhood of $\{x_1, x_2, \dots, x_u\}$ such that $\bigcap_{n=1}^{\infty} \overline{J_n} = \{x_1, x_2, \dots, x_u\}$. For each n let G_n be the approximating distribution function of the set of all points of B whose orbits omit J_n . In Theorem 3 it is proved that $\lim_{n \rightarrow \infty} G_n(t) = G(t)$ for all t where G is right continuous.

The authors like to thank Franz Hofbauer for useful discussions on the topic of this paper, in particular for a lot of useful suggestions concerning the proofs.

1. The distribution function of a dynamical system

Assume that $M \neq \emptyset$ is a metric space (with metric d), and let $T : M \rightarrow M$ be a map. If $x, y \in M$, then set

$$(1) \quad \begin{aligned} F_{x,y}(t) &:= \liminf_{n \rightarrow \infty} \frac{1}{n} \text{card}\{k \in \{0, 1, \dots, n-1\} : d(T^k(x), T^k(y)) \leq t\} \\ &\quad \text{and} \\ U_{x,y}(t) &:= \limsup_{n \rightarrow \infty} \frac{1}{n} \text{card}\{k \in \{0, 1, \dots, n-1\} : d(T^k(x), T^k(y)) \leq t\} \end{aligned}$$

for $t \in \mathbb{R}$, $t > 0$. Obviously both $F_{x,y}$ and $U_{x,y}$ are increasing functions with values in $[0, 1]$, and $F_{x,y} \leq U_{x,y}$. Moreover, note that $F_{x,x}(t) = U_{x,x}(t) = 1$ for all $t > 0$. We define the *lower distribution function* F of the dynamical system (M, T) by

$$(2) \quad F(t) := \inf \{F_{x,y}(t) : x, y \in M \text{ with } U_{x,y}(s) = 1 \text{ for all } s > 0\}$$

for $t \in \mathbb{R}$, $t > 0$. It is obvious that $F : \{t \in \mathbb{R} : t > 0\} \rightarrow [0, 1]$ is increasing. If $A \subseteq M$ is T -invariant, then we call the lower distribution function F of $(A, T|_A)$ the lower distribution function of A . Throughout this paper we will call the lower distribution function simply the distribution function.

The distribution function has been introduced by Bert Schweizer and Jaroslav Smital in [12] for continuous maps. A dynamical system is said to be distributionally chaotic, if the distribution function F satisfies $F(t) < 1$ for some $t > 0$. For continuous interval maps it is proved in [12] that distributional chaos is equivalent to positive topological entropy and to the existence of a basic set. In the case of maps on a graph this has been proved by Roman Hric and Michal Málek in [6]. This result is no longer true in higher dimensions as shown by Marta Štefánková in [1] for two dimensional skew products.

A map $T : [0, 1] \rightarrow [0, 1]$ is called *piecewise monotonic* with respect to $c_0 = 0 < c_1 < \dots < c_N = 1$, if T is continuous and strictly monotonic on (c_{j-1}, c_j) for $j \in \{1, 2, \dots, N\}$. Note that it may be discontinuous in $E := \{c_0, c_1, \dots, c_N\}$. If T is not continuous, then it is much more difficult to deal with the distribution function as defined in (2). Therefore we use another approach in order to overcome these problems.

Set $\mathcal{Z} := \{(c_{j-1}, c_j) : 1 \leq j \leq N\}$ and for $n \in \mathbb{N}$ set $\mathcal{Z}_n := \bigvee_{j=0}^{n-1} T^{-j} \mathcal{Z} = \left\{ Z = \bigcap_{j=0}^{n-1} T^{-j} Z_j : Z \neq \emptyset, Z_0, Z_1, \dots, Z_{n-1} \in \mathcal{Z} \right\}$. Furthermore, for $n \in \mathbb{N}$ let C_n be the union of $\{c_0, c_1, \dots, c_N\}$ and those intervals in \mathcal{Z}_n having an

endpoint in $\{c_0, c_1, \dots, c_N\}$. Observe that $C_1 = [0, 1] \supseteq C_2 \supseteq C_3 \supseteq \dots$. Let $A \subseteq [0, 1]$ be closed and T -invariant. For $n \in \mathbb{N}$ define

$$(3) \quad A_n := A \setminus \left(\bigcup_{k=0}^{\infty} T^{-k}(C_n) \right).$$

Note that A_n is closed for any n , and $A_1 = \emptyset \subseteq A_2 \subseteq A_3 \subseteq \dots$. If there exists an n with $A_n \neq \emptyset$, then we define the *approximating distribution function* G of A by

$$(4) \quad \begin{aligned} G(t) &:= \inf \{F_{x,y}(t) : x, y \in A_n \text{ for some } n \\ &\quad \text{and } U_{x,y}(s) = 1 \text{ for all } s > 0\} \end{aligned}$$

for $t \in \mathbb{R}$, $t > 0$. Again it is obvious that $G : \{t \in \mathbb{R} : t > 0\} \rightarrow [0, 1]$ is increasing.

We call a closed T -invariant set $A \subseteq [0, 1]$ *topologically transitive*, if the map $T|_A$ is topologically transitive, this means there exists an element in A whose orbit is dense in A . If there does not exist a topologically transitive set A_0 with $A \subsetneq A_0$, then A is called *maximal topologically transitive*. A closed T -invariant set $B \subseteq [0, 1]$ is called a *basic set*, if B is maximal topologically transitive, and $h_{\text{top}}(T|_B) > 0$. One calls a closed T -invariant set $A \subseteq [0, 1]$ *mixing*, if for any open U with $A \cap U \neq \emptyset$ there is an $n \in \mathbb{N}$ with $T^n(A \cap U) = A$.

Finally we define a topology on the family of all maps $f : [0, 1] \rightarrow [0, 1]$. Suppose that $g : [0, 1] \rightarrow [0, 1]$ and let $\varepsilon > 0$. Then a map $h : [0, 1] \rightarrow [0, 1]$ is called to be in the ε -*neighbourhood* of g , if for all $x \in [0, 1]$ there is a $y \in [0, 1]$ such that $|x - y| < \varepsilon$ and $|h(x) - g(y)| < \varepsilon$. Moreover, we say that a map $h : [0, 1] \rightarrow [0, 1]$ is in the *upper ε -neighbourhood* of g , if for all $x \in [0, 1]$ there is a $y \in [0, 1]$ with $|x - y| < \varepsilon$ and $h(x) < g(y) + \varepsilon$.

2. Alternative definitions of the approximating distribution function

Let $T : [0, 1] \rightarrow [0, 1]$ be a piecewise monotonic map and let $A \subseteq [0, 1]$ be closed and T -invariant. For $n \in \mathbb{N}$ define

$$(5) \quad \mathcal{R}_n(A) := \{(x, y) \in A_n \times A_n : U_{x,y}(s) = 1 \text{ for all } s > 0\},$$

where A_n is defined as in (3). Now set

$$(6) \quad \mathcal{R}(A) := \bigcup_{n=1}^{\infty} \mathcal{R}_n(A).$$

If there exists an n with $A_n \neq \emptyset$, then we obtain by (4), (5) and (6) that

$$(7) \quad G(t) := \inf_{(x,y) \in \mathcal{R}(A)} F_{x,y}(t)$$

for all $t \in \mathbb{R}$ with $t > 0$, where G is the approximating distribution function of A . We give two more similar definitions which will be useful later. For these definitions we assume that there exists an n with $A_n \neq \emptyset$. Set $\mathcal{Q}(A) := \bigcup_{n=1}^{\infty} A_n \times A_n$ and define

$$(8) \quad L(t) := \inf_{(x,y) \in \mathcal{Q}(A)} F_{x,y}(t)$$

for all $t > 0$. Obviously L is increasing and $L(t) \leq G(t)$ for all $t > 0$. For $n \in \mathbb{N}$ let $P_n(A)$ be the set of all periodic points in A having period n . If for a $q \in \mathbb{N}$ there is an $n \geq q$ with $P_n(A) \neq \emptyset$, then set $\mathcal{P}_q(A) := \bigcup_{n=q}^{\infty} P_n(A) \times P_n(A)$, and define

$$(9) \quad H_q(t) := \inf_{(x,y) \in \mathcal{P}_q(A)} F_{x,y}(t)$$

for all $t > 0$. Also in this case it is obvious that H_q is increasing.

As introduced above let \mathcal{Z} be the collection of intervals of monotonicity of T . We call \mathcal{Y} a *refinement* of \mathcal{Z} , if \mathcal{Y} consists of finitely many pairwise disjoint open intervals such that for every $Y \in \mathcal{Y}$ there is a $Z \in \mathcal{Z}$ with $Y \subseteq Z$, and $[0, 1] \subseteq \bigcup_{Y \in \mathcal{Y}} \overline{Y}$. In order to find suitable approximations of basic sets, we define Markov subsets. Suppose that $X \subseteq [0, 1]$ is nonempty, closed and T -invariant. We call \mathcal{Y} a Markov partition refining \mathcal{Z} of X , if \mathcal{Y} is a refinement of \mathcal{Z} such that for every $Y \in \mathcal{Y}$ with $Y \cap X \neq \emptyset$ there are $Y_1, Y_2, \dots, Y_r \in \mathcal{Y}$ satisfying $\overline{T(Y \cap X)} = \overline{T(Y)} \cap X = \bigcup_{j=1}^r \overline{Y_j} \cap X$. A nonempty set $X \subseteq [0, 1]$ is called a *Markov subset*, if X is compact, T -invariant, disjoint from C_n for some $n \geq 1$, and has a finite Markov partition refining \mathcal{Z} .

In the proofs of our next results we need as our main tool the Markov diagram of T . It has been introduced by Franz Hofbauer and describes the orbit structure of T . A description of the Markov diagram can be found e.g. in [5]. Now we recall its definition and its most important properties.

Assume that \mathcal{Y} is a refinement of \mathcal{Z} . Let D be a nonempty open subinterval of an element of \mathcal{Y} . A nonempty $C \subseteq [0, 1]$ is called *successor* of D , if there exists a $Y \in \mathcal{Y}$ with $C = T(D) \cap Y$, and we write $D \rightarrow C$. Obviously every successor C of D is again an open subinterval of an element of \mathcal{Y} . Let \mathcal{D} be the smallest set with $\mathcal{Y} \subseteq \mathcal{D}$ and such that $D \in \mathcal{D}$ and $D \rightarrow C$ imply $C \in \mathcal{D}$. Then $(\mathcal{D}, \rightarrow)$ is called the *Markov diagram* of T with respect to \mathcal{Y} . The set \mathcal{D} is at most countable and its elements

are nonempty open subintervals of elements of \mathcal{Y} . Define $\mathcal{D}(0) := \mathcal{Y}$, and for $n \in \mathbb{N}$ set $\mathcal{D}(n) := \{D \in \mathcal{D} : \exists C \in \mathcal{D}(n-1) \text{ with } C \rightarrow D\}$. Then $\mathcal{D}(0) \subseteq \mathcal{D}(1) \subseteq \mathcal{D}(2) \subseteq \dots \subseteq \mathcal{D}$ and $\mathcal{D} = \bigcup_{n=0}^{\infty} \mathcal{D}(n)$.

We call $D_0 \rightarrow D_1 \rightarrow D_2 \rightarrow \dots$ an *infinite path* in $(\mathcal{D}, \rightarrow)$, if $D_j \in \mathcal{D}$ for all $j \geq 0$ and $D_{j-1} \rightarrow D_j$ for all $j \in \mathbb{N}$. An infinite path $D_0 \rightarrow D_1 \rightarrow D_2 \rightarrow \dots$ is called *periodic*, if there is an $n \geq 1$ such that $D_{n+k} = D_k$ for all $k \geq 0$. Let $\mathcal{C} \subseteq \mathcal{D}$. Then $D_0 \rightarrow D_1 \rightarrow \dots \rightarrow D_n$ is called a *finite path* in \mathcal{C} , if $D_j \in \mathcal{C}$ for all $j \in \{0, 1, \dots, n\}$ and $D_{j-1} \rightarrow D_j$ for all $j \in \{1, 2, \dots, n\}$. The set \mathcal{C} is called *irreducible*, if for every $C, D \in \mathcal{C}$ there exists a finite path $D_0 \rightarrow D_1 \rightarrow \dots \rightarrow D_n$ in \mathcal{C} with $D_0 = C$ and $D_n = D$. If \mathcal{C} is irreducible and $C \in \mathcal{C}$, then define $N(C)$ as the set of all n such that there is a path $D_0 \rightarrow D_1 \rightarrow \dots \rightarrow D_n$ in \mathcal{C} with $D_0 = C$ and $D_n = C$. Then the greatest common divisor of $N(C)$ equals the greatest common divisor of $N(D)$ for any $D \in \mathcal{C}$. If the greatest common divisor of $N(C)$ equals 1, then \mathcal{C} is called *aperiodic*. We call an irreducible $\mathcal{C} \subseteq \mathcal{D}$ *maximal irreducible*, if there is no irreducible $\mathcal{C}' \subseteq \mathcal{D}$ with $\mathcal{C} \subsetneq \mathcal{C}'$. An infinite path $D_0 \rightarrow D_1 \rightarrow D_2 \rightarrow \dots$ *represents* $x \in [0, 1]$, if $T^j(x) \in \overline{D_j}$ for all $j \geq 0$.

For a maximal irreducible $\mathcal{C} \subseteq \mathcal{D}$ let $K(\mathcal{C})$ be the set of all x such that x is represented by an infinite path in \mathcal{C} and x is not contained in the interior of an interval I satisfying that $T^n|_I$ is monotonic for all n . Suppose that B is a basic set of T . By Theorem 11 of [5] there exists a maximal irreducible $\mathcal{C} \subseteq \mathcal{D}$ such that $B = K(\mathcal{C})$. For a $\mathcal{C}' \subseteq \mathcal{C}$ define $K(\mathcal{C}')$ as the set of all $x \in K(\mathcal{C})$ which can be represented by an infinite path in \mathcal{C}' . If \mathcal{C}' is irreducible, then $K(\mathcal{C}')$ is topologically transitive. Moreover, for an irreducible \mathcal{C}' the set $K(\mathcal{C}')$ is mixing if and only if \mathcal{C}' is aperiodic.

As we will also need the notion of a *variant* $(\mathcal{A}, \rightarrow)$ of the Markov diagram as introduced in [7] (cf. also [10]), we briefly describe this concept. It is an orientated graph together with a function $A : \mathcal{A} \rightarrow \mathcal{D}$ such that $c \rightarrow d$ in \mathcal{A} implies $A(c) \rightarrow A(d)$ in \mathcal{D} . Moreover, for each $c \in \mathcal{A}$ the map A is bijective from $\{d \in \mathcal{A} : c \rightarrow d \text{ in } \mathcal{A}\}$ to $\{D \in \mathcal{D} : A(c) \rightarrow D \text{ in } \mathcal{D}\}$. We can write $\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{A}(n)$ with $\mathcal{A}(0) \subseteq \mathcal{A}(1) \subseteq \mathcal{A}(2) \subseteq \dots \subseteq \mathcal{A}$ such that $A(\mathcal{A}(n)) = \mathcal{D}(n)$ for all $n \geq 0$. For an irreducible $\mathcal{C} \subseteq \mathcal{A}$ we set $K(\mathcal{C}) := K(A(\mathcal{C}))$, and a path $d_0 \rightarrow d_1 \rightarrow d_2 \dots$ in \mathcal{A} represents x , if $A(d_0) \rightarrow A(d_1) \rightarrow A(d_2) \dots$ represents x . Furthermore the construction in [7] shows that for every irreducible $\mathcal{C} \subseteq \mathcal{A}$ having an element with more than one successor in \mathcal{C} the property that $A(\mathcal{C})$ is aperiodic implies that \mathcal{C} is aperiodic. The *versions of the Markov diagram* introduced in [8] (and also used in [9]) are essentially the same as variants, but contain also the orbits of some single points. Since these additional orbits of single points are not interesting for our purpose we can work with variants. Then in the conclusions of Lemma 2 in [8] and Lemma 1 in [9] we can work with variants

instead of versions, and the conclusions concerning $A(c)$ having only one element can be omitted.

Remark. Consider a finite union X of closed subintervals of $[0, 1]$ and a map $T : X \rightarrow [0, 1]$ such that there exists a finite family \mathcal{Z} of pairwise disjoint open intervals with $\bigcup_{Z \in \mathcal{Z}} \overline{Z} = X$ and $T|_Z$ is continuous and strictly monotonic for all $Z \in \mathcal{Z}$. Then one can define the Markov diagram (and also variants and versions of the Markov diagram) of T with respect to a refinement \mathcal{Y} of \mathcal{Z} in the same way as above (see also for example [7]). This can also be seen by extending T to a map on $[0, 1]$ defining it so on each maximal open subinterval I of $[0, 1] \setminus X$ that $TI \subseteq I$ and each point in I is attracted by a fixed point. Hence all results apply also in this slightly more general case.

Lemma 1. *Suppose that $T : [0, 1] \rightarrow [0, 1]$ is a piecewise monotonic map, assume that B is a mixing basic set of T , and let $m \in \mathbb{N}$. Moreover, let $x_1, x_2, \dots, x_r \in B_m$. Then there exists an integer $n \geq m$ and there exists a mixing Markov subset X of B_n with $x_1, x_2, \dots, x_r \in X$.*

Proof. Without loss of generality we may assume that $m \geq 2$. Since our assumptions imply that $x_1, x_2, \dots, x_r \in B$, we can find a refinement \mathcal{Y} of \mathcal{Z} such that $\bigcap_{n=1}^{\infty} \overline{Y_n(x_j)} = \{x_j\}$ for every $j \in \{1, 2, \dots, r\}$, where $\mathcal{Y}_n := \bigvee_{j=0}^{n-1} T^{-j}\mathcal{Y}$ and $Y_n(x_j)$ is an element of \mathcal{Y}_n with $x_j \in \overline{Y_n(x_j)}$ and $Y_n(x_j) \cap B \neq \emptyset$.

Set $C_\infty := \bigcap_{n=1}^{\infty} C_n$. Then C_∞ is a union of at most $N + 1$ intervals, and $\{c_0, c_1, \dots, c_N\} \subseteq C_\infty$. Hence there are $0 \leq a_1 \leq a_2 < a_3 \leq a_4 < \dots < a_{2v-1} \leq a_{2v} \leq 1$ such that $C_\infty = \bigcup_{u=1}^v [a_{2u-1}, a_{2u}]$. Define (cf. (1.2) of [9])

$$X_\infty := \bigcap_{k=0}^{\infty} \overline{\left[0, 1\right] \setminus \left(\bigcup_{u=1}^v T^{-k}(a_{2u-1}, a_{2u})\right)}.$$

If $I \subseteq C_\infty$ is a nonempty open interval which does not contain an element of $\{c_0, c_1, \dots, c_N\}$, then T^n is strictly monotonic on I for each n , and therefore it cannot contain a point in a basic set of T . Hence $B \subseteq X_\infty$. For sufficiently large n the set C_n is a union of v pairwise disjoint open intervals. Therefore there are $0 = a_1^{(n)} < a_2^{(n)} < a_3^{(n)} < a_4^{(n)} < \dots < a_{2v-1}^{(n)} < a_{2v}^{(n)} = 1$ such that $C_n = \bigcup_{u=1}^v (a_{2u-1}^{(n)}, a_{2u}^{(n)})$. Moreover, for any $l \in \{1, 2, \dots, 2v\}$ we have $\lim_{n \rightarrow \infty} a_l^{(n)} = a_l$. This means that “ C_n converges to C_∞ ” in the sense described in [9]. Set $X_n := [0, 1] \setminus (\bigcup_{k=0}^{\infty} T^{-k}(C_n))$ (note that this corresponds to (1.2) of [9]).

Assume that $n \in \mathbb{N} \cup \{\infty\}$. Next we define the Markov diagram $(\tilde{\mathcal{D}}_n, \rightarrow)$ of $T|_{X_n}$ with respect to \mathcal{Y}_m . To this end set $\tilde{\mathcal{Y}} := \{Y \setminus \overline{C_n} : Y \in \mathcal{Y}_m\}$. For a nonempty open subinterval D of an element of $\tilde{\mathcal{Y}}$ write $D \rightarrow C$, if there exists

a $Y \in \tilde{\mathcal{Y}}$ with $C = T(D) \cap Y$. Now let $\tilde{\mathcal{D}}_n$ be the smallest set containing $\tilde{\mathcal{Y}}$ and with the property that $D \in \tilde{\mathcal{D}}_n$ and $D \rightarrow C$ imply $C \in \tilde{\mathcal{D}}_n$.

Let $(\tilde{\mathcal{D}}_\infty, \rightarrow)$ be the Markov diagram of $T|_{X_\infty}$ with respect to \mathcal{Y}_m . Since $B \subseteq X_\infty$ is mixing there exists a maximal irreducible and aperiodic $\mathcal{C} \subseteq \tilde{\mathcal{D}}_\infty$ such that $B = K(\mathcal{C})$.

Now let \mathcal{E} be the set of all $Y \in \mathcal{Y}_m$ with $Y \cap B_m \neq \emptyset$. Fix a $Y \in \mathcal{E}$. As $Y \cap C_m = \emptyset$ we get that $Y \in \tilde{\mathcal{D}}_\infty$. Moreover, using again $Y \cap C_m = \emptyset$, one gets that $T(Y) \in \mathcal{Y}_{m-1}$ and its closure is the union of closures of elements of \mathcal{Y}_m . This means that all successors of Y having nonempty intersection with B_m are again in \mathcal{E} . Therefore x_j can be represented by an infinite path $C_0^{(j)} \rightarrow C_1^{(j)} \rightarrow C_2^{(j)} \rightarrow \dots$ in \mathcal{E} , if $j \in \{1, 2, \dots, r\}$. Choose a $j \in \{1, 2, \dots, r\}$. Since $x_j \in B$ we can represent x_j by an infinite path $D_0^{(j)} \rightarrow D_1^{(j)} \rightarrow D_2^{(j)} \rightarrow \dots$ in \mathcal{C} . By Theorem 1 in [5] there is an n_j such that $D_n^{(j)} = C_n^{(j)}$ for all $n \geq n_j$. Hence $\mathcal{E}_j := \{D_0^{(j)}, D_1^{(j)}, D_2^{(j)}, \dots\}$ is a finite subset of \mathcal{C} .

Choose a $C \in \bigcup_{j=1}^r \mathcal{E}_j$. As \mathcal{C} is aperiodic there exists an s , and there exist paths $E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_s$ and $D_0 \rightarrow D_1 \rightarrow \dots \rightarrow D_{s+1}$ in \mathcal{C} with $E_0 = E_s = D_0 = D_{s+1} = C$. Let $\mathcal{C}' \subseteq \mathcal{C}$ be a finite and irreducible set containing $\bigcup_{j=1}^r \mathcal{E}_j \cup \{E_0, E_1, \dots, E_s\} \cup \{D_0, D_1, \dots, D_{s+1}\}$. Since both $E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_s$ and $D_0 \rightarrow D_1 \rightarrow \dots \rightarrow D_{s+1}$ are paths in \mathcal{C}' beginning and ending in C the set \mathcal{C}' is aperiodic.

By Lemma 2 in [10] and Lemma 1 in [9] there exists an $n \geq m$ such that the following properties hold. There exist variants $(\mathcal{A}, \rightarrow)$ and $(\tilde{\mathcal{A}}, \rightarrow)$ of $\tilde{\mathcal{D}}_\infty$, resp. of $\tilde{\mathcal{D}}_n$, there exist $\tilde{\mathcal{C}} \subseteq \tilde{\mathcal{A}}$ and an irreducible and aperiodic $\tilde{\mathcal{C}}' \subseteq \mathcal{A}$ with $A(\tilde{\mathcal{C}}') = \mathcal{C}'$, and there exists a bijective function $\varphi : \tilde{\mathcal{C}} \rightarrow \tilde{\mathcal{C}}'$ such that for $c, d \in \tilde{\mathcal{C}}$ we have $c \rightarrow d$ in $(\tilde{\mathcal{A}}, \rightarrow)$ if and only if $\varphi(c) \rightarrow \varphi(d)$ in $(\mathcal{A}, \rightarrow)$. Moreover, as $C_\infty \subseteq C_n$ we get that $\tilde{A}(c) \subseteq A(\varphi(c))$. The set $\tilde{\mathcal{C}}$ is irreducible and aperiodic, because $\tilde{\mathcal{C}}'$ is irreducible and aperiodic. There exists a maximal irreducible $\tilde{\mathcal{B}} \subseteq \tilde{\mathcal{D}}_n$ with $\tilde{A}(\tilde{\mathcal{C}}) \subseteq \tilde{\mathcal{B}}$. Then the corresponding set $K(\tilde{\mathcal{B}})$ is a subset of B_n .

Define $X := K(\tilde{\mathcal{C}})$ (the set of all points in $K(\tilde{\mathcal{B}})$ which can be represented by a path in $\tilde{\mathcal{C}}$). Then $X \subseteq B_n$ is mixing, as $\tilde{\mathcal{C}}$ is aperiodic. Assume that $j \in \{1, 2, \dots, r\}$. We get that $d_0^{(j)} \rightarrow d_1^{(j)} \rightarrow d_2^{(j)} \rightarrow \dots$ is a path in $\tilde{\mathcal{C}}'$ representing x_j . Moreover, $\bigcap_{u=0}^\infty A(d_u^{(j)}) = \{x_j\}$ by the choice of \mathcal{Y} . Then $\varphi^{-1}(d_0^{(j)}) \rightarrow \varphi^{-1}(d_1^{(j)}) \rightarrow \varphi^{-1}(d_2^{(j)}) \rightarrow \dots$ is a path in $\tilde{\mathcal{C}}$. As this path represents an $x \in X$, and $\tilde{A}(c) \subseteq A(\varphi(c))$ we get that $x_j \in X$.

Because of the finiteness of $\tilde{\mathcal{C}}$ the set E of all endpoints of elements of $\tilde{A}(\tilde{\mathcal{C}})$ is finite, and also the set

$$E' := E \cup \{x : x \text{ is an endpoint of an element of } \mathcal{Y}\}$$

is finite. Let \mathcal{Y}' be the collection of all open intervals having both endpoints in E' and not containing an element of E' . Then \mathcal{Y}' is a Markov partition of X refining \mathcal{Z} . This implies that X is a Markov subset. \square

Lemma 2. *Let $T : [0, 1] \rightarrow [0, 1]$ be a piecewise monotonic map. Suppose that X is a mixing Markov subset of $[0, 1]$. Then the following properties hold.*

(a) *For every $\varepsilon > 0$ there is an $M \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, any k points $x_1, x_2, \dots, x_k \in X$, any nonnegative integers $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$ with $a_j - b_{j-1} \geq M$ for $1 \leq j \leq k$ and any integer $p \geq M + b_k - a_1$ there exists a point $x \in X$ of period p satisfying $|T^n(x) - T^{n-a_j}(x_j)| \leq \varepsilon$ for all $n \in \{a_j, a_{j+1}, \dots, b_j\}$ and all $j \in \{1, 2, \dots, k\}$.*

(b) *Suppose that $\varepsilon > 0$. Then there exists an $M \in \mathbb{N}$ such that for any points $x_1, x_2, x_3, \dots \in X$ and any integers $0 \leq a_1 \leq b_1 < a_2 \leq b_2 < a_3 \leq b_3 < \dots$ with $a_j - b_{j-1} \geq M$ for all $j \geq 2$ there exists a point $x \in X$ with $|T^n(x) - T^{n-a_j}(x_j)| \leq \varepsilon$ for all $n \in \{a_j, a_{j+1}, \dots, b_j\}$ and all $j \geq 1$.*

Proof. Fix $\varepsilon > 0$ and define $B_\varepsilon(x) := (x - \varepsilon, x + \varepsilon)$. Moreover, let \mathcal{Y} be a Markov partition of X refining \mathcal{Z} . Now we show that there is an $M \in \mathbb{N}$ such that $T^M(X \cap B_\varepsilon(x)) = X$ for all $x \in X$.

Set $\mathcal{Y}_n = \bigvee_{j=0}^{n-1} T^{-j}\mathcal{Y}$ for $n \geq 1$. By the compactness of X there exists a $k \in \mathbb{N}$ such that for each $x \in X$ the set $B_\varepsilon(x)$ contains an element of \mathcal{Y}_k (notice that the interior of a periodic or wandering interval does not belong to X). Since X is a mixing Markov subset, for every $Y \in \mathcal{Y}_k$ there is an $n(Y)$ such that $\overline{T^n(Y \cap X)} = X$ for all $n \geq n(Y)$. Set $M := \max\{n(Y) : Y \in \mathcal{Y}_k\}$. Hence $X = \overline{T^n(Y \cap X)} \subseteq \overline{T^M(X \cap B_\varepsilon(x))} \subseteq X$ for any $x \in X$.

Now one can prove (a) in the same way as in [4] (see the proof of Theorem 8.7 in [4]).

In order to prove (b) we use (a). If $k \in \mathbb{N}$, then there is a (periodic) point $y_k \in X$ satisfying $|T^n(y_k) - T^{n-a_j}(x_j)| \leq \varepsilon$ for $a_j \leq n \leq b_j$ and $1 \leq j \leq k$. Let x be a limit point of the sequence $(y_k)_{k \geq 1}$. Then x satisfies (b), since T is continuous on X . \square

For $w \in [0, 1] \setminus \bigcup_{j=0}^{\infty} T^{-j}\{c_0, c_1, \dots, c_N\}$ let $Z_n(w)$ be the unique interval in \mathcal{Z}_n containing w . We have $Z_{n+1}(w) \subseteq Z_n(w)$ for $n \geq 1$ and $\bigcap_{n=1}^{\infty} Z_n(w)$

is a single point or an interval. If x and y are in $[0, 1]$, if a and b are integers with $0 \leq a < b$ and if $t \in \mathbb{R}$, $t > 0$, then define

$$(10) \quad N_{x,y}(a, b, t) := \text{card} \{j \in \{a, a+1, \dots, b-1\} : |T^j(x) - T^j(y)| \leq t\} .$$

Lemma 3. *Assume that B is a mixing basic set or a mixing Markov subset of the piecewise monotonic map T and fix a $q \in \mathbb{N}$. Let $t > 0$ and $\varepsilon \in (0, t)$. Then there is an integer $p \geq q$ and there are u and v in $P_p(B)$ such that $\frac{1}{p}N_{u,v}(0, p, t - \varepsilon) < L(t) + \varepsilon$. In particular, $F_{u,v}(t - \varepsilon) < L(t) + \varepsilon$.*

Proof. By the definition of L there is a $(x, y) \in \mathcal{Q}(B)$ with $F_{x,y}(t) < L(t) + \frac{\varepsilon}{2}$. Then there is an $m \geq 1$ with $x, y \in B_m$ because of the definition of $\mathcal{Q}(B)$. If B is a mixing basic set then by Lemma 1 there is a mixing Markov subset X of B containing x and y . Otherwise set $X = B$, which is a mixing Markov subset. In particular, X satisfies the requirements of Lemma 2. Let M be the constant for $\frac{\varepsilon}{2}$ in (a) of Lemma 2. Because of $F_{x,y}(t) = \liminf_{n \rightarrow \infty} \frac{1}{n}N_{x,y}(0, n, t)$ there is a $b > \max\{q, \frac{2M}{\varepsilon}\}$ with $\frac{1}{b}N_{x,y}(0, b, t) < L(t) + \frac{\varepsilon}{2}$. Set $p := b + M$. By (a) of Lemma 2 there are points $u, v \in P_p(B)$ with $|T^j(u) - T^j(x)| \leq \frac{\varepsilon}{2}$ and $|T^j(v) - T^j(y)| \leq \frac{\varepsilon}{2}$ for $0 \leq j \leq b$. This implies that $|T^j(x) - T^j(y)| \leq t$ holds, whenever $0 \leq j \leq b$ and $|T^j(u) - T^j(v)| \leq t - \varepsilon$, and hence we get $N_{u,v}(0, b, t - \varepsilon) \leq N_{x,y}(0, b, t)$. Therefore we have

$$\begin{aligned} \frac{1}{p}N_{u,v}(0, p, t - \varepsilon) &\leq \frac{1}{p}N_{u,v}(0, b, t - \varepsilon) + \frac{1}{p}N_{u,v}(b, p, t - \varepsilon) \leq \\ &\leq \frac{b}{p}(L(t) + \frac{\varepsilon}{2}) + \frac{p-b}{p} < L(t) + \varepsilon . \end{aligned}$$

This shows the first assertion, since $p > q$.

As u and v have period p , it follows that

$$\frac{1}{kp}N_{u,v}(0, kp, t - \varepsilon) = \frac{1}{p}N_{u,v}(0, p, t - \varepsilon)$$

for all $k \geq 1$. This implies $F_{u,v}(t - \varepsilon) \leq \frac{1}{p}N_{u,v}(0, p, t - \varepsilon)$ and hence we obtain $F_{u,v}(t - \varepsilon) < L(t) + \varepsilon$. \square

Note that for maps $T : [0, 1] \rightarrow [0, 1]$ one always has $F_{x,y}(t) = 1$ for $t \geq 1$. Therefore $F(t) = U(t) = L(t) = H_q(t) = 1$ for any $t \geq 1$. This means that one may consider these functions as functions from $[0, 1] \rightarrow [0, 1]$, if one is only interested in the non-trivial part of them.

Lemma 4. *Suppose that A is a mixing basic set or a mixing Markov subset of the piecewise monotonic map T . Then for every $\varepsilon > 0$ there is $(u, v) \in \mathcal{R}(A)$ such that $F_{u,v}$ is in the upper ε -neighbourhood of L .*

Proof. Choose $t_0 = 0 < t_1 < \dots < t_r = 1$ with $t_j - t_{j-1} < \frac{\varepsilon}{2}$ for $1 \leq j \leq r$. By the definition of L there are $(x_j, y_j) \in \mathcal{Q}(A)$ with $F_{x_j, y_j}(t_j) < L(t_j) + \frac{\varepsilon}{4}$ for $1 \leq j \leq r$. Using the definition of $\mathcal{Q}(A)$ there is an $m \geq 1$ such that x_j and y_j are in A_m for $1 \leq j \leq r$. If A is a mixing basic set we apply Lemma 1 and find a Markov subset X of A with x_j and y_j in X for $1 \leq j \leq r$. Otherwise A is a Markov subset, and we set $X = A$. Let α be the minimal distance in the Hausdorff metric between $U \cap X$ and $V \cap X$ for two different $U, V \in \mathcal{Z}$. Since X is a Markov subset there is an $n \geq 1$ with $X \subseteq A_n$ and hence $\alpha > 0$. As X is uncountable, there is a $w \in X$ with $\lim_{n \rightarrow \infty} |Z_n(w)| = 0$, where $|I|$ denotes the length of the interval I . Set $u_l(n) := \max \{k \leq n : |T^j(Z_n(w))| < \frac{1}{l} \text{ for } 0 \leq j \leq k\}$. Since T is continuous on the intervals in \mathcal{Z} , we have that $\lim_{n \rightarrow \infty} u_l(n) = \infty$ for all $l \geq 1$.

Let M be the constant in (b) of Lemma 2 for the set X and for $\min \{\frac{\varepsilon}{4}, \frac{\alpha}{2}\}$ instead of ε . Set $x_{j+lr} := x_j$, $y_{j+lr} = y_j$ and $t_{j+lr} = t_j$ for $l \geq 1$ and $1 \leq j \leq r$. We choose integers $a_1 \leq b_1 < c_1 \leq d_1 < a_2 \leq b_2 < c_2 \leq d_2 < \dots$ in the following way. Define $a_1 := 0$. If a_k is defined, choose $b_k > \frac{4a_k}{\varepsilon}$ such that $\frac{1}{b_k - a_k} N_{x_k, y_k}(0, b_k - a_k, t_k) < L(t_k) + \frac{\varepsilon}{4}$ holds. This is possible, since we have $F_{x_k, y_k}(t_k) < L(t_k) + \frac{\varepsilon}{4}$. Then set $c_k := b_k + M$, choose d_k such that $u_k(d_k - c_k) \geq (k-1)c_k$, and set $a_{k+1} = d_k + M$.

By Lemma 2 there are $u, v \in X$ satisfying

$$(11) \quad |T^n(u) - T^{n-a_k}(x_k)| \leq \frac{\varepsilon}{4} \quad \text{and} \quad |T^n(v) - T^{n-a_k}(y_k)| \leq \frac{\varepsilon}{4}$$

for $a_k \leq n \leq b_k$, $k \geq 1$,

$$(12) \quad |T^n(u) - T^{n-c_k}(w)| \leq \frac{\alpha}{2} \quad \text{and} \quad |T^n(v) - T^{n-c_k}(w)| \leq \frac{\alpha}{2}$$

for $c_k \leq n \leq d_k$, $k \geq 1$. Now (11) and the definition of b_k give

$$\begin{aligned} \frac{1}{b_k} N_{u,v}(0, b_k, t_k - \frac{\varepsilon}{2}) &\leq \frac{a_k}{b_k} + \frac{1}{b_k} N_{x_k, y_k}(0, b_k - a_k, t_k) < \\ &< \frac{\varepsilon}{4} + L(t_k) + \frac{\varepsilon}{4} = L(t_k) + \frac{\varepsilon}{2}. \end{aligned}$$

Using the definition of t_k for $k > r$ this implies $F_{u,v}(t_j - \frac{\varepsilon}{2}) \leq L(t_j) + \frac{\varepsilon}{2} < L(t_j) + \varepsilon$ for $1 \leq j \leq r$. If $t \in (0, 1 - \frac{\varepsilon}{2}]$, then there is a $j \in \{1, 2, \dots, r\}$ with $t \in (t_j - \varepsilon, t_j - \frac{\varepsilon}{2}]$, which implies $F_{u,v}(t) \leq F_{u,v}(t_j - \frac{\varepsilon}{2}) < L(t_j) + \varepsilon$. For $t \in (1 - \frac{\varepsilon}{2}, 1]$ we have $F_{u,v}(t) \leq 1 = L(1) < L(1) + \varepsilon$. Hence for every $t \in (0, 1]$ there is a t_j with $|t - t_j| < \varepsilon$ and $F_{u,v}(t) < L(t_j) + \varepsilon$. This means that $F_{u,v}$ is in the upper ε -neighbourhood of L .

It follows from (12) and from the definition of α that $T^n(u)$ and $T^n(v)$ are in the same element of \mathcal{Z} as $T^{n-c_k}(w)$ for $c_k \leq n \leq d_k$. This implies that

$T^{c_k}(u)$ and $T^{c_k}(v)$ are in $Z_{d_k-c_k}(w)$. Because of $u_k(d_k - c_k) \geq (k-1)c_k$ we get $|T^n(u) - T^n(v)| < \frac{1}{k}$ for $c_k \leq n \leq kc_k$. This implies that $\frac{1}{kc_k}N_{u,v}(0, kc_k, \frac{1}{k}) \geq \frac{(k-1)c_k}{kc_k} = 1 - \frac{1}{k}$. Since this holds for all $k \geq 1$, we get $U_{u,v}(t) = 1$ for all $t \in (0, 1]$. Since X is a Markov subset there is an $n \geq 1$ with $X \subseteq A_n$. Hence we have also proved that $(u, v) \in \mathcal{R}(A)$. \square

If the piecewise monotonic map T has points in E which are eventually periodic or are attracted by periodic orbits, then let γ be the maximum of the periods of these periodic points. Otherwise set $\gamma := 0$.

Proposition 1. *Fix $q > \gamma$ and let B be a mixing basic set or a mixing Markov subset of the piecewise monotonic map T . Then $H_q(t) = L(t) = G(t)$ for all points $t \in (0, 1)$, where L is right continuous.*

Proof. Since $\mathcal{R}(B) \subseteq \mathcal{Q}(B)$ we have $L(t) \leq G(t)$ for all $t \in (0, 1]$. It follows from Lemma 4 that G is contained in every upper ε -neighbourhood of L . The intersection of all these upper ε -neighbourhoods is $\{(a, b) : 0 < a < 1, b \leq \lim_{z \rightarrow a^+} L(z)\}$, which must then contain the points $(t, G(t))$ for $0 < t < 1$. Therefore, we have $L(t) = G(t)$ for all points $t \in (0, 1)$, where L is right continuous.

By the definition of γ , for a periodic point x whose period is larger than γ there is an n such that x and an element of E cannot lie in the closure of the same element of \mathcal{Z}_n . This implies $\mathcal{P}_q(B) \subseteq \mathcal{Q}(B)$, and hence $L(t) \leq H_q(t)$ for all $t \in (0, 1]$. Choose $t \in (0, 1)$ such that L is right continuous in t . For every $\varepsilon \in (0, 1 - t)$ we have that $H_q(t) < L(t + \varepsilon) + \varepsilon$ by Lemma 3, which implies $H_q(t) \leq L(t)$. \square

3. The approximating distribution function of continuous piecewise monotonic maps

For piecewise monotonic maps we work with the approximating distribution function G instead of the distribution function F . Therefore we will prove in our next result that for continuous piecewise monotonic transformations these two functions essentially coincide. More exactly, they coincide in all points where F is right continuous.

Theorem 1. *Let $T : [0, 1] \rightarrow [0, 1]$ be a continuous and piecewise monotonic map with $h_{\text{top}}(T) > 0$. Suppose that B is a mixing basic set of T and let F be the minimal distribution function of T on B . Then the approximating distribution function G of T on B coincides with F in all points, where F is right continuous.*

Proof. Choose an arbitrary $\varepsilon > 0$ and fix some $q > \gamma$. Then there is a $\delta > 0$ such that $F(t) \leq F(s) < F(t) + \varepsilon$ for all $s \in [t, t + \delta)$. Now we choose an $s \in [t, t + \delta)$ such that F is continuous at s and L is right continuous at s . Then by Theorem 4.3 in [2] there is a pair of periodic points $u, v \in B$ with $F_{u,v}(s) < F(t) + \varepsilon$. Using the specification property we may assume that the periods of u and v are larger than q . Hence $H_q(s) \leq F_{u,v}(s)$. As L is right continuous at s Proposition 1 implies that $G(s) = H_q(s)$. Therefore we get $F(t) \leq G(t) \leq G(s) = H_q(s) \leq F_{u,v}(s) < F(t) + \varepsilon$. Since $\varepsilon > 0$ was arbitrary this implies $F(t) = G(t)$. \square

4. Stability results for the approximating distribution function of piecewise monotonic maps

Next we show that for (not necessarily continuous) piecewise monotonic maps the approximating distribution function is upper semi-continuous. If $\varepsilon > 0$ then a piecewise monotonic map \tilde{T} with respect to $\tilde{c}_0 = 0 < \tilde{c}_1 < \dots < \tilde{c}_{\tilde{N}} = 1$ is called to be ε -close to a piecewise monotonic map T with respect to $c_0 = 0 < c_1 < \dots < c_N = 1$, if $\tilde{N} = N$ (this means they have the same number of intervals of monotonicity) and $\tilde{T}|_{(\tilde{c}_{j-1}, \tilde{c}_j)}$ is in an ε -neighbourhood of $T|_{(c_{j-1}, c_j)}$ for all $j \in \{1, 2, \dots, N\}$.

Theorem 2. *Let $T : [0, 1] \rightarrow [0, 1]$ be a piecewise monotonic map satisfying $h_{\text{top}}(T) > 0$. Assume that B is a mixing basic set of T and let G be the approximating distribution function of T on B . Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every piecewise monotonic map \tilde{T} which is δ -close to T has a mixing basic set \tilde{B} satisfying that \tilde{G} is in an upper ε -neighbourhood of T , where \tilde{G} is the approximating distribution function of \tilde{T} on \tilde{B} .*

Proof. Fix an $\varepsilon > 0$ and a $q > \gamma$. Since G is increasing we can choose $a_0 = 0 < t_1 < a_1 < t_2 < a_2 < \dots < a_{k-1} < t_k < a_k = 1$ with $a_j - a_{j-1} < \frac{\varepsilon}{5}$, $G(t) > G(t_j) - \frac{\varepsilon}{3}$ for all $t \in (a_{j-1}, a_j]$ and L is right continuous at t_j for $1 \leq j \leq k$. Then $G(t_j) = H_q(t_j)$ for $1 \leq j \leq k$ by Proposition 1. Hence there exist periodic points x_j and y_j of period at least q with $F_{x_j, y_j}(t_j) < G(t_j) + \frac{\varepsilon}{3}$. We choose a refinement \mathcal{Y} of \mathcal{Z} with $|\mathcal{Y}| < \frac{\varepsilon}{5}$, and $\bigcap_{n=1}^{\infty} \overline{Y_n(x_j)} = \{x_j\}$ and $\bigcap_{n=1}^{\infty} \overline{Y_n(y_j)} = \{y_j\}$ for every $j \in \{1, 2, \dots, k\}$, where $\mathcal{Y}_n := \bigvee_{j=0}^{n-1} T^{-j} \mathcal{Y}$ and $Y_n(x)$ denotes an element of \mathcal{Y}_n with $x \in \overline{Y_n(x)}$ and $Y_n(x) \cap B \neq \emptyset$. From Theorem 8 in [5] we get that the points x_j and y_j can be represented by periodic paths in the Markov diagram $(\mathcal{D}, \rightarrow)$ of T with respect to \mathcal{Y} . Furthermore there exists an aperiodic maximal irreducible $\mathcal{C} \subseteq \mathcal{D}$ with $B =$

$K(\mathcal{C})$. Therefore using also Lemma 5 of [7] there is an $r \in \mathbb{N}$ such that any $x \in M$ can be represented by a periodic path in \mathcal{D}_r and for every variant $(\mathcal{A}, \rightarrow)$ of \mathcal{D} there is an aperiodic irreducible $\mathcal{C}_1 \subseteq \mathcal{A}_r$ with $A(\mathcal{C}_1) \subseteq \mathcal{C}$ and each $x \in M$ is represented by a periodic path in \mathcal{C}_1 .

By Lemma 6 in [7] there exists a $\delta > 0$ such that every piecewise monotonic map \tilde{T} which is δ -close to T satisfies the following properties. There exists a variant $(\mathcal{A}, \rightarrow)$ of the Markov diagram of T , a finite partition $\tilde{\mathcal{Y}}$ refining the partition of intervals of monotonicity of \tilde{T} , a variant $(\tilde{\mathcal{A}}, \rightarrow)$ of the Markov diagram of \tilde{T} with respect to $\tilde{\mathcal{Y}}$, and an injective function $\varphi : \mathcal{A}_r \rightarrow \tilde{\mathcal{A}}_r$ with $c \rightarrow d$ in \mathcal{A} is equivalent to $\varphi(c) \rightarrow \varphi(d)$ in $\tilde{\mathcal{A}}$ and $\tilde{A}(\varphi(c))$ is $\frac{\varepsilon}{5}$ -close to $A(c)$ in the Hausdorff-metric for all $c \in \mathcal{A}_r$.

Then $\varphi(\mathcal{C}_1)$ is an aperiodic irreducible subset of $\tilde{\mathcal{A}}_r$. Hence it is contained in an aperiodic maximal irreducible $\tilde{\mathcal{C}} \subseteq \tilde{\mathcal{A}}$ and $\tilde{B} := \tilde{K}(\tilde{\mathcal{C}})$ is a mixing basic set of \tilde{T} . Let $x \in M$. Then there is an $n \geq 1$ such that x is represented by a path $c_0 \rightarrow c_1 \rightarrow c_2 \rightarrow \dots$ in $\mathcal{C}_1 \subseteq \mathcal{A}_r$ with $c_{n+k} = c_k$ for all k . Therefore $\varphi(c_0) \rightarrow \varphi(c_1) \rightarrow \varphi(c_2) \rightarrow \dots$ is a periodic path in $\varphi(\mathcal{C}_1) \subseteq \tilde{\mathcal{A}}_r$. Hence it represents an $\tilde{x} \in \tilde{B}$, and we obtain that $\tilde{T}^n \tilde{x} = \tilde{x}$. Moreover we have $|\tilde{x} - x| < \frac{2\varepsilon}{5}$. Since $\tilde{T}^s \tilde{x} \in \varphi(Y(T^s x))$ we get also $|\tilde{T}^s \tilde{x} - T^s x| < \frac{2\varepsilon}{5}$ for all $s \geq 0$. This implies that $\tilde{F}_{\tilde{x}_j, \tilde{y}_j}(t_j - \frac{4\varepsilon}{5}) \leq F_{x_j, y_j}(t_j)$ for all $j \in \{1, 2, \dots, k\}$. Using Lemma 1 and Lemma 2 we obtain that there are $u_j, v_j \in \tilde{B}$ with $(u_j, v_j) \in \tilde{\mathcal{Q}}(\tilde{B})$ and $\tilde{F}_{u_j, v_j}(t_j - \frac{4\varepsilon}{5}) < \tilde{F}_{\tilde{x}_j, \tilde{y}_j}(t_j - \frac{4\varepsilon}{5}) + \frac{\varepsilon}{3}$. Therefore

$$\begin{aligned} \tilde{L}\left(t_j - \frac{4\varepsilon}{5}\right) &\leq \tilde{F}_{u_j, v_j}\left(t_j - \frac{4\varepsilon}{5}\right) < \tilde{F}_{\tilde{x}_j, \tilde{y}_j}\left(t_j - \frac{4\varepsilon}{5}\right) + \frac{\varepsilon}{3} \leq \\ &\leq F_{x_j, y_j}(t_j) + \frac{\varepsilon}{3} < G(t_j) + \frac{2\varepsilon}{3} \end{aligned}$$

for all $j \in \{1, 2, \dots, k\}$. Now let $t \in (0, 1)$. Choose $j \in \{1, 2, \dots, k\}$ so that $a_{j-1} < t \leq a_j$. Note that $a_j < t_j + \frac{\varepsilon}{5}$. Then there exists an $s > t$ with $s - \frac{\varepsilon}{5} < t_j$ such that \tilde{L} is right continuous at $s - \varepsilon$. By Proposition 1 we get $\tilde{G}(s - \varepsilon) = \tilde{L}(s - \varepsilon)$. Observe that $t - \varepsilon < s - \varepsilon < t_j - \frac{4\varepsilon}{5}$ and $G(t_j) < G(t) + \frac{\varepsilon}{3}$. Therefore we obtain

$$\tilde{G}(t - \varepsilon) \leq \tilde{G}(s - \varepsilon) = \tilde{L}(s - \varepsilon) \leq \tilde{L}\left(t_j - \frac{4\varepsilon}{5}\right) < G(t_j) + \frac{2\varepsilon}{3} < G(t) + \varepsilon.$$

This shows that \tilde{G} is in an upper ε -neighbourhood of G . \square

In our next result we show that the approximating distribution function does not change essentially if one adds artificially intervals of monotonicity. Observe that there are at most countably many points where G is not right

continuous, since G is increasing. Hence $G_n(t) \rightarrow G(t)$ for all t where G is right continuous implies that $\lim_{n \rightarrow \infty} G_n(t) = G(t)$ for all $t \in \mathbb{R} \setminus E$ where E is an at most countable set (note that the distribution functions are equal to 0 for $t \leq 0$ and equal to 1 for $t \geq 1$). In particular, $G_n \rightarrow G$ almost everywhere (with respect to the Lebesgue measure).

Theorem 3. *Suppose that $T : [0, 1] \rightarrow [0, 1]$ is a piecewise monotonic map with $h_{\text{top}}(T) > 0$, and assume that B is a mixing basic set of T . Let G be the approximating distribution function of T on B . Moreover, suppose that $u \in \mathbb{N}$ and $x_1, x_2, \dots, x_u \in [0, 1]$, and for $n \geq 1$ let $J_n \subseteq [0, 1]$ be a union of at most u pairwise disjoint open intervals such that $J_{n+1} \subseteq J_n$ holds for all $n \geq 1$ and $\bigcap_{n=1}^{\infty} \overline{J_n} = \{x_1, x_2, \dots, x_u\}$. Assume that G_n is the approximating distribution function of T on $\bigcap_{k=0}^{\infty} \overline{B \setminus (T^{-k}(J_n))}$. Then $\lim_{n \rightarrow \infty} G_n(t) = G(t)$ for all $t \in (0, 1)$ where G is right continuous.*

Proof. It is obvious that $G_n(t) \geq G(t)$ for all n and all t . Moreover, for any t the sequence $(G_n(t))$ is decreasing. Hence it remains to show that every $t \in (0, 1)$ where G is right continuous, and for every $\varepsilon > 0$ there is an n with $G_n(t) < G(t) + \varepsilon$. For $n \in \mathbb{N}$ set $\tilde{B}_n := \bigcap_{k=0}^{\infty} \overline{B \setminus (T^{-k}(J_n))}$. Denote the endpoints of intervals of monotonicity of T by c_0, c_1, \dots, c_N . Furthermore let L_n be the function defined in (8) for \tilde{B}_n . Again it is obvious that the sequence $(L_n(t))$ is decreasing for any t and $L_n(t) \geq L(t)$ for any t and any n .

Next we define a refinement \mathcal{Y} of \mathcal{Z} . Denote by \mathcal{Y} the family of all maximal open subintervals of $[0, 1] \setminus \{c_0, c_1, \dots, c_N, x_1, x_2, \dots, x_u\}$. If a point in $\{c_0, c_1, \dots, c_N, x_1, x_2, \dots, x_u\}$ is eventually periodic or attracted by a periodic orbit, then let γ_0 be the maximum of the periods of these periodic points. Otherwise set $\gamma_0 := 0$. Now fix $q > \gamma_0$.

Let $t \in (0, 1)$ be so that G is right continuous at t and let $\varepsilon > 0$. As G is right continuous at t there is a $\delta > 0$ with $G(s) < G(t) + \frac{\varepsilon}{2}$ for all $s \in (t, t + \delta)$. Choose $t_1 \in (t, t + \delta)$ so that L is right continuous at t_1 . From Proposition 1 we obtain that $G(t_1) = H_q(t_1)$. Therefore there exist periodic points $x, y \in B$ whose periods are at least q , such that

$$F_{x,y}(t_1) < H_q(t_1) + \frac{\varepsilon}{2} = G(t_1) + \frac{\varepsilon}{2} < G(t) + \varepsilon.$$

By the choice of q there is an m with $x, y \notin \bigcup_{k=0}^{\infty} T^{-k}(C_m)$. Moreover, again by the choice of q there is an $l \in \mathbb{N}$ with $x \in \tilde{B}_l$ (otherwise $T^k x \in J_l$ for some k) and $y \in \tilde{B}_l$. Hence $(x, y) \in \mathcal{Q}(\tilde{B}_l)$.

Using Theorem 8 of [5] we obtain that the x and y can be represented by periodic paths in the Markov diagram $(\mathcal{D}, \rightarrow)$ of T with respect to \mathcal{Y} .

Moreover, there is an aperiodic maximal irreducible $\mathcal{C} \subseteq \mathcal{D}$ with $B = K(\mathcal{C})$. Therefore using also Lemma 5 of [7] there exists an r such that x and y can be represented by a periodic path in \mathcal{D}_r and for every variant $(\mathcal{A}, \rightarrow)$ of \mathcal{D} there is an aperiodic irreducible $\mathcal{C}_1 \subseteq \mathcal{A}_r$ with $A(\mathcal{C}_1) \subseteq \mathcal{C}$ and x and y can be represented by a periodic path in \mathcal{C}_1 . For $n \in \mathbb{N}$ define $X_n := \bigcap_{k=0}^{\infty} [0, 1] \setminus (T^{-k}(J_n))$.

Because of our assumptions “ J_n converges to $\{x_1, x_2, \dots, x_u\}$ ” in the sense described in [9]. Hence Lemma 2 in [10] and Lemma 1 in [9] yield the existence of an $n \geq l$ and variants $(\mathcal{A}, \rightarrow)$ of \mathcal{D} and $(\tilde{\mathcal{A}}, \rightarrow)$ of the Markov diagram of $T|_{X_n}$ and an injective function $\varphi: \mathcal{A}_r \rightarrow \tilde{\mathcal{A}}_r$ with $c \rightarrow d$ in \mathcal{A} is equivalent to $\varphi(c) \rightarrow \varphi(d)$ in $\tilde{\mathcal{A}}$. Moreover, in this case we get $\tilde{A}(\varphi(c)) \subseteq A(c)$ for all $c \in \mathcal{A}_r$. Then there is an aperiodic irreducible $\mathcal{C}_1 \subseteq \mathcal{A}_r$ with $A(\mathcal{C}_1) \subseteq \mathcal{C}$ and x and y can be represented by a periodic path in \mathcal{C}_1 . Therefore $\varphi(\mathcal{C}_1)$ is aperiodic and irreducible, and x and y can be represented by a path in $\varphi(\mathcal{C}_1)$. This yields that $\varphi(\mathcal{C}_1)$ is contained in an aperiodic maximal irreducible $\tilde{\mathcal{C}} \subseteq \tilde{\mathcal{A}}$ and therefore $\tilde{K}(\tilde{\mathcal{C}})$ is a mixing basic set of $T|_{X_n}$. By Lemma 1 it contains an aperiodic Markov subset $X \subseteq \tilde{B}_n$ with $x, y \in X$. Then $(x, y) \in \mathcal{Q}(X)$. Denote by \tilde{G} , respectively \tilde{L} , the functions defined in (4), respectively (8), for the set X . We get $\tilde{L}(t_1) \leq F_{x,y}(t_1) < G(t) + \varepsilon$. Now choose an s with $t < s < t_1$ such that \tilde{L} is right continuous at s . As \tilde{L} is increasing we get $\tilde{L}(s) \leq \tilde{L}(t_1) < G(t) + \varepsilon$. Furthermore Proposition 1 implies that $\tilde{G}(s) = \tilde{L}(s)$, which leads to $\tilde{G}(s) < G(t) + \varepsilon$. Since $X \subseteq \tilde{B}_n$ we get $G_n(s) \leq \tilde{G}(s)$. Now $G_n(t) \leq G_n(s) \leq \tilde{G}(s) < G(t) + \varepsilon$ completes the proof. \square

References

- [1] M. Babilonová, *Distributional chaos for triangular maps*, Ann. Math. Sil. **13** (1999), 33–38.
- [2] F. Balibrea, B. Schweizer, A. Sklar, J. Smítal, *Generalized specification property and distributional chaos*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), 1683–1694.
- [3] F. Balibrea, J. Smítal, M. Štefánková, *The three versions of distributional chaos*, Chaos Solitons Fractals **23** (2005), 1581–1583.
- [4] A. Blokh, *The ‘spectral’ decomposition for one-dimensional maps*, in *Dynamics Reported, Expositions in Dynamical Systems, vol. 4* (eds.: C. K. R. T. Jones, U. Kirchner, H. O. Walther), Springer, Berlin, 1995, pp. 1–59.
- [5] F. Hofbauer, *Piecewise invertible dynamical systems*, Probab. Theory Related Fields **72** (1986), 359–386.

- [6] R. Hric, M. Málek, *Omega limit sets and distributional chaos on graphs*, Topology Appl. **153** (2006), 2469–2475.
- [7] P. Raith, *Continuity of the Hausdorff dimension for piecewise monotonic maps*, Israel J. Math. **80** (1992), 97–133.
- [8] P. Raith, *Continuity of the Hausdorff dimension for invariant subsets of interval maps*, Acta Math. Univ. Comenian. **63** (1994), 39–53.
- [9] P. Raith, *The behaviour of the nonwandering set of a piecewise monotonic interval map under small perturbations*, Math. Bohem. **122** (1997), 37–55.
- [10] P. Raith, *The dynamics of piecewise monotonic maps under small perturbations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **24** (1997), 783–811.
- [11] B. Schweizer, A. Sklar, J. Smítal, *Distributional (and other) chaos and its measurement*, Real Anal. Exchange **26** (2000/2001), 495–524.
- [12] B. Schweizer, J. Smítal, *Measures of chaos and a spectral decomposition of dynamical systems on the interval*, Trans. Amer. Math. Soc. **344** (1994), 737–754.

MICHAL MÁLEK

Matematický ústav, Slezská univerzita v Opavě, Na Rybníčku 1, 746 01 Opava, Czech Republic

e-mail address: `michal.malek@math.slu.cz`

PETER RAITH

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

e-mail address: `peter.raith@univie.ac.at`